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Construction Of Almost Unbiased Estimator for Population Median Using Neutrosophic Information

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Abstract: This paper introduces the development of an almost unbiased estimator for estimating the unknown population median of the primary variable. The proposed estimator leverages neutrosophic auxiliary information and employs simple random sampling without replacement (SRSWOR). In order to establish the efficacy of the proposed method, we derive the mathematical formulations for the mean square error (MSE), bias, and the minimum MSE of the estimator, providing approximations up to the first order. These derivations allow for a comprehensive analysis of the estimator's performance and its suitability for accurate population median estimation. To validate the theoretical results, we conduct an empirical study using two real-world datasets, ensuring that the proposed estimator's behavior aligns with theoretical predictions in practical scenarios. The study shows that the proposed estimator remains nearly unbiased, with minimal bias when approximated to the first order. This result further demonstrates that the estimator performs robustly across various data conditions. In comparison to existing estimators, the proposed estimator outperforms the others in terms of efficiency, as evidenced by the MSE and PRE values derived. The proposed method not only minimizes bias but also provides more accurate population median estimates with reduced estimation error, making it a more reliable tool in the context of uncertain or incomplete data, where traditional estimators might fall short. By bridging the gap between classical estimation techniques and modern methods that account for uncertainty, the proposed estimator offers a significant advancement in the field of statistical estimation, particularly in real-world applications involving uncertain datasets. The findings presented in this study contribute to the growing body of knowledge in statistical estimation, particularly in the use of neutrosophic information for enhancing estimator accuracy. Furthermore, the results provide a valuable foundation for future research aimed at developing more efficient and reliable statistical estimators for a variety of practical applications.

Keyword: Neutrosophic auxiliary information, Population median, Almost unbiased estimators, Mean square error, Percent relative efficiency, Exponential estimator, Logarithmic estimator.

1. Introduction

Median estimators in sampling are statistical methods used to estimate the population median based on sample data. These estimators provide a central measure of location, especially useful in situations where the data may contain

outliers or be skewed. These estimators depend on the population's characteristics and the sampling strategy. The median is a more accurate measure of central tendency than other measures for highly skewed distributions, like individual consumption or income, where extreme values significantly affect the mean. Many studies have made significant contribution in estimating the median of a finite population. [1] has studied about sample median estimator in case of various sampling methods. [2] suggested an idea about estimating the median of study variable using known median of auxiliary variable. [3] proposed a family of estimator for population median of study variable using auxiliary variable. Several researchers have proposed estimators for unknown population median [4-10].

The neutrosophic statistical technique is used to analyze data that contains uncertainty and ambiguity, making it suitable for handling imprecise or indeterminate data. This technique is generalized form of fuzzy concept. An uncertain or imprecise idea is commonly referred to as a fuzzy concept. The concept of fuzzy logic is proposed by L.A. Zadeh [11]. In order to properly handle uncertainty and imprecision, fuzzy concepts are frequently utilized in domains such as artificial intelligence and fuzzy logic. Fuzzy ideas can be helpful in situations that require flexibility and approximation, but they can become problematic when precision and accuracy are needed. For example, in medical analysis, vague terms like "moderate pain" or "mild fever" can help doctors quickly figure out how sick a patient is without having to take exact measurements. But in pharmaceuticals, where exact doses are essential, using vague terms can lead to mistakes and bad results. Fuzzy concept mainly deals with degrees of truth and falsity. On the other hand, neutrosophic concept is given by Florentin Smarandache[12] which is generalization of the fuzzy concept. It seeks to deal with imprecision, ambiguity, and uncertainty more deeply. Neutrosophic logic incorporates an additional component to model indeterminacy. This makes it especially appropriate in cases when there is insufficient, inaccurate, or incomplete information.

[13] constructed an almost unbiased estimator in case of neutrosophic study for population mean. [14] developed robust estimators for the neutrosophic finite median using auxiliary variables to improve accuracy in uncertain data analysis.. [15] suggested a generalized class of estimators for the population median, utilizing auxiliary information to improve estimation accuracy. [16] suggested a unique estimation for the finite population mean using the auxiliary variable's median. [17] promoted the use of Neutrosophic statistics for analyzing uncertain systems.[18-22] and more Researchers suggested about neutrosophic concept.

