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Pentapartitioned Single-Valued Neutrosophic Z-Numbers: Theory and Applications

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Abstract. This paper introduces a novel framework, Pentapartitioned Single-Valued Neutrosophic Z-Numbers (PSVNZNs), to model uncertainty in decision-making problems. By integrating the five-component uncertainty representation of Pentapartitioned Neutrosophic Sets (PNSs) with the reliability aspect of Z-numbers, PSVNZNs offer a comprehensive approach to tackle complex decision scenarios. A novel scoring function and various aggregation operators, including weighted arithmetic, geometric, and hybrid averages, are developed to facilitate decision-making processes. The proposed PSVNZN framework adheres to the properties of linear averaging operators. To demonstrate its practical application, a multi-criteria decision-making (MCDM) problem involving electric vehicle selection is considered. A graphical representation of the obtained scores further enhances the clarity and interpretability of the results.

Keywords: Pentapartitioned neutrosophic sets, Z numbers, Aggregation operator, Score function, MCDM, Electric vehicle.

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1. Introduction

Fuzzy set theory, pioneered by Zadeh [29] in 1985, provides a mathematical framework for representing and manipulating imprecise or vague information. By allowing elements to have degrees of membership within a set, fuzzy sets effectively capture the inherent ambiguity present in many real-world phenomena. This powerful concept has found widespread application across diverse domains, including industrial engineering [3], risk assessment [14], system analysis [4] and predictive modeling, where it enables more robust and nuanced decision-making processes. Although fuzzy set theory effectively addresses uncertainty, its limitations have spurred the development of various extensions.

To address the limitations of fuzzy set theory, several extensions have been developed. These include: Atanassov's intuitionistic fuzzy sets (IFS) [1], proposed in 1986, which incorporate both membership and non-membership degrees; Pawlak's rough sets [19], introduced in 1982, which provide a framework for dealing with uncertainty and vagueness in data analysis; Molodtsov's soft sets [18], proposed in 1999, which offer a general mathematical framework for handling uncertainty; and Goguen's L-fuzzy sets [11], which extend the concept of fuzzy sets by allowing membership degrees to be defined over a more general lattice structure. While IFS offer enhanced flexibility, the inherent dependence of the hesitation margin on membership and non-membership degrees limits their capacity to represent independent uncertainty.

To accommodate complex forms of uncertainty, authors have developed models with independent hesitation parameters. Smarandache [21] introduced neutrosophic sets (NSs), a powerful generalization of fuzzy sets, characterized by three independent membership functions. Wang et al. [24] made a significant contribution by introducing single-valued neutrosophic sets (SVNSs). Shapique and Mathivadhana [17] introduced modified neutrosophic numbers to analyse integral equations. While SVNSs provide greater flexibility than IFSs, the meaning of indeterminacy in SVNSs remains unclear. Chatterjee et al. [7] addressed this by introducing Quadripartitioned SVNS (QSVNSs), which differentiate between contradiction and ignorance within the indeterminacy component.

Building upon the foundations of neutrosophic logic, Smarandache [22] introduced five-valued neutrosophic logic, which further subdivides the indeterminacy component into 'unknown', 'ignorance', and 'contradiction'. Mallick and Pramanik [15] expanded upon this concept by proposing Pentapartitioned neutrosophic sets, which incorporate an additional 'unknown' component. This extension enables a more comprehensive approach to representing uncertainty and incompleteness in information, accommodating complex decision-making scenarios where traditional models may fall short.

In decision-making theory, models that can efficiently manage and process large amounts of uncertain information are highly valued. QSVNSs have become a well-established mathematical tool for handling complex data. QSVNSs classify information into four primary components: truth, contradiction, ignorance, and falsity, providing a structured approach to analysing uncertainty across multiple dimensions. However, specific decision-making contexts involve an additional element of 'unknown' that QSVNSs do not fully capture. To address this limitation, Pentapartitioned Single-Valued Neutrosophic Sets (PSVNSs) introduce an additional component, 'unknown'. Thus Pentapartitioned framework offers a distinct advantage in scenarios where the unknown plays a significant role, enabling a more nuanced understanding of uncertainty by addressing ambiguity beyond the scope of the other four categories.

To expand the potential of these sets, a quantitative assessment of reliability can be assigned to each individual neutrosophic component. Zadeh's Z-numbers [28] offer a suitable framework for this scenario. A Z-number is composed of two fuzzy numbers: the first component defines the range of possible values for an uncertain variable, while the second component represents the degree of confidence in the accuracy of this information. Reliable information is essential for effective decision-making, as relying on vague or uncertain data can lead to financial losses, wasted resources, and inefficiencies. By integrating Z-numbers into PSVNSs, decision-makers gain a dual advantage: they can quantify both cognitive information and its associated reliability. This combination makes Z-numbers and PSVNSs a powerful tool for capturing and managing imprecise, ambiguous information with greater accuracy. A substantial advancement arises from integrating Z-numbers with PSVNSs, allowing for the representation of truth, contradiction, ignorance, unknown, and falsity degrees, each with an associated reliability measure. This innovative framework enables decision-makers to manage highly uncertain information more effectively, capturing the complex nature of uncertainty through the five-dimensional structure of PSVNSs.

Integrating Z-numbers into the PNSVNSs framework can significantly improve the accuracy and reliability of multi-criteria decision-making (MCDM) when dealing with complex uncertainty. Recognizing the potential of combining Z-numbers with PNSVNSs to enhance flexibility and reliability, a novel hybrid mathematical framework is introduced: Pentapartitioned Single-Valued Neutrosophic Z-numbers (PNSVNZNs). PNSVNZNs represent Pentapartitioned neutrosophic values augmented with Pentapartitioned reliability measures, providing a more comprehensive structure for managing uncertainty. This paper presents the definitions and operations of PNSVNZNs, including a score function for ranking. Additionally, three aggregation operators for PNSVNZNs are explored: PNSVNZN Weighted Arithmetic Averaging, PNSVNZN Weighted Geometric Averaging, and PNSVNZN Weighted Hybrid Averaging. These operators are thoroughly validated against essential properties. To demonstrate the practical application of PNSVNZNs, an MCDM algorithm is developed that leverages the proposed score function and aggregation operators to rank alternative solutions. This framework empowers decision-makers to tackle complex scenarios involving imprecise and uncertain information across five components—truth, contradiction, ignorance, unknown, and falsity—providing a comprehensive tool for high-stakes decision-making under uncertainty.

The paper is structured as follows. Section 1.1 provides a literature review. Section 1.2 outlines the contributions of this work. Section 2 presents the preliminaries and introduces the proposed framework. Section 3 defines and investigates the properties of three aggregation operators. Section 4 applies the proposed framework to a MCDM problem. Section 5 evaluates electric vehicles (EVs) using the proposed score function. Finally, Section 6 concludes the paper.

1.1. *Literature review*

In recent years, there has been a significant increase in interest in Z numbers, resulting in extensive research in both qualitative and quantitative areas. Researchers have examined various aspects of Z numbers, including their utility, distance measures, arithmetic operations, and applications in decision-making processes. Kang et al. [12] introduced an innovative concept of total utility for Z numbers, which serves as a powerful tool for representing human knowledge. This measure allows for a comprehensive assessment of Z numbers, facilitating their ranking and simplifying decision-making. Importantly, the proposed method is derived directly from the Z number format, thus avoiding subjective judgments. Shen and Wang [23] developed a new comprehensive distance measure for Z numbers, which is instrumental in representing uncertain information. Their measure takes into account both the reliability and uncertainty components of Z numbers by utilizing fuzzy cut-set theory and Hellinger distance. This contribution enhances the theoretical framework of Z numbers and provides a valuable tool for decision-making under uncertainty. Aliev et al. [2] presented a general method for constructing Z number functions and arithmetic operations, essential for addressing both probabilistic and fuzzy uncertainties in human-centric fields such as economics and decision-making. Yang and Wang [26] proposed a novel Multi-Criteria Decision-Making (MCDM) method based on discrete Z numbers and the Stochastic Multi-criteria Acceptability Analysis (SMAA) technique. This approach is particularly beneficial for managing uncertain and incomplete information in decision-making scenarios.

Several authors have explored various applications of Z-numbers such as renewable energy selection, multi-criteria group decision-making, and healthcare waste management. Chatterjee and Kar [8] proposed a novel method for selecting renewable energy alternatives based on Z-numbers and the COPRAS framework. Peng et al. [20] introduced a new multi-criteria group decision-making (MCGDM) method that utilizes uncertain Z-numbers and cloud models. Chai et al. [9] presented Z-uncertain probabilistic linguistic variables (Z-UPLVs) to overcome the limitations of traditional probabilistic linguistic term sets in capturing the credibility of information. Z-UPLVs take into account both linguistic evaluations and probabilistic distributions, as well as the reliability of information. Furthermore Chen et al. [10] proposed a new group decision-making framework for selecting the best healthcare treatment. This framework employs Z-numbers to represent expert opinions and their reliability, alongside the TODIM method, which accounts for bounded rationality to enhance decision-making. Borah and Dutta [5] analysed mask selection problem using quadripartitioned single-valued neutrosophic Z-numbers.

In recent years, researchers have developed several powerful aggregation operators designed to combine and analyze neutrosophic information. Wei and Wei [25] proposed a novel approach to address single-valued neutrosophic multiple attribute decision-making (MADM) problems. They developed single-valued neutrosophic Dombi prioritized aggregation operators. Zhao et al. [27] proposed a new method for multi-attribute group decision-making problems with single-valued neutrosophic numbers (SVNNs). They introduced several power Heronian aggregation operators for SVNNs. These operators consider the interrelationship between attributes and can handle uncertainty and imprecision in decision-making. Khan et al. [13] developed a new methodology to evaluate air pollution by employing neutrosophic cubic sets (NCSs). The authors also introduced several aggregation operators specifically designed for NCSs to facilitate the integration of neutrosophic information. Muniba et al. [16] developed a climate change prediction model by applying the neutrosophic soft set function integrated with aggregate operators, utilizing the MCDM technique to address uncertainties and enhance prediction accuracy. Baser and Ulucay [6] introduced the concept of energy, lower energy, and upper energy of neutrosophic soft sets based on the singular values of their matrix representation. By analyzing the product of a set and its transpose under a specific norm, they developed a quantitative method for decision-making in uncertain environments.

The primary goal is to enhance the reliability of MCDM processes, particularly in situations marked by complex uncertainty. Traditional MCDM seeks to identify the best alternative from a set based on established criteria. However, when faced with complex uncertainty, represented by PNSVNs—including truth, contradiction, ignorance, the unknown, and falsity—the decision-making process becomes more complicated. The varying perspectives of multiple decision-makers, combined with such uncertainty, can result in suboptimal outcomes unless reliability is explicitly addressed.

1.2. *Scope of the Proposed work*

The scope of the proposed work are as follows:

- The notation of Pentapartitioned Single-Valued Neutrosophic Z-Numbers (PSVNZN) is introduced to model uncertainty and imprecision in decision-making scenarios.

- A novel score function for PSVNZNs is proposed to rank alternatives effectively. Additionally, weighted averaging and geometric aggregation operators for PSVNZNs are introduced to aggregate information from multiple sources.
- A novel MCDM framework based on PSVNZNs is developed to address the complex and uncertain nature of EV selection. This framework leverages the proposed score function and aggregation operators to rank EV alternatives based on multiple criteria, such as battery range, charging time, initial cost, and environmental impact.
- The proposed work is illustrated by means of a case study on EV selection. Various uncertain factors, such as the battery range, charging time, safety features and maintenance are considered. By employing PSVNZNs, these uncertainties can be effectively modelled and informed decisions can be made.

In summary, our research significantly advances the field of decision-making under uncertainty by introducing the innovative concept of PSVNZNs and developing a robust MCDM framework. This framework empowers decision-makers to handle complex and uncertain information, leading to more accurate and reliable decisions in the context of EV selection and other real-world applications.

2. Preliminaries

Definition 2.1. [15] A pentapartitioned single-valued neutrosophic set (PSVNS), denoted by A, maps each element $x \in X$ to five independent membership functions: truth $T_A(x)$, contradiction $C_A(x)$, ignorance $I_A(x)$, unknown $U_A(x)$, and falsity $F_A(x)$. These functions $T_A, C_A, I_A, U_A, F_A : X \to [0, 1]$ collectively satisfy the condition:

$$0 \leq T_A(x) + C_A(x) + I_A(x) + U_A(x) + F_A(x) \leq 5$$

For simplicity, we represent a PSVNS A in X as $A = (T_A, C_A, I_A, U_A, F_A)$.

