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Optimizing of Universal DEA Model with Multi-Level Integration under Neutrosophic Environment

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Abstract. This research introduces a comprehensive Universal Data Envelopment Analysis (DEA) model to handle real-world problems fraught with uncertainty of every operational facet. Apart from all this, this framework has a feature of embracing many level variations: deterministic, stochastic, fuzzy, and neutrosophic, in both input and output variables, unlike the traditional DEA models, which are limited to deterministic. Besides accommodating different orientation types detected in models such as input-oriented and output-oriented, it also incorporates variable and constant return types of scale. The main objective of this study is to develop a flexible DEA model that can measure and rank, under uncertain conditions, the performance efficiency of organizations having comparable input-output structures. The proposed model seeks to identify inefficient organizations so as to improve specifically on those areas to have a much better outcome in overall efficiency. This provides decision-makers with very strong and versatile options for efficient measures, even in the context of heterogeneous and imprecise data. The paradigm for this study lies in bringing together several ways of handling uncertainty under one umbrella. This study is much better than the conventional DEA models. Possible restrictions, however, on applicability in very large data sets and computation time complexities would need further probing.

Keywords: Data Envelopment Analysis; Neutrosophic Analysis; Efficiency Assessment; Uncertainty Integration; Multi-Level Uncertainty; Healthcare Facility; Triangular neutrosophic number

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1. Introduction

In these days, managers are now required not only to make collaborative decisions, but also interpret abundant data related to their business or organization. The biggest difficulty is finding valuable insight from this flood of data, and this insight can be used to improve organizational performance. Efficiency measurement represents a vital domain of interest within the complex nexus of dimensions that characterize organizational performance. One of the most rising challenges across organizations is questioning them on whether they believe themselves to be more efficient than one or more industry competitors. Performance measurement has evolved into necessary means of answering questions about how to measure productivity, including efficiency, effectiveness and accountability. Traditional performance measurement systems tend to present an incomplete picture of organizational performance leading to the risk of managers missing some important areas for development. In this sense efficiency is measured against established goals, whether it relates to the level of output produced, profit achieved, or cost avoided [1].

Data Envelopment Analysis (DEA) is a nonparametric method in operations research that is used to evaluate the relative efficiency and performance of diverse entities, such as firms, organizations, or functionally similar but independent operational units, often referred to as Decision-Making Units (DMUs) [2, 3]. These DMUs are usually specified by a vector of outputs and inputs. The main focus of a DEA investigation is to determine the relative efficiency of each DMU with respect to its peers. Based on this assessment, all entities are grouped in "efficient" or "inefficient" and it determines how much input/output slack is needed to convert an inefficient DMU into an efficient one. Radial distance from inefficient (DMU) to efficiency frontier indicates how much more efficient DMUs must be. An interesting property of the DEA algorithm is the fact that it can be applied without having an explicit specification (including a functional form) for the production function. Overall, the DEA model has the advantage of the ability to treat multiple inputs and outputs [4], which can improve the flexibility of this approach. Nevertheless, one important shortcoming of classical DEA models is that they do not deal with the uncertainty variations on input and output variables. Such a constraint makes the ordinary DEA models exceedingly exposed to prima facie uncertainty variables such as statistical error, data source scarcity, and usual randomness/vagueness/neutrosophicness of all economic phenomena. Thus, the use of classical DEA approaches to assess efficiency is very sensitive to variations in uncertainty, since such differences might cause DMU's changes between efficient and inefficient groups.

Recent research dealt with randomness of data within the DEA framework, and important steps have been taken. Among the output-oriented stochastic DEA models, two streams of approach are maintained. First, it is assumed that all variables are uniformly random [4, 5, 6], while in the second, only the output variables are random with inputs considered as deterministic [7, 8, 9]. Similarly, the input-oriented stochastic DEA models have also developed in two ways: one with outputs being random and inputs deterministic [10]. These models share common assumptions: they assume independence between the same random variables across different DMUs and identical distribution patterns for all stochastic variables. Expanding on these works, a few researchers have advanced novel robust stochastic DEA methodologies, which enhances the capability of the field to address uncertainty further. Researchers have recently studied randomness of data in DEA context, and development of the field further into important promises. Under the output-oriented stochastic DEA modeling, however, two streams of approach have been held. In the first, all variables are considered random [4, 5, 6]; in the second, the random ones are only considered output variables, while inputs are deterministic [7, 8, 9]. Along the same lines have come up these input-oriented stochastic DEA-models, one treating output random and inputs as deterministic [10]. These families of models share common assumptions: they assume the independence of the same random variables across different DMUs and by symmetry for all stochastic variables, the models are assumed to share the same distribution.

A few researchers built upon these works to develop novel robust stochastic DEA methodologies that further empower the field to address uncertainty. El-Demerdash et al. [11, 12] proposed a Stochastic Input-Oriented DEA model for evaluating the efficiency of decision-making units under uncertainty in input variables and/or output variables to provide realistic and reliable assessments of efficiency in environments where input values are subject to variability. Montazeri [13] furthered the construction of fuzzy stochastic DEA model to portray the synergistic relevance of joining fuzziness and randomness better to counter uncertainty in efficiency assessment. Sihotang et al. [14] modified DEA model for stochastic decision-making in production and supply chain planning under uncertain operational conditions. The reliability of efficiency measuring was improved by Syhotong et al. [15] providing for stochastic DEA considerations which in turn would stabilize DEA results when operating in unstable data conditions. In this respect, DEA models have been developed utilizing fuzzy logical approaches to capture the inherent vagueness of the variables. The development work follows two major avenues: First the classical fuzzy DEA models. Saati, et al., [16] developed a fuzzy DEA model to assess the efficiency and ranking of DMUs under uncertainty and emphasized handling imprecise inputs and outputs, offering a more flexible and realistic evaluation framework. Liu, et al., [17] applied fuzzy DEA to evaluate product design schemes, considering the vagueness inherent in design parameters and enable more accurate and practical assessments in the early stages of product development. Chiang and Che [18] proposed a fuzzy robust evaluation model combining Bayesian Belief Networks and weight-restricted DEA for ranking New Product Development project to manage uncertainty and enhance decision making in project selection. Khoshfetrat and Daneshvar [19] introduced fuzzy DEA models by refining the frontier structure to enhance discriminatory power and provide more precise efficiency analysis under fuzzy data conditions.