This paper is organized into sections. Section 1 provides an introduction and a comprehensive review of the relevant literature. Section 2 discusses the observations and key terminologies related to neutrosophic statistics. Section 3 outlines some existing neutrosophic estimators. Section 4 presents the proposed almost unbiased estimator. Section 5 includes the empirical study to validate the theoretical concepts. Section 6 and section 7 present the results of the proposed estimators along with a detailed discussion and conclusion.

2. Symbolic representations for neutrosophic data

Various neutrosophic observations were provided, such as quantitative information suggesting that a number might fall inside a vague range (a,b). There are multiple ways to express the interval value of a neutrosophic number. [13] and [14] describe neutrosophic interval values as $Z_n = Z_L + Z_U I_N$ where $I_N \in \left[I_L, I_U\right]$. Z_L and Z_U represent the lower and upper values of the neutrosophic variable $Z_N \,.\, I_N$ reflecting the indeterminacy level with range 0 to 1. Here we see that the neutrosophic observations show as in an interval form; hence further calculation will provide us with an interval instead of a point value.

Let a population consists $U_N = (U_{1N}, \mathrm{U}_{2N}, \mathrm{U}_{3N}, ...\mathrm{U}_{\mathrm{NN}})$, N_N units. Let $y_{\mathrm{N}} \in [y_L, y_U]$ and $x_N \in [x_{L,} x_U]$ be the i[th] sampled values of the neutrosophic study variable y_{iN} and the auxiliary variables x_N respectively.

Let a sample of size $n_N \in (n_L, n_U)$ is chosen from U_N. The sample and population medians of the neutrosophic study and the auxiliary variable are denoted by $\hat{M}_{yN}$ and $\hat{M}_{xN}$, and M_{yN} and M_{xN} with probability density functions of $f_{yN}(M_{yN})$ and $f_{xN}(M_{xN})$.

Where,

$\hat{M}_{yN} \in \left(\hat{M}_{yL}, \hat{M}_{yU}\right)$ and $\hat{M}_{xN} \in \left(\hat{M}_{xL}, \hat{M}_{xU}\right)$.

The correlation coefficient between the M_{yN} and M_{xN} is represented by ρ_{yxN} and is defined as $\rho\left(M_{yN}, M_{xN}\right) = \left(4P_{11}(y_N, x_N) - 1\right)$, where, $P_{11}(y_N, x_N) = P(y_N \le M_{yN} \cap x_N \le M_{xN})$.

To derive expressions for bias and MSE, we have used the following error terms

$$e_{yN} = \left(\frac{\hat{M}_{yN} - M_{yN}}{M_{yN}}\right),\ e_{xN} = \left(\frac{\hat{M}_{xN} - M_{xN}}{M_{xN}}\right),\ e_{yn} \in \left[e_{yL}, e_{yU}\right],\ e_{xn} \in \left[e_{xL}, e_{xU}\right]$$

$$E(e_{yN}^2) = \lambda_N C_{MyN}^2\ ,\ E(e_{xN}^2) = \lambda_N C_{MxN}^2,\ E(e_{yN} e_{xN}) = \lambda_N C_{MyxN},$$

$$E(e_{yN}) = E(e_{xN}) = 0\ .$$

$$C_{M_{yxN}} = \rho_{yxN} C_{M_{yN}} C_{M_{xN}},\ C_{MyN} = \frac{1}{M_{yN} f_{yN}(M_{yN})}\ ,\ C_{MxN} = \frac{1}{M_{xN} f_{xN}(M_{xN})},$$