Example 2.1 $A = (0.9, 0.6, 0.5, 0.3, 0.1)$.

Definition 2.2. [28] A Z-number is an ordered pair $Z = (S_1, S_2)$, where S_1 is a fuzzy set representing constraints on an uncertain variable, and S_2 is a fuzzy set representing the certainty or reliability of S_1.

Example 2.2 $Z = (0.9, 0.6)$.

Definition 2.3. (PSVNZN) A pentapartitioned single-valued neutrosophic Z-number (PSVNZN) set, denoted P_Z in X, is represented as:

$$P_Z = [T_P(\xi, \zeta)(x), C_P(\xi, \zeta)(x), G_P(\xi, \zeta)(x), U_P(\xi, \zeta)(x), F_P(\xi, \zeta)(x)],$$

where T_P, C_P, G_P, U_P, and F_P are membership functions over $x \in X$, parameterized by ξ and ζ. Each component $T_P(\xi,\zeta)(x)$, $C_P(\xi,\zeta)(x)$, $G_P(\xi,\zeta)(x)$, $U_P(\xi,\zeta)(x)$, and $F_P(\xi,\zeta)(x)$ is an ordered pair of functions $X \to [0,1]^2$ and is defined as:

$$T_P(\xi,\zeta)(x) = (T_P^{\xi}(x), T_P^{\zeta}(x)),$$

$$C_P(\xi,\zeta)(x) = (C_P^{\xi}(x), C_P^{\zeta}(x)),$$

$$G_P(\xi,\zeta)(x) = (G_P^{\xi}(x), G_P^{\zeta}(x)),$$

$$U_P(\xi,\zeta)(x) = (U_P^{\xi}(x), U_P^{\zeta}(x)),$$

$$F_P(\xi,\zeta)(x) = (F_P^{\xi}(x), F_P^{\zeta}(x)).$$

The components $T_P^{\xi}(x), C_P^{\xi}(x), G_P^{\xi}(x), U_P^{\xi}(x), F_P^{\xi}(x)$ satisfies the conditions:

$$0 \leq T_P^{\xi}(x) + C_P^{\xi}(x) + G_P^{\xi}(x) + U_P^{\xi}(x) + F_P^{\xi}(x) \leq 5,$$

$$0 \leq T_P^{\zeta}(x) + C_P^{\zeta}(x) + G_P^{\zeta}(x) + U_P^{\zeta}(x) + F_P^{\zeta}(x) \leq 5.$$

Here, ξ denotes the PSVN values for the element x in the universe, while ζ represents the measure of reliability associated with these values. For simplicity, the PSVNZN P_Z is expressed as:

$$P_Z = \left[(T_P^{\xi}, T_P^{\zeta}), (C_P^{\xi}, C_P^{\zeta}), (G_P^{\xi}, G_P^{\zeta}), (U_P^{\xi}, U_P^{\zeta}), (F_P^{\xi}, F_P^{\zeta})\right].$$

Example 2.3 $P_Z = [(0.9, 0.8), (0.7, 0.8), (0.5, 0.6), (0.8, 0.8), (0.5, 0.4)]$.

Definition 2.4. (Null PSVNZN) A null PSVNZN set P_Z is one where, for each $x \in X$, the membership values are:

$$T_P^{\xi} = T_P^{\zeta} = 0, \quad C_P^{\xi} = C_P^{\zeta} = 0, \quad G_P^{\xi} = G_P^{\zeta} = 1, \quad U_P^{\xi} = U_P^{\zeta} = 1, \quad F_P^{\xi} = F_P^{\zeta} = 1,$$

Example 2.4 $P_Z = [(0,0), (0,0), (1,1), (1,1), (1,1)]$

Definition 2.5. (Absolute PSVNZN)A absolute PSVNZN set P_Z is one where, for each $x \in X$, the membership values are:

$$T_P^{\xi} = T_P^{\zeta} = 1, \quad C_P^{\xi} = C_P^{\zeta} = 1, \quad G_P^{\xi} = G_P^{\zeta} = 0, \quad U_P^{\xi} = U_P^{\zeta} = 0, \quad F_P^{\xi} = F_P^{\zeta} = 0,$$

Example 2.5 $P_Z = [(1,1), (1,1), (0,0), (0,0), (0,0)]$

Definition 2.6. (Complement) The complement of a PSVNZN set P_z is defined as:

$$P_z' = \left[\left(F_P^{\xi}, F_P^{\zeta}\right), \left(U_P^{\xi}, U_P^{\zeta}\right), \left(G_P^{\xi}, G_P^{\zeta}\right), \left(C_P^{\xi}, C_P^{\zeta}\right), \left(T_P^{\xi}, T_P^{\zeta}\right)\right]$$

Example 2.6 If $P_Z = [(0.9, 0.8), (0.7, 0.8), (0.5, 0.6), (0.8, 0.8), (0.5, 0.4)]$ then
$P_Z' = [(0.5, 0.4), (0.8, 0.8), (0.5, 0.6), (0.7, 0.8), (0.9, 0.8)]$.

Definition 2.7. (Inclusion) Consider two PSVNZNs

$$P_{Z_1} = \left[\left(T_P^{\xi_1}, T_P^{\zeta_1} \right), \left(C_P^{\xi_1}, C_P^{\zeta_1} \right), \left(G_P^{\xi_1}, G_P^{\zeta_1} \right), \left(U_P^{\xi_1}, U_P^{\zeta_1} \right), \left(F_P^{\xi_1}, F_P^{\zeta_1} \right) \right]$$

and

$$P_{Z_2} = \left[\left(T_P^{\xi_2}, T_P^{\zeta_2} \right), \left(C_P^{\xi_2}, C_P^{\zeta_2} \right), \left(G_P^{\xi_2}, G_P^{\zeta_2} \right), \left(U_P^{\xi_2}, U_P^{\zeta_2} \right), \left(F_P^{\xi_2}, F_P^{\zeta_2} \right) \right].$$

Then, for every $x \in X$, P_{Z_1} is said to be included in P_{Z_2}, denoted as $P_{Z_1} \subseteq P_{Z_2}$, if

$$T_P^{\xi_1} \leq T_P^{\xi_2}, \quad C_P^{\xi_1} \leq C_P^{\xi_2}, \quad G_P^{\xi_1} \leq G_P^{\xi_2}, \quad U_P^{\xi_1} \leq U_P^{\xi_2}, \quad F_P^{\xi_1} \leq F_P^{\xi_2}$$

and

$$T_P^{\zeta_1} \leq T_P^{\zeta_2}, \quad C_P^{\zeta_1} \leq C_P^{\zeta_2}, \quad G_P^{\zeta_1} \leq G_P^{\zeta_2}, \quad U_P^{\zeta_1} \leq U_P^{\zeta_2}, \quad F_P^{\zeta_1} \leq F_P^{\zeta_2}.$$

Example 2.7

$$P_{Z_1} = [(0.6, 0.8), (0.4, 0.7), (0.3, 0.6), (0.5, 0.9), (0.2, 0.5)]$$

and

$$P_{Z_2} = [(0.7, 0.9), (0.5, 0.8), (0.4, 0.7), (0.6, 1.0), (0.3, 0.6)].$$

For every $x \in X$, we check whether $P_{Z_1} \subseteq P_{Z_2}$:

(1) **First Components:**

$$T_P^{\xi_1} = 0.6 \leq 0.7 = T_P^{\xi_2}, \quad T_P^{\zeta_1} = 0.8 \leq 0.9 = T_P^{\zeta_2}.$$

(2) **Second Components:**

$$C_P^{\xi_1} = 0.4 \leq 0.5 = C_P^{\xi_2}, \quad C_P^{\zeta_1} = 0.7 \leq 0.8 = C_P^{\zeta_2}.$$

(3) **Third Components:**

$$G_P^{\xi_1} = 0.3 \leq 0.4 = G_P^{\xi_2}, \quad G_P^{\zeta_1} = 0.6 \leq 0.7 = G_P^{\zeta_2}.$$

(4) **Fourth Components:**

$$U_P^{\xi_1} = 0.5 \leq 0.6 = U_P^{\xi_2}, \quad U_P^{\zeta_1} = 0.9 \leq 1.0 = U_P^{\zeta_2}.$$

(5) **Fifth Components:**

$$F_P^{\xi_1} = 0.2 \leq 0.3 = F_P^{\xi_2}, \quad F_P^{\zeta_1} = 0.5 \leq 0.6 = F_P^{\zeta_2}.$$

Since all the conditions hold, we conclude that:

$$P_{Z_1} \subseteq P_{Z_2}.$$

Definition 2.8. (Equality) Two PSVNZNs

$$P_{Z_1} = \left[\left(T_P^{\xi_1}, T_P^{\zeta_1}\right), \left(C_P^{\xi_1}, C_P^{\zeta_1}\right), \left(G_P^{\xi_1}, G_P^{\zeta_1}\right), \left(U_P^{\xi_1}, U_P^{\zeta_1}\right), \left(F_P^{\xi_1}, F_P^{\zeta_1}\right)\right]$$

and

$$P_{Z_2} = \left[\left(T_P^{\xi_2}, T_P^{\zeta_2}\right), \left(C_P^{\xi_2}, C_P^{\zeta_2}\right), \left(G_P^{\xi_2}, G_P^{\zeta_2}\right), \left(U_P^{\xi_2}, U_P^{\zeta_2}\right), \left(F_P^{\xi_2}, F_P^{\zeta_2}\right)\right]$$

is said to be equal, in the universe X, if

$$T_P^{\xi_1} = T_P^{\xi_2}, \quad C_P^{\xi_1} = C_P^{\xi_2}, \quad G_P^{\xi_1} = G_P^{\xi_2}, \quad U_P^{\xi_1} = U_P^{\xi_2}, \quad F_P^{\xi_1} = F_P^{\xi_2}$$

and

$$T_P^{\zeta_1} = T_P^{\zeta_2}, \quad C_P^{\zeta_1} = C_P^{\zeta_2}, \quad G_P^{\zeta_1} = G_P^{\zeta_2}, \quad U_P^{\zeta_1} = U_P^{\zeta_2}, \quad F_P^{\zeta_1} = F_P^{\zeta_2}.$$

We can also express this as $P_{Z_1} = P_{Z_2} \Leftrightarrow P_{Z_1} \subseteq P_{Z_2}$ and $P_{Z_1} \supseteq P_{Z_2}$.

Example 2.8

$$P_{Z_1} = [(0.7, 0.8), (0.6, 0.7), (0.9, 0.95), (0.5, 0.6), (0.8, 0.85)]$$

and

$$P_{Z_2} = [(0.7, 0.8), (0.6, 0.7), (0.9, 0.95), (0.5, 0.6), (0.8, 0.85)].$$

Definition 2.9. (Union) Let

$$P_{Z_1} = \left[\left(T_P^{\xi_1}, T_P^{\zeta_1}\right), \left(C_P^{\xi_1}, C_P^{\zeta_1}\right), \left(G_P^{\xi_1}, G_P^{\zeta_1}\right), \left(U_P^{\xi_1}, U_P^{\zeta_1}\right), \left(F_P^{\xi_1}, F_P^{\zeta_1}\right)\right]$$

and

$$P_{Z_2} = \left[\left(T_P^{\xi_2}, T_P^{\zeta_2}\right), \left(C_P^{\xi_2}, C_P^{\zeta_2}\right), \left(G_P^{\xi_2}, G_P^{\zeta_2}\right), \left(U_P^{\xi_2}, U_P^{\zeta_2}\right), \left(F_P^{\xi_2}, F_P^{\zeta_2}\right)\right]$$

be two PSVNZNs. Then $P_{Z_1} \cup P_{Z_2}$ is defined by

$$P_{Z_1} \cup P_{Z_2} = \Big[\left(T_P^{\xi_1} \vee T_P^{\xi_2}, T_P^{\zeta_1} \vee T_P^{\zeta_2}\right), \left(C_P^{\xi_1} \vee C_P^{\xi_2}, C_P^{\zeta_1} \vee C_P^{\zeta_2}\right), \left(G_P^{\xi_1} \wedge G_P^{\xi_2}, G_P^{\zeta_1} \wedge G_P^{\zeta_2}\right), \left(U_P^{\xi_1} \wedge U_P^{\xi_2}, U_P^{\zeta_1} \wedge U_P^{\zeta_2}\right), \left(F_P^{\xi_1} \wedge F_P^{\xi_2}, F_P^{\zeta_1} \wedge F_P^{\zeta_2}\right)\Big].$$