The input-output-oriented fuzzy DEA methods were presented as the second route. Kakao and Liu [20] were the pioneers in presenting the mathematical programming approach for ranking the DMUs within the framework of fuzzy DEA. They emphasized effective discrimination between DMUs and had a mechanism for handling imprecise input-output data very well. Liu [21] put forward the fuzzy DEA/Analytic Hierarchy Approach for determining flexible manufacturing systems in uncertain conditions and situations. It aimed at supporting decision-making by combining the analysis of efficiency with preference-based ranking. Azadeh et al. [22] discussed hybrid methodologies combining neural networks through fuzzy DEA for optimal location for solar plants, all under uncertainty and complexity, while handling multiple uncertainties concerning the technical, economic, and environmental parameters involved. Zerafat Angiz, et al. [23] presented a discrete fuzzy DEA model for evaluating the efficiency of DMUs under imprecise data while retaining robustness in the efficiency assessment process. Kaleibar, et al., [24] modified a centralized resource allocation model under a variable return to scale DEA model using fuzzy data to better accommodate a more flexible and fair resource allocation among DMUs in uncertain environments. with routes establishing that all variables, inputs and/or outputs are considered fuzzy. These models are distinguished further by their assumption that all fuzzy variables have some standard, common individual membership function. This includes the most outstanding contributions, one being that of Hatami-Marbini et al. [25] which provided an adaptable but otherwise flexible cross-efficiency evaluation methodology through a fuzzy output-oriented DEA model conceived specifically for supplier performance assessment. Or Tharwat et al. [26, 27] developed the models about input-output orientation. However, Mohanta & Sharanappa [28] achieved this application to fuzzy DEA: the spherical fuzzy DEA, which can represent uncertainty of decision makers regarding assessment of data. Stanojević, B., & Stanojević, M. [29] presented an empirical Monte Carlo simulation to visualize fuzzy efficiencies within the wholly fuzzy DEA models.

The recent development on DEA methodology to deal with neutrosophic uncertainty of the novel approach in the efficiency analysis. Edalatpanah [30] was the first one who introduced neutrosophic components into inputs as well as outputs. Furthermore, Abdelfattah [31] worked by incorporating triangular neutrosophic variables to derive a more developed model capturing degrees of truth, indeterminacy, and falsity within values of data. The field progressed even further with many critical contributions in the field. Kahraman et al. [32] integrated methods of DEA to Neutrosophic Analytic Hierarchy Process to improve performance assessment. Yang et al. [33] proceeded further by considering all variables in the form of single-valued neutrosophic triangular number representation and Mao et al. [34] developed an efficient neutrosophic DEA model that has been uniquely considered on undesirable outputs based on an aggregation operator approach. Other fairly recent works are the input-oriented neutrosophic DEA model by El-Demerdash et al. [35], and the approach to model uncertainty of both input and output data by Farnam et al. [36]. A particularly comprehensive development was that of Almutairi et al. [37], who presented a model that could do the following: determining whether variables should be treated as deterministic or neutrosophic, accommodating both input and output orientations, and handling multiple types of returns to scale (constant and variable).

These advancements in DEA methodologies, incorporating stochastic, fuzzy or neutrosophic elements, represent a significant shift in the way decision-making units are evaluated, offering more nuanced and comprehensive approaches to handle uncertainty in real-world data. Even though the above literature survey in DEA has yielded new models, these models still exhibit certain limitations. Firstly, the DEA models that have been developed lack the generality needed to effectively handle both deterministic variables and variables characterized by various forms of uncertainty. Secondly, the existing DEA models that do address uncertainty in variables typically categorize all variables, whether they are inputs or outputs, into a single category of uncertainty, ~~be~~ either in their randomness, vagueness, or neutrosophic nature. Thirdly, the DEA models; that are introduced to handle different orientation types, whether input or output, and to manage distinct return-to-scale types, such as constant or variable; remain relatively scarce. Lastly, a number of these newly developed models are specifically tailored to address certain applications, limiting their versatility and adaptability across different domains.

So, the main objective of this study is to create a comprehensive DEA model capable of accommodating a wide spectrum of variable types, including deterministic, stochastic, fuzzy, and neutrosophic variations, applied to both input and output parameters. This model will also address various orientation types, whether output-oriented or input-oriented, and consider different return-to-scale scenarios, encompassing both constant return to scale (CRS) and variable return to scale (VRS). The ultimate goal is to utilize this model to assess and compare the relative efficiency of a collection of organizations that share common input-output characteristics. Furthermore, the model aims to provide actionable recommendations to organizations that are found to be inefficient, enabling them to enhance their overall performance.

This paper is structured as follows: In the second section, we outline the basic mathematical framework of DEA. In the third section, we introduce our novel contribution: a generalized DEA model that incorporates multi-level uncertainty integration. In the fourth section, we show the practical applicability of our proposed model by implementing it in a case study. Section fifth introduces the comparison of the proposed approach eith the existing method and the sensitivity analysis. Finally, in the sixth section, we conclude the paper with a comprehensive summary of our findings and discuss promising avenues for future research.

2. Basic Data Envelopment Analysis Mathematical Models

Arguably, there are two DEA models. The CRS model [2] that a given amount of input will yield a proportional increase in output. The second category is known as VRS model which is a refinement by Banker et al. [38] where one would not expect a constant relationship between changes in inputs and outputs. The VRS is advanced as a refinement of the CRS model in a way that all the efficient DMUs are covered with a convex curve representing efficient frontiers. Further, there are also output oriented and input-oriented DEA model's classification. Input-oriented DEA model's frontier is determined by reducing the scale of an input in proportion to its amount that is used to produce a given amount of output for a DMU. In a similar manner to input oriented models, output-oriented DEA models seek the maximum possible increase of output under an unchanged amount of input. So, we presented two basic distinct models for this study, an output-oriented CRS model presented in Model (M-1), and an input-oriented CRS model presented in Model (M-2).

A basic CRS Output-Oriented DEA model

$$
\begin{aligned}
& Max\ \emptyset \\
& s.t. \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \geq \emptyset y_{pk} \quad ,\forall k = 1 \ldots \\
& \textstyle\sum_{i=1}^{n} \lambda_i x_{ij} \ \leq\ x_{pj} \quad ,\forall j = 1 \ldots .v \\
& \lambda_i \ \geq 0, (i = 1,2, \ldots, n)
\end{aligned}
\qquad (M-1)
$$

A basic CRS Input-Oriented DEA model

$$
\begin{aligned}
& Min\ \theta \\
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \leq\ \theta x_{pj} \quad ,\forall j = 1 \ldots .v \\
& \textstyle\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad ,\forall k = 1 \ldots w \\
& \lambda_i \ \geq 0, (i = 1,2, \ldots, n)
\end{aligned}
\qquad (M-2)
$$

where *k*: number of outputs (1 to 'w'); *j:* number of inputs (1 to 'v'); *i*: number of DMUs (1 to 'n'); y_{ik} is the amount of output *k* for DMU *i*, x_{ij} is the amount of input *j* for DMU *i*, and λ_i is the weight given to DMU *i*.

