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Improved neutrosophic exponential-ratio estimator of population mean with simple random sampling

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Abstract: In this paper we present a neutrosophic exponential-ratio estimator for calculating the population mean using simple random sampling. In sampling methods classical statistics always depends on exact and complete data, but when we are dealing with unclear data these all become insufficient. By managing ambiguous and indeterminate data, neutrosophic statistics an extension of fuzzy and classical statistics addresses this drawback. The bias and mean square error (MSE) of proposed estimator are derived up to the first approximation order. Comparative study shows that it is more efficient than existing estimators, especially when we are working with data that is imprecise or of the neutrosophic kind. The proposed approach produces interval-based estimations in contrast to traditional estimators, which summarizes the unknown population mean with minimal MSE, improving reliability. The effectiveness of the estimator is confirmed by simulations and neutrosophic data sets, highlighting its potential in situations where uncertainty is common in the real world.

Keywords: Neutrosophic statistics, population mean, exponential-ratio estimator, bias, mean squared error, simple random sampling.

1. Introduction

In sample surveys, the statistical technique that works with indeterminate data and inference techniques with varying degrees of indeterminacy is called neutrophilic statistics. Based on interval analysis, it is an expansion of interval statistics. All sets and intervals, including finite discrete sets, are covered by neutrophilic statistics, which is based on set analysis. Classical and interval statistics assume that each member of the sample or population belongs 100% of the time, but neutrosophic statistics can account for individuals who are partially belonging or not belonging. The accuracy of data obtained by neutrosophic statistics is higher than that of classical and interval statistics. It works with data and inference methods and is more flexible than traditional statistics. Neutrosophic and interval statistics overlap if all sets are intervals, each individual is 100% representative of the sample or population, and there is only one probability distribution curve. However, more non-classical statistical techniques are required than classical ones since our environment contains more unclear data.

In traditional statistics, data are known and created by discrete numerical values. Numerous writers developed several estimators for calculating the finite population mean under classical statistics when auxiliary data was included. Instead of using the study variable alone, the study revealed that when there is a good correlation between the study variable and the auxiliary variable, we obtain considerably lower sampling error for the ratio and, as a result, we may use less sampling for ratio estimation. One paper [1] included a thorough overview of ratio estimation, along with examples and its features. Alternatively, the ratio estimation approach lowers the sample size while maintaining the same level of precision [2]. Classic ratio, regression, and exponential approaches are easy to compute population mean estimates. These methods use auxiliary variable parameters like coefficients of variation, skewness, and kurtosis, and require knowledge of the auxiliary variable's population characteristics.

The author of [3] considered supplementary data and suggested the best linear unbiased (BLU) estimator, a typical regression estimate of the population mean of the study variable. The traditional ratio estimate of population mean under SRS was presented by [2]. Using supplementary data, [4] proposed a product and ratio estimate of the population mean under SRS. A ratio type estimate for the population mean under SRS was proposed by [5]. An exponential ratio estimate of population mean under SRS was suggested by [6]. Using known auxiliary variable parameters, [7-10] proposed various modified ratio estimators of population mean under SRS. A population mean estimator of the generalized exponential ratio type was proposed by [11] and improved by [12] using [13] method. While [14] considered the co-efficient of variation and median of an auxiliary variable and developed an estimation procedure of population mean, [15] used the coefficient of skewness of the auxiliary variable and presented a ratio method of estimation of population mean. A modified ratio estimator of population mean was developed by [16] using a linear combination of the quartile deviation and the co-efficient of skewness. [17] proposed a modified ratio estimator of population mean using the size of the sample selected from the population. An improved ratio cum exponential ratio estimate of population mean was proposed by [18]. To create a new ratio estimator for predicting population mean under SRS, [19] used a coefficient of variation, correlation coefficient, and regression coefficient. In order to estimate population mean using SRS, [20] proposed a few improved ratio estimators. Using a variety of auxiliary data, [21] created a class of population mean estimators under SRS. Neutrosophic statistics specifically addresses unpredictability and indeterminacy. [22] introduced the idea of neutrosophic statistics, which was then expanded upon by [23,24], who saw it as a generalization of both classical and neutrosophic statistics. Additionally, [25] provided a basic overview of neutrosophic statistics. For the first time in the history of neutrosophic statistics, [26] have created the neutrosophic ratio-type estimators to estimate the mean of the inadequate population using auxiliary information to resolving the problem of estimating the population mean of neutrosophic data. Using Neutrosophic studies and an auxiliary variable, [27] provides a Neutrosophic exponential-type estimator for population mean estimation in the presence of uncertainty.