Example 2.9 Let two PSVNZNs be defined as follows:

$$P_{Z_1} = [(0.7, 0.8), (0.6, 0.7), (0.9, 0.95), (0.5, 0.6), (0.8, 0.85)]$$

and

$$P_{Z_2} = [(0.6, 0.75), (0.5, 0.7), (0.85, 0.9), (0.6, 0.65), (0.75, 0.8)].$$

Now, we compute the union $P_{Z_1} \cup P_{Z_2}$ using the following operations for each pair of components:

$$T_P^{\xi_1} \vee T_P^{\xi_2} = \max(0.7, 0.6) = 0.7, \quad T_P^{\zeta_1} \vee T_P^{\zeta_2} = \max(0.8, 0.75) = 0.8,$$
$$C_P^{\xi_1} \vee C_P^{\xi_2} = \max(0.6, 0.5) = 0.6, \quad C_P^{\zeta_1} \vee C_P^{\zeta_2} = \max(0.7, 0.7) = 0.7,$$
$$G_P^{\xi_1} \wedge G_P^{\xi_2} = \min(0.9, 0.85) = 0.85, \quad G_P^{\zeta_1} \wedge G_P^{\zeta_2} = \min(0.95, 0.9) = 0.9,$$
$$U_P^{\xi_1} \wedge U_P^{\xi_2} = \min(0.5, 0.6) = 0.5, \quad U_P^{\zeta_1} \wedge U_P^{\zeta_2} = \min(0.6, 0.65) = 0.6,$$
$$F_P^{\xi_1} \wedge F_P^{\xi_2} = \min(0.8, 0.75) = 0.75, \quad F_P^{\zeta_1} \wedge F_P^{\zeta_2} = \min(0.85, 0.8) = 0.8.$$

Thus, the union of the two PSVNZNs $P_{Z_1} \cup P_{Z_2}$ is given by:

$$P_{Z_1} \cup P_{Z_2} = \left[(0.7, 0.8), (0.6, 0.7), (0.85, 0.9), (0.5, 0.6), (0.75, 0.8)\right].$$

Definition 2.10. (Intersection) Let

$$P_{Z_1} = \left[\left(T_P^{\xi_1}, T_P^{\zeta_1}\right), \left(C_P^{\xi_1}, C_P^{\zeta_1}\right), \left(G_P^{\xi_1}, G_P^{\zeta_1}\right), \left(U_P^{\xi_1}, U_P^{\zeta_1}\right), \left(F_P^{\xi_1}, F_P^{\zeta_1}\right)\right]$$

and

$$P_{Z_2} = \left[\left(T_P^{\xi_2}, T_P^{\zeta_2}\right), \left(C_P^{\xi_2}, C_P^{\zeta_2}\right), \left(G_P^{\xi_2}, G_P^{\zeta_2}\right), \left(U_P^{\xi_2}, U_P^{\zeta_2}\right), \left(F_P^{\xi_2}, F_P^{\zeta_2}\right)\right]$$

two PSVNZNs. Then $P_{Z_1} \cap P_{Z_2}$ is defined by

$$P_{Z_1} \cap P_{Z_2} = \Bigg[\left(T_P^{\xi_1} \wedge T_P^{\xi_2}, T_P^{\zeta_1} \wedge T_P^{\zeta_2}\right), \left(C_P^{\xi_1} \wedge C_P^{\xi_2}, C_P^{\zeta_1} \wedge C_P^{\zeta_2}\right), \left(G_P^{\xi_1} \vee G_P^{\xi_2}, G_P^{\zeta_1} \vee G_P^{\zeta_2}\right), \left(U_P^{\xi_1} \vee U_P^{\xi_2}, U_P^{\zeta_1} \vee U_P^{\zeta_2}\right), \left(F_P^{\xi_1} \vee F_P^{\xi_2}, F_P^{\zeta_1} \vee F_P^{\zeta_2}\right) \Bigg].$$

Example 2.10 Let two PSVNZNs be defined as follows:

$$P_{Z_1} = \left[(0.7, 0.8), (0.6, 0.7), (0.9, 0.95), (0.5, 0.6), (0.8, 0.85)\right]$$

and

$$P_{Z_2} = \left[(0.6, 0.75), (0.5, 0.7), (0.85, 0.9), (0.6, 0.65), (0.75, 0.8)\right].$$

Now, we compute the intersection $P_{Z_1} \cap P_{Z_2}$ using the following operations for each pair of components:

$$T_P^{\xi_1} \wedge T_P^{\xi_2} = \min(0.7, 0.6) = 0.6, \quad T_P^{\zeta_1} \wedge T_P^{\zeta_2} = \min(0.8, 0.75) = 0.75,$$
$$C_P^{\xi_1} \wedge C_P^{\xi_2} = \min(0.6, 0.5) = 0.5, \quad C_P^{\zeta_1} \wedge C_P^{\zeta_2} = \min(0.7, 0.7) = 0.7,$$
$$G_P^{\xi_1} \vee G_P^{\xi_2} = \max(0.9, 0.85) = 0.9, \quad G_P^{\zeta_1} \vee G_P^{\zeta_2} = \max(0.95, 0.9) = 0.95,$$
$$U_P^{\xi_1} \vee U_P^{\xi_2} = \max(0.5, 0.6) = 0.6, \quad U_P^{\zeta_1} \vee U_P^{\zeta_2} = \max(0.6, 0.65) = 0.65,$$
$$F_P^{\xi_1} \vee F_P^{\xi_2} = \max(0.8, 0.75) = 0.8, \quad F_P^{\zeta_1} \vee F_P^{\zeta_2} = \max(0.85, 0.8) = 0.85.$$

Thus, the intersection of the two PSVNZNs $P_{Z_1} \cap P_{Z_2}$ is given by:

$$P_{Z_1} \cap P_{Z_2} = \left[(0.6, 0.75), (0.5, 0.7), (0.9, 0.95), (0.6, 0.65), (0.8, 0.85)\right].$$

Definition 2.11. (Arithmetic operations on PSVNZNs)

- Addition

$$P_{Z_1} \oplus P_{Z_2} = \begin{bmatrix} \left(T_P^{\xi_1} + T_P^{\xi_2} - T_P^{\xi_1} T_P^{\xi_2}, T_P^{\zeta_1} + T_P^{\zeta_2} - T_P^{\zeta_1} T_P^{\zeta_2}\right), \\ \left(C_P^{\xi_1} + C_P^{\xi_2} - C_P^{\xi_1} C_P^{\xi_2}, C_P^{\zeta_1} + C_P^{\zeta_2} - C_P^{\zeta_1} C_P^{\zeta_2}\right), \\ \left(G_P^{\xi_1} G_P^{\xi_2}, G_P^{\zeta_1} G_P^{\zeta_2}\right), \left(U_P^{\xi_1} . U_P^{\xi_2}, U_P^{\zeta_1} . U_P^{\zeta_2}\right), \\ \left(F_P^{\xi_1} F_P^{\xi_2}, F_P^{\zeta_1} F_P^{\zeta_2}\right) \end{bmatrix}$$

- Multiplication

$$P_{Z_1} \otimes P_{Z_2} = \begin{bmatrix} \left(T_P^{\xi_1} T_P^{\xi_2}, T_P^{\zeta_1} T_P^{\zeta_2}\right), \left(C_P^{\xi_1} C_P^{\xi_2}, C_P^{\zeta_1} C_P^{\zeta_2}\right), \\ \left(G_P^{\xi_1} + G_P^{\xi_2} - G_P^{\xi_1} G_P^{\xi_2}, G_P^{\zeta_1} + G_P^{\zeta_2} - G_P^{\zeta_1} G_P^{\zeta_2}\right), \\ \left(U_P^{\xi_1} + U_P^{\xi_2} - U_P^{\xi_1} U_P^{\xi_2}, U_P^{\zeta_1} + U_P^{\zeta_2} - U_P^{\zeta_1} U_P^{\zeta_2}\right), \\ \left(F_P^{\xi_1} + F_P^{\xi_2} - F_P^{\xi_1} F_P^{\xi_2}, F_P^{\zeta_1} + F_P^{\zeta_2} - F_P^{\zeta_1} F_P^{\zeta_2}\right) \end{bmatrix}$$

- For any scalar a

$$aP_Z = \begin{bmatrix} \left(1 - \left(1 - T_P^{\xi}\right)^a, 1 - \left(1 - T_P^{\zeta}\right)^a\right), \left(1 - \left(1 - C_P^{\xi}\right)^a, 1 - \left(1 - C_P^{\zeta}\right)^a\right), \\ \left(\left(G_P^{\xi}\right)^a, \left(G_P^{\zeta}\right)^a\right), \left(\left(U_P^{\xi}\right)^a, \left(U_P^{\zeta}\right)^a\right), \left(\left(F_P^{\xi}\right)^a, \left(F_P^{\zeta}\right)^a\right) \end{bmatrix}$$

- For any scalar a

$$P_Z^a = \begin{bmatrix} \left(\left(T_P^{\xi}\right)^a, \left(T_P^{\zeta}\right)^a\right), \left(\left(C_P^{\xi}\right)^a, \left(C_P^{\zeta}\right)\right)^a, \left(1 - \left(1 - G_P^{\xi}\right)^a, 1 - \left(1 - G_P^{\zeta}\right)^a\right), \\ \left(1 - \left(1 - U_P^{\xi}\right)^a, 1 - \left(1 - U_P^{\zeta}\right)^a\right), \left(1 - \left(1 - F_P^{\xi}\right)^a, 1 - \left(1 - F_P^{\zeta}\right)^a\right) \end{bmatrix}$$

Clearly, the results of the above operations are still PSVNZNs.

Example 2.11 If $P_{Z_1} = [(0.5, 0.6), (0.4, 0.5), (0.8, 0.85), (0.6, 0.7), (0.7, 0.75)]$, $P_{Z_2} = [(0.3, 0.4), (0.5, 0.6), (0.7, 0.8), (0.65, 0.75), (0.6, 0.7)]$ and $a = 2$, then

$$P_{Z_1} \oplus P_{Z_2} = [(0.65, 0.76), (0.7, 0.8), (0.56, 0.6375), (0.39, 0.525), (0.42, 0.525)].$$

$$P_{Z_1} \otimes P_{Z_2} = [(0.15, 0.24), (0.2, 0.3), (0.94, 0.9625), (0.86, 0.925), (0.88, 0.925)].$$

$$2P_{Z_1} = [(0.75, 0.84), (0.64, 0.75), (0.64, 0.7225), (0.36, 0.49), (0.49, 0.5625)],$$

$$(P_Z)^2 = [(0.25, 0.36), (0.16, 0.25), (0.96, 0.9775), (0.84, 0.91), (0.91, 0.9375)].$$

Definition 2.12. (Score Function) For comparing and ranking PSVNZNs $P_{Z_i}, i = 1, 2$ we define a score function as follows:

$$S(P_{Z_i}) = T_P^{\xi_i} T_P^{\zeta_i} + C_P^{\xi_i} C_P^{\zeta_i} - G_P^{\xi_i} G_P^{\zeta_i} - U_P^{\xi_i} U_P^{\zeta_i} - F_P^{\xi_i} F_P^{\zeta_i}, i = 1, 2$$

and $S(P_{Z_i}) \in [-3, 2]$ Thus, between two $P_{Z_i}, i = 1, 2$ the ranking between them can be found as,

- If $S(P_{Z_1}) > S(P_{Z_2})$ then $P_{Z_1} > P_{Z_2}$
- If $S(P_{Z_1}) < S(P_{Z_2})$ then $P_{Z_1} < P_{Z_2}$
- If $S(P_{Z_1}) = S(P_{Z_2})$ then $P_{Z_1} = P_{Z_2}$

Example 2.12. Assume two PSVNZNs

$$P_{Z_1} = (0.9, 0.8), (0.5, 0.8), (0.9, 0.4), (0.1, 0.5), (0.7, 0.6)$$

and

$$P_{Z_2} = (0.5, 0.4), (0.7, 0.8), (0.4, 0.3), (0.5, 0.8), (0.1, 0.6).$$

Then,

$$S(P_{Z_1}) = 0.9 \times 0.8 + 0.5 \times 0.8 - 0.9 \times 0.4 - 0.1 \times 0.5 - 0.7 \times 0.6 = 0.29$$

and

$$S(P_{Z_2}) = 0.5 \times 0.4 + 0.7 \times 0.8 - 0.4 \times 0.3 - 0.5 \times 0.8 - 0.1 \times 0.6 = 0.18.$$

Since $S(P_{Z_1}) > S(P_{Z_2})$, their ranking is $P_{Z_1} > P_{Z_2}$.