The models can be written as VRS models instated of CRS ones by adding a new constraint $(\sum_{i=1}^{n} \lambda_i = 1)$ to both (M-1) and (M-2) models. This constraint is formed as a combination convex of DMUs with positive λ's in the optimal solution.

3. Developing A Generalized Data Envelopment Analysis Model

A major development in DEA methodology is the formulation of more general and flexible models. This section introduces a new method that outperforms the traditional DEA models: the Generalized Mathematical Model for DEA. This model permits the incorporation of stochastic, fuzzy, and neutrosophic measures to assess performance efficiency under uncertainty. Great adaptability is one of the major strengths of this model. It moves effortlessly between input-oriented and output-oriented analyses, offering the means of relative efficiency measurement. Besides that, the model enables both constant return to scale and variable returns to scale assumptions, hence it can cope with efficiency analysis for any kind of operating scale.

This section will aim at an analysis of the changes in basic rules of this new DEA model regarding the patterns graph transformation using parametric techniques and thus increasing the robustness of the models applied even in the case of complex, uncertain and multi-dimensional situations. In this context, whereby managers are becoming more and more concerned with the framework design of organizational decision support systems, this model appears to be well suited for the practical applications for real life situational performance measurement and decision making. The subsequent remarks and propositions aim to establish a comprehensive DEA model for assessing the relative efficiency of each DMU while accommodating variables of distinct natures (deterministic, fuzzy, stochastic, and neutrosophic) independently. In addition, it accommodates the orientation types and allows for different kinds of return to scale.

Remark 1: Consider a versatile DEA model capable of addressing both input- and output-oriented problems, accommodating both CRS and VRS. The following model (M-3) represents a generalized adaptation of the traditional DEA model

$$
\begin{aligned}
& Min\ \ \xi_M \emptyset - (1-\xi_M)\theta \\
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \le\ \xi_M x_{pj} +\ (1-\xi_M)\theta x_{pj} \qquad ,\forall j = 1 \dots v \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \ge \xi_M \emptyset y_{pk} + (1-\xi_M)\ y_{pk} \qquad ,\forall k = 1 \dots w \\
& \xi_R \left[\sum_{i=1}^{n} \lambda_i - 1 \right] = 0 \\
& \quad \lambda_i\ \ge 0, (i = 1,2, \dots, n) \qquad\qquad (M-3)
\end{aligned}
$$

where ξ_M is the model type variable for orientation, which is defined as:

$$\xi_M = \begin{cases} 1 & if\ the\ model\ is\ Output\ oriented \\ 0 & if\ the\ model\ is\ Input\ oriented \end{cases}$$

and ξ_R is the model type variable for return to scale, which is defined as:

$$\xi_R = \begin{cases} 1 & if\ the\ model\ is\ variable\ reurn\ to\ scale \\ 0 & if\ the\ model\ is\ constant\ reurn\ to\ scale \end{cases}$$

As a result, we can identify three distinct cases:

- $\xi_M + \xi_R = 2$, represents a output-oriented VRS DEA model.
- $\xi_M + \xi_R = 1 \begin{cases} If\ \xi_M = 1, represents\ an\ output - oriented\ CRS\ \ DEA\ model. \\ \ If\ \xi_R = 1,\ represents\ a\ input - oriented\ VRS\ \ \ DEA\ model. \end{cases}$
- $\xi_M + \xi_R = 0$, represents an input-oriented CRS DEA model.

a- First stage - handle fuzzy variation: In this part, we have developed fuzzy generalized DEA models, which are unable to handle certain input or output variables. All other variables are considered deterministic in nature.

Remark 2: If either output and/or input are considered as fuzzy variables, then the fuzzy equivalent generalized DEA model to evaluate the p^{th} DMU efficiency level presented in Model (M – 3) can be formulated as follows:

$$
\begin{aligned}
& Min\ \ \xi_M \emptyset - (1-\xi_M)\theta \\
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \le\ \xi_M x_{pj} +\ (1-\xi_M)\theta x_{pj} \qquad ,\forall j \in J_D \\
& \sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \ \le\ \xi_M \tilde{x}_{pj} +\ (1-\xi_M)\theta \tilde{x}_{pj} \qquad ,\forall j \in J_F
\end{aligned}
$$

$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \qquad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \xi_M \emptyset \tilde{y}_{pk} + (1-\xi_M)\, \tilde{y}_{pk} \qquad ,\forall k \in K_F$$
$$\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0$$
$$\lambda_i \geq 0, (i = 1,2, \dots, n)\,, \qquad (M-4)$$

where $\tilde{x}_{ij}$ is the fuzzy input j for DMU i, $\tilde{y}_{ik}$ is the fuzzy output k *for* DMU i, J_D is the deterministic inputs set, J_F is the fuzzy inputs set, J are all the inputs set, K_D is the deterministic outputs set, K_F is the fuzzy outputs set, and K are all the outputs set. Taking into consideration, $J_D \cup J_F = J$ and $K_D \cup K_F = K$.

We can note that, when input, output, or both observations are treated as fuzzy variables, the deterministic inequalities are transformed into corresponding fuzzy inequalities. This transformation utilizes principles from fuzzy set theory as outlined in [39].

Proposition 1: considering fuzzy input variables ($\tilde{y}_{ik} \in K_F$) have triangular membership functions, then the crisp equivalent linear model for fuzzy generalized DEA model in the context of Model (*M - 4*) can be expressed as follows:

$$Min \;\; \xi_M \emptyset - (1-\xi_M)\theta$$

$$s.t.$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \xi_M x_{pj} + (1-\xi_M)\theta x_{pj} \qquad ,\forall j \in J_D$$
$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \xi_M \tilde{x}_{pj} + (1-\xi_M)\theta \tilde{x}_{pj} \qquad ,\forall j \in J_F$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \qquad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \xi_M \emptyset \tilde{y}_{pk} + (1-\xi_M)\, \tilde{y}_{pk} \qquad ,\forall k \in K_F$$
$$\alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{L} \leq \tilde{y}_{ik} \leq \alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{U} \qquad ,\forall k \in K_F$$
$$\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0$$
$$\lambda_i \geq 0, (i = 1,2, \dots, n)\,, \qquad (M-5)$$

where y_{ik}^{L} is the fuzzy output variable k for DMU i lower value, y_{ik}^{M} is the fuzzy output variable k for DMU i median value, y_{ik}^{U} is the fuzzy output variable k for DMU i upper value, and α is the fuzzy variables α-cut level.

