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Generalized Neutrosophic Sets and Its Application in KU-Algebras

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Abstract. The KU-algebras are given a $GNeuS$. In KU-algebras, the notions of $GNeuSubAlg$'s and GNI's are introduced, and related properties are explored. The $GNeuSubAlg/GNI$ is studied. There is a discussion of the relationship between $GNeuSubAlg$ and GNI. Conditions for a $GNeuSubAlg$ to be a GNI are given in a KU-algebra. There are also conditions for a $GNeuS$ to be a GNI. We investigate the homomorphic image and preimage of the GNI.

Keywords: $GNeuS$, $GNeuSubAlg$, GNI.

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1. Introduction

In 1965, Zadeh [11] developed the fuzzy set and introduced the degree of membership/truth (t). In 1986, Atanassov cite9 established the degree of non membership/falsehood (f) as a generalisation of fuzzy sets and developed the intuitionistic fuzzy set. In 1995, Smarandache [7] developed the neutrosophic set on three components (t, i, f)=(truth, indeterminacy, falsehood) and introduced the degree of indeterminacy/neutrality (i) as an independent component. Smarandache [7, 8] introduced the notion of neutrosophic set (NS), which is a more general platform that extends the concepts of classic set and fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set. Neutrosophic set theory is used in several parts (see http://fs.gallup.unm.edu/neutrosophy.htm for more information).

The notions of neutrosophic ideals of neutrosophic KU-algebras were introduced by Bijan Davvaz et al. [3]. Single valued neutrosophic sub-implicative ideals of KU-algebras were studied by Abd El-Baseer and Mostafa [1]. Vasu and Ramesh Kumar [10] introduced the concepts of neutrosophic implicative N-ideals in KU-algebras.

In this paper, we consider a generalisation of Smarandache's NS's. The concept of $GNeuS$'s is introduced, and it is used to KU-algebras. In KU-algebras, we present the concepts of $GNeuSubAlg$ and GNI's, as well as their related characteristics. The relationship between $GNeuSubAlg$ and GNI is examined, as well as characterizations of $GNeuSubAlg$. In a KU-algebra, we define conditions for a $GNeuSubAlg$ to be a GNI. We also discuss the homomorphic image and preimage of a GNI, as well as the conditions for a $GNeuS$ to be a GNI.

2. Preliminaries

We let $L(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. A KU-algebra (briefly, KU-alg) [5,6] on a system $P = (P, \diamond, 0) \in L(\tau)$ satisfies

(KU1) $(k_{01} \diamond k_{02}) \diamond ((k_{02} \diamond k_{03}) \diamond (k_{01} \diamond k_{03})) = 0$,

(KU2) $k_{01} \diamond 0 = 0$,

(KU3) $0 \diamond k_{01} = k_{01}$,

(KU4) $k_{01} \diamond k_{02} = 0$ & $k_{02} \diamond k_{01} = 0$ implies $k_{01} = k_{01}$,

(KU5) $k_{01} \diamond k_{01} = 0$, $\forall\ k_{01}, k_{02}, k_{03} \in P$.

Also a binary relation $\leq$ by putting $k_{01} \leq k_{02} \Leftrightarrow k_{02} \diamond k_{01} = 0$, $\forall\ k_{01}, k_{02} \in P$.

In a KU-algebra P, the following hold:

(KU1′) $(k_{02} \diamond k_{03}) \diamond (k_{01} \diamond k_{03}) \leq (k_{01} \diamond k_{02})$,

(KU2′) $0 \leq k_{01}$,

(KU3′) $k_{01} \leq k_{02}$, $k_{02} \leq k_{01}$ implies $k_{01} = k_{02}$,

(KU4′) $k_{02} \diamond k_{01} \leq k_{01}$.

Theorem 2.1. [4] A KU-alg P satisfies the following axioms.: $\forall\ k_{01}, k_{02}, k_{03} \in P$,

(i) $k_{01} \leq k_{02}$ imply $k_{02} \diamond k_{03} \leq k_{01} \diamond k_{03}$,

(ii) $k_{01} \diamond (k_{02} \diamond k_{03}) = k_{02} \diamond (k_{01} \diamond k_{03})$, $\forall\ k_{01}, k_{02}, k_{03} \in P$,

(iii) $((k_{02} \diamond k_{01}) \diamond k_{01}) \leq k_{02}$,

(iv) $(((k_{02} \diamond k_{01}) \diamond k_{01}) \diamond k_{01}) = (k_{02} \diamond k_{01})$.

Definition 2.2. [5,6] A non-empty subset S of a KU-algebra P is called a KU-subalgebra (briefly. KU-subalg) of P if $l_{11} \diamond l_{22} \in S\ \forall\ l_{11}, l_{22} \in S$.

Definition 2.3. [5,6] A subset S of a KU-alg P is called an ideal of P if it satisfies the following:

(I1) $0 \in S$,

(I2) $(\forall\ k_{01}, k_{02} \in P)$ $(k_{01} \diamond k_{02} \in S, k_{01} \in S \Rightarrow k_{01} \in S)$.

For any family $\{c_k \mid k \in \Lambda\}$ of real numbers, we define

$$\vee\{c_k \mid k \in \Lambda\} := \begin{cases} \max\{c_k \mid k \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{c_k \mid k \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\wedge\{c_k \mid k \in \Lambda\} := \begin{cases} \min\{c_k \mid k \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{c_k \mid k \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $c_1 \vee c_2$ and $c_1 \wedge c_2$ instead of $\{c_k \mid k \in \Lambda\}$ and $\wedge\{c_k \mid k \in \Lambda\}$, respectively.

By a fuzzy set in a nonempty set P we mean a function $\mu : P \to [0, 1]$, and the complement of μ, denoted by μ^c, is the fuzzy set in P given by $\mu^c(l) = 1 - \mu(l)$ for all $l \in P$. A fuzzy set μ in a KU-alg P is called a fuzzy subalgebra of P if $\mu(l * n) \geq \mu(l) \wedge \mu(n)$ for all $l, n \in P$. A fuzzy set μ in a KU-alg P is called a fuzzy ideal of P if

$$(\forall\, l \in P)(\mu(0) \geq \mu(l)), \tag{1}$$

$$(\forall\, l, n \in P)(\mu(l) \geq \mu(n * l) \wedge \mu(n)) \tag{2}$$

Let P be a non-empty set. A neutrosophic set (NS) in P [7] is a structure of the form:

$$L := \{\langle l; L_T(l), L_I(l), L_F(l)\rangle \mid l \in P\}$$

where $L_T : P \to [0, 1]$ is a truth membership function, $L_I : P \to [0, 1]$ is an indeterminate membership function, and $L_F : P \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $L = (L_T, L_I, L_F)$ for the neutrosophic set

$$L := \{\langle l; L_T(l), L_I(l), L_F(l)\rangle \mid l \in P\}.$$

Definition 2.4. [9] A generalized neutrosophic set $(GNeuS)$ in a non-empty set P is a structure of the form:

$$L := \{\langle l; L_T(l), L_{IT}(l), L_{IF}(l), L_F(l)\rangle \mid l \in P, L_{IT}(l) + L_{IF}(l) \leq 1\}$$

where $L_T : P \to [0, 1]$ is a truth membership function, $L_F : P \to [0, 1]$ is a false membership function, $L_{IT} : P \to [0, 1]$ is an indeterminate membership function which is familiar with truth membership function, and $L_{IF} : P \to [0, 1]$ is an indeterminate membership function which is familiar with false membership function.