3. Operators of PSVNZNs

This section introduces three weighted aggregation operators designed specifically for combining information represented by PSVNZNs.

3.1. *PSVNZN weighted arithmetic averaging operator(PSVNZNWAA)*

Definition 3.1. Let $(P_{Z_1}, P_{Z_2}, ..., P_{Z_n})$ be a group of PSVNZNs. Then the PSVNZNWAA operator is given by $PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}) = \sum_{i=1}^{n} \varepsilon_i P_{Z_i}$ where ε_i represents the weight assigned to P_{Z_i}. The weights ε_i satisfies the conditions $0 \leq \varepsilon_i \leq 1$ and $\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n = 1$.

Theorem 1. Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$, $i = 1, 2, 3, ...n$ be the set of PSVNZNs. Then the aggregated value of the operator PSVNZNWAA is defined as $PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}) = \sum_{i=1}^{n} \varepsilon_i P_{Z_i}$

$$= \left[\begin{array}{l} \left(1 - \prod_{i=1}^{n}\left(1 - T_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(1 - \prod_{i=1}^{n}\left(1 - C_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{array}\right] \quad (1)$$

Theorem 1 can be proved using mathematical induction.

Proof. Setting $n = 2$ in Equation (1), we get

$$\sum_{i=1}^{2}\varepsilon_i P_{Z_i} = \left[\begin{array}{l} 1-\left(1-T_P^{\xi_1}\right)^{\varepsilon_1}+1-\left(1-T_P^{\xi_2}\right)^{\varepsilon_2}-\left(1-\left(1-T_P^{\xi_1}\right)^{\varepsilon_1}\right)\left(1-\left(1-T_P^{\xi_2}\right)^{\varepsilon_2}\right), \\ 1-\left(1-T_P^{\zeta_1}\right)^{\varepsilon_1}+1-\left(1-T_P^{\zeta_2}\right)^{\varepsilon_2}-\left(1-\left(1-T_P^{\zeta_1}\right)^{\varepsilon_1}\right)\left(1-\left(1-T_P^{\zeta_2}\right)^{\varepsilon_2}\right), \\ 1-\left(1-C_P^{\xi_1}\right)^{\varepsilon_1}+1-\left(1-C_P^{\xi_2}\right)^{\varepsilon_2}-\left(1-\left(1-C_P^{\xi_1}\right)^{\varepsilon_1}\right)\left(1-\left(1-C_P^{\xi_2}\right)^{\varepsilon_2}\right), \\ 1-\left(1-C_P^{\zeta_1}\right)^{\varepsilon_1}+1-\left(1-C_P^{\zeta_2}\right)^{\varepsilon_2}-\left(1-\left(1-C_P^{\zeta_1}\right)^{\varepsilon_1}\right)\left(1-\left(1-C_P^{\zeta_2}\right)^{\varepsilon_2}\right), \\ \left(\left(G_P^{\xi_1}\right)^{\varepsilon_1}\left(G_P^{\xi_2}\right)^{\varepsilon_2},\left(G_P^{\zeta_1}\right)^{\varepsilon_1}\left(G_P^{\zeta_2}\right)^{\varepsilon_2}\right),\left(\left(U_P^{\xi_1}\right)^{\varepsilon_1}\left(U_P^{\xi_2}\right)^{\varepsilon_2},\left(U_P^{\zeta_1}\right)^{\varepsilon_1}\left(U_P^{\zeta_2}\right)^{\varepsilon_2}\right), \\ \left(\left(F_P^{\xi_1}\right)^{\varepsilon_1}\left(F_P^{\xi_2}\right)^{\varepsilon_2},\left(F_P^{\zeta_1}\right)^{\varepsilon_1}\left(F_P^{\zeta_2}\right)^{\varepsilon_2}\right) \end{array}\right] \tag{2}$$

After some simplification, Equation (2) yields

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{2}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{2}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(1-\prod_{i=1}^{2}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{2}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{2}\left(G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{2}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{2}\left(U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{2}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{2}\left(F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{2}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{array}\right]$$

Let $n = m$ be true, then $PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_m}) = \sum_{i=1}^{m}\varepsilon_i P_{Z_i}$

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{m}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(1-\prod_{i=1}^{m}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{m}\left(G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m}\left(U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m}\left(F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{array}\right]$$

Setting $n = m + 1$ in Equation (1).

$$PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_{m+1}}) = \sum_{i=1}^{m}\varepsilon_i P_{Z_i} \oplus \varepsilon_{m+1} P_{Z_{m+1}}$$

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{m}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(1-\prod_{i=1}^{m}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right) \\ \left(\prod_{i=1}^{m}\left(G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m}\left(U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m}\left(F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{array}\right]$$
$$\oplus \left[\begin{array}{l} \left(1-\left(1-T_P^{\xi_{m+1}}\right)^{\varepsilon_{m+1}}, 1-\left(1-T_P^{\zeta_{m+1}}\right)^{\varepsilon_{m+1}}\right), \left(1-\left(1-C_P^{\xi_{m+1}}\right)^{\varepsilon_{m+1}}, 1-\left(1-C_P^{\zeta_{m+1}}\right)^{\varepsilon_{m+1}}\right), \\ \left(\left(G_P^{\xi_{m+1}}\right)^{\varepsilon_{m+1}}, \left(G_P^{\zeta_{m+1}}\right)^{\varepsilon_{m+1}}\right), \left(\left(U_P^{\xi_{m+1}}\right)^{\varepsilon_{m+1}}, \left(U_P^{\zeta_{m+1}}\right)^{\varepsilon_{m+1}}\right), \left(\left(F_P^{\xi_{m+1}}\right)^{\varepsilon_{m+1}}, \left(F_P^{\zeta_{m+1}}\right)^{\varepsilon_{m+1}}\right) \end{array}\right]$$

After some manipulation, we get

$$=\begin{bmatrix} \left(1-\prod_{i=1}^{m+1}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i},1-\prod_{i=1}^{m+1}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(1-\prod_{i=1}^{m+1}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i},1-\prod_{i=1}^{m+1}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{m+1}\left(G_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{m+1}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(\prod_{i=1}^{m+1}\left(U_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{m+1}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(\prod_{i=1}^{m+1}\left(F_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{m+1}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right). \end{bmatrix}.$$

It is observed that Equation (1) holds for $n \in Z^+$.

3.2. *Properties of $PSVNZNWAA$*

The PSVNZNWAA operator exhibits the following properties:

Theorem 2 (Idempotency).
Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$, be a set of PSVNZNs. If $P_{Z_i} = P_Z, i = 1, 2, ...n$ then $PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_m}) = P_Z$.

Proof. If $P_{Z_i} = P_Z, i = 1, 2, ...n$, then $PSVNZNWAA(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}) = \sum_{i=1}^{n} \varepsilon_i P_{Z_i}$

$$=\begin{bmatrix} \left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i},1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i},1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right),\left(\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i},\prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

After some simple manipulation, we get

$$=\begin{bmatrix} \left(1-\left(1-T_P^{\xi}\right)^{\sum_{i=1}^{n}\varepsilon_i},1-\left(1-T_P^{\zeta}\right)^{\sum_{i=1}^{n}\varepsilon_i}\right),\left(1-\left(1-C_P^{\xi}\right)^{\sum_{i=1}^{n}\varepsilon_i},1-\left(1-C_P^{\zeta}\right)^{\sum_{i=1}^{n}\varepsilon_i}\right), \\ \left(\left(G_P^{\xi}\right)^{\sum_{i=1}^{n}\varepsilon_i},\left(G_P^{\zeta}\right)^{\sum_{i=1}^{n}\varepsilon_i}\right),\left(\left(U_P^{\xi}\right)^{\sum_{i=1}^{n}\varepsilon_i},\left(U_P^{\zeta}\right)^{\sum_{i=1}^{n}\varepsilon_i}\right),\left(\left(F_P^{\xi}\right)^{\sum_{i=1}^{n}\varepsilon_i},\left(F_P^{\zeta}\right)^{\sum_{i=1}^{n}\varepsilon_i}\right) \end{bmatrix}$$

$$= \left[\left(T_P^{\xi}, T_P^{\zeta}\right), \left(C_P^{\xi}, C_P^{\zeta}\right), \left(G_P^{\xi}, G_P^{\zeta}\right), \left(U_P^{\xi}, U_P^{\zeta}\right), \left(F_P^{\xi}, F_P^{\zeta}\right)\right] = P_Z.$$

Theorem 3 (Boundedness).
Let $P_{Z_i} = \left[(T_P^{\xi_i}, T_P^{\zeta_i}), (C_P^{\xi_i}, C_P^{\zeta_i}), (G_P^{\xi_i}, G_P^{\zeta_i}), (U_P^{\xi_i}, U_P^{\zeta_i}), (F_P^{\xi_i}, F_P^{\zeta_i})\right], i = 1, 2, ...n$ be PSVNZNs. Define,

$$\begin{aligned} P_{Z(\min)} &= \left[\min_i\left(T_{P_i}\left(\xi,\zeta\right)\right),\min_i\left(C_{P_i}\left(\xi,\zeta\right)\right),\min_i\left(G_{P_i}\left(\xi,\zeta\right)\right),\min_i\left(U_{P_i}\left(\xi,\zeta\right)\right),\min_i\left(F_{P_i}\left(\xi,\zeta\right)\right)\right] \\ &= \left[\left(\min_i\left(T_P^{\xi_i}\right),\min_i\left(T_P^{\zeta_i}\right)\right),\left(\min_i\left(C_P^{\xi_i}\right),\min_i\left(C_P^{\zeta_i}\right)\right),\left(\min_i\left(G_P^{\xi_i}\right),\min_i\left(G_P^{\zeta_i}\right)\right),\right. \\ &\quad \left.\left(\min_i\left(U_P^{\xi_i}\right),\min_i\left(U_P^{\zeta_i}\right)\right),\left(\min_i\left(F_P^{\xi_i}\right),\min_i\left(F_P^{\zeta_i}\right)\right)\right] \end{aligned}$$

and

$$P_{Z(\max)} = \left[\max_i \left(T_{P_i}\left(\xi,\zeta\right)\right), \max_i \left(C_{P_i}\left(\xi,\zeta\right)\right), \max_i \left(G_{P_i}\left(\xi,\zeta\right)\right), \max_i \left(U_{P_i}\left(\xi,\zeta\right)\right), \max_i \left(F_{P_i}\left(\xi,\zeta\right)\right)\right]$$
$$= \left[\left(\max_i\left(T_P^{\xi_i}\right), \max_i\left(T_P^{\zeta_i}\right)\right), \left(\max_i\left(C_P^{\xi_i}\right), \max_i\left(C_P^{\zeta_i}\right)\right), \left(\max_i\left(G_P^{\xi_i}\right), \max_i\left(G_P^{\zeta_i}\right)\right),\right.$$
$$\left.\left(\max_i\left(U_P^{\xi_i}\right), \max_i\left(U_P^{\zeta_i}\right)\right), \left(\max_i\left(F_P^{\xi_i}\right), \max_i\left(F_P^{\zeta_i}\right)\right)\right]$$

then

$$P_{z(\min)} \leqslant PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant P_{z(\max)}$$

Proof. Since $P_{z(\min)}$ and $P_{z(\max)}$ represents the minimum and maximum of PSVNZN respectively, then

$$P_{z(\min)} \leqslant P_{Z_i} \leqslant P_{z(\max)}.$$

Introducing the weighted sum on both sides of the above inequality, we get

$$\sum_{i=1}^{n} \varepsilon_i P_{Z(\min)} \leqslant \sum_{i=1}^{n} \varepsilon_i P_{Z_i} \leqslant \sum_{i=1}^{n} \varepsilon_i P_{Z(\max)}.$$

Using the idempotency property, the above expression becomes

$$P_{Z(\min)} \leqslant \sum_{i=1}^{n} \varepsilon_i P_{Z_i} \leqslant P_{Z(\max)}.$$

Thus

$$P_{z(\min)} \leqslant PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant P_{z(\max)}.$$