Proof: Consider an input fuzzy numbers triangular membership function that are used for explaining input fuzzy for *(M-5)* as follows:

$$\mu_{\tilde{y}_{ik}} = \begin{cases} 0\,, & y_{ik} \leq y_{ik}^{L} \\ \frac{y_{ik}-y_{ik}^{L}}{y_{ik}^{M}-y_{ik}^{L}}\,, & y_{ik}^{L} \leq y_{ik} \leq y_{ik}^{M} \\ \frac{y_{i}^{U}-y_{ik}}{y_{ik}^{U}-y_{ik}^{M}}\,, & y_{ik}^{M} \leq y_{ik} \leq y_{ik}^{U} \\ 0\,, & y_{ik} \geq y_{ik}^{U} \end{cases} \qquad (1)$$

$$\tilde{y}_{ik} = (y_{ik}^L, y_{ik}^M, y_{ik}^U), \quad 0 \le y_{ik}^L \le y_{ik}^M \le y_{ik}^U \ \rightarrow \tilde{y}_{ik} \in [y_{ik}^L, y_{ik}^U] \tag{2}$$

$\mu_{\tilde{y}_{ik}}$ as express in eq. (1) defines the α-cuts of outputs $\tilde{y}_{ik}$ for triangular fuzzy numbers of arithmetic operations. This approach yields intervals representing lower and upper bounds at different α-levels. The following outlines the application of α-cut interval operations to fuzzy inputs.

$$\mu_{\tilde{y}_{ik}} \ge \alpha \frac{y_{ik} - y_{ik}^L}{y_{ik}^M - y_{ik}^L} \ge \alpha \ \frac{y_{ik}^U - y_{ik}}{y_{ik}^U - y_{ik}^M} \ge \alpha \tag{3}$$

$$\tilde{y}_{ik} \in [\alpha y_{ik}^M + (1-\alpha) y_{ik}^L, \alpha y_{ik}^M + (1-\alpha) y_{ik}^U \ ] \tag{4}$$

Proposition 2: considering fuzzy input variables ($\tilde{x}_{ij} \in J_F$) have triangular membership functions, then the crisp equivalent linear model for fuzzy generalized DEA model in the context of Model (*M - 5*) can be expressed as follows:

$$Min \ \xi_M \emptyset - (1-\xi_M)\theta$$

$$\begin{aligned}
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \le \ \xi_M x_{pj} + \ (1-\xi_M)\theta x_{pj} && ,\forall j \in J_D \\
& \sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \ \le \ \xi_M \tilde{x}_{pj} + \ (1-\xi_M)\theta \tilde{x}_{pj} && ,\forall j \in J_F \\
& \alpha x_{ij}^M + (1-\alpha) x_{ij}^L \le \tilde{x}_{ij} \le \ \alpha x_{ij}^M + (1-\alpha) x_{ij}^U && ,\forall j \in J_F \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \ge \xi_M \emptyset y_{pk} + (1-\xi_M) \ y_{pk} && ,\forall k \in K_D \\
& \sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \ge \xi_M \emptyset \tilde{y}_{pk} + (1-\xi_M) \ \tilde{y}_{pk} && ,\forall k \in K_F \\
& \alpha y_{ik}^M + (1-\alpha) y_{ik}^L \le \tilde{y}_{ik} \le \ \alpha y_{ik}^M + (1-\alpha) y_{ik}^U && ,\forall k \in K_F \\
& \xi_R \left[\sum_{i=1}^{n} \lambda_i - 1 \right] = 0 \\
& \qquad \lambda_i \ \ge 0, (i = 1,2, \dots, n) && (M-6)
\end{aligned}$$

where x_{ij}^L is the input fuzzy variable *j* for DMU i lower value, x_{ij}^M is the input fuzzy variable *j* for DMU i median value, x_{ij}^U is the input fuzzy variable *j* for DMU i upper value, and α is the level of α-cut for fuzzy variables *j*

Proof: Consider an input fuzzy numbers triangular membership function that are used for explaining input fuzzy for *(M-6)* as follows:

$$\mu_{\tilde{x}_{ij}} = \begin{cases} 0 \ , & x_{ij} \le \ x_{ij}^L \\ \frac{x_{ij} - x_{ij}^L}{x_{ij}^M - x_{ij}^L} \ , & x_{ij}^L \le \ x_{ij} \le x_{ij}^M \\ \frac{x_{ij}^U - x_{ij}}{x_{ij}^U - x_{ij}^M} \ , & x_{ij}^M \le \ x_{ij} \le x_{ij}^U \\ 0 \ , & x_{ij} \ge \ x_{ij}^U \end{cases} \tag{5}$$

$$\tilde{x}_{ij} = \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right), \quad 0 \le x_{ij}^L \le x_{ij}^M \le x_{ij}^U \ \rightarrow \tilde{x}_{ij} \in \left[x_{ij}^L, x_{ij}^U\right] \tag{6}$$

$\mu_{\tilde{x}_{ij}}$ as express in eq. (5) defines the α-cuts of inputs $\tilde{x}_{ij}$ for triangular fuzzy numbers of arithmetic operations. This approach yields intervals representing lower and upper bounds at different α-levels. The following outlines the application of α-cut interval operations to fuzzy inputs.

$$\mu_{\tilde{x}_{ij}} \geq \alpha \frac{x_{ij}-x_{ij}^{L}}{x_{ij}^{M}-x_{ij}^{L}} \geq \alpha \ \frac{x_{ij}^{U}-x_{ij}}{x_{ij}^{U}-x_{ij}^{M}} \geq \alpha \tag{7}$$

$$\tilde{x}_{ij} \in \left[\alpha x_{ij}^{M} + (1-\alpha)\, x_{ij}^{L}, \alpha x_{ij}^{M} + (1-\alpha)\, x_{ij}^{U}\right] \tag{8}$$

b- **Second Stage - handle stochastic variations:** In this part, we have constructed stochastic generalized DEA model, which integrates stochastic elements for any certain either output and/or input variable. However, the remaining variables are treated to be deterministic.

Proposition 4: assuming that either output and/or input are random variables, then the equivalent generalized chance-constraint DEA model to evaluate the efficiency level for DMU p within Model (*M - 3*) can be formulated as follows:

$$Min \ \ \xi_M \emptyset - (1-\xi_M)\theta$$

$$\begin{aligned}
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \leq \ \xi_M x_{pj} + \ (1-\xi_M)\theta x_{pj} && ,\forall j \in J_D \\
& pr \ \left\{ \sum_{i=1}^{n} \lambda_i x_{ij} \ \leq \ \xi_M x_{pj} + \ (1-\xi_M)\theta x_{pj} \right\} \ \geq (1-\eta_j) && ,\forall j \in J_S \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} && ,\forall k \in K_D \\
& pr \left\{ \sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \right\} \ \geq (1-\eta_k) && ,\forall k \in K_S \\
& \xi_R \left[\sum_{i=1}^{n} \lambda_i - 1 \right] = 0 \\
& \lambda_i \ \geq 0, (i = 1,2, \ldots, n)\ ,
\end{aligned} \tag{$M-7$}$$

where η_j is the level of significance for stochastic input variable j, η_k is the level of significance for stochastic output variable k, J_S is the stochastic inputs set, J: all inputs sets, $J_D \cup J_S = J$, K_S is the stochastic outputs set, and K is the all outputs set, $K_D \cup K_S = K$.