3. Applications in KU-algs

Unless otherwise stated, let P signify a KU-alg in the following.

Definition 3.1. A $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P is called a generalized neutrosophic subalgebra (briefly, $GNeuSubAlg$) of P if the following conditions are valid.

$$(\forall\ \zeta_{01}, \zeta_{02} \in P) \begin{pmatrix} L_T(\zeta_{01} * \zeta_{02}) \geq L_T(\zeta_{01}) \wedge L_T(\zeta_{02}) \\ L_{IT}(\zeta_{01} * \zeta_{02}) \geq L_{IT}(\zeta_{01}) \wedge L_{IT}(\zeta_{02}) \\ L_{IF}(\zeta_{01} * \zeta_{02}) \leq L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02}) \\ L_F(\zeta_{01} * \zeta_{02}) \leq L_F(\zeta_{01}) \vee L_F(\zeta_{02}) \end{pmatrix}. \tag{3}$$

Example 3.2. Consider a set $P = \{ll_0, ll_a, ll_b, ll_c\}$ with the binary operator $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$".

$*$	ll_0	ll_a	ll_b	ll_c
ll_0	ll_0	ll_a	ll_b	ll_c
ll_a	ll_0	ll_0	ll_a	ll_c
ll_b	ll_0	ll_0	ll_0	ll_c
ll_c	ll_0	ll_a	ll_b	ll_0

$(P;\ *, 0)$ is a KU-alg. Then the $GNeuS$

$L = \{\langle ll_0; 0.7, 0.6, 0.3, 0.2\rangle, \langle ll_a; 0.3, 0.5, 0.4, 0.6\rangle, \langle ll_b; 0.1, 0.6, 0.3, 0.4\rangle, \langle ll_c; 0.5, 0.5, 0.4, 0.7\rangle\}$ in P is a $GNeuSubAlg$ of P.

Given a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P and $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$, consider the following sets.

$$U(T, \lambda_T) := \{\zeta_{01} \in P \mid L_T(\zeta_{01}) \geq \lambda_T\},$$

$$U(IT, \lambda_{IT}) := \{\zeta_{01} \in P \mid L_{IT}(\zeta_{01}) \geq \lambda_{IT}\},$$

$$L(F, \mu_F) := \{\zeta_{01} \in P \mid L_F(\zeta_{01}) \leq \mu_F\},$$

$$L(IF, \mu_{IF}) := \{\zeta_{01} \in P \mid L_{IF}(\zeta_{01}) \leq \mu_{IF}\}.$$

Theorem 3.3. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of P, then the set $U(T, \lambda_T)$, $U(IT, \lambda_{IT}), L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are subalg's of P $\forall\ \lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$ whenever they are non-empty.

Proof. Assume that $U(T, \lambda_T), U(IT, \lambda_{IT}), L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are nonempty $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$. Let $\zeta_{01}, \zeta_{02} \in P$. If $\zeta_{01}, \zeta_{02} \in U(T, \lambda_T)$, then $L_T(\zeta_{01}) \geq \lambda_T$ and $L_T(\zeta_{02}) \geq \lambda_T$. It follows that

$$L_T(\zeta_{01} * \zeta_{02}) \geq L_T(\zeta_{01}) \wedge L_T(\zeta_{02}) \geq \lambda_T$$

and so that $\zeta_{01} * \zeta_{02} \in U(T, \lambda_T)$. Hence $U(T, \lambda_T)$ is a subalg of P. Similarly, if $\zeta_{01}, \zeta_{02} \in U(IT, \lambda_{IT})$, then $\zeta_{01} * \zeta_{02} \in U(IT, \lambda_{IT})$, that is, $U(IT, \lambda_{IT})$ is a subalg of P. Suppose that $\zeta_{01}, \zeta_{02} \in L(F, \mu_F)$. Then $L_F(\zeta_{01}) \leq \mu_F$ and $L_F(\zeta_{02}) \leq \mu_F$, which imply that

$$L_F(\zeta_{01} * \zeta_{02}) \leq L_F(\zeta_{01}) \vee L_F(\zeta_{02}) \leq \mu_F,$$

that is, $\zeta_{01} * \zeta_{02} \in L(F, \mu_F)$. Hence $L(F, \mu_F)$ is a subalg of P. Similarly we can verify that $L(IF, \mu_{IF})$ is a subalg of P. Ξ

Corollary 3.4. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of P, then the set

$$L(\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF}) := \{\zeta_{01} \in P \mid L_T(\zeta_{01}) \geq \lambda_T,\ L_{IT}(\zeta_{01}) \geq \lambda_{IT}, L_F(\zeta_{01}) \leq \mu_F, L_{IF}(\zeta_{01}) \leq \mu_{IF}\}$$

is a subalg of P $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$.

Proof. Straightforward. Ξ

Theorem 3.5. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in $P \ni U(T, \lambda_T), U(IT, \lambda_{IT})$, $L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are subalg's of P $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$ whenever they are non-empty. Then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of P.

Proof. Assume that $U(T, \lambda_T), U(IT, \lambda_{IT}), L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are subalg's $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$. If there exist $\zeta_{01}, \zeta_{02} \in P \ni$

$$L_T(\zeta_{01} * \zeta_{02}) < L_T(\zeta_{01}) \wedge L_T(\zeta_{02}),$$

then $\zeta_{01}, \zeta_{02} \in U(T, t_\zeta)$ and $\zeta_{01} * \zeta_{02} \notin U(T, t_\zeta)$ for $t_\zeta = L_T(\zeta_{01}) \wedge L_T(\zeta_{02})$. This is a contradiction, and so

$$L_T(\zeta_{01} * \zeta_{02}) \geq L_T(\zeta_{01}) \wedge L_T(\zeta_{02})$$

$\forall$ $\zeta_{01}, \zeta_{02} \in P$. Similarly, we can prove

$$L_{IT}(\zeta_{01} * \zeta_{02}) \geq L_{IT}(\zeta_{01}) \wedge L_{IT}(\zeta_{02})$$

$\forall$ $\zeta_{01}, \zeta_{02} \in P$. Suppose that

$$L_{IF}(\zeta_{01} * \zeta_{02}) > L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02})$$

for some $\zeta_{01}, \zeta_{02} \in P$. Then there exists $f_\eta \in [0, 1) \ni$

$$L_{IF}(\zeta_{01} * \zeta_{02}) > f_\eta \geq L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02}),$$

which induces a contradiction since $\zeta_{01}, \zeta_{02} \in L(IF, f_\eta)$ and $\zeta_{01} * \zeta_{02} \notin L(IF, f_\eta)$. Thus

$$L_{IF}(\zeta_{01} * \zeta_{02}) \leq L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02})$$

$\forall$ $\zeta_{01}, \zeta_{02} \in P$. Similar way shows that

$$L_F(\zeta_{01} * \zeta_{02}) \leq L_F(\zeta_{01}) \vee L_F(\zeta_{02})$$

$\forall$ $\zeta_{01}, \zeta_{02} \in P$. Therefore $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of P. Ξ

We have the following theorem since $[0, 1]$ is a completely distributive lattice (abbreviated as CDL) under the standard ordering.

Theorem 3.6. The family of $GNeuSubAlg$'s of P forms a CDL under the inclusion.

Proposition 3.7. Every $GNeuSubAlg$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ of P satisfies the following claims:

(i) $(\forall\ \zeta_{01} \in P)(L_T(0) \geq L_T(\zeta_{01}), L_{IT}(0) \geq L_{IT}(\zeta_{01}))$,

(ii) $(\forall\ \zeta_{01} \in P)(L_{IF}(0) \leq L_{IF}(\zeta_{01}), L_F(0) \leq L_F(\zeta_{01}))$.