Theorem 4 (Monotonicity). Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right] \text{ and}$$
$$P_{Z_i}^{*} = \left[\left(T_P^{\xi_i *}, T_P^{\zeta_i *}\right), \left(C_P^{\xi_i *}, C_P^{\zeta_i *}\right), \left(G_P^{\xi_i *}, G_P^{\zeta_i *}\right), \left(U_P^{\xi_i *}, U_P^{\zeta_i *}\right), \left(F_P^{\xi_i *}, F_P^{\zeta_i *}\right)\right],$$

be two sets of PSVNZNs. If $P_{Z_i} \leq P_{Z_i}^{*}$ then

$$PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant PSVNZNWAA\left(P_{Z_1}^{*}, P_{Z_2}^{*}, ..., P_{Z_n}^{*}\right)$$

Proof. Since

$$P_{Z_i} \leqslant P_{Z_i}^{*},$$

then

$$\sum_{i=1}^{n} \varepsilon_i P_{Z_i} \leqslant \sum_{i=1}^{n} \varepsilon_i P_{Z_i}^{*}$$

From the above it is concluded that

$$PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant PSVNZNWAA\left(P_{Z_1}^{*}, P_{Z_2}^{*}, ..., P_{Z_n}^{*}\right).$$

Theorem 5 (Shift Invariant). Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

and

$$\gamma = \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$$

be two PSVNZNs. Then

$$PSVNZNWAA\left(P_{Z_1} \oplus \gamma, P_{Z_2} \oplus \gamma, ..., P_{Z_n} \oplus \gamma\right) = PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \oplus \gamma$$

Proof.

$$P_{Z_i} \oplus \gamma = \begin{bmatrix} \left(T_P^{\xi_i} + T_\beta^{\xi} - T_P^{\xi_i} T_\beta^{\xi}, T_P^{\zeta_i} + T_\beta^{\zeta} - T_P^{\zeta_i} T_\beta^{\zeta}\right), \\ \left(C_P^{\xi_i} + C_\beta^{\xi} - C_P^{\xi_i} C_\beta^{\xi}, C_P^{\zeta_i} + C_\beta^{\zeta} - C_P^{\zeta_i} C_\beta^{\zeta}\right), \\ \left(G_P^{\xi_i} G_\beta^{\xi}, G_P^{\zeta_i}.G_\beta^{\zeta}\right), \left(U_P^{\xi_i} U_\beta^{\xi}, U_P^{\zeta_i} U_\beta^{\zeta}\right), \\ \left(F_P^{\xi_i} F_\beta^{\xi}, F_P^{\zeta_i} F_\beta^{\zeta}\right) \end{bmatrix}$$

We know that

$$PSVNZWAA\left(P_{Z_1} \oplus \gamma, P_{Z_2} \oplus \gamma, ..., P_{Z_n} \oplus \gamma\right) = \sum_{i=1}^{n} \varepsilon_i \left(P_{Z_i} \oplus \gamma\right)$$

$$= \begin{bmatrix} \left(1 - \prod_{i=1}^{n}\left(1 - T_P^{\xi_i} - T_\gamma^{\xi} + T_P^{\xi_i} T_\gamma^{\xi}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - T_P^{\zeta_i} - T_\gamma^{\zeta} + T_P^{\zeta_i} T_\gamma^{\zeta}\right)^{\varepsilon_i}\right), \\ \left(1 - \prod_{i=1}^{n}\left(1 - C_P^{\xi_i} - C_\gamma^{\xi} + C_P^{\xi_i} C_\gamma^{\xi}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - C_P^{\zeta_i} - C_\gamma^{\zeta} + C_P^{\zeta_i} C_\gamma^{\zeta}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(G_P^{\xi_i}.G_\gamma^{\xi}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(G_P^{\zeta_i}.G_\gamma^{\zeta}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(U_P^{\xi_i}.U_\gamma^{\xi}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(U_P^{\zeta_i}.U_\gamma^{\zeta}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(F_P^{\xi_i}.F_\gamma^{\xi}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(F_P^{\zeta_i}.F_\gamma^{\zeta}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(1 - \prod_{i=1}^{n}\left(1 - T_P^{\xi_i}\right)^{\varepsilon_i} + T_\gamma^{\xi} \prod_{i=1}^{n}\left(1 - T_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - T_P^{\zeta_i}\right)^{\varepsilon_i} + T_\gamma^{\zeta} \prod_{i=1}^{n}\left(1 - T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1 - \prod_{i=1}^{n}\left(1 - C_P^{\xi_i}\right)^{\varepsilon_i} + C_\gamma^{\xi} \prod_{i=1}^{n}\left(1 - C_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - C_P^{\zeta_i}\right)^{\varepsilon_i} + C_\gamma^{\zeta} \prod_{i=1}^{n}\left(1 - C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}.G_\gamma^{\xi}, \prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}.G_\gamma^{\zeta}\right), \left(\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}.U_\gamma^{\xi}, \prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}.U_\gamma^{\xi}\right), \\ \left(\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}.F_\gamma^{\xi}, \prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}.F_\gamma^{\zeta}\right) \end{bmatrix}$$

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{array} \right]$$
$$\oplus \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$$

$= PSVNZWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \oplus \gamma.$

Theorem 6 (Homogeneity).

Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ and $\gamma = \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$ be two PSVNZNs. Then $PSVNZWAA\left(\gamma P_{Z_1}, \gamma P_{Z_2}, ..., \gamma P_{Z_n}\right) = \gamma PSVNZWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right)$

Proof.

$$PSVNZWAA\left(\gamma P_{Z_1}, \gamma P_{Z_2}, ..., \gamma P_{Z_n}\right) = \sum_{i=1}^{n} \varepsilon_i \left(\gamma P_{Z_i}\right)$$

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(1-T_P^{\xi_i}\right)^{\gamma}\right)\right)^{\varepsilon_i}, 1-\prod_{i=1}^{n}\left(1-\left(1-\left(1-T_P^{\zeta_i}\right)^{\gamma}\right)\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(1-C_P^{\xi_i}\right)^{\gamma}\right)\right)^{\varepsilon_i}, 1-\prod_{i=1}^{n}\left(1-\left(1-\left(1-C_P^{\zeta_i}\right)^{\gamma}\right)\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(\left(G_P^{\xi_i}\right)^{\gamma}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(\left(U_P^{\zeta_i}\right)^{\gamma}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(\left(U_P^{\xi_i}\right)^{\gamma}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(\left(U_P^{\zeta_i}\right)^{\gamma}\right)^{\varepsilon_i}\right), \\ \left(\prod_{i=1}^{n}\left(\left(F_P^{\xi_i}\right)^{\gamma}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(\left(F_P^{\zeta_i}\right)^{\gamma}\right)^{\varepsilon_i}\right) \end{array} \right]$$

$$= \left[\begin{array}{l} \left(1-\prod_{i=1}^{n}\left(\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\gamma}, 1-\prod_{i=1}^{n}\left(\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\gamma}\right), \\ \left(1-\prod_{i=1}^{n}\left(\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\gamma}, 1-\prod_{i=1}^{n}\left(\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\gamma}\right), \\ \left(\prod_{i=1}^{n}\left(\left(G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\gamma}, \prod_{i=1}^{n}\left(\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\gamma}\right), \\ \left(\prod_{i=1}^{n}\left(\left(U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\gamma}, \prod_{i=1}^{n}\left(\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\gamma}\right), \\ \left(\prod_{i=1}^{n}\left(\left(F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\gamma}, \prod_{i=1}^{n}\left(\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\gamma}\right) \end{array} \right]$$

Thus, we conclude that

$PSVNZWAA\left(\gamma P_{Z_1}, \gamma P_{Z_2}, ..., \gamma P_{Z_n}\right) = \gamma PSVNZWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right)$

3.3. *PSVNZN weighted geometric averaging operator (PSVNZNWGA)*

Definition 14. Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ be a set of PSVNZNs. Then, the PSVNZN weighted geometric averaging operator is defined as

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \prod_{i=1}^{n} P_{Z_i}^{\varepsilon_i}$$

where ε_i is the weight of $P_{Z_i}, i = 1, 2, ...n$ and satisfies the condition $0 \leqslant \varepsilon_i \leqslant 1$ and $\sum_{i=1}^{n} \varepsilon_i = 1$.

Theorem 7. Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$, $i = 1, 2, 3, ...n$ be PSVNZNs. Then, the collected value of the PSVNZNWGA operator is also a PSVNZN which is defined by the following formula.

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \begin{bmatrix} \left(\prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{n}\left(C_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1 - \prod_{i=1}^{n}\left(1 - G_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1 - \prod_{i=1}^{n}\left(1 - U_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1 - \prod_{i=1}^{n}\left(1 - F_P^{\xi_i}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

Proof. Setting $n = 2$, we get

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}\right) = P_{Z_1}^{w_1} \otimes P_{Z_2}^{w_2}$$

$$= \begin{bmatrix} \left(\left(T_P^{\xi_1}\right)^{\varepsilon_1}\left(T_P^{\xi_2}\right)^{\varepsilon_2}, \left(T_P^{\zeta_1}\right)^{\varepsilon_1}\left(T_P^{\zeta_2}\right)^{\varepsilon_2}\right), \left(\left(C_P^{\xi_1}\right)^{\varepsilon_1}\left(C_P^{\xi_2}\right)^{\varepsilon_2}, \left(C_P^{\zeta_1}\right)^{\varepsilon_1}\left(C_P^{\zeta_2}\right)^{\varepsilon_2}\right), \\ \left(1 - \left(1 - G_P^{\xi_1}\right)^{\varepsilon_1} + 1 - \left(1 - G_P^{\xi_2}\right)^{\varepsilon_2} - \left(1 - \left(1 - G_P^{\xi_1}\right)^{\varepsilon_1}\right)\left(1 - \left(1 - G_P^{\xi_2}\right)^{\varepsilon_2}\right)\right), \\ \left(1 - \left(1 - U_P^{\zeta_1}\right)^{\varepsilon_1} + 1 - \left(1 - U_P^{\zeta_2}\right)^{\varepsilon_2} - \left(1 - \left(1 - U_P^{\zeta_1}\right)^{\varepsilon_1}\right)\left(1 - \left(1 - U_P^{\zeta_2}\right)^{\varepsilon_2}\right)\right), \\ \left(1 - \left(1 - F_P^{\xi_1}\right)^{\varepsilon_1} + 1 - \left(1 - F_P^{\xi_2}\right)^{\varepsilon_2} - \left(1 - \left(1 - F_P^{\xi_1}\right)^{\varepsilon_1}\right)\left(1 - \left(1 - F_P^{\xi_2}\right)^{\varepsilon_2}\right)\right) \end{bmatrix}$$

After some simplification, the equation reduces to

$$= \begin{bmatrix} \left(\prod_{i=1}^{2}\left(T_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{2}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{2}\left(C_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{2}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right) \\ \left(1-\prod_{i=1}^{2}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{2}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{2}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{2}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{2}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{2}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

Let $n = m$

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_m}\right) = \prod_{i=1}^{m} P_{Z_i}^{\varepsilon_i}$$

$$= \begin{bmatrix} \left(\prod_{i=1}^{m}\left(T_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m}\left(C_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right) \\ \left(1-\prod_{i=1}^{m}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{m}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{m}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

Setting $n = m + 1$, we get

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_{m+1}}\right) = \prod_{i=1}^{m} P_{Z_i}^{\varepsilon_i} \otimes P_{Z_{m+1}}^{\varepsilon_{m+1}}$$

$$= \begin{bmatrix} \left(\prod_{i=1}^{m+1}\left(T_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m+1}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right), \left(\prod_{i=1}^{m+1}\left(C_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{m+1}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right) \\ \left(1-\prod_{i=1}^{m+1}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m+1}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{m+1}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m+1}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right), \\ \left(1-\prod_{i=1}^{m+1}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}, 1-\prod_{i=1}^{m+1}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right) \end{bmatrix}$$

3.4. *Properties of PSVNZNWGA operator*

In this section the properties of PSVNZNWGA operator such as Idempotency, Monotonicity, Shift Invariant, Homogeneity are presented.

Theorem 8 (Idempotency) Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

be a set of PSVNZNs. If $P_{Z_i} = P_Z, i = 1, 2, ...n$ then $PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = P_Z$.