Detailed proof is presented at [40].

Proposition 5: Assume the random input variable ($x_{ij} \in J_s$) is normally distributed, then the deterministic nonlinear equivalent model, for generalized stochastic DEA model presented in (M-7) for p^{th} DMU, is as follows:

$$Min \ \ \xi_M \emptyset - (1-\xi_M)\theta$$

$$\begin{aligned}
& s.t. \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \ \leq \ \xi_M x_{pj} + \ (1-\xi_M)\theta x_{pj} && ,\forall j \in J_D \\
& \sum_{i=1}^{n} \lambda_i \mu_{ij} - \xi_M \mu_{pj} - (1-\xi_M)\theta \mu_{pj} \leq e_j \sqrt{\left(\lambda_p - (1-\xi_M)\theta\right)^2 \sigma_{pj}^2 + \sum_{i=1\ i\neq p}^{n} \lambda_i^2 \sigma_{ij}^2} && ,\forall j \in \ J_S \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} && ,\forall k \in K_D
\end{aligned}$$

$$pr\left\{\sum_{i=1}^{n}\lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk}\right\} \geq (1-\eta_k) \quad ,\forall k \in K_S$$

$$\xi_R\left[\sum_{i=1}^{n}\lambda_i - 1\right] = 0$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n)\ , \qquad (M-8)$$

Proof: assume that each input $x_{ij} \in J_s$ is normally distributed with a mean μ_{ij} and variance σ^2_{ij}. Now, define the random variable for all input $j \in J_S$

$$u_j = \sum_{i=1}^{n}\lambda_i x_{ij} - \xi_M x_{pj} - (1-\xi_M)\theta x_{pj} \tag{9}$$

with mean and variance:

$$E(u_j) = \sum_{i=1}^{n}\lambda_i \mu_{ij} - \xi_M \mu_{pj} - (1-\xi_M)\theta\mu_{pj} \equiv \mu_{u_j} \quad , \tag{10}$$

$$var(u_j) = \left(\lambda_p - (1-\xi_M)\theta\right)^2\sigma^2_{pj} + \sum_{i=1\ i\neq p}^{n}\lambda_i^2\sigma^2_{ij} \equiv \sigma^2_{u_j} \quad . \tag{11}$$

Since the $x_{ij}'s$ are normally distributed, $x_{ij} \sim N(\mu_{u_j}, \sigma^2_{u_j})$, z_j denote the equivalent standardized normal value for the variable u_j, as follows,

$$z_j = \frac{u_j - \mu_{u_j}}{\sigma_{u_j}} \quad . \tag{12}$$

Hence,

$$pr\left\{\sum_{i=1}^{m}\lambda_i x_{ij} \leq \xi_M x_{pj} + (1-\xi_M)\theta x_{pj}\right\} = pr\{u_j \leq 0\} = pr\left\{z_j \leq \frac{-\mu_{u_j}}{\sigma_{u_j}}\right\} \quad . \tag{13}$$

Using the normal distribution symmetry property,

$$pr\left\{z_j \leq \frac{-\mu_{u_j}}{\sigma_{u_j}}\right\} = pr\left\{z_j \geq \frac{\mu_{u_j}}{\sigma_{u_j}}\right\} = 1 - \varphi\left(\frac{\mu_{u_j}}{\sigma_{u_j}}\right) \quad , \tag{14}$$

where $\varphi(\,)$ denotes the distribution function for cumulative standard. The equivalent restriction for the chance-constrained DEA problem inequality as follows:

$$1 - \varphi\left(\frac{\mu_{u_j}}{\sigma_{u_j}}\right) \geq \left(1-\alpha_j\right) \quad , \tag{15}$$

$$\varphi\left(\frac{\mu_{u_j}}{\sigma_{u_j}}\right) \leq \varphi\left(e_j\right) \quad . \tag{16}$$

$\varphi(e)$ is determined by referencing the standard normal distribution table. Therefore, the final equation can be written as

$$\mu_{u_j} \leq e_j\sigma_{u_j}\ . \tag{17}$$

So, the next step is to substitute Eq. (10) and (11) in Eq. (17), i.e.,

$$\sum_{i=1}^{n} \lambda_i \mu_{ij} - \xi_M \mu_{pj} - (1-\xi_M)\theta\mu_{pj} \le e_j \sqrt{\left(\lambda_p - (1-\xi_M)\theta\right)^2 \sigma_{pj}^2 + \sum_{i=1\ i\ne p}^{n} \lambda_i^2 \sigma_{ij}^2} \quad . \tag{18}$$

Proposition 6: Assume both the random output and input variables ($y_{ik} \in K_s$, $x_{ij} \in J_s$) are normally distributed, then the deterministic nonlinear equivalent model, for generalized stochastic DEA model presented in (M-8) for the p^{th} DMU, is as follows:

$$\begin{aligned}
&Min\ \ \xi_M \emptyset - (1-\xi_M)\theta \\
&s.t. \\
&\sum_{i=1}^{n} \lambda_i x_{ij} \le \xi_M x_{pj} + (1-\xi_M)\theta x_{pj} \qquad , \forall j \in J_D \\
&\sum_{i=1}^{n} \lambda_i \mu_{ij} - \xi_M \mu_{pj} - (1-\xi_M)\theta\mu_{pj} \le e_j \sqrt{\left(\lambda_p - (1-\xi_M)\theta\right)^2 \sigma_{pj}^2 + \sum_{i=1\ i\ne p}^{n} \lambda_i^2 \sigma_{ij}^2} \qquad , \forall j \in J_S \\
&\sum_{i=1}^{n} \lambda_i y_{ik} \ge \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \qquad , \forall k \in K_D \\
&\sum_{i=1}^{n} \lambda_i \mu_{ik} - \xi_M \emptyset \mu_{pk} - (1-\xi_M)\mu_{pk} \ge e_k \sqrt{(\lambda_p - \xi_M \emptyset)^2 \sigma_{pk}^2 + \sum_{i=1\ i\ne p}^{n} \lambda_i^2 \sigma_{ik}^2}\ , \forall k \in K_s \\
&\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0 \\
&\quad \lambda_i \ge 0, (i = 1,2, \dots, n)\ ,
\end{aligned} \tag{M-9}$$