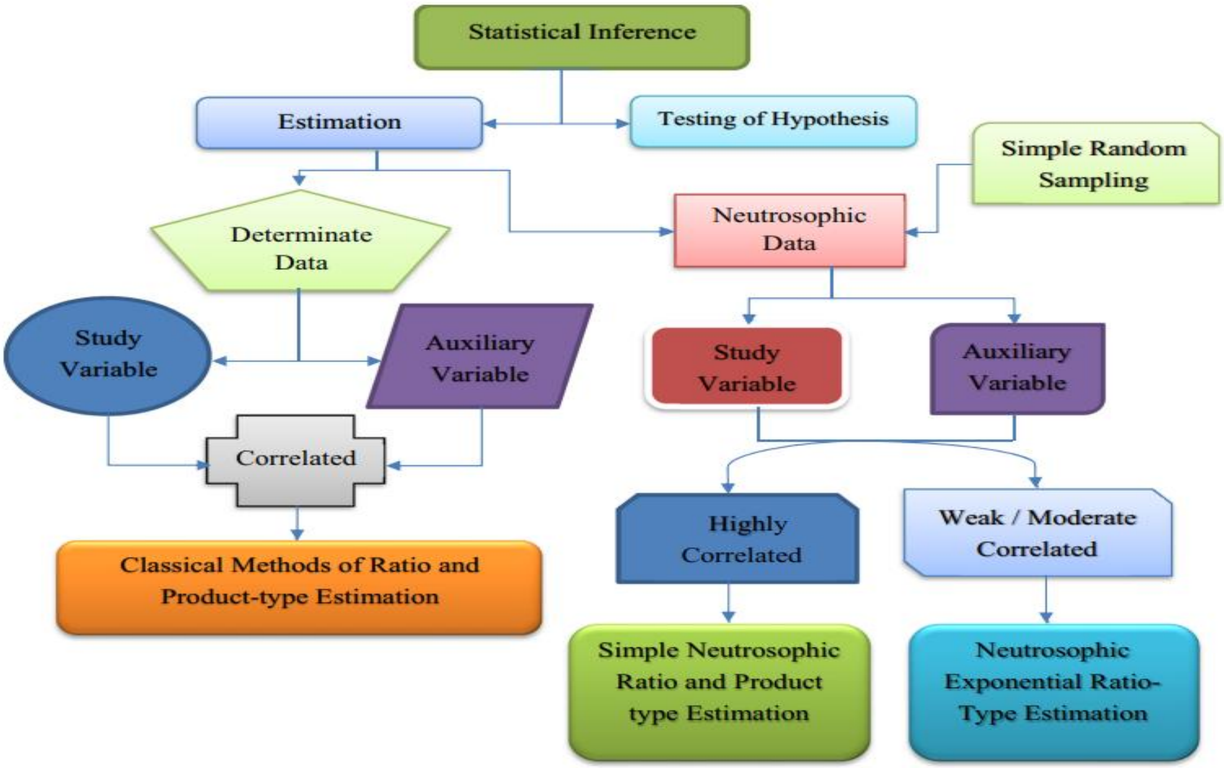
$$C_{M_{yxN}} \in \left[C_{M_{yxL}}, C_{M_{yxU}}\right], \rho_{yxN} \in \left[\rho_{yxL}, \rho_{yxU}\right], C_{M_{yN}} \in \left[C_{M_{yL}}, C_{M_{yU}}\right] \text{and}\ C_{M_{xN}} \in \left[C_{M_{xL}}, C_{MU}\right]$$

$$\lambda_N = \left(\frac{1}{4}\left\{\frac{1}{n_N} - \frac{1}{N_N}\right\}\right) \text{Where}\ \lambda_n = \frac{\left(1 - f_n\right)}{n_N},\ \text{where}\ \lambda_n \in \left(\lambda_L, \lambda_U\right),\ n_N \in \left(n_L, n_U\right).$$

Similarly, in neutrosophic terms, MSE and PRE are given as,

$$Bias(\bar{y}_n) \in \left[Bias_L, Bias_L\right]\ ,\ MSE\left(\bar{y}_n\right) \in \left[MSE_L, MSE_U\right] \text{and}\ PRE(\bar{y}_n) \in \left[PRE_L, PRE_U\right].$$

The flow chart as used by [20] shows the work flow of the parameter estimation.



3. Existing estimators

Motivated by [1], estimator of neutrosophic population median is given as,

$$t_{0N} = \hat{M}_{yN} \tag{1.1}$$

$$Bias(t_{0N}) = 0 \tag{1.2}$$

The mean square error of the estimator t_{0N} is given as-

$$MSE(t_{0N}) = M_{yN}^2 \lambda_N C_{M_{yN}}^2 \,. \tag{1.3}$$

Inspired by [3], We have introduced the neutrosophic exponential estimator as follows,

$$t_{1N} = \hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)^{a} \exp\left(\frac{\mathrm{b}(\hat{M}_{xN} - M_{xN})}{\hat{M}_{xN} + M_{xN}}\right) \tag{1.4}$$

Bias of the estimator t_{1N} is given as,

$$Bias(t_{1N}) = M_{yN}\lambda_N\left[\left(\left(\frac{a(a-1)}{2}\right) + \frac{ab}{2} - \frac{b}{4} + \frac{b^2}{8}\right)C_{MxN}^2 + aC_{MyxN}\right] \tag{1.5}$$