Theorem 9 (Boundedness). $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ be a set of PSVNZNs. If

$$\begin{aligned} P_{Z(\min)} &= \left[\min_i \left(T_{P_i}(\xi, \zeta)\right), \min_i \left(C_{P_i}(\xi, \zeta)\right), \min_i \left(G_{P_i}(\xi, \zeta)\right), \min_i \left(U_{P_i}(\xi, \zeta)\right), \min_i \left(F_{P_i}(\xi, \zeta)\right)\right] \\ &= \left(\min_i \left(T_P^{\xi_i}\right), \min_i \left(T_P^{\zeta_i}\right)\right), \left(\min_i \left(C_P^{\xi_i}\right), \min_i \left(C_P^{\zeta_i}\right)\right), \left(\min_i \left(G_P^{\xi_i}\right), \min_i \left(G_P^{\zeta_i}\right)\right), \\ &\quad \left(\min_i \left(U_P^{\xi_i}\right), \min_i \left(U_P^{\zeta_i}\right)\right), \left(\min_i \left(F_P^{\xi_i}\right), \min_i \left(F_P^{\zeta_i}\right)\right) \Big] \end{aligned}$$

and

$$\begin{aligned} P_{Z(\max)} &= \left[\max_i \left(T_{P_i}(\xi, \zeta)\right), \max_i \left(C_{P_i}(\xi, \zeta)\right), \max_i \left(G_{P_i}(\xi, \zeta)\right), \max_i \left(U_{P_i}(\xi, \zeta)\right), \max_i \left(F_{P_i}(\xi, \zeta)\right)\right] \\ &= \left(\max_i \left(T_P^{\xi_i}\right), \max_i \left(T_P^{\zeta_i}\right)\right), \left(\max_i \left(C_P^{\xi_i}\right), \max_i \left(C_P^{\zeta_i}\right)\right), \left(\max_i \left(G_P^{\xi_i}\right), \max_i \left(G_P^{\zeta_i}\right)\right), \\ &\quad \left(\max_i \left(U_P^{\xi_i}\right), \max_i \left(U_P^{\zeta_i}\right)\right), \left(\max_i \left(F_P^{\xi_i}\right), \max_i \left(F_P^{\zeta_i}\right)\right) \Big] \end{aligned}$$

then

$$P_{z(\min)} \leqslant PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant P_{z(\max)}$$

Theorem 10 (Monotonicity). Let P_{Z_i} and $P^*_{Z_i}$ be two groups of PSVNZNs. If $P_{Z_i} \leq P^*_{Z_i}$ then $PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant PSVNZNWGA\left(P^*_{Z_1}, P^*_{Z_2}, ..., P^*_{Z_n}\right)$.

Theorem 11 (Shift Invariant). Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

and

$$\gamma = \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$$

be two PSVNZNs. Then

$$PSVNZNWGA\left(P_{Z_1} \oplus \gamma, P_{Z_2} \oplus \gamma, ..., P_{Z_n} \oplus \gamma\right) = PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \oplus \gamma$$

Theorem 12 (Homogeneity). Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

and

$$\gamma = \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$$

be two PSVNZNs. Then

$$PSVNZNWGA\left(\gamma P_{Z_1}, \gamma P_{Z_2}, ..., \gamma P_{Z_n}\right) = \gamma PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right).$$

The proof of Theorems 8–12 is similar to that of Theorems 2–6.

3.5. *PSVNZN weighted hybrid averaging operator (PSVNZNWHA)*

Definition 14. Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ be a set of PSVNZNs. Then, for any $\upsilon \in [0,1]$ the PSVNZNWHA operator is defined as

$$PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \left(\sum_{i=1}^{n} \varepsilon_i P_{Z_i}\right)^{\nu} \otimes \left(\prod_{i=1}^{n} P_{Z_i}^{\varepsilon_i}\right)^{1-\nu}$$

where ε_i is the weight of $P_{Z_i}, i = 1, 2, 3...n$ and satisfies the condition $0 \leqslant \varepsilon_i \leqslant 1$ and $\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n = 1$.

Theorem 13. Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ be a set of PSVNZNs. Then, the collected value of the PSVNZNWGA operator is also a PSVNZN, which is expressed by the following formula: $PSVNZNWGA\left(P_Z\right)$

$$= \begin{bmatrix} \left(\left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}, \left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right), \\ \left(\left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}, \left(1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right), \\ \left(1-\left(1-\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}, 1-\left(1-\prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right), \\ \left(1-\left(1-\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}, 1-\left(1-\prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right), \\ \left(1-\left(1-\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}, 1-\left(1-\prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right) \end{bmatrix}$$

The proof of Theorem is similar to that of Theorem 7.

3.6. *Properties of PSVNZNWHA operator*

This section presents the properties of PSVNZNWHA operator such as Idempotency, Monotonicity, Shift Invariant and Homogeneity.

Theorem 14 (Idempotency)
Let $P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$ be a set of PSVNZNs. If $P_{Z_i} = P_Z, i = 1, 2, ...n$ then $PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = P_Z$.

Theorem 15 (Boundedness).$P_{Z_i} = \left[(T_P^{\xi_i}, T_P^{\zeta_i}), (C_P^{\xi_i}, C_P^{\zeta_i}), (G_P^{\xi_i}, G_P^{\zeta_i}), (U_P^{\xi_i}, U_P^{\zeta_i}), (F_P^{\xi_i}, F_P^{\zeta_i})\right]$ be a group of PSVNZNs. If

$$\begin{aligned} P_{Z(\min)} &= \left[\min_i \left(T_{P_i}(\xi, \zeta)\right), \min_i \left(C_{P_i}(\xi, \zeta)\right), \min_i \left(G_{P_i}(\xi, \zeta)\right), \min_i \left(U_{P_i}(\xi, \zeta)\right), \min_i \left(F_{P_i}(\xi, \zeta)\right)\right] \\ &= \left[\left(\min_i \left(T_P^{\xi_i}\right), \min_i \left(T_P^{\zeta_i}\right)\right), \left(\min_i \left(C_P^{\xi_i}\right), \min_i \left(C_P^{\zeta_i}\right)\right), \left(\min_i \left(G_P^{\xi_i}\right), \min_i \left(G_P^{\zeta_i}\right)\right), \right. \\ &\quad \left. \left(\min_i \left(U_P^{\xi_i}\right), \min_i \left(U_P^{\zeta_i}\right)\right), \left(\min_i \left(F_P^{\xi_i}\right), \min_i \left(F_P^{\zeta_i}\right)\right)\right]. \end{aligned}$$

and

$$\begin{aligned} P_{Z(\max)} &= \left[\max_i \left(T_{P_i}(\xi, \zeta)\right), \max_i \left(C_{P_i}(\xi, \zeta)\right), \max_i \left(G_{P_i}(\xi, \zeta)\right), \max_i \left(U_{P_i}(\xi, \zeta)\right), \max_i \left(F_{P_i}(\xi, \zeta)\right)\right] \\ &= \left[\left(\max_i \left(T_P^{\xi_i}\right), \max_i \left(T_P^{\zeta_i}\right)\right), \left(\max_i \left(C_P^{\xi_i}\right), \max_i \left(C_P^{\zeta_i}\right)\right), \left(\max_i \left(G_P^{\xi_i}\right), \max_i \left(G_P^{\zeta_i}\right)\right), \right. \\ &\quad \left. \left(\max_i \left(U_P^{\xi_i}\right), \max_i \left(U_P^{\zeta_i}\right)\right), \left(\max_i \left(F_P^{\xi_i}\right), \max_i \left(F_P^{\zeta_i}\right)\right)\right]. \end{aligned}$$

then

$$P_{z(\min)} \leqslant PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant P_{z(\max)}$$

Theorem 16 (Monotonicity). Let P_{Z_i} and $P^*_{Z_i}$ be two groups of PSVNZNs. If $P_{Z_i} \leq P^*_{Z_i}$ then $PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \leqslant PSVNZNWHA\left(P^*_{Z_1}, P^*_{Z_2}, ..., P^*_{Z_n}\right)$.

Theorem 17 (Shift Invariant) Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

and $\gamma = \left[\left(T_\gamma^{\xi}, T_\gamma^{\zeta}\right), \left(C_\gamma^{\xi}, C_\gamma^{\zeta}\right), \left(G_\gamma^{\xi}, G_\gamma^{\zeta}\right), \left(U_\gamma^{\xi}, U_\gamma^{\zeta}\right), \left(F_\gamma^{\xi}, F_\gamma^{\zeta}\right)\right]$ be two PSVNZNs. Then

$$PSVNZNWHA\left(P_{Z_1} \oplus \gamma, P_{Z_2} \oplus \gamma, ..., P_{Z_n} \oplus \gamma\right) = PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) \oplus \gamma$$

Theorem 18 (Homogeneity). Let

$$P_{Z_i} = \left[\left(T_P^{\xi_i}, T_P^{\zeta_i}\right), \left(C_P^{\xi_i}, C_P^{\zeta_i}\right), \left(G_P^{\xi_i}, G_P^{\zeta_i}\right), \left(U_P^{\xi_i}, U_P^{\zeta_i}\right), \left(F_P^{\xi_i}, F_P^{\zeta_i}\right)\right]$$

and

$$\gamma = \left[\left(T_\gamma^\xi, T_\gamma^\zeta\right), \left(C_\gamma^\xi, C_\gamma^\zeta\right), \left(G_\gamma^\xi, G_\gamma^\zeta\right), \left(U_\gamma^\xi, U_\gamma^\zeta\right), \left(F_\gamma^\xi, F_\gamma^\zeta\right)\right]$$

be two PSVNZNs. Then

$$PSVNZNWHA\left(\gamma P_{Z_1}, \gamma P_{Z_2}, ..., \gamma P_{Z_n}\right) = \gamma PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right).$$

The proof of Theorems 14–17 is similar to that of Theorems 2–6.

3.7. *Mathematical relationship between the operators*

Theorem 19. The PSVNZNWHA operator is a generalization of the PSVNZNWGA and PSVNZNWAA operators. The aggregated value produced by the PSVNZNWHA operator is bounded by the values obtained from the PSVNZNWGA and PSVNZNWAA operators. The parameter $\upsilon \in [0,1]$ controls the degree of influence of each operator. Specifically, when $\upsilon = 1$, the PSVNZNWHA operator reduces to the PSVNZNWAA operator, and when $\upsilon = 0$, it becomes the PSVNZNWGA operator.Mathematically, we can express this relationship as:

$$\begin{aligned} S\left(PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right)\right) &\leqslant S\left(PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right)\right) \\ &\leqslant S\left(PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right)\right) \end{aligned}$$

Proof. We know that

$$\prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i} \text{where} \prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i} \geqslant 0 \,\text{and}\, 1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1$$

and

$$\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i} \text{where} \prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i} \geqslant 0 \,\text{and}\, 1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1.$$

Raising the power $1-\upsilon$ on both sides of the above expression we get

$$\left(\prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon}, \left(\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon}$$

The above equation reduces to

$$\left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right),$$

$$\left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right).$$

Similarly, we can write

$$\prod_{i=1}^{n}\left(C_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i},$$

$$\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i},$$

$$\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i},$$

$$\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i} \leqslant 1-\prod_{i=1}^{n}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}.$$

By replacing $T_P^{\xi_i}=C_P^{\xi_i}$ and $T_P^{\zeta_i}=C_P^{\zeta_i}$, we get

$$\left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)$$

$$\left(1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \leqslant \left(1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)$$

Replacing $1-T_P^{\xi_i}$ and $1-T_P^{\zeta_i}$ by $G_P^{\xi_i}$ and $G_P^{\zeta_i}$, we get

$$1-\left(1-\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}$$

$$1-\left(1-\prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}$$

Similarly we can obtain

$$1-\left(1-\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i},$$

$$1-\left(1-\prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i},$$

$$1-\left(1-\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i},$$

$$1-\left(1-\prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\upsilon}\left(\prod_{i=1}^{n}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\upsilon} \geqslant \prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}.$$

Using the score function $S\left(PSVNZNWAA\left(P_{Z_1},P_{Z_2},...,P_{Z_n}\right)\right)$

$$= \left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)\left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)+\left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)\left(1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)$$
$$-\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}\prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}-\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}\prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}-\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}\prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}$$
$$\geqslant \left(\left(1-\prod_{i=1}^{n}\left(1-T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(T_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}\left(1-\prod_{i=1}^{n}\left(1-T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(T_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)$$
$$+\left(1-\prod_{i=1}^{n}\left(1-C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(C_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}\left(1-\prod_{i=1}^{n}\left(1-C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(C_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}$$
$$-\left(1-\left(1-\prod_{i=1}^{n}\left(G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-G_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)\left(1-\left(1-\prod_{i=1}^{n}\left(G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-G_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)$$
$$-\left(1-\left(1-\prod_{i=1}^{n}\left(U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-U_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)\left(1-\left(1-\prod_{i=1}^{n}\left(U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-U_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)$$
$$-\left(1-\left(1-\prod_{i=1}^{n}\left(F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-F_P^{\xi_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right)\left(1-\left(1-\prod_{i=1}^{n}\left(F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{\nu}\left(\prod_{i=1}^{n}\left(1-F_P^{\zeta_i}\right)^{\varepsilon_i}\right)^{1-\nu}\right).$$