Proof: consider each output $y_{ik} \in K_s$ to be normally distributed, $y_{ik} \sim N\,(\mu_{ik}, \sigma_{ik}^2)$. so, we can define $u_k$as a random variable for all output variables taking into consideration $k \in K_s$

$$u_k = \sum_{i=1}^{n} \lambda_i y_{ik} - \xi_M \emptyset y_{pk} - (1-\xi_M)\, y_{pk} \tag{19}$$

with mean and variance:

$$E(u_k) = \sum_{i=1}^{n} \lambda_i \mu_{ik} - \xi_M \emptyset \mu_{pk} - (1-\xi_M)\, \mu_{pk} \equiv \mu_{uk} \tag{20}$$

$$var(u_k) = (\lambda_p - \xi_M \emptyset)^2 \sigma_{pk}^2 + \sum_{i=1\ i\ne p}^{n} \lambda_i^2 \sigma_{ik}^2 \equiv \sigma_{u_k}^2 \tag{21}$$

Since $y_{ik}{}'s$ are normally distributed, $y_{ik} \sim N\,(\mu_{u_k}, \sigma_{u_k}^2)$, ~~so~~ let z_k denote the equivalent standardized normal value for the variable u_k, as follows,

$$z_k = \frac{u_k - \mu_{u_k}}{\sigma_{u_k}} \tag{22}$$

Hence,

$$pr\left\{\sum_{i=1}^{n} \lambda_i y_{ik} \ge \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk}\right\} = pr\{u_k \ge 0\} = pr\left\{z_k \ge \frac{-\mu_{u_k}}{\sigma_{u_k}}\right\} \tag{23}$$

Using the normal distribution symmetry property,

$$pr\left\{z_k \geq \frac{-\mu_{u_k}}{\sigma_{u_k}}\right\} = pr\left\{z_k \leq \frac{\mu_{u_k}}{\sigma_{u_k}}\right\} = \varphi\left(\frac{\mu_{u_k}}{\sigma_{u_k}}\right) \quad (24)$$

where $\varphi(\,)$ denotes the distribution function for cumulative standard. The equivalent restriction for the chance-constrained DEA problem inequality [40] as follows:

$$\varphi\left(\frac{\mu_{u_k}}{\sigma_{u_k}}\right) \geq (1-\alpha_k) \quad (25)$$

$$\varphi\left(\frac{\mu_{u_k}}{\sigma_{u_k}}\right) \geq \varphi(e_k) \quad (26)$$

The value of $\varphi(e)$ is determined by referencing the standard normal distribution table. Therefore, the final equation can be written as

$$\mu_{u_k} \geq e_k \sigma_{u_k} \quad (27)$$

Substitution equations (20) and (21) in equation (27) leads to

$$\sum_{i=1}^{n} \lambda_i \mu_{ik} - \xi_M \emptyset \mu_{pk} - (1-\xi_M)\mu_{pk} \geq e_k \sqrt{(\lambda_p - \xi_M \emptyset)^2 \sigma_{pk}^2 + \sum_{i=1\ i\neq p}^{n} \lambda_i^2 \sigma_{ik}^2}\ , \forall k \in\ K_s \quad (28)$$

c- **Third stage - handle neutrosophic variation:** In this part, we make the attempt to construct neutrosophic generalized DEA model with two assumptions; some specific variables either input and/or output are neutrosophic in nature while the remaining are deterministic under the first assumption.

- **Definition 1**[37]**:** Let U be the field of all objects and let u be an arbitrary object element in U, $u \in U$. A neutrosophic set of $(\widetilde{W}^N)$ is expressed as $\widetilde{W}^N = \{\langle u : T_{\widetilde{W}^N}(u), I_{\widetilde{W}^N}(u), F_{\widetilde{W}^N}(u)\rangle, u \in U, T_{\widetilde{W}^N}(u), I_{\widetilde{W}^N}(u), F_{\widetilde{W}^N}(u) \in]0^-, 1^+[\}$, where $T_{\widetilde{W}^N}(u), I_{\widetilde{W}^N}(u)$, and $F_{\widetilde{W}^N}(u)$ are membership functions representing truth, indeterminacy, and falsity, respectively.
- **Definition 2** [37]**:** the general formulation of the neutrosophic set single value $(\widetilde{W}^{SVN})$ concerning to non-empty set u is $\widetilde{W}^{SVN} = \{\langle u, T_{\widetilde{W}^N}(u), I_{\widetilde{W}^N}(u), F_{\widetilde{W}^N}(u)\rangle, u \in U\}$. In this representation, $T_{\widetilde{W}^N}(u), I_{\widetilde{W}^N}(u), F_{\widetilde{W}^N}(u)$ all take the values $0^- \leq T_{\widetilde{W}^N}(u) + I_{\widetilde{W}^N}(u) + F_{\widetilde{W}^N}(u) \leq 3^+$, such that u is an element of the set U.
- **Definition 3** [37]**:** let $\tilde{v}^{TN} = \langle (a,b,c); L_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, F_{\tilde{v}^{TN}}\rangle$ be a triangular neutrosophic single valued (SVTFN), Then SVTFN as Score function

$$SF(\tilde{v}^{TN}) = \left(\frac{1}{4}(a+2b+c)\right)\left(\frac{1}{3}(2 + L_{\tilde{v}^{TN}} - \delta_{\tilde{v}^{TN}} - F_{\tilde{v}^{TN}})\right) \quad (29)$$

Remark 3: If certain variable either output and/or input are considered as neutrosophic variables, then the equivalent neutrosophic generalized DEA model to evaluate the ***pth*** DMU relative efficiency level in model (M – 3) can be formulated as follows:

$$Min\ \ \xi_M \emptyset - (1-\xi_M)\theta$$
$$s.t.$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \ \leq\ \xi_M x_{pj} +\ (1-\xi_M)\theta x_{pj} \qquad ,\forall j \in J_D$$
$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij}^{\ TN} \ \leq\ \xi_M \tilde{x}_{pj}^{\ TN} +\ (1-\xi_M)\theta \tilde{x}_{pj}^{\ TN} \qquad ,\forall j \in J_N$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\ y_{pk} \qquad ,\forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik}{}^{TN} \geq \xi_M \emptyset \tilde{y}_{pk}{}^{TN} + (1-\xi_M)\, \tilde{y}_{pk}{}^{TN} \qquad ,\forall k \in K_N$$
$$\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0$$
$$\lambda_i \geq 0, (i = 1,2, \dots, n)\,, \qquad (M-10)$$

where J_N is the neutrosophic inputs set, J is all the inputs set, $J_D \cup J_N = J$, and K_N is the neutrosophic outputs set, and K is all the outputs set, where $K_D \cup K_N = K$.