Proof. Since $\zeta_{01} * \zeta_{01} = 0\ \forall\ \zeta_{01} \in P$, it is straightforward. Ξ

Theorem 3.8. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in P. If there exists a sequence $\{c_k\}$ in $P \ni \lim_{k\to\infty} L_T(c_k) = 1 = \lim_{k\to\infty} L_{IT}(c_k)$ and $\lim_{k\to\infty} L_F(c_k) = 0 = \lim_{k\to\infty} L_{IF}(c_k)$, then $L_T(0) = 1 = L_{IT}(0)$ and $L_F(0) = 0 = L_{IF}(0)$.

Proof. Using Proposition 3.7, we know that $L_T(0) \geq L_T(c_k), L_{IT}(0) \geq L_{IT}(c_k), L_{IF}(0) \leq L_{IF}(c_k)$ and $L_F(0) \leq L_F(c_k)$ for every positive integer k. It follows that

$$1 \geq L_T(0) \geq \lim_{k\to\infty} L_T(c_k) = 1,$$

$$1 \geq L_{IT}(0) \geq \lim_{k\to\infty} L_{IT}(c_k) = 1,$$

$$0 \leq L_{IF}(0) \leq \lim_{k\to\infty} L_{IF}(c_k) = 0,$$

$$0 \leq L_F(0) \leq \lim_{k\to\infty} L_F(c_k) = 0.$$

Thus $L_T(0) = 1 = L_{IT}(0)$ and $L_F(0) = 0 = L_{IF}(0)$. Ξ

Proposition 3.9. If every $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P satisfies:

$$(\forall\ \zeta_{01}, \zeta_{02} \in P) \begin{pmatrix} L_T(\zeta_{01} * \zeta_{02}) \geq L_T(\zeta_{02}), L_{IT}(\zeta_{01} * \zeta_{02}) \geq L_{IT}(\zeta_{02}) \\ L_{IF}(\zeta_{01} * \zeta_{02}) \leq L_{IF}(\zeta_{02}), L_F(\zeta_{01} * \zeta_{02}) \leq L_F(\zeta_{02}) \end{pmatrix}, \tag{4}$$

then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is constant on P.

Proof. Using (KU3) and (4), we have $L_T(\zeta_{01}) = L_T(0 * \zeta_{01}) \geq L_T(0), L_{IT}(\zeta_{01}) = L_{IT}(0 * \zeta_{01}) \geq L_{IT}(0)$, $L_{IF}(\zeta_{01}) = L_{IF}(0 * \zeta_{01}) \leq L_{IF}(0)$, and $L_F(\zeta_{01}) = L_F(0 * \zeta_{01}) \leq L_F(0)$. It follows from Proposition 3.7 that $L_T(\zeta_{01}) = L_T(0), L_{IT}(\zeta_{01}) = L_{IT}(0), L_{IF}(\zeta_{01}) = L_{IF}(0)$ and $L_F(\zeta_{01}) = L_F(0)\ \forall\ \zeta_{01} \in X$. Hence $L = (L_T, L_{IT}, L_{IF}, L_F)$ is constant on P. Ξ

A mapping $h : P \to Q$ of KU-algs is called a homomorphism (breifly, *homo*) if $h(\zeta_{01} * \zeta_{02}) = h(\zeta_{01}) * h(\zeta_{02})\ \forall\ \zeta_{01}, \zeta_{02} \in P$. Note that if $h : P \to Q$ is a *homo*, then $h(0) = 0$. Let $h : P \to Q$ be a *homo* of KU-algs. For any $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in Q, we define a new $GNeuS$ $L^h = (L^h_T, L^h_{IT}, L^h_{IF}, L^h_F)$ in P, which is called the induced $GNeuS$, by

$$(\forall\ \zeta_{01} \in P) \begin{pmatrix} L^h_T(\zeta_{01}) = L_T(h(\zeta_{01})), L^h_{IT}(\zeta_{01}) = L_{IT}(h(\zeta_{01})) \\ L^h_{IF}(\zeta_{01}) = L_{IF}(h(\zeta_{01})), L^h_F(\zeta_{01}) = L_F(h(\zeta_{01})) \end{pmatrix} \tag{5}$$

Theorem 3.10. Let $h : P \to Q$ be a *homo* of KU-algs. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in Q is a $GNeuSubAlg$ of Q, then the induced $GNeuS$ $L^h = (L^h_T, L^h_{IT}, L^h_{IF}, L^h_F)$ in P is a $GNeuSubAlg$ of P.

Proof. For any $\zeta_{01}, \zeta_{02} \in P$, we have

$$\begin{aligned} L_T^h(\zeta_{01} * \zeta_{02}) =& L_T(h(\zeta_{01} * \zeta_{02})) = L_T(h(\zeta_{01}) * h(\zeta_{02})) \\ \geq& L_T(h(\zeta_{01})) \wedge L_T(h(\zeta_{02})) = L_T^h(\zeta_{01}) \wedge L_T^h(\zeta_{02}), \\ L_{IT}^h(\zeta_{01} * \zeta_{02}) =& L_{IT}(h(\zeta_{01} * \zeta_{02})) = L_{IT}(h(\zeta_{01}) * h(\zeta_{02})) \\ \geq& L_{IT}(h(\zeta_{01})) \wedge L_{IT}(h(\zeta_{02})) = L_{IT}^h(\zeta_{01}) \wedge L_{IT}^h(\zeta_{02}), \\ L_{IF}^h(\zeta_{01} * \zeta_{02}) =& L_{IF}(h(\zeta_{01} * \zeta_{02})) = L_{IF}(h(\zeta_{01}) * h(\zeta_{02})) \\ \leq& L_{IF}(h(\zeta_{01})) \vee L_{IF}(h(\zeta_{02})) = L_{IF}^h(\zeta_{01}) \vee L_{IF}^h(\zeta_{02}), \end{aligned}$$

and

$$\begin{aligned} L_F^h(\zeta_{01} * \zeta_{02}) =& L_F(h(\zeta_{01} * \zeta_{02})) = L_F(h(\zeta_{01}) * h(\zeta_{02})) \\ \leq& L_F(h(\zeta_{01})) \vee L_F(h(\zeta_{02})) = L_F^h(\zeta_{01}) \vee L_F^h(\zeta_{02}). \end{aligned}$$

Therefore $L^h = (L_T^h, L_{IT}^h, L_{IF}^h, L_F^h)$ is a $GNeuSubAlg$ of P. Ξ

Theorem 3.11. Let $h : P \to Q$ be an onto *homo* of KU-algs and let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in Q. If the induced $GNeuS$ $L^h = (L_T^h, L_{IT}^h, L_{IF}^h, L_F^h)$ in P is a $GNeuSubAlg$ of P, then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of Q.