Similarly, we can prove

$$S\left(PSVNZNWGA\left(P_{Z_1},P_{Z_2},...,P_{Z_n}\right)\right)\leqslant S\left(PSVNZNWHA\left(P_{Z_1},P_{Z_2},...,P_{Z_n}\right)\right).$$

4. MCDM application based PSVNZN Approach and score function

This section presents the application of our proposed $PSVNZN_{WAA}$, $PSVNZN_{WGA}$, and $PSVNZN_{WHA}$ operators, in conjunction with a novel score function, to address MCDM problems. The proposed framework is specifically designed to accommodate decision scenarios where information is expressed in terms of PSVNZN, each associated with a corresponding reliability measure. Consider an MCDM problem involving a set of m alternatives, denoted as $A=\{A_1,A_2,...,A_m\}$, evaluated against a set of n criteria, $C=\{C_1,C_2,...,C_n\}$. Decision-makers assign weights ε_i to each criterion C_i $(i=1,2,...,n)$ using a weight vector $W=\{w_1,w_2,...,w_n\}$. The performance of each alternative A_j $(j=1,2,...,m)$ with respect to each criterion C_i is assessed by decision-makers using ordered pairs of fuzzy values. The given information can be represented by an PSVNZN, as follows:

$$P_{Z_{ji}}=\left[\left(T_P^{\xi_{ji}},T_P^{\zeta_{ji}}\right),\left(C_P^{\xi_{ji}},C_P^{\zeta_{ji}}\right),\left(G_P^{\xi_{ji}},G_P^{\zeta_{ji}}\right),\left(U_P^{\xi_{ji}},U_P^{\zeta_{ji}}\right),\left(F_P^{\xi_{ji}},F_P^{\zeta_{ji}}\right)\right]$$

To facilitate the decision-making process, the decision-makers' assessments of the alternatives are structured into a decision matrix, denoted as $P_Z=[P_{Z_{ji}}]_{m\times n}$. Each element $P_{Z_{ji}}$ within this matrix represents the evaluation of alternative A_j with respect to criterion C_i and is expressed using the innovative concept of PSVNZNs.

Algorithm:

(1) Employ one of the aggregation operators to aggregate the individual criteria evaluations $P_{Z_{ji}}$ for each alternative A_j, resulting in an overall performance P_{Z_j}.

- $PSVNZNWAA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \sum_{i=1}^{n} \varepsilon_i P_{Z_i}$
- $PSVNZNWGA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \prod_{i=1}^{n} P_{Z_i}^{\varepsilon_i}$
- $PSVNZNWHA\left(P_{Z_1}, P_{Z_2}, ..., P_{Z_n}\right) = \left(\sum_{i=1}^{n} \varepsilon_i P_{Z_i}\right)^{v} \otimes \left(\prod_{i=1}^{n} P_{Z_i}^{\varepsilon_i}\right)^{1-v}$,

(2) Calculate the score value of each alternative P_{Z_j} using the proposed score function.

$$S\left(P_{Z_i}\right) = T_P^{\xi_i} T_P^{\zeta_i} + C_P^{\xi_i} C_P^{\zeta_i} - G_P^{\xi_i} G_P^{\zeta_i} - U_P^{\xi_i} U_P^{\zeta_i} - F_P^{\xi_i} F_P^{\zeta_i}$$

(3) Rank the alternatives based on their calculated score values, with the highest score indicating the best alternative.

5. An illustrative example : Evaluation of EV

This section presents a MCDM problem which deals with the performance of 6 EV models EV_1, EV_2, EV_3, EV_4, EV_5, and EV_6. To evaluate these EVs, four critical criteria have been identified : C1-Battery Range, C2-Charging Time, C3-Safety Features, C4- Maintenance Cost.

	C_1	C_2	C_3	C_4
EV_1	(0.7, 0.8), (0.7, 0.7), (0.1, 0.2), (0.2, 0.4), (0.3, 0.5)	(0.8, 0.8), (0.5, 0.8), (0.5, 0.6), (0.4, 0.3), (0.6, 0.5)	(0.6, 0.7), (0.7, 0.8), (0.5, 0.6), (0.3, 0.2), (0.5, 0.4)	(0.5, 0.6), (0.6, 0.7), (0.4, 0.5), (0.3, 0.4), (0.2, 0.1)
EV_2	(0.8, 0.9), (0.7, 0.6), (0.4, 0.3), (0.2, 0.2), (0.3, 0.4)	(0.7, 0.9), (0.5, 0.7), (0.4, 0.4), (0.3, 0.2), (0.4, 0.3)	(0.4, 0.5), (0.3, 0.4), (0.7, 0.6), (0.7, 0.9), (0.8, 0.9)	(0.3, 0.2), (0.2, 0.3), (0.7, 0.6), (0.7, 0.9), (0.5, 0.6)
EV_3	(0.3, 0.4), (0.5, 0.5), (0.8, 0.9), (0.8, 0.9), (0.7, 0.8)	(0.2, 0.4), (0.4, 0.6), (0.7, 0.7), (0.8, 0.9), (0.5, 0.6)	(0.8, 0.7), (0.7, 0.6), (0.2, 0.3), (0.3, 0.3), (0.4, 0.5)	(0.7, 0.6), (0.6, 0.5), (0.1, 0.2), (0.2, 0.2), (0.3, 0.4)
EV_4	(0.2, 0.3), (0.4, 0.5), (0.7, 0.8), (0.9, 0.9), (0.8,0.9)	(0.3, 0.3), (0.4, 0.5), (0.7, 0.5), (0.7, 0.5), (0.6,0.4)	(0.6, 0.6), (0.6, 0.4), (0.3, 0.4), (0.3, 0.2), (0.4,0.5)	(0.5, 0.5), (0.6, 0.4), (0.3, 0.4), (0.3, 0.2), (0.7,0.8)
EV_5	(0.7, 0.6), (0.7, 0.7), (0.4, 0.6), (0.6, 0.5), (0.8,0.9)	(0.6, 0.6), (0.7, 0.6), (0.4, 0.6), (0.6, 0.9), (0.5,0.4)	(0.3, 0.2), (0.4, 0.3), (0.7, 0.9), (0.9, 0.9), (0.8,0.5)	(0.2, 0.1), (0.3, 0.2), (0.7, 0.9), (0.7, 0.7), (0.8,0.9)
EV_6	(0.2, 0.4), (0.4, 0.5), (0.7, 0.8), (0.8, 0.9), (0.5,0.8)	(0.3, 0.4), (0.5, 0.6), (0.6, 0.5), (0.7, 0.9), (0.8,0.9)	(0.2, 0.2), (0.3, 0.3), (0.6, 0.8), (0.7, 0.8), (0.5,0.6)	(0.3, 0.3), (0.4, 0.4), (0.7, 0.9), (0.8, 0.9), (0.5,0.8)

Table 1. Preference Ratings of EV Alternatives in Terms of PSVNZNs

The weight vector is given by $\varepsilon = (0.4, 0.2, 0.3, 0.1)$ and $\upsilon = 0.5$ The preference information for each EV $(EV_1, EV_2, EV_3, EV_4, EV_5, EV_6)$ with respect to each criterion (C_1, C_2, C_3, C_4) is presented in Table 1 in the form of PSVNZNs. This matrix encapsulates the expert evaluations and preferences for each EV-criterion pair, enabling a comprehensive comparison. By applying our proposed scoring function, we calculated the scores for each EV under three different aggregation scenarios. The results are summarized in Table 3. The ranking order of the 6 alternatives $EV_1 > EV_2 > EV_3 > EV_4 > EV_5 > EV_6$. Our analysis reveals that EV_1 consistently achieves the highest score across all scenarios. This indicates that EV_1 is the most reliable and suitable choice among the evaluated EVs.