Remark 4: Assume that each input $\tilde{x}_{ij}{}^{TN} \in J_N$ and $\tilde{y}_{ik}{}^{TN} \in K_N$ are triangular neutrosophic variables, then the deterministic equivalent linear model explained in the neutrosophic generalized DEA model (M-10) for ***p^th^*** DMU is given below based on a single valued triangular neutrosophic set definition (as shown in definition 3):

$$Min\ \xi_M \emptyset - (1-\xi_M)\theta$$
$$s.t.$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \xi_M x_{pj} + (1-\xi_M)\theta x_{pj} \qquad ,\forall j \in J_D$$
$$\sum_{i=1}^{n} \lambda_i\, SF(\tilde{x}_{ij}{}^{TN}) \leq \xi_M\, SF(\tilde{x}_{pj}{}^{TN}) + (1-\xi_M)\theta\, SF(\tilde{x}_{pj}{}^{TN}) \qquad ,\forall j \in J_N$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \qquad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i\, SF\big(\tilde{y}_{ik}{}^{TN}\big) \geq \xi_M \emptyset\, SF(\tilde{y}_{pk}{}^{TN}) + (1-\xi_M)\, SF(\tilde{y}_{pk}{}^{TN}) \qquad ,\forall k \in K_N$$
$$\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0$$
$$\lambda_i \geq 0, (i = 1,2, \dots, n)\,, \qquad (M-11)$$

Finally, the proposed generalized DEA model, designed to accommodate various types of variable variations (deterministic, fuzzy, stochastic, and neutrosophic), is demonstrated in model (M-12). This model integrates the formulations from Models (M-3), (M-6), (M-9), and (M-11) into a unified framework represented by model (M-12).

$$Min\ \xi_M \emptyset - (1-\xi_M)\theta$$
$$s.t.$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \xi_M x_{pj} + (1-\xi_M)\theta x_{pj} \qquad ,\forall j \in J_D$$
$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \xi_M \tilde{x}_{pj} + (1-\xi_M)\theta \tilde{x}_{pj} \qquad ,\forall j \in J_F$$
$$\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \leq \tilde{x}_{ij} \leq \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{U} \qquad ,\forall j \in J_F$$
$$\sum_{i=1}^{n} \lambda_i \mu_{ij} - \xi_M \mu_{pj} - (1-\xi_M)\theta \mu_{pj} \leq e_j \sqrt{\big(\lambda_p - (1-\xi_M)\theta\big)^2 \sigma_{pj}^2 + \sum_{i=1\, i\neq p}^{n} \lambda_i^2 \sigma_{ij}^2} \qquad ,\forall j \in J_S$$

$$\sum_{i=1}^{n} \lambda_i \, SF(\tilde{x}_{ij}{}^{TN}) \le \xi_M \, SF(\tilde{x}_{pj}{}^{TN}) + (1-\xi_M)\theta \, SF(\tilde{x}_{pj}{}^{TN}) \qquad , \forall j \in J_N$$

$$\sum_{i=1}^{n} \lambda_i y_{ik} \ge \xi_M \emptyset y_{pk} + (1-\xi_M)\, y_{pk} \qquad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \ge \xi_M \emptyset \tilde{y}_{pk} + (1-\xi_M)\, \tilde{y}_{pk} \qquad , \forall k \in K_F$$

$$\alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{L} \le \tilde{y}_{ik} \le \alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{U} \qquad , \forall k \in K_F$$

$$\sum_{i=1}^{n} \lambda_i \mu_{ik} - \xi_M \emptyset \mu_{pk} - (1-\xi_M)\mu_{pk} \ge e_k \sqrt{(\lambda_p - \xi_M \emptyset)^2 \sigma_{pk}^2 + \sum_{i=1\, i\ne p}^{n} \lambda_i^2 \sigma_{ik}^2} \; , \forall k \in K_s$$

$$\sum_{i=1}^{n} \lambda_i \, SF\left(\tilde{y}_{ik}{}^{TN}\right) \ge \xi_M \emptyset \, SF(\tilde{y}_{pk}{}^{TN}) + (1-\xi_M)\, SF(\tilde{y}_{pk}{}^{TN}) \qquad , \forall k \in K_N$$

$$\xi_R \left[\sum_{i=1}^{n} \lambda_i - 1\right] = 0$$

$$\lambda_i \ge 0, (i = 1,2,\ldots,n)\,, \qquad (M-12)$$

4. Case Study

In this case study, we apply the proposed robust and versatile DEA framework in the assessment of relative efficiency performance for hospitals. Hospitals are quite complex organizations; therefore, the level of uncertainty surrounding some or all the inputs and/or outputs can provide suitable grounds for illustrating the versatility of the DEA model. Their operational inputs and outputs are usually driven by factors that can be deterministic, stochastic, fuzzy, or neutrosophic; therefore, the efficiency assessment approach should be flexible and robust. The case selected involves a comparison of five comparable hospitals, namely H_1, H_2, H_3, H_4, and H_5, operating within similar contexts and providing comparable healthcare services. Various inputs form parts of the analysis, such as the number of doctors, number of nurses, annual budget, and medical equipment available. In addition, outputs to be measured will include the number of patients treated, patient satisfaction rate, and treatment success rate. These reflect both certainty and uncertainty, classified as follows: deterministic variables include the Number of doctors, number of nurses; stochastic variables include Patient satisfaction rate, treatment success rate; fuzzy variables are medical equipment; neutrosophic variables are Annual budget, number of patients treated. It, therefore, gives an indication that this model can handle a wide range of data sets. These input and output variables data are presented in the following Table 1and Table 2, respectively.

Table 1 Data for the hospital's **input variables**

Hospital	Number of Doctors	Number of nurses	Annual Budget (millions)	Medical Equipment
H_1	20	50	$\langle(4.8, 5, 5.2); 0.9,0.4,0.1\rangle$	(0.4, 0.6, 0.8)
H_2	15	40	$\langle(3.8, 4, 5.5); 0.9,0.7,0.1\rangle$	(0.2, 0.3, 0.4)
H_3	30	70	$\langle(5.5, 6, 6.3); 0.9, 0.4, 0.1\rangle$	(0.55, 0.6, 0.85)
H_4	18	45	$\langle(4.5, 5, 5.5); 0.8, 0.5, 0.1\rangle$	(0.2, 0.5, 0.7)
H_5	25	60	$\langle(4.2,4.7,5.2); 0.9, 0.4, 0.1\rangle$	(0.15, 0.4, 0.7)

Table 2 Data for the hospital's **output variables**

Hospital	Number of Patients Treated (thousands)	Patient Satisfaction Rate % (μ, σ^2)	Treatment Success Rate % (μ, σ^2)
H_1	$\langle(1.8, 2, 2.1); 1.0, 0.0,0.0\rangle$	(85, 5)	(90, 3)
H_2	$\langle(1.3, 1.4, 1.5); 1.0,0.0,0.0\rangle$	(80, 6)	(85, 4)
H_3	$\langle(2.8, 3, 3.2); 1.0,0.0,0.0\rangle$	(90, 4)	(95, 2)
H_4	$\langle(2.3, 2.45, 2.6); 1.0,0.0,0.0\rangle$	(82, 4)	(88, 3)
H_5	$\langle(1.9,2 , 2.2); 1.0,0.0,0.0\rangle$	(87, 4)	(92, 2)

A proposed algorithm to address the problem under investigation can be implemented in two distinct scenarios, both of which may indicate how the departments under consideration may be considered to be relatively efficient. Scenario 1: A generalized VRS output-oriented VRS DEA is executed as specified in Model ***M-12***, with $\xi_M + \xi_R = 2$. Scenario 2: A generalized VRS input-oriented DEA is executed as specified in Model ***M-12***, with $\xi_M + \xi_R = 1$, with $\xi_R = 1$.