Proof. Let $\zeta_{01}, \zeta_{02} \in Q$. Then $h(r) = \zeta_{01}$ and $h(s) = \zeta_{02}$ for some $r, s \in P$. Then

$$\begin{aligned} L_T(\zeta_{01} * \zeta_{02}) =& L_T(h(r) * h(s)) = L_T(h(r * s)) = L_T^h(r * s) \\ \geq& L_T^h(r) \wedge L_T^h(s) = L_T(h(r)) \wedge L_T(h(s)) \\ =& L_T(\zeta_{01}) \wedge L_T(\zeta_{02}), \\ L_{IT}(\zeta_{01} * \zeta_{02}) =& L_{IT}(h(r) * h(s)) = L_{IT}(h(r * s)) = L_{IT}^h(r * s) \\ \geq& L_{IT}^h(r) \wedge L_{IT}^h(s) = L_{IT}(h(r)) \wedge L_{IT}(h(s)) \\ =& L_{IT}(\zeta_{01}) \wedge L_{IT}(\zeta_{02}), \\ L_{IF}(\zeta_{01} * \zeta_{02}) =& L_{IF}(h(r) * h(s)) = L_{IF}(h(r * s)) = L_{IF}^h(r * s) \\ \leq& L_{IF}^h(r) \vee L_{IF}^h(s) = L_{IF}(h(r)) \vee L_{IF}(h(s)) \\ =& L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02}), \end{aligned}$$

and

$$\begin{aligned} L_F(\zeta_{01} * \zeta_{02}) =& L_F(h(r) * h(s)) = L_F(h(r * s)) = L_F^h(r * s) \\ \leq& L_F^h(r) \vee L_F^h(s) = L_F(h(r)) \vee L_F(h(s)) \\ =& L_F(\zeta_{01}) \vee L_F(\zeta_{02}). \end{aligned}$$

Hence $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of Q. Ξ

Definition 3.12. A $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P is called a generalized neutrosophic ideal (briefly, GNI) of P if the following conditions are valid.

$$(\forall\ \zeta_{01} \in P) \begin{pmatrix} L_T(0) \geq L_T(\zeta_{01}), L_{IT}(0) \geq L_{IT}(\zeta_{01}) \\ L_{IF}(0) \leq L_{IF}(\zeta_{01}), L_F(0) \leq L_F(\zeta_{01}) \end{pmatrix}, \tag{6}$$

$$(\forall\ \zeta_{01}, \zeta_{02} \in P) \begin{pmatrix} L_T(\zeta_{01}) \geq L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}) \\ L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}) \\ L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}) \\ L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}) \end{pmatrix}. \tag{7}$$

Example 3.13. Consider a set $P = \{ll_0, ll_1, ll_2, ll_3, ll_4\}$ with the binary operation $*$ which is given in Table 2. Then

Table 2: Cayley table for the binary operation "$*$".

$*$	ll_0	ll_1	ll_2	ll_3	ll_4
ll_0	ll_0	ll_1	ll_2	ll_3	ll_4
ll_1	ll_0	ll_0	ll_2	ll_3	ll_4
ll_2	ll_0	ll_1	ll_0	ll_3	ll_3
ll_3	ll_0	ll_0	ll_2	ll_0	ll_2
ll_4	ll_0	ll_0	ll_0	ll_0	ll_0

$(P;\ *, 0)$ is a KU-alg. Let

$L = \{\langle ll_0; 0.8, 0.9, 0.1, 0.2\rangle, \langle ll_1; 0.7, 0.8, 0.2, 0.3\rangle, \langle ll_2; 0.5, 0.6, 0.3, 0.7\rangle, \langle ll_3; 0.3, 0.5, 0.4, 0.5\rangle, \langle ll_4; 0.3, 0.5, 0.4, 0.7\rangle\}$.

be a $GNeuS$ in P. By routine calculations, we know that L is a GNI of P.

Lemma 3.14. Every GNI $L = (L_T, L_{IT}, L_{IF}, L_F)$ of P satisfies:

$$(\forall\ \zeta_{01}, \zeta_{02} \in P) \left(\zeta_{01} \leq \zeta_{02} \Rightarrow \begin{cases} L_T(\zeta_{01}) \geq L_T(\zeta_{02}), L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02}) \\ L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02}), L_F(\zeta_{01}) \leq L_F(\zeta_{02}) \end{cases} \right). \tag{8}$$

Proof. Let $\zeta_{01}, \zeta_{02} \in P$ be $\ni \zeta_{01} \leq \zeta_{02}$. Then $\zeta_{02} * \zeta_{01} = 0$, and so

$$L_T(\zeta_{01}) \geq L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}) = L_T(0) \wedge L_T(\zeta_{02}) = L_T(\zeta_{02}),$$

$$L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}) = L_{IT}(0) \wedge L_{IT}(\zeta_{02}) = L_{IT}(\zeta_{02}),$$

$$L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}) = L_{IF}(0) \vee L_{IF}(\zeta_{02}) = L_{IF}(\zeta_{02}),$$

$$L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}) = L_F(0) \vee L_F(\zeta_{02}) = L_F(\zeta_{02}).$$

This completes the proof. Ξ

Lemma 3.15. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a GNI of P. If the inequality $\zeta_{02} * \zeta_{01} \leq \zeta_{03}$ holds in P, then $L_T(\zeta_{01}) \geq L_T(\zeta_{02}) \wedge L_T(\zeta_{03}), L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02}) \wedge L_{IT}(\zeta_{03})$, $L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02}) \vee L_{IF}(\zeta_{03})$ and $L_F(\zeta_{01}) \leq L_F(\zeta_{02}) \vee L_F(\zeta_{03})$.

Proof. Let $\zeta_{01}, \zeta_{02}, \zeta_{03} \in P$ be $\ni \zeta_{02} * \zeta_{01} \leq \zeta_{03}$, Then $\zeta_{03} * (\zeta_{02} * \zeta_{01}) = 0$, and so

$$\begin{aligned} L_T(\zeta_{01}) &\geq \bigwedge\{L_T(\zeta_{02} * \zeta_{01}), L_T(\zeta_{02})\} \\ &\geq \bigwedge\{\bigwedge\{L_T(\zeta_{03} * (\zeta_{02} * \zeta_{01})), L_T(\zeta_{03})\}, L_T(\zeta_{02})\} \\ &= \bigwedge\{\bigwedge\{L_T(0), L_T(\zeta_{03})\}, L_T(\zeta_{02})\} \\ &= \bigwedge\{L_T(\zeta_{02}), L_T(\zeta_{03})\}, \\ L_{IT}(\zeta_{01}) &\geq \bigwedge\{L_{IT}(\zeta_{02} * \zeta_{01}), L_{IT}(\zeta_{02})\} \\ &\geq \bigwedge\{\bigwedge\{L_{IT}(\zeta_{03} * (\zeta_{02} * \zeta_{01})), L_{IT}(\zeta_{03})\}, L_{IT}(\zeta_{02})\} \\ &= \bigwedge\{\bigwedge\{L_{IT}(0), L_{IT}(\zeta_{03})\}, L_{IT}(\zeta_{02})\} \\ &= \bigwedge\{L_{IT}(\zeta_{02}), L_{IT}(\zeta_{03})\}, \\ L_{IF}(\zeta_{01}) &\leq \bigvee\{L_{IF}(\zeta_{02} * \zeta_{01}), L_{IF}(\zeta_{02})\} \\ &\leq \bigvee\{\bigvee\{L_{IF}(\zeta_{03} * (\zeta_{02} * \zeta_{01})), L_{IF}(\zeta_{03})\}, L_{IF}(\zeta_{02})\} \\ &= \bigvee\{\bigvee\{L_{IF}(0), L_{IF}(\zeta_{03})\}, L_{IF}(\zeta_{02})\} \\ &= \bigvee\{L_{IF}(\zeta_{02}), L_{IF}(\zeta_{03})\}, \end{aligned}$$

and

$$\begin{aligned} L_F(\zeta_{01}) &\leq \bigvee\{L_F(\zeta_{02} * \zeta_{01}), L_F(\zeta_{02})\} \\ &\leq \bigvee\{\bigvee\{L_F(\zeta_{03} * (\zeta_{02} * \zeta_{01})), L_F(\zeta_{03})\}, L_F(\zeta_{02})\} \\ &= \bigvee\{\bigvee\{L_F(0), L_F(\zeta_{03})\}, L_F(\zeta_{02})\} \\ &= \bigvee\{L_F(\zeta_{02}), L_F(\zeta_{03})\}. \end{aligned}$$