The MSE of the estimator t_{1N} is,

$$MSE(t_{1N})=M_{yN}^{2}\lambda_N\left[C_{MyN}^{2}+\left(a+\frac{b}{2}\right)^{2}C_{MxN}^{2}+2\left(a+\frac{b}{2}\right)C_{MyxN}\right]. \quad (1.6)$$

Inspired by[23], a neutrosophic log type estimator is proposed as

$$t_{2N}=\hat{M}_{yN}\left(1+\log\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\right) \quad (1.7)$$

Bias of the estimator t_{2N} is given as

$$Bias(t_{2N})=M_{yN}\left[\lambda_N C_{MyxN}-\frac{1}{2}C_{MxN}^{2}\right] \quad (1.8)$$

The MSE of the t_{2N} estimator is given as

$$MSE(t_{2N})=M_{yN}^{2}\lambda_N\left[C_{MyN}^{2}+C_{MxN}^{2}+2C_{MyxN}^{2}\right]. \quad (1.9)$$

4. Proposed estimators

Motivated by [1], [3] and [23], we have proposed improved neutrosophic estimators for estimating the neutrosophic finite population median. Let t_{0N} , t_{1N} and t_{2N} $\in$ L, Here, L represents the set of all possible estimators used for estimating the neutrosophic population median.

By definition, the set L is linear variety [24-25] if,

$$t_{hN}=\sum_{i=0}^{2}\alpha_{iN}t_{iN}\in \text{L} \quad (2.1)$$

For

$$\sum_{i=0}^{2}\alpha_{iN}=1,\ \alpha_{iN}\in \text{R}. \quad (2.2)$$

where $\alpha_{iN}\,(i=0,1,2)$ is the statistical constants and R denotes the set of real numbers.

The proposed neutrosophic estimator is given as, (where a and b are neutrosophic constants).

$$t_{hN}=\alpha_{0N}\hat{M}_{yN}+\alpha_{1N}\hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)^{a}\exp\left(\frac{\text{b}(\hat{M}_{xN}-M_{xN})}{(\hat{M}_{xN}+M_{xN})}\right)+\alpha_{2N}\,\hat{M}_{yN}\left(1+\log\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\right) \quad (2.3)$$

To obtain the bias and MSE of the estimator t_{hN} , we write t_{hN} in the form of error terms such as,

$$t_{hN}=M_{yN}(1+e_{yN})\left\{\alpha_{0N}+\alpha_{1N}(1+e_{xN})^{a}\exp\left(\frac{b}{2}e_{xN}-\frac{b}{4}e_{xN}^{2}\right)+\alpha_{2N}\left(1+\log(1+e_{xN})\right)\right\} \quad (2.4)$$

Expending the right-hand side of equation (2.4) and retaining terms up to second powers of e’s, we have,

$$t_{hN} = M_{yN}\left[\begin{array}{l} 1+e_{yN}+\left(\alpha_{1N}a+\alpha_{1N}\dfrac{b}{2}+\alpha_{2N}\right)e_{xN}+\left(\alpha_{1N}a+\alpha_{1N}\dfrac{b}{2}+\alpha_{2N}\right)e_{xN}e_{yN}+ \\ \left(\dfrac{a(a-1)}{2!}\alpha_{1N}+\dfrac{ab}{2}\alpha_{1N}-\dfrac{b}{4}\alpha_{1N}+\dfrac{b^2}{8}\alpha_{1N}-\dfrac{1}{2}\alpha_{2N}\right)e_{xN}^2 \end{array}\right] \tag{2.5}$$

Subtracting M_{yN} in equation (2.5) and then taking expectation of both sides, we get the bias of the estimator t_{hN}, up to the first order of approximation as,

$$t_{hN}-M_{yN} = M_{yN}\left[e_{yN}+H_N e_{xN}+H_N e_{xN}e_{yN}+\left(\frac{a(a-1)}{2!}\alpha_{1N}+\frac{ab}{2}\alpha_{1N}-\frac{b}{4}\alpha_{1N}+\frac{b^2}{8}\alpha_{1N}-\frac{1}{2}\alpha_{2N}\right)e_{xN}^2\right] \tag{2.6}$$

Let $\left(\alpha_{1N}a+\alpha_{1N}\dfrac{b}{2}+\alpha_{2N}\right)=H_N$.