Inspired from all previous work, we suggested an improved neutrosophic exponential-ratio estimator which extend the work of [26] to estimate population mean with auxiliary information, addressing sample uncertainty. Although several estimators have been developed based on classical statistics, but they do not meet the requirements arising from uncertainty and indeterminacy of a real data. While some neutrosophic statistics have been introduced for estimating the parameters, such estimators either do not achieve high MSE and PRE efficiencies or are not sufficiently adaptive to the auxiliary information. There are few studies in the literature considering exponential-ratio estimators in neutrosophic environment. These limitations open up a research gap to propose various efficient neutrosophic estimators that can provide better estimations as compared to existing neutrosophic methods and also in situations having imperfect information or interval values.

To fill this gap, the current study suggests a new neutrosophic exponential-ratio estimator for estimating population mean under simple random sampling. To optimize performance, the proposed method combines existing estimators by taking advantage of the coefficient of variation as auxiliary variable, and a balancing parameter is introduced. The new estimator is derived in a way that can be handled analytically and theoretically compared with existing neutrosophic estimator and classical estimator. The practical utility of the new construction is showcased with a real dataset and benchmark simulation study where it is associated with reductions in mean squared error and percentage relative efficiency.

There are several essential differences between the proposed estimator and the already available classical and neutrosophic estimators. The first is that the classical estimators assume that the data is precise and do not consider the uncertainty or indeterminacy that often arises in real-life conditions. The proposed estimator is developed based on neutrosophic statistics, allowing it to be suitable for processing imprecise and interval-valued data. The second is that most of the already developed neutrosophic estimators are either performable with the use of only simple ratio or product form or do not use auxiliary variable for the estimator construction. To generalize, a novel method introduces an exponential-ratio form of the new estimator, which includes the coefficient of variation of an auxiliary variable and applies a balancing parameter that optimizes efficiency. Such design not only increases the accuracy of the estimation but also enhances flexibility in an uncertain environment, making the estimator more robust and applicable. Additionally, we compare our neutrosophic exponential ratio estimator to other Existing estimators which demonstrate its superiority under certain conditions and efficiency criteria are developed, as described in Section 3. Section 4 presents quantifiable data on the efficacy of the recommended strategies. Finally, Section 5 provides classical and neutrosophic real and simulated data sets, section 6 focused on conclusion.

2. Materials & Methodology

Consider a finite population of $'N'$ units $K = (K_1, K_2, \ldots \ldots . K_N)$ that contains a neutrosophic random sample of size $n_N \in [n_L, n_U]$. For the the *i*[th] *(i=1,2,…….N)* unit from *P*, let $y_{N(i)}$ and $x_{N(i)}$ stand for the study variable and auxiliary variable.

Let $y_{N(i)}$ is the *i*[th] representative observation of our neutrosophic data, originating from $y_{N(i)} \in [y_L, y_U]$ and in the same way for the auxiliary variable $x_{N(i)} \in [x_L, x_U]$. Let $\bar{y}_{N(i)} \in [\bar{y}_L, \bar{y}_U]$ is our key neutrosophic variable and $\bar{x}_{N(i)} \in [\bar{x}_L, \bar{x}_U]$ is associated with our study variable and serves as our auxiliary neutrosophic variable $\bar{y}_N$. Furthermore, $Y_{N(i)} \in [Y_L, Y_U]$ and $X_{N(i)} \in [X_L, X_U]$ are the statistics for the neutrosophic data. Also let $C_{yN} \in [C_{yL}, C_{yU}]$ and $C_{xN} \in [C_{xL}, C_{xU}]$ are coefficients of variation for neutrosophic Y_N, X_N and $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$ is the association between Y_N and X_N that is neutrosophic.