This completes the proof. Ξ

Proposition 3.16. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a GNI of P. If the inequality

$$c_n * (\cdots * (c_2 * (c_1 * \zeta_{01})) \cdots) = 0$$

holds in P, then

$$\begin{aligned} L_T(\zeta_{01}) &\geq \bigwedge\{L_T(c_k) \mid k = 1, 2, \cdots, n\}, \\ L_{IT}(\zeta_{01}) &\geq \bigwedge\{L_{IT}(c_k) \mid k = 1, 2, \cdots, n\}, \\ L_{IF}(\zeta_{01}) &\leq \bigwedge\{L_{IF}(c_k) \mid k = 1, 2, \cdots, n\}, \\ L_F(\zeta_{01}) &\leq \bigwedge\{L_F(c_k) \mid k = 1, 2, \cdots, n\}. \end{aligned}$$

Proof. Using induction on n and Lemmas makes it simple 3.14 & 3.15 Ξ

Theorem 3.17. In a KU-alg P, every GNI is a $GNeuSubAlg$.

Proof. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a GNI of a KU-alg P. Since $\zeta_{02} * \zeta_{01} \leq \zeta_{01}$ $\forall$ $\zeta_{01}, \zeta_{02} \in P$, we have $L_T(\zeta_{02} * \zeta_{01}) \geq L_T(\zeta_{01}), L_{IT}(\zeta_{02} * \zeta_{01}) \geq L_{IT}(\zeta_{01})$, $L_{IF}(\zeta_{02} * \zeta_{01}) \leq L_{IF}(\zeta_{01})$ and $L_F(\zeta_{02} * \zeta_{01}) \leq L_F(\zeta_{01})$ by Lemma 3.14 It follows from (7) that

$$L_T(\zeta_{02} * \zeta_{01}) \geq L_T(\zeta_{01}) \geq L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}) \geq L_T(\zeta_{01}) \wedge L_T(\zeta_{02}),$$

$$L_{IT}(\zeta_{02} * \zeta_{01}) \geq L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}) \geq L_{IT}(\zeta_{01}) \wedge L_{IT}(\zeta_{02}),$$

$$L_{IF}(\zeta_{02} * \zeta_{01}) \leq L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}) \leq L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02}),$$

and

$$L_F(\zeta_{02} * \zeta_{01}) \leq L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}) \leq L_F(\zeta_{01}) \vee L_F(\zeta_{02}).$$

Therefore $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a $GNeuSubAlg$ of P. Ξ

The converse of Theorem 3.17 is not true. For example, the $GNeuSubAlg$ L in Example 3.2 is not a GNI of P since

$$L_{IT}(ll_a) = 0.5 \ngeq 0.6 = L_{IT}(ll_b * ll_a) \wedge L_{IT}(ll_b).$$

We give a condition for a $GNeuSubAlg$ to be a GNI.

Theorem 3.18. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuSubAlg$ of $P \ni$

$$L_T(\zeta_{01}) \geq L_T(\zeta_{02}) \wedge L_T(\zeta_{03}),$$

$$L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02}) \wedge L_{IT}(\zeta_{03}),$$

$$L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02}) \vee L_{IF}(\zeta_{03}),$$

$$L_F(\zeta_{01}) \leq L_F(\zeta_{02}) \vee L_F(\zeta_{03})$$

$\forall$ $\zeta_{01}, \zeta_{02}, \zeta_{03} \in P$ satisfying the inequality $\zeta_{02} * \zeta_{01} \leq \zeta_{03}$. Then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P.

Proof. Recall that $L_T(0) \geq L_T(\zeta_{01}), L_{IT}(0) \geq L_{IT}(\zeta_{01}), L_{IF}(0) \leq L_{IF}(\zeta_{01})$ and $L_F(0) \leq L_F(\zeta_{01})$ $\forall$ $\zeta_{01} \in P$ by Proposition 3.7 Let $\zeta_{01}, \zeta_{02} \in P$. Since $(\zeta_{02} * \zeta_{01}) * \zeta_{01} \leq \zeta_{02}$, it follows from the hypothesis that

$$L_T(\zeta_{01}) \geq L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}),$$

$$L_{IT}(\zeta_{01}) \geq L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}),$$

$$L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}),$$

$$L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}).$$

Hence $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P. Ξ

Theorem 3.19. A $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P is a GNI of P iff the fuzzy sets L_T, L_{IT}, L_{IF}^c and L_F^c are fuzzy ideals of P.

Proof. Assume that $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P. Clearly, L_T and L_{IT} are fuzzy ideals of P. For every $\zeta_{01}, \zeta_{02} \in P$, we have

$$\begin{aligned} L^c_{IF}(0) =& 1 - L_{IF}(0) \geq 1 - L_{IF}(\zeta_{01}) = L^c_{IF}(\zeta_{01}), \\ L^c_F(0) =& 1 - L_F(0) \geq 1 - L_F(\zeta_{01}) = L^c_F(\zeta_{01}), \\ L^c_{IF}(\zeta_{01}) =& 1 - L_{IF}(\zeta_{01}) \geq 1 - L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}) \\ =& \bigwedge\{1 - L_{IF}(\zeta_{02} * \zeta_{01}), 1 - L_{IF}(\zeta_{02})\} \\ =& \bigwedge\{L^c_{IF}(\zeta_{02} * \zeta_{01}), L^c_{IF}(\zeta_{02})\} \end{aligned}$$

and

$$\begin{aligned} L^c_F(\zeta_{01}) =& 1 - L_F(\zeta_{01}) \geq 1 - L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}) \\ =& \bigwedge\{1 - L_F(\zeta_{02} * \zeta_{01}), 1 - L_F(\zeta_{02})\} \\ =& \bigwedge\{L^c_F(\zeta_{02} * \zeta_{01}), L^c_F(\zeta_{02})\}. \end{aligned}$$

Therefore L_T, L_{IT}, L^c_{IF} and L^c_F are fuzzy ideals of P.

Conversely, let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in P for which L_T, L_{IT}, L^c_{IF} and L^c_F are fuzzy ideals of P. For every $\zeta_{01} \in P$, we have $L_T(0) \geq L_T(\zeta_{01}), L_{IT}(0) \geq L_{IT}(\zeta_{01})$,

$$1 - L_{IF}(0) = L^c_{IF}(0) \geq L^c_{IF}(\zeta_{01}) = 1 - L_{IF}(\zeta_{01}), \text{ that is, } L_{IF}(0) \leq L_{IF}(\zeta_{01})$$

and

$$1 - L_F(0) = L^c_F(0) \geq L^c_F(\zeta_{01}) = 1 - L_F(\zeta_{01}), \text{ that is, } L_F(0) \leq L_F(\zeta_{01}).$$