This will provide an LP formulation to evaluate, in relative terms, the efficiency level of each hospital. In each case, the five models for the five different hospitals are solved using GAMS programming language. The results of the relative efficiency estimated for each hospital involved are presented in Table 3 below.

Table 3 Hospital Relative efficiency Results for each Scenario

Hospital	Scenario 1	Scenario 2
H_1	100%	100%
H_2	36%	44%
H_3	100%	100%
H_4	44%	56%
H_5	80%	66%

5. Results Comparison and Sensitivity Analysis

Results analysis reveal that the developed models return promising and comparable results. While VRS DEA models consistently identify the efficient and inefficient DMUs without regard to orientation, either to output or input, efficiency scores for inefficient DMUs could be different. This is so because changes in inputs and outputs do not always relate proportionally. Table 3 shows the relative efficiencies score for each hospital in two scenarios. These efficiency scores for the two scenarios have the following implications for the performance of the five hospitals: Hospitals H_1 and H_3 achieved efficient scores (1.0) in both scenarios, indicating they are operating at optimal efficiency levels, and These hospitals serve as benchmarks for the others in the study. For the remaining hospitals, hospital H_2 shows the lowest efficiency scores (0.36 and 0.44), hospital H_4 demonstrates moderate inefficiency (0.44 and 0.56), and hospital H_5 shows better performance but still room for improvement (0.80 and 0.66). We notice that the input-oriented scenario (Scenario 2) generally yielded higher efficiency scores than the output-oriented scenario, this means that hospitals might have more control over their input utilization than their output generation. The variation in scores between the two scenarios depicts the flexibility and adaptability of the proposed DEA model in capturing different efficiency dimensions. Finally, we can recommend:

- Benchmark Hospitals (H_1 and H_3): these hospitals represent best practices and can serve as a model for others. It is true that their current practices need to be maintained, but continuous monitoring is recommended to uphold their efficiency.
- Inefficient Hospitals (H_2, H_4, and H_5): H_2 requires major improvements in both input utilization and output generation. Efforts should emphasize enhancements in operational processes, staff productivity, and patient care outcomes. H_4: the amelioration in Scenario 2 suggests that it needs to focus on its input optimization. It

should try investing in training, utilizing equipment, and budget allocation. H_5: does better on both H_2 and H_4, but it must balance input optimization with output improvement. To enhance efficiency, it may use strategies that streamline operations and reallocate resources.

6. Conclusion and Future works

This research developed and demonstrated a strong and flexible DEA framework that had already addressed complex challenges in efficiency assessment under different forms of uncertainty. By implementation on the case study of hospitals, we successfully validated its capabilities to treat deterministic, stochastic, fuzzy, and neutrosophic variables simultaneously in input and output parameters, which was considered an advance in the methodology of DEA. The proposed framework has several key strengths and contributions. First, the ability to handle multiple types of data uncertainty within a model represents a substantial improvement over the traditional DEA approach. Second, the flexibility in handling both input-oriented and output-oriented scenarios, with different returns to scale assumptions, makes the framework a comprehensive tool for efficiency assessment across diverse organizational contexts. The validation of the practical applicability of our framework, as obtained from the hospital case study, indicates that there are significant differences in the efficiency scores of the analyzed units, ranging from 0.36 to 1.0. This application shows that the model is able to highlight both benchmark performers and opportunities for improvement, thus providing valuable insights for decision-makers in performance management and resource allocation.

Moreover, the capability of the framework to provide tangible scores about efficiency while considering multiple forms of uncertainty gives this model considerable strength in terms of supporting any decision. Such features are helpful in complex organizations where conventional deterministic approaches would probably miss operational nuances. The future avenues of this research open with the possibility of additional types and sources of uncertainties or data that may be incorporated within the framework. Longitudinal studies could also be done to assess the change in efficiencies over time and adapt the model to an application in an industrial context; maintain robustness in dealing with uncertainty. The contribution of this research to the field of efficiency assessment is, therefore, immense with an all-encompassing, flexible, and reliable DEA framework. That is where the strength of the model lies: it can handle different forms of uncertainty while retaining practical applicability; hence, it is a useful tool for decision-makers in various sectors, especially in complex organizational settings where uncertainty may be inherent in operational processes.

The aim of this research is developed a DEA model that bear its novelty and generality in considering the different sources of uncertainty—deterministic, fuzzy, stochastic, or neutrosophic—in a common framework. However, its application imposes several limitations. Among these restrictions, enlarged computational complexity can be put as a foremost one. The conjoining of all sorts of evidence in a unified framework LP-based DEA model will put a lot of computational capabilities in stake especially for its generalization to large-scale problems. Another area of concern is scalability. This model has been tried only on a limited number of five hospitals, with performance and scalability to larger databases with many DMUs with complex interdependencies needing to be further duly checked. Also, even if neutrosophic data lend high expressiveness to modeling uncertainty, soul-searching interpretability may prove tough to those practitioners placed in the context of actual healthcare setups, where data clarity and verification must precede everything else. Lastly, a static analysis is conducted with the current model, while the dynamic aspect providing support for performance adjustment commensurate to changing time is the most crucial ingredient in the long-term assessment of care efficiency.

Finally, future research would include working around the above limits to make the model more useful. Creating dynamic DEA models under a neutrosophic framework may be one of them, whereby performance would be captured in time-series trends. Another possible extension could be a generalization of the model to cover multi-sector efficiency analysis, like in education, energy, or even transportation, where uncertainty mostly is manifested in inputs and outputs. Moreover, it could also include machine learning or metaheuristic algorithms in optimizing scale-large problems and increasing computational efficiencies. Future work could also involve developing decision-support frameworks or visualization tools to enhance interpretability of complex fuzzy and neutrosophic results to end-users. Lastly, comparative studies among different paradigms of uncertainty models would go a long way in adding credibility to the robustness and reliability of the proposed framework across different real-world applications.

Data Availability

In this article, no data was used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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