Assume that the neutrosophic mean error terms $\Delta_{yN} \in [\Delta_{yL}, \Delta_{yU}]$ and $\Delta_{xN} \in [\Delta_{xL}, \Delta_{xU}]$. Let the study's neutrosophic sets be calculated as

$$\bar{y}_{N(i)} = \bar{Y}_N + \Delta_{yN(i)} \text{ and } \bar{x}_{N(i)} = \bar{X}_N + \Delta_{xN(i)}$$

such that

$$E(\Delta_{yN}) = E(\Delta_{yN}) = 0;$$
$$E(\Delta_{yN}^2) = \delta_N C_{yN}^2;\ E(\Delta_{xN}^2) = \delta_N C_{xN}^2;$$
$$E(\Delta_{yN}\Delta_{xN}) = \delta_N C_{yxN};$$

Where $C_{yxN} = \rho_{yxn} C_{yN} C_{xN};\ \delta_N = \left[\frac{1}{n_N} - \frac{1}{N_N}\right];\ \delta_N \in [\delta_L, \delta_U];$

$$\Delta_{yN}^2 \in [\Delta_{yL}^2, \Delta_{yU}^2];\ \Delta_{xN}^2 \in [\Delta_{xL}^2, \Delta_{xU}^2];\ \Delta_{yxN} \in [\Delta_{yxL}, \Delta_{yxU}];$$
$$S_{yN} \in [S_{yL}, S_{yU}];\ S_{xN} \in [S_{xL}, S_{xU}];\ S_{yxN} \in [S_{yxL}, S_{yxU}];$$
$$C_{yN} \in [C_{yL}, C_{yU}];\ C_{xN} \in [C_{xL}, C_{xU}];\ C_{yxN} \in [C_{yxL}, C_{yxU}];$$
$$\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}] \text{ respectively.}$$

We may use a range of existing estimators, such as the unbiased estimator, ratio estimator, and exponential estimator for population mean estimation under indeterminacy, to estimate the performance of our suggested estimator. These estimators are represented as follows

Typically, the unbiased neutrosophic estimator is provided by

$$\hat{Y}_N = \bar{y}_n$$

With the MSE,

$$MSE\ (\hat{Y}_N) = \delta_N \bar{Y}_N^2 C_{yN}^2 \tag{1}$$

[26] suggested neutrosophic ratio estimator for determining the finite population mean in the presence of auxiliary variables is as follows:

$$\hat{Y}_{RN} = \bar{X}_N\left(\frac{\bar{y}_N}{\bar{x}_N}\right)$$

$$\hat{Y}_{RN} = \bar{X}_N\left(\frac{\bar{Y}_N + \Delta_{yN}}{\bar{X}_N + \Delta_{xN}}\right)$$

Where, $\hat{Y}_{RN} \in [\hat{Y}_{RL}, \hat{Y}_{RU}]$, $\Delta_{yN} \in [\Delta_{yL}, \Delta_{yU}]$ and $\Delta_{xN} \in [\Delta_{xL}, \Delta_{xU}]$

The bias and MSE of the $\hat{Y}_{RN}$ up to the first order of approximation are given below,

$$Bias(\hat{Y}_{RN}) = \delta_N \bar{Y}_N[C_{xN}^2 - C_{yxN}]$$

$$MSE(\hat{Y}_{RN}) = \delta_N \bar{Y}_N^2[C_{yN}^2 + C_{xN}^2 - 2C_{yxN}] \quad (2)$$

Where, $\delta_N \in [\delta_{NL}, \delta_{NU}]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$

The mean for a finite population in the presence of auxiliary variables is estimated by [26] using a neutrosophic exponential-type estimator inspired from [6]:

$$\hat{Y}_{expN} = \bar{y}_N exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right)$$

$$\hat{Y}_{expN} = (\bar{Y}_N + \Delta_{yN}) exp\left(-\frac{\Delta_{xN}}{2\bar{X}_N}\left(1 + \frac{\Delta_{xN}}{2\bar{X}_N}\right)^{-1}\right)$$

Where $\hat{Y}_{expN} \in [\hat{Y}_{expL}, \hat{Y}_{expU}]$, $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$, $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$, $\Delta_{yN} \in [\Delta_{yL}, \Delta_{yU}]$, $\Delta_{xN} \in [\Delta_{xL}, \Delta_{xU}]$.