$$Bias(t_{hN}) = M_{yN}\left[\left(\frac{a(a-1)}{2!}\alpha_{1N}+\frac{ab}{2}\alpha_{1N}-\frac{b}{4}\alpha_{1N}+\frac{b^2}{8}\alpha_{1N}-\frac{1}{2}\alpha_{2N}\right)\lambda_N C_{MxN}^2+H_N\lambda_N C_{MyxN}\right] \tag{2.7}$$

From equation (2.6), we have

$$t_{hN}-M_{yN} = M_{yN}(e_{yN}+H_N e_{xN}). \tag{2.8}$$

where, we take

$$\left(\alpha_{1N}a+\alpha_{1N}\frac{b}{2}+\alpha_{2N}\right)=H_N. \tag{2.9}$$

Squaring both sides of equation (2.8) and then taking expectations, we get the MSE of the neutrosophic estimator t_{hN}, up to the first order of approximation, as

$$MSE(t_{hN}) = M_{yN}^2\lambda_N\left[C_{MyN}^2+H_N^2C_{MxN}^2+2H_N C_{MyxN}\right]. \tag{2.10}$$

Differentiate equation (2.10) with respect to H_N and put equal to zero, we get the value of H_N.

MSE (t_{hN}) is minimum, when

$$H_N = \frac{-\rho_{yxN}C_{MyN}}{C_{MxN}}. \tag{2.11}$$

Putting this value of $H_N = \dfrac{-\rho_{yxN}C_{M_{yN}}}{C_{M_{xN}}}$ in equation (2.10) we get the Min. MSE of the neutrosophic estimator t_{hN} as-

$$Min.MSE(t_{hN}) = M_{yN}^2\lambda_N C_{MyN}^2(1-\rho_{yxN}^2). \tag{2.12}$$

From equation (2.2) and (2.9), we have only two equations in three unknowns. It is not possible to find the unique values for α_{iN} 's , (i = 0, 1, 2). To get unique values of α_{iN} 's, we shall impose the linear restriction.

$$\sum_{i=0}^{2}\alpha_{iN}B(t_{iN})=0\,. \tag{2.13}$$

Such that,

$$\alpha_{0N}B(t_{0N})+\alpha_{1N}B(t_{1N})+\alpha_{2N}B(t_{2N})=0 \tag{2.14}$$

where $B(t_{iN})$ denotes the bias in the i^{th} estimator.

Equations (2.2), (2.9) and (2.13) can be written in the matrix form as,

$$\begin{bmatrix}1 & 1 & 1\\ 0 & \left(a+\dfrac{b}{2}\right) & 1\\ 0 & B(t_{1N}) & B(t_{2N})\end{bmatrix}\begin{bmatrix}\alpha_{0N}\\ \alpha_{1N}\\ \alpha_{2N}\end{bmatrix}=\begin{bmatrix}1\\ H_N\\ 0\end{bmatrix} \tag{2.15}$$

From the system of equations (2.15), we get the unique values of α_i 's (i =0, 1, 2) as,

$$\alpha_{0N}=\frac{\left(a+\dfrac{b}{2}\right)B(t_{2N})-B(t_{1N})-H_NB(t_{2N})+H_NB(t_{1N})}{\left(a+\dfrac{b}{2}\right)B(t_{2N})-B(t_{1N})}\,. \tag{2.16}$$

$$\alpha_{1N}=\frac{H_NB(t_{2N})}{\left(a+\dfrac{b}{2}\right)B(t_{2N})-B(t_{1N})}\,. \tag{2.17}$$

$$\alpha_{2N}=\frac{-H_NB(t_{1N})}{\left(a+\dfrac{b}{2}\right)B(t_{2N})-B(t_{1N})}\,. \tag{2.18}$$

such that

$$\alpha_{0N}+\alpha_{1N}+\alpha_{2N}=1\,. \tag{2.19}$$

Use of these α_{iN} 's (i =0, 1, 2) in equations, remove the bias up to terms of order $o(n^{-1})$.

Member of Estimators on different values of scalars and constant.