Let $\zeta_{01}, \zeta_{02} \in P$. Then

$$\begin{aligned} L_T(\zeta_{01}) \geq& L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}), \\ L_{IT}(\zeta_{01}) \geq& L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}), \\ 1 - L_{IF}(\zeta_{01}) =& L^c_{IF}(\zeta_{01}) \geq L^c_{IF}(\zeta_{02} * \zeta_{01}) \wedge L^c_{IF}(\zeta_{02}) \\ =& \bigwedge\{1 - L_{IF}(\zeta_{02} * \zeta_{01}), 1 - L_{IF}(\zeta_{02})\} \\ =& 1 - \bigvee\{L_{IF}(\zeta_{02} * \zeta_{01}), L_{IF}(\zeta_{02})\}, \end{aligned}$$

and

$$\begin{aligned} 1 - L_F(\zeta_{01}) =& L^c_F(\zeta_{01}) \geq L^c_F(\zeta_{02} * \zeta_{01}) \wedge L^c_F(\zeta_{02}) \\ =& \bigwedge\{1 - L_F(\zeta_{02} * \zeta_{01}), 1 - L_F(\zeta_{02})\} \\ =& 1 - \bigvee\{L_F(\zeta_{02} * \zeta_{01}), L_F(\zeta_{02})\}, \end{aligned}$$

that is, $L_{IF}(\zeta_{01}) \leq L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02})$ and $L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02})$. Hence $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P. Ξ

Theorem 3.20. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in P is a GNI of P, then $\Box L = (L_T, L_{IT}, L^c_{IT}, L^c_T)$ and $\Diamond L = (L^c_{IF}, L^c_F, L_F, L_{IF})$ are GNI's of P.

Proof. Assume that $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P and let $\zeta_{01}, \zeta_{02} \in P$. Note that $\Box L = (L_T, L_{IT}, L^c_{IT}, L^c_T)$ and $\Diamond L = (L^c_{IF}, L^c_F, L_F, L_{IF})$ are $GNeuS$'s in P. Let $\zeta_{01}, \zeta_{02} \in P$. Then

$$\begin{aligned}
L^c_{IT}(\zeta_{02} * \zeta_{01}) =& 1 - L_{IT}(\zeta_{02} * \zeta_{01}) \leq 1 - \bigwedge\{L_{IT}(\zeta_{01}), L_{IT}(\zeta_{02})\} \\
=& \bigvee\{1 - L_{IT}(\zeta_{01}), 1 - L_{IT}(\zeta_{02})\} \\
=& \bigvee\{L^c_{IT}(\zeta_{01}), L^c_{IT}(\zeta_{02})\}, \\
L^c_T(\zeta_{02} * \zeta_{01}) =& 1 - L_T(\zeta_{02} * \zeta_{01}) \leq 1 - \bigwedge\{L_T(\zeta_{01}), L_T(\zeta_{02})\} \\
=& \bigvee\{1 - L_T(\zeta_{01}), 1 - L_T(\zeta_{02})\} \\
=& \bigvee\{L^c_T(\zeta_{01}), L^c_T(\zeta_{02})\}, \\
L^c_{IF}(\zeta_{02} * \zeta_{01}) =& 1 - L_{IF}(\zeta_{02} * \zeta_{01}) \geq 1 - \bigvee\{L_{IF}(\zeta_{01}), L_{IF}(\zeta_{02})\} \\
=& \bigwedge\{1 - L_{IF}(\zeta_{01}), 1 - L_{IF}(\zeta_{02})\} \\
=& \bigwedge\{L^c_{IF}(\zeta_{01}), L^c_{IF}(\zeta_{02})\}
\end{aligned}$$

and

$$\begin{aligned}
L^c_F(\zeta_{02} * \zeta_{01}) =& 1 - L_F(\zeta_{02} * \zeta_{01}) \geq 1 - \bigvee\{L_F(\zeta_{01}), L_F(\zeta_{02})\} \\
=& \bigwedge\{1 - L_F(\zeta_{01}), 1 - L_F(\zeta_{02})\} \\
=& \bigwedge\{L^c_F(\zeta_{01}), L^c_F(\zeta_{02})\}.
\end{aligned}$$

Therefore $\Box L = (L_T, L_{IT}, L^c_{IT}, L^c_T)$ and $\Diamond L = (L^c_{IF}, L^c_F, L_F, L_{IF})$ are GNI's of P. Ξ

Theorem 3.21. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P, then the set $U(T, \lambda_T)$, $U(IT, \lambda_{IT}), L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are ideals of P $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$ whenever they are non-empty.

Proof. Assume that $U(T, \lambda_T), U(IT, \lambda_{IT}), L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are nonempty $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$. It is clear that $0 \in U(T, \lambda_T), 0 \in U(IT, \lambda_{IT})$, $0 \in L(F, \mu_F)$ and $0 \in L(IF, \mu_{IF})$. Let $\zeta_{01}, \zeta_{02} \in P$. If $\zeta_{02} * \zeta_{01} \in U(T, \lambda_T)$ and $\zeta_{02} \in U(T, \lambda_T)$, then $L_T(\zeta_{02} * \zeta_{01}) \geq \lambda_T$ and $L_T(\zeta_{02}) \geq \lambda_T$. Hence

$$L_T(\zeta_{01}) \geq L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}) \geq \lambda_T,$$

and so $\zeta_{01} \in U(T, \lambda_T)$. Similarly, if $\zeta_{02} * \zeta_{01} \in U(IT, \lambda_T)$ and $\zeta_{02} \in U(IT, \lambda_T)$, then $\zeta_{01} \in U(IT, \lambda_T)$. If $\zeta_{02} * \zeta_{01} \in L(F, \mu_F)$ and $\zeta_{02} \in L(F, \mu_F)$, then $L_F(\zeta_{02} * \zeta_{01}) \leq \mu_F$ and $L_F(\zeta_{02}) \leq \mu_F$. Hence

$$L_F(\zeta_{01}) \leq L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}) \leq \mu_F,$$

and so $\zeta_{01} \in L(F, \mu_F)$. Similarly, if $\zeta_{02} * \zeta_{01} \in L(IF, \mu_{IF})$ and $\zeta_{02} \in L(IF, \mu_{IF})$, then $\zeta_{01} \in L(IF, \mu_{IF})$. This completes the proof. Ξ

Theorem 3.22. Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in $P \ni U(T, \lambda_T), U(IT, \lambda_{IT})$, $L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are ideals of P $\forall$ $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$. Then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P.

Proof. Let $\lambda_T, \lambda_{IT}, \mu_F, \mu_{IF} \in [0, 1]$ be $\ni U(T, \lambda_T), U(IT, \lambda_{IT})$, $L(F, \mu_F)$ and $L(IF, \mu_{IF})$ are ideals of P. For any $\zeta_{01} \in P$, let $L_T(\zeta_{01}) = \lambda_T, L_{IT}(\zeta_{01}) = \lambda_{IT}, L_{IF}(\zeta_{01}) = \mu_{IF}$ and $L_F(\zeta_{01}) = \mu_F$. Since $0 \in U(T, \lambda_T), 0 \in U(IT, \lambda_{IT}), 0 \in L(F, \mu_F)$ and $0 \in L(IF, \mu_{IF})$, we have $L_T(0) \geq \lambda_T = L_T(\zeta_{01}), L_{IT}(0) \geq \lambda_{IT} = L_{IT}(\zeta_{01}), L_{IF}(0) \leq \mu_{IF} = L_{IF}(\zeta_{01})$ and $L_F(0) \leq \mu_F = L_F(\zeta_{01})$. If there exist $r, s \in P \ni L_T(s * r) < L_T(r) \wedge L_T(s)$, then $r, s \in U(T, \lambda_0)$ and $s * r \notin U(T, \lambda_0)$ where $\lambda_0 := L_T(r) \wedge L_T(s)$. This is a contradiction, and hence $L_T(\zeta_{02} * \zeta_{01}) \geq L_T(\zeta_{01}) \wedge L_T(\zeta_{02})$ $\forall$ $\zeta_{01}, \zeta_{02} \in P$. Similarly, we can verify $L_{IT}(\zeta_{02} * \zeta_{01}) \geq L_{IT}(\zeta_{01}) \wedge L_{IT}(\zeta_{02})$ $\forall$ $\zeta_{01}, \zeta_{02} \in P$. Suppose that $L_{IF}(s * r) > L_{IF}(r) \vee L_{IF}(s)$ for some $r, s \in P$. Taking $\mu_0 := L_{IF}(r) \vee L_{IF}(s)$ induces $r, s \in L(IF, \mu_{IF})$ and $s * r \notin L(IF, \mu_{IF})$, a contradiction. Thus $L_{IF}(\zeta_{02} * \zeta_{01}) \leq L_{IF}(\zeta_{01}) \vee L_{IF}(\zeta_{02})$ $\forall$ $\zeta_{01}, \zeta_{02} \in P$. Similarly we have $L_F(\zeta_{02} * \zeta_{01}) \leq L_F(\zeta_{01}) \vee L_F(\zeta_{02})$ $\forall$ $\zeta_{01}, \zeta_{02} \in P$. Consequently, $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P. Ξ