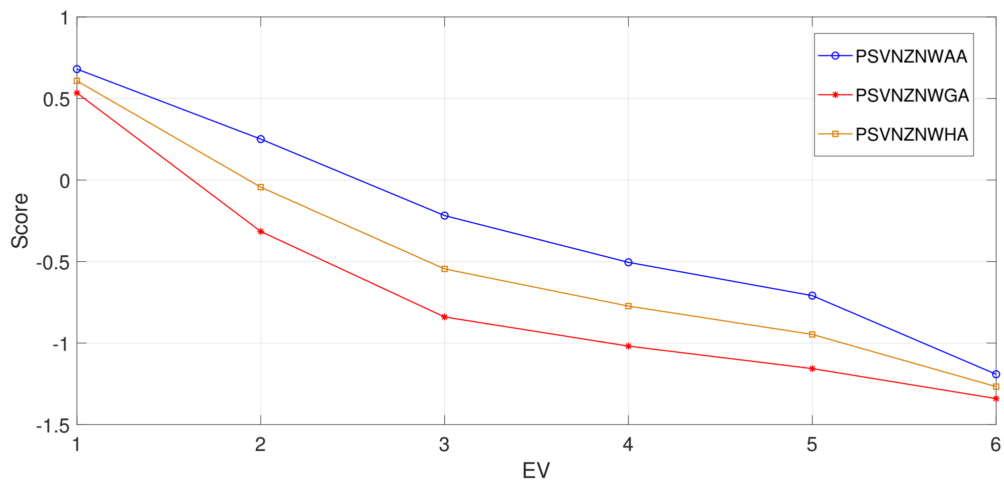

FIGURE 1. Score Values for EVs using Different Operators

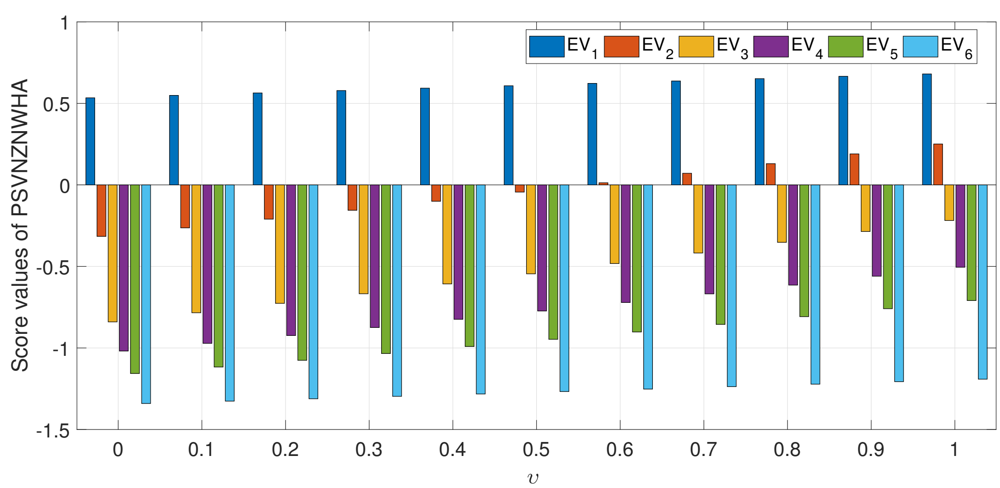

FIGURE 2. Score Values of PSVNZNWHA vs υ

EV \Operators	PSVNZNWAA	PSVNZNWGA	PSVNZNWHA
EV_1	(0.6826, 0.7579), (0.6580, 0.7551), (0.2569, 0.3797), (0.2702, 0.3067), (0.3857, 0.3981)	(0.6637, 0.7468), (0.6444, 0.7483), (0.3558, 0.4603), (0.2840, 0.3254), (0.4266, 0.4399)	(0.6731, 0.7523), (0.6512, 0.7517), (0.3081, 0.4214), (0.2771, 0.3162), (0.4065, 0.4194)
EV_2	(0.6582, 0.8005), (0.5274, 0.5490), (0.5004, 0.4193), (0.3580, 0.3650), (0.4488, 0.5016),	(0.5736, 0.6491), (0.4478, 0.5112), (0.5453, 0.4574), (0.4739, 0.6518), (0.5493, 0.6529)	(0.6144, 0.7208), (0.4860, 0.5298), (0.5233, 0.4387), (0.4188, 0.5298), (0.5016, 0.5840)
EV_3	(0.5464, 0.5320), (0.5649, 0.5528), (0.4174, 0.5296), (0.5189, 0.5569), (0.5084, 0.6120)	(0.4041, 0.4927), (0.5387, 0.5477), (0.6179, 0.7250), (0.6655, 0.7793), (0.5548, 0.6625)	(0.4699, 0.5120), (0.5517, 0.5502), (0.5282, 0.6404), (0.5988, 0.6873), (0.5321, 0.6381)
EV_4	(0.3964, 0.4278), (0.4898, 0.4622), (0.4988, 0.5519), (0.5515, 0.4384), (0.6053, 0.6340)	(0.3305, 0.3887), (0.4704, 0.4573), (0.5790, 0.6272), (0.7287, 0.6830), (0.6674, 0.7514)	(0.3619, 0.4078), (0.4800, 0.4597), (0.5406, 0.5913), (0.6512, 0.5781), (0.6377, 0.6984),
EV_5	(0.5480, 0.4659), (0.5980, 0.5480), (0.5004, 0.7056), (0.6881, 0.6938), (0.7282, 0.6340),	(0.4644, 0.3607), (0.5437, 0.4644), (0.5453, 0.7703), (0.7436, 0.7875), (0.7598, 0.7514)	(0.5045, 0.4100), (0.5702, 0.5045), (0.5233, 0.7400), (0.7172, 0.7449), (0.7445, 0.6984)
EV_6	(0.2314, 0.3358), (0.3941, 0.4613), (0.5264, 0.7369), (0.7483, 0.8688), (0.5493, 0.7513)	(0.2259, 0.3157), (0.3837, 0.4351), (0.5903, 0.7759), (0.7551, 0.8769), (0.5837, 0.7856)	(0.2286, 0.3256), (0.3889, 0.4480), (0.5595, 0.7571), (0.7517, 0.8729), (0.5668, 0.7691)

Table 2. Aggregated values of the EV

5.1. *Sensitive analysis*

This section analysis two key factors. Firstly, variation in the score values of the operator PSVNZNWHA for different values of the parameter υ. Secondly, a change in the criteria weights affects the score values for each EV are aggregated under the operators. Table 4 reveals the variation in PSVNZNWHA score values for different EVs as the parameter υ ranges from 0 to 1. EV_1 demonstrates a clear positive trend, with its PSVNZNWHA values

	Operators		
Score values	$PSVNZNWAA$	$PSVNZNWGA$	$PSVNZNWHA$
$S(EV_1)$	0.680273072	0.534038504	0.607969219
$S(EV_2)$	0.250801858	-0.31563944	-0.044048216
$S(EV_3)$	-0.218197152	-0.839919065	-0.545217105
$S(EV_4)$	-0.504936422	-1.018728374	-0.773173466
$S(EV_5)$	-0.709114935	-1.156474936	-0.946955968
$S(EV_6)$	-1.191194441	-1.340429022	-1.267111309

TABLE 3. Score Values of the operators

υ	$S(EV_1)$	$S(EV_2)$	$S(EV_3)$	$S(EV_4)$	$S(EV_5)$	$S(EV_6)$	Rank
0	0.5340	-0.3156	-0.8399	-1.0187	-1.1565	-1.3404	
0.1	0.5489	-0.2633	-0.7839	-0.9717	-1.1167	-1.3260	
0.2	0.5638	-0.2100	-0.7264	-0.9236	-1.0758	-1.3114	
0.3	0.5786	-0.1556	-0.6674	-0.8745	-1.0340	-1.2968	
0.4	0.5933	-0.1003	-0.6070	-0.8243	-0.9910	-1.2820	$EV_1 > EV_2 > EV_3 > EV_4 > EV_5 > EV_6$
0.5	0.6080	-0.0440	-0.5452	-0.7732	-0.9470	-1.2671	
0.6	0.6226	0.0132	-0.4821	-0.7211	-0.9018	-1.2521	
0.7	0.6371	0.0713	-0.4177	-0.6681	-0.8554	-1.2371	
0.8	0.6516	0.1303	-0.3522	-0.6144	-0.8079	-1.2219	
0.9	0.6660	0.1902	-0.2856	-0.5599	-0.7591	-1.2066	
1	0.6803	0.2508	-0.2182	-0.5049	-0.7091	-1.1912	

TABLE 4. Score values of PSVNZNWHA for various υ with weights $\varepsilon = (0.4, 0.2, 0.3, 0.1)$

consistently increasing as υ grows, indicating strong and improving performance. EV_2 begins with negative values for PSVNZNWHA at lower υ values, transitions to positive values around $\upsilon = 0.6$, and continues to show steady improvement thereafter.

In contrast, EV_3, EV_4, EV_5, and EV_6 maintain predominantly negative PSVNZNWHA values across the range of υ. However, the magnitude of negativity for these models decreases slightly as υ increases, indicating marginal improvement. Among these, EV_6 performs the worst, with the most negative PSVNZNWHA values throughout the range, while EV_3, EV_4,

	Operators		
Score Values	$PSVNZNWAA$	$PSVNZNWGA$	$PSVNZNWHA$
$S(EV_1)$	0.539492046	0.401504237	0.470465988
$S(EV_2)$	-0.209214306	-0.787338533	-0.520011862
$S(EV_3)$	0.1831887	-0.316442285	-0.072382084
$S(EV_4)$	-0.161957724	-0.445789819	-0.304691556
$S(EV_5)$	-0.992134345	-1.353170293	-1.183130234
$S(EV_6)$	-1.136478808	-1.314663194	-1.227332371

TABLE 5. Score Values of the operators

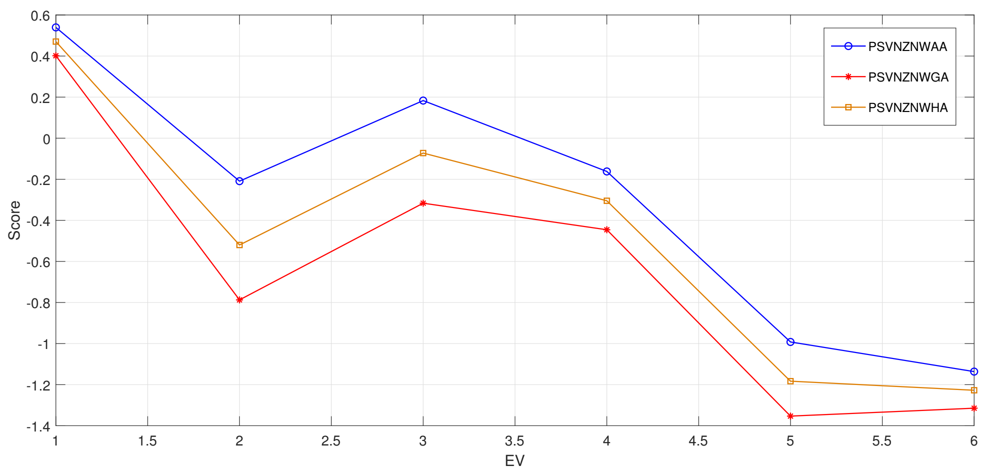

FIGURE 3. Score Values for EVs using PSVNZNWAA, PSVNZNWGA & PSVNZNWHA

and EV_5 show slightly better but still negative results. Overall, EV_1 is the best performer, exhibiting a steady upward trend in PSVNZNWHA values, followed by EV_2, which shows moderate improvement. The remaining models lag behind significantly.

Table 5 is presented with the weight vector $\varepsilon = (0.1, 0.3, 0.4, 0.2)$ and $\upsilon = 0.5$, yielding the ranking order $EV_1 > EV_3 > EV_4 > EV_2 > EV_5 > EV_6$. Across both Table 3 and Table 4, EV_1 consistently ranks first, demonstrating its robustness and superiority regardless of the weight distribution for criteria. This consistency highlights EV_1 as the most favorable alternative among the six options. However, the rankings of mid-tier alternatives (EV_2, EV_3, and EV_4) show significant variability. In Table 3, where the first criterion has the highest weight (0.4),

υ	$S(\text{EV}_1)$	$S(\text{EV}_2)$	$S(\text{EV}_3)$	$S(\text{EV}_4)$	$S(\text{EV}_5)$	$S(\text{EV}_6)$	Rank
0	0.4015	-0.7873	-0.3164	-0.4458	-1.3532	-1.3147	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_6 > \text{EV}_5$
0.1	0.4153	-0.7371	-0.2689	-0.4178	-1.3207	-1.2975	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_6 > \text{EV}_5$
0.2	0.4291	-0.6853	-0.2207	-0.3897	-1.2875	-1.2802	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_6 > \text{EV}_5$
0.3	0.4429	-0.6319	-0.1718	-0.3614	-1.2535	-1.2627	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.4	0.4567	-0.5768	-0.1224	-0.3331	-1.2187	-1.2451	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.5	0.4705	-0.5200	-0.0724	-0.3047	-1.1831	-1.2273	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.6	0.4843	-0.4615	-0.0219	-0.2762	-1.1467	-1.2094	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.7	0.4981	-0.4012	0.0290	-0.2477	-1.1094	-1.1914	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.8	0.5119	-0.3390	0.0802	-0.2191	-1.0712	-1.1732	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
0.9	0.5257	-0.2751	0.1316	-0.1905	-1.0322	-1.1549	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$
1	0.5395	-0.2092	0.1832	-0.1620	-0.9921	-1.1365	$\text{EV}_1 > \text{EV}_3 > \text{EV}_4 > \text{EV}_2 > \text{EV}_5 > \text{EV}_6$

TABLE 6. Score values of PSVNZNWHA for various υ with weights $W = (0.1, 0.3, 0.4, 0.2)$

EV_2 ranks second, leveraging its strong performance in that criterion. Conversely, in Table 4, where the third criterion has the highest weight (0.4), EV_3 overtakes EV_2 to secure the second position. This shift illustrates the sensitivity of the rankings to the weight distribution of criteria. Meanwhile, EV_5 and EV_6 consistently rank at the bottom in both tables, reflecting their inability to compete across all criteria configurations.

From Table 6 it is observed that, across all values of υ, EV_1 consistently achieves the highest score, indicating its dominance under the given weight distribution $\varepsilon = (0.1, 0.3, 0.4, 0.2)$. The impact of υ on the rankings is evident:

$$\text{Ranking of } EV = \begin{cases} EV_1 > EV_3 > EV_4 > EV_2 > EV_6 > EV_5, & \text{for } \upsilon \leq 0.2, \\ EV_1 > EV_3 > EV_4 > EV_2 > EV_5 > EV_6, & \text{for } \upsilon \geq 0.3. \end{cases}$$

The scores of EV_1, EV_3, EV_4, and EV_2 increase with increasing υ, suggesting a positive influence of υ on their performance. In contrast, the scores of EV_5 and EV_6 decrease as υ increases, indicating a negative effect. Notably, the rankings of EV_1, EV_3, EV_4, and EV_2 remain stable across all values of υ, demonstrating consistency in their comparative positions.

The score difference between EV_1 and the other events widens with increasing υ, underscoring its robustness across the parameter range. Additionally, the scores of EV_5 and EV_6 are closely spaced, particularly for $\upsilon \geq 0.3$, leading to a change in their rankings. The weight distribution $\varepsilon = (0.1, 0.3, 0.4, 0.2)$ appears to favor EV_1, which remains the top performer for all

υ. These observations highlight the stability of EV_1 and the dynamic behavior of lower-ranked events like EV_5 and EV_6 as υ varies.

These results underscore the importance of weight selection in multi-criteria decision-making, as changes in criteria prioritization significantly influence the ranking of alternatives. While mid-tier alternatives are sensitive to weight variations, EV_1 emerges as the most robust and reliable choice across all configurations. This analysis highlights the need for careful consideration of criteria weights to ensure alignment with decision-making objectives.

6. Conclusion

This research introduces a novel decision-making framework utilizing PSVNZNs. By integrating the granular uncertainty representation of PNSs with the reliability of Z-numbers, PSVNZNs offer a powerful tool for addressing complex decision-making challenges. The proposed score function and aggregation operators—weighted arithmetic, geometric, and hybrid averages—provide reliable methods for evaluating and aggregating information. These operators have been rigorously validated to ensure robustness and consistency. The practical application of the framework is demonstrated through a case study on electric vehicle selection, showcasing the ability of PSVNZNs to handle complex uncertainties and imprecisions effectively.

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