Members of estimators	α_0	α_1	α_2	a	b
$t_{0N} = \hat{M}_{yN}$	1	0	0	0	0
$t_{1N} = \hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)$	0	1	0	1	0
$t_{2N} = \hat{M}_{yN}\left(\frac{M_{xN}}{\hat{M}_{xN}}\right)$	0	1	0	-1	0
$t_{3N} = \hat{M}_{yN}\exp\left(\frac{\hat{M}_{xN} - M_{xN}}{\hat{M}_{xN} + M_{xN}}\right)$	0	1	0	0	1
$t_{4N} = \hat{M}_{yN}\exp\left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}}\right)$	0	1	0	0	-1
$t_{5N} = \hat{M}_{yN}\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\exp\left(\frac{\hat{M}_{xN} - M_{xN}}{\hat{M}_{xN} + M_{xN}}\right)$	0	1	0	1	1
$t_{6N} = \hat{M}_{yN}\left(\frac{M_{xN}}{\hat{M}_{xN}}\right)\exp\left(\frac{M_{xN} - \hat{M}_{xN}}{\hat{M}_{xN} + M_{xN}}\right)$	0	1	0	-1	-1
$t_{7N} = \hat{M}_{yN}\left(1 + \log\left(\frac{\hat{M}_{xN}}{M_{xN}}\right)\right)$	0	0	1	0	0

Bias and MSE of the members of the estimators.

Bias of the Estimators	MSE of the Estimators
$Bias(t_{0N}) = 0$	$MSE(t_{0N}) = M_{yN}^2 \lambda_N C_{yN}^2$
$Bias(t_{1N}) = M_{yN} \lambda_N C_{M_{yxN}}$	$MSE(t_{1N}) = M_{yN}^2 \lambda_N \left(C_{M_{yN}}^2 + C_{M_{xN}}^2 + 2C_{M_{yxN}}\right)$
$Bias(t_{2N}) = M_{yN} \lambda_N \left(C_{M_{yN}}^2 - C_{M_{yxN}}\right)$	$MSE(t_{2N}) = M_{yN}^2 \lambda_N \left(C_{M_{yN}}^2 + C_{M_{xN}}^2 - 2C_{M_{yxN}}\right)$

$Bias(t_{3N})=M_{yN}\lambda_N\left[\frac{1}{2}C_{M_{yxN}}-\frac{1}{8}C^2_{M_{xN}}\right]$	$MSE(t_{3N})=M^2_{yN}\lambda_N\left(C^2_{M_{yN}}+\frac{1}{4}C^2_{M_{xN}}+C_{M_{yxN}}\right)$
$Bias(t_{4N})=M_{yN}\lambda_N\left(\frac{3}{8}C^2_{M_{xN}}-\frac{1}{2}C_{M_{yxN}}\right)$	$MSE(t_{4N})=M^2_{yN}\lambda_N\left(C^2_{M_{yN}}+\frac{1}{4}C^2_{M_{xN}}-C_{M_{yxN}}\right)$
$Bias(t_{5N})=M_{yN}\lambda_N\left(\frac{3}{8}C^2_{M_{xN}}+\frac{3}{2}C_{M_{yxN}}\right)$	$MSE(t_{5N})=M^2_{yN}\lambda_N\left(C^2_{M_{yN}}+\frac{9}{4}C^2_{M_{xN}}+3C_{M_{yxN}}\right)$
$Bias(t_{6N})=M_{yN}\lambda_N\left(\frac{15}{8}C^2_{M_{xN}}-\frac{3}{2}C_{M_{yxN}}\right)$	$MSE(t_{6N})=M^2_{yN}\lambda_N\left(C^2_{M_{yN}}+\frac{9}{4}C^2_{M_{xN}}-3C_{M_{yxN}}\right)$
$Bias(t_{7N})=M_{yN}\left[\lambda_N C_{M_{yxN}}-\frac{1}{2}C^2_{M_{xN}}\right]$	$MSE(t_{7N})=M^2_{yN}\lambda_N\left[C^2_{M_{yN}}+C^2_{M_{xN}}+2C^2_{M_{yxN}}\right]$
If a=1 and b=1	
$Bias(t_{hN})=M_{yN}\lambda_N\left[\left(\frac{3}{2}\alpha_{1N}+\alpha_{2N}\right)C_{M_{yxN}}+\left(\frac{3}{8}\alpha_{1N}-\frac{1}{2}\alpha_{2N}\right)C^2_{M_{xN}}\right]$	$Min.MSE(t_{hN})=M^2_{yN}\lambda_N C^2_{M_{yN}}(1-\rho^2_{yxN})$
If a= -1 and b= -1	
$Bias(t_{hN})=M_{yN}\lambda_N\left[\left(-\frac{3}{2}\alpha_{1N}+\alpha_{2N}\right)C_{M_{yxN}}+\left(\frac{15}{8}\alpha_{1N}-\frac{1}{2}\alpha_{2N}\right)C^2_{M_{xN}}\right]$	$Min.MSE(\mathrm{t_{hN}})=M^2_{yN}\lambda_N C^2_{M_{yN}}(1-\rho^2_{yxN})$

5. Empirical Study

In our study, we considered the stock price as a neutrosophic auxiliary variable, given that it changes from its opening price at the start of the trading session to its closing price at the end. Throughout the trading day, the price fluctuates between the highest price and the lowest price of the day.