Let Λ be a nonempty subset of $[0, 1]$.

Theorem 3.23. Let $\{I_{(q_2)} \mid q_2 \in \Lambda\}$ be a collection of ideals of $P \ni$

(i) $P = \bigcup_{q_2 \in \Lambda} I_{(q_2)}$,

(ii) $(\forall\ r_1, q_2 \in \Lambda)(r_1 > q_2 \Leftrightarrow I_{r_1} \subset I_{(q_2)})$.

Let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in P given as follows:

$$(\forall\ \zeta_{01} \in P) \begin{pmatrix} L_T(\zeta_{01}) = \bigvee\{q_2 \in \Lambda \mid \zeta_{01} \in I_{(q_2)} = L_{IT}(\zeta_{01})\} \\ L_{IF}(\zeta_{01}) = \bigwedge\{q_2 \in \Lambda \mid \zeta_{01} \in I_{(q_2)} = L_F(\zeta_{01})\} \end{pmatrix} \tag{9}$$

Then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of P.

Proof. According to Theorem 3.22 it is sufficient to show that $U(T, q_2), U(IT, q_2)$, $L(F, r_1)$ and $L(IF, r_1)$ are ideals of P for every $q_2 \in [0, L_T(0) = L_{IT}(0)]$ and $r_1 \in [L_{IF}(0) = L_F(0), 1]$. In order to prove $U(T, q_2)$ and $U(IT, q_2)$ are ideals of P, we consider two cases:

(i) $q_2 = \bigvee\{q_1 \in \Lambda \mid q_1 < q_2\}$,

(ii) $q_2 \neq \bigvee\{q_1 \in \Lambda \mid q_1 < q_2\}$.

For the first case, we have

$$\zeta_{01} \in U(T, q_2) \Leftrightarrow (\forall\ q_1 < q_2)(\zeta_{01} \in I_{q_1}) \Leftrightarrow \zeta_{01} \in \bigcap_{q_1 < q_2} I_{q_1},$$

$$\zeta_{01} \in U(IT, q_2) \Leftrightarrow (\forall\ q_1 < q_2)(\zeta_{01} \in I_{q_1}) \Leftrightarrow \zeta_{01} \in \bigcap_{q_1 < q_2} I_{q_1}.$$

Hence $U(T, q_2) = \bigcap_{q_1 < q_2} I_{q_1} = U(IT, q_2)$, and so $U(T, q_2)$ and $U(IT, q_2)$ are ideals of P. For the second case, we claim that $U(T, q_2) = \bigcup_{q_1 \geq q_2} I_{q_1} = U(IT, q_2)$. If $\zeta_{01} \in \bigcup_{q_1 \geq q_2} I_{q_1}$, then $\zeta_{01} \in I_{q_1}$ for some $q_1 \geq q_2$. It follows that $L_{IT}(\zeta_{01}) = L_T(\zeta_{01}) \geq q_1 \geq q_2$ and so that $\zeta_{01} \in U(T, q_2)$ and $\zeta_{01} \in U(IT, q_2)$. This shows that $\bigcup_{q_1 \geq q_2} I_{q_1} \subseteq U(T, q_2) = U(IT, q_2)$. Now, assume that $\zeta_{01} \notin \bigcup_{q_1 \geq q_2} I_{q_1}$. Then $\zeta_{01} \notin I_{q_1}\ \forall\ q_1 \geq q_2$. Since $q_2 \neq \bigvee\{q_1 \in \Lambda \mid q_1 < q_2\}$, there exists $\epsilon > 0$ $\ni (q_2 - \epsilon, q_2) \cap \Lambda = \emptyset$. Hence $\zeta_{01} \notin I_{q_1}\ \forall\ q_1 > q_2 - \epsilon$, which means that if $\zeta_{01} \in I_{q_1}$, then $q_1 \leq q_2 - \epsilon$. Thus $L_{IT}(\zeta_{01}) = L_T(\zeta_{01}) \leq q_2 - \epsilon < q_2$, and so $\zeta_{01} \notin U(T, q_2) = U(IT, q_2)$. Therefore $U(T, q_2) = U(IT, q_2) \subseteq \bigcup_{q_1 \geq q_2} I_{q_1}$. Consequently, $U(T, q_2) = U(IT, q_2) = \bigcup_{q_1 \geq q_2} I_{q_1}$ which is an ideal of P. Next we show that $L(F, r_1)$ and $L(IF, r_1)$ are ideals of P. We consider two cases as follows:

(iii) $r_1 = \bigwedge\{r_2 \in \Lambda \mid r_1 < r_2\}$,

(iv) $r_1 \neq \bigwedge\{r_2 \in \Lambda \mid r_1 < r_2\}$.

Case (iii) implies that

$$\zeta_{01} \in L(IF, r_1) \Leftrightarrow (\forall\ r_1 < r_2)(\zeta_{01} \in I_{r_2}) \Leftrightarrow \zeta_{01} \in \bigcap_{r_1 < r_2} I_{r_2},$$

$$\zeta_{01} \in U(F, r_1) \Leftrightarrow (\forall\ r_1 < r_2)(\zeta_{01} \in I_{r_2}) \Leftrightarrow \zeta_{01} \in \bigcap_{r_1 < r_2} I_{r_2}.$$

It follows that $L(IF, r_1) = L(F, r_1) = \bigcap_{r_1 < r_2} I_{r_2}$, which is an ideal of P. Case (iv) induces $(r_1, r_1 + \epsilon) \cap \Lambda = \emptyset$ for some $\epsilon > 0$. If $\zeta_{01} \in \bigcup_{r_1 \geq r_2} I_{r_2}$, then $\zeta_{01} \in I_{r_2}$ for some $r_2 \leq r_1$, and so $L_{IF}(\zeta_{01}) = L_F(\zeta_{01}) \leq r_2 \leq r_1$, that is, $\zeta_{01} \in L(IF, r_1)$ and $\zeta_{01} \in L(F, r_1)$. Hence $\bigcup_{r_1 \geq r_2} I_{r_2} \subseteq L(IF, r_1) = L(F, r_1)$. If $\zeta_{01} \notin \bigcup_{r_1 \geq r_2} I_{r_2}$, then $\zeta_{01} \notin I_{r_2}\ \forall\ r_2 \leq r_1$ which implies that $\zeta_{01} \notin I_{r_2}\ \forall$ $r_2 \leq r_1 + \epsilon$, that is, if $\zeta_{01} \in I_{r_2}$ then $r_2 \geq r_1 + \epsilon$. Hence $L_{IF}(\zeta_{01}) = L_F(\zeta_{01}) \geq r_1 + \epsilon > r_1$, and so $\zeta_{01} \notin L(L_{IF}, r_1) = L(L_F, r_1)$. Hence $L(L_{IF}, r_1) = L(L_F, r_1) = \bigcup_{r_1 \geq r_2} I_{r_2}$ which is an ideal of P. This completes the proof. Ξ

Theorem 3.24. Let $h : P \to Q$ be a *homo* of KU-algs. If a $GNeuS$ $L = (L_T, L_{IT}, L_{IF}, L_F)$ in Q is a GNI of Q, then the new $GNeuS$ $L^h = (L^h_T, L^h_{IT}, L^h_{IF}, L^h_F)$ in P is a GNI of P.