Population 1: We retrieved stock price data from June 1, 2022, to July 29, 2022, as sourced from [14],
The neutrosophic auxiliary variable X_N stands for the opening price, and the study variable Y_N represents the low and high prices.
Furthermore, the formula for calculating Percent Relative Efficiency is as follows:

$$PRE=\frac{Var(\hat{M}_{0N})}{MSE(estimator)}\times 100 \qquad (6.1)$$

Table 1. Descriptive statistics for population 1.

Values	Population 1	Values	Population 1
N_N	41	ρ_{yxN}	[0.95121, 0.95121]
n_N	[8, 8]	C_{MyN}	[0.14920, 0.15220]
M_{yN}	[1133, 1156.5]	C_{MxN}	[0.15368, 0.15368]

M_{xN}	[1149.5, 1149.5]	QA_N	[1143.5, 1143.5]
$f_{xN}(M_{xN})$	[0.005660, 0.005660]	TM_N	[1146.5, 1146.5]
$f_{yN}\left(M_{yN}\right)$	[0.005798, 0.005795]	DM_N	[1156.778, 1156.778]

Table 2. The neutrosophic MSE, together with the PRE, is evaluated for both the existing and newly proposed estimators.

Estimator	MSE	I_N	PRE
t_{0N}	[747.9443, 748.8745]	[0, 0.0012]	[100, 100]
t_{1N}	[2947.2458, 3010.8489]	[0, 0.0211]	[24.8725, 25.3777]
t_{2N}	[73.7648, 75.9446]	[0, 0.0287]	[986.0802, 1013.9578]
t_{3N}	[1656.9548, 1681.2311]	[0, 0.0144]	[44.5432, 45.1397]
t_{4N}	[213.7790, 220.2143]	[0, 0.0301]	[339.6439, 350.3032]
t_{5N}	[4618.8174, 4737.7279]	[0, 0.0251]	[15.8066, 16.1934]
t_{6N}	[308.5959, 335.3713]	[0, 0.0798]	[223.2971, 242.3701]
t_{7N}	[2947.2458, 3010.8489]	[0, 0.0211]	[24.8725, 25.3777]
t_{hN}	**[71.2039, 71.2925]**	**[0, 0.0012]**	**[1050.4253, 1050.4253]**

Population 2: We retrieved stock price data from February 1, 2022, to July 29, 2022, as sourced from [14], The neutrosophic auxiliary variable X_N stands for the opening price, and the study variable Y_N represents the low and high prices.

Table 3. Descriptive statistics for population 2.

Values	Population 2	Values	Population 2
N_N	124	ρ_{yxN}	[0.67741, 0.67741]
n_N	[45, 45]	C_{MyN}	[0.36725, 0.36233]
M_{yN}	[122.215, 128.775]	C_{MxN}	[0.37393, 0.37393]
M_{xN}	[125.103, 125.103]	QA_N	[127.7385, 127.7385]
$f_{xN}(M_{xN})$	[0.02137, 0.02137]	TM_N	[126.4208, 126.4208]
$f_{yN}\left(M_{yN}\right)$	[0.02143, 0.02228]	DM_N	[130.1744, 130.1744]

Table 4. The neutrosophic MSE, together with the PRE, is evaluated for both the existing and newly proposed estimators.

Estimator	MSE	I_N	PRE
t_{0N}	[7.1303, 7.7056]	[0, 0.0747]	[100, 100]

t_{1N}	[24.3582, 26.6863]	[0, 0.0872]	[28.8746, 29.2725]
t_{2N}	[4.6863, 5.1385]	[0, 0.0880]	[149.9564, 152.1500]
t_{3N}	[13.8962, 15.1442]	[0, 0.0824]	[50.8812, 51.3108]
t_{4N}	[4.0603, 4.3703]	[0, 0.0709]	[175.6094, 176.3151]
t_{5N}	[38.5162, 42.3318]	[0, 0.0901]	[18.2028, 18.5124]
t_{6N}	[9.0084, 10.0102]	[0, 0.1001]	[76.9774, 79.1514]
t_{7N}	[24.3582, 26.6863]	[0, 0.0872]	[28.8746, 29.2725]
t_{hN}	**[3.8583, 4.1696]**	**[0, 0.0747]**	**[184.8034, 184.8034]**

For the particular case of a = 1 and b = 1

Table 5: Bias values of the existing and proposed estimators.

Estimators	**Population 1 Bias**	**Population 2 Bias**
t_{0N}	**[0, 0]**	**[0, 0]**
t_{5N}	[1.203456107, 1.209282309]	[0.083042, 0.086647]
t_{7N}	[0.297519605, 0.290933806]	[0.009967, 0.009998]
t_{hN}	**[0, 0]**	**[0, 0]**

Table 6: Values of scalars α_{iN}'s (i = 0, 1, 2) for which bias of t_{hN} reduces to zero.

Scalars	**Population 1**	**Population 2**
α_0	[2.097293, 2.127130562]	[1.702021, 1.714189]
α_1	[0.347623, 0.37016219]	[0.091251, 0.097761]
α_2	[-1.49729275, -1.44492]	[-0.81195, -0.79327]
H_N	[-0.94205, -0.92348]	[-0.66531, -0.6564]

For the particular case of a = -1 and b = -1

Table 7: Bias values of the existing and proposed estimators.

Estimators	**Population 1 Bias**	**Population 2 Bias**
t_{0N}	**[0, 0]**	**[0, 0]**

t_{6N}	[0.310897, 0.336481]	[0.053046, 0.056745775]
t_{7N}	[0.290934, 0.29752]	[0.00996, 0.00999]
t_{hN}	**[0, 0]**	**[0, 0]**

Table 8:Values of scalars $\alpha_{iN}'s$ (i = 0, 1, 2) for which bias of t_{hN} reduces to zero.

Scalars	Population 1	Population 2
α_0	[1.016644, 1.054422]	[1.420906, 1.428268]
α_1	[0.347623, 0.370162]	[0.091251, 0.097761]
α_2	[-0.40205, -0.38681]	[-0.51952, -0.51867]
H_N	[-0.94205, -0.92348]	[-0.66531, -0.6564]

6. Discussion

Table 2 presents the values of MSE and PRE for both the existing and proposed estimators for population 1. From the table, it is evident that the MSE of the proposed estimator is the lowest, ranging between [71.2039, 71.2925], indicating that it has minimal estimation error and is highly reliable for the given dataset. The corresponding higher PRE values of [1050.4253, 1050.4253] further highlight that the proposed estimator is more efficient compared to the baseline estimator, thus making it the most efficient estimator.

Similarly, for Population 2, as shown in Table 4, the proposed estimator exhibits the smallest MSE value of [3.8583, 4.1696] when compared to all other estimators, indicating minimal estimation error. The higher PRE value of [184.8034, 184.8034] indicates that the proposed estimator is more efficient than the baseline estimators.

The range of indeterminacy, denoted as I_N , in the tables for Population 1 is [0, 0.0012] and for Population 2 is [0, 0.0747]. These values are the smallest among the estimators, indicating that the proposed estimator provides more precise estimates with minimal uncertainty or variability.

Additionally, Table 6 and Table 8 present the scalar values $\alpha_{iN}'s$ (i = 0, 1, 2) for which the bias of t_{hN} reduces to zero, further confirming the effectiveness of the proposed estimator.

7. Conclusion

This paper introduces almost unbiased estimators that utilize known neutrosophic auxiliary information to estimate the unknown population median of primary variable. This article is based on three existing estimators, and the bias and mean squared error (MSE) of the proposed estimator are derived using the error term up to the first-order approximation. To support the theoretical findings, empirical studies were conducted using two real datasets of retrieved stock price rates. The results provide strong evidence that the proposed estimator outperforms the existing estimators. A table of scalar values is included, demonstrating cases where the proposed estimator reduces to zero. Additionally, the paper highlights that neutrosophic estimators offer superior accuracy in estimating population medians, particularly when the data of the study variable are vague or nondeterministic. Therefore, neutrosophic estimators prove to be a reliable choice for handling uncertain data.

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