Proof. We first have

$$L^h_T(0) = L_T(h(0)) = L_T(0) \geq L_T(h(\zeta_{01})) = L^h_T(\zeta_{01}),$$

$$L^h_{IT}(0) = L_{IT}(h(0)) = L_{IT}(0) \geq L_{IT}(h(\zeta_{01})) = L^h_{IT}(\zeta_{01}),$$

$$L^h_{IF}(0) = L_{IF}(h(0)) = L_{IF}(0) \leq L_{IF}(h(\zeta_{01})) = L^h_{IF}(\zeta_{01}),$$

$$L^h_F(0) = L_F(h(0)) = L_F(0) \leq L_F(h(\zeta_{01})) = L^h_F(\zeta_{01})$$

$\forall\ \zeta_{01} \in P$. Let $\zeta_{01}, \zeta_{02} \in P$. Then

$$\begin{aligned}
L_T^h(\zeta_{01}) =& L_T(h(\zeta_{01})) \geq L_T(h(\zeta_{02}) * h(\zeta_{01})) \wedge L_T(h(\zeta_{02})) \\
=& L_T(h(\zeta_{02} * \zeta_{01})) \wedge L_T(h(\zeta_{02})) \\
=& L_T^h(\zeta_{02} * \zeta_{01}) \wedge L_T^h(\zeta_{02}), \\
L_{IT}^h(\zeta_{01}) =& L_{IT}(h(\zeta_{01})) \geq L_{IT}(h(\zeta_{02}) * h(\zeta_{01})) \wedge L_{IT}(h(\zeta_{02})) \\
=& L_{IT}(h(\zeta_{02} * \zeta_{01})) \wedge L_{IT}(h(\zeta_{02})) \\
=& L_{IT}^h(\zeta_{02} * \zeta_{01}) \wedge L_{IT}^h(\zeta_{02}), \\
L_{IF}^h(\zeta_{01}) =& L_{IF}(h(\zeta_{01})) \leq L_{IF}(h(\zeta_{02}) * h(\zeta_{01})) \vee L_{IF}(h(\zeta_{02})) \\
=& L_{IF}(h(\zeta_{02} * \zeta_{01})) \vee L_{IF}(h(\zeta_{02})) \\
=& L_{IF}^h(\zeta_{02} * \zeta_{01}) \vee L_{IF}^h(\zeta_{02})
\end{aligned}$$

and

$$\begin{aligned}
L_F^h(\zeta_{01}) =& L_F(h(\zeta_{01})) \leq L_F(h(\zeta_{02}) * h(\zeta_{01})) \vee L_F(h(\zeta_{02})) \\
=& L_F(h(\zeta_{02} * \zeta_{01})) \vee L_F(h(\zeta_{02})) \\
=& L_F^h(\zeta_{02} * \zeta_{01}) \vee L_F^h(\zeta_{02}).
\end{aligned}$$

Therefore $L^h = (L_T^h, L_{IT}^h, L_{IF}^h, L_F^h)$ in P is a GNI of P. Ξ

Theorem 3.25. Let $h : P \to Q$ be an onto *homo* of KU-algs and let $L = (L_T, L_{IT}, L_{IF}, L_F)$ be a $GNeuS$ in Q. If the induced $GNeuS$ $L^h = (L_T^h, L_{IT}^h, L_{IF}^h, L_F^h)$ in P is a GNI of P, then $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of Q.

Proof. For any $\zeta_{01} \in Q$, there exists $r \in P \ni h(r) = \zeta_{01}$. Then

$$\begin{aligned}
&L_T(0) = L_T(h(0)) = L_T^h(0) \geq L_T^h(r) = L_T(h(r)) = L_T(\zeta_{01}), \\
&L_{IT}(0) = L_{IT}(h(0)) = L_{IT}^h(0) \geq L_{IT}^h(r) = L_{IT}(h(r)) = L_{IT}(\zeta_{01}), \\
&L_{IF}(0) = L_{IF}(h(0)) = L_{IF}^h(0) \leq L_{IF}^h(r) = L_{IF}(h(r)) = L_{IF}(\zeta_{01}), \\
&L_F(0) = L_F(h(0)) = L_F^h(0) \leq L_F^h(r) = L_F(h(r)) = L_F(\zeta_{01}).
\end{aligned}$$

Let $\zeta_{01}, \zeta_{02} \in Q$. Then $h(r) = \zeta_{01}$ and $h(s) = \zeta_{02}$ for some $r, s \in P$. It follows that

$$\begin{aligned}
L_T(\zeta_{01}) = L_T(h(r)) &= L_T^h(r) \\
&\geq L_T^h(s * r) \wedge L_T^h(s) \\
&= L_T(h(s * r)) \wedge L_T(h(s)) \\
&= L_T(h(s) * h(r)) \wedge L_T(h(s)) \\
&= L_T(\zeta_{02} * \zeta_{01}) \wedge L_T(\zeta_{02}), \\
L_{IT}(\zeta_{01}) &= L_{IT}(h(r)) = L_{IT}^h(r) \\
&\geq L_{IT}^h(s * r) \wedge L_{IT}^h(s) \\
&= L_{IT}(h(s * r)) \wedge L_{IT}(h(s)) \\
&= L_{IT}(h(s) * h(r)) \wedge L_{IT}(h(s)) \\
&= L_{IT}(\zeta_{02} * \zeta_{01}) \wedge L_{IT}(\zeta_{02}), \\
L_{IF}(\zeta_{01}) &= L_{IF}(h(r)) = L_{IF}^h(r) \\
&\leq L_{IF}^h(s * r) \vee L_{IF}^h(s) \\
&= L_{IF}(h(s * r)) \vee L_{IF}(h(s)) \\
&= L_{IF}(h(s) * h(r)) \vee L_{IF}(h(s)) \\
&= L_{IF}(\zeta_{02} * \zeta_{01}) \vee L_{IF}(\zeta_{02}),
\end{aligned}$$

and

$$\begin{aligned}
L_F(\zeta_{01}) &= L_F(h(r)) = L_F^h(r) \\
&\leq L_F^h(s * r) \vee L_F^h(s) \\
&= L_F(h(s * r)) \vee L_F(h(s)) \\
&= L_F(h(s) * h(r)) \vee L_F(h(s)) \\
&= L_F(\zeta_{02} * \zeta_{01}) \vee L_F(\zeta_{02}).
\end{aligned}$$

Therefore $L = (L_T, L_{IT}, L_{IF}, L_F)$ is a GNI of Q. Ξ

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