The bias and MSE of $\hat{Y}_{expN}$ up to first-degree of approximation are

$$Bias(\hat{Y}_{expN}) = \delta_N \bar{Y}_N\left[\frac{3}{8}C_{xN}^2 - \frac{1}{2}C_{yxN}\right]$$

$$MSE(\hat{Y}_{expN}) = \delta_N \bar{Y}_N^2\left[C_{yN}^2 + \frac{1}{4}C_{xN}^2 - C_{yxN}\right] \quad (3)$$

Where $\delta_N \in [\delta_L, \delta_U]$, $n_N \in [n_L, n_U]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$.

Inspired by reference [9], [26] have created an altered version of the neutrosophic ratio estimator, utilizing the coefficient of variation as an auxiliary variable.

$$\hat{Y}_{SDrN} = \bar{y}_N\left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}}\right)$$

$$\hat{Y}_{SDrN} = (\bar{Y}_N + \Delta_{yN})\left(1 + \frac{\Delta_{xN}}{\bar{X}_N + C_{xN}}\right)^{-1}\left(\frac{\bar{X}_N + C_{xN}}{\bar{X}_N + C_{xN} + \Delta_{xN}}\right)$$

Where $\hat{Y}_{SDrN} \in [\hat{Y}_{SDrL}, \hat{Y}_{SDrU}]$, $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$, $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$, $\Delta_{yN} \in [\Delta_{yL}, \Delta_{yU}]$, $\Delta_{xN} \in [\Delta_{xL}, \Delta_{xU}]$, $C_{xN} \in [C_{xL}, C_{xU}]$

For the Bias and MSE of $\hat{Y}_{SDrN}$ up to the first order of approximation we get,

$$Bias(\hat{Y}_{SDrN}) = \delta_N \bar{Y}_N\left[\left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)C_{yxN}\right]$$

$$MSE(\hat{Y}_{SDrN}) = \delta_N \bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)C_{yxN}\right] \quad (4)$$

Where $\delta_N \in [\delta_L, \delta_U]$, $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $n_N \in [n_L, n_U]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$.

Inspired by [10], employing the neutrosophic ratio-type estimator provided as well as the coefficient of variation and kurtosis

$$\hat{Y}_{US1rN} = \bar{y}_N\left(\frac{\bar{X}_N\beta_{2(x)N} + C_{xN}}{\bar{x}_N\beta_{2(x)N} + C_{xN}}\right)$$

$$\hat{Y}_{US1rN} = (\bar{Y}_N + \Delta_{yN})\left(1 + \frac{\beta_{2(x)N}\Delta_{xN}}{\bar{X}_N\beta_{2(x)N} + C_{xN}}\right)^{-1}$$

$$\hat{Y}_{US2rN} = \bar{y}_N\left(\frac{\bar{X}_N C_{xN} + \beta_{2(x)N}}{\bar{x}_N C_{xN} + \beta_{2(x)N}}\right)$$

$$\hat{Y}_{US2rN} = (\bar{Y}_N + \Delta_{yN})\left(1 + \frac{C_{xN}\Delta_{xN}}{\bar{x}_N C_{xN} + \beta_{2(x)N}}\right)^{-1}$$

Where $\hat{Y}_{US1rN} \in [\hat{Y}_{US1rN}, \hat{Y}_{US1rN}]$, $\hat{Y}_{US2rN} \in [\hat{Y}_{US2rN}, \hat{Y}_{US2rN}]$, $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$, $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$, $\Delta_{yN} \in [\Delta_{yL}, \Delta_{yU}]$, $\Delta_{xN} \in [\Delta_{xL}, \Delta_{xU}]$, $C_{xN} \in [C_{xL}, C_{xU}]$, $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$

For the Bias and MSE of $\hat{Y}_{SKrN}$ up to the first order of approximation we get,

$$Bias(\hat{Y}_{US1rN}) = \delta_N \bar{Y}_N \left[\left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}}\right)C_{yxN}\right]$$

$$MSE(\hat{Y}_{US1rN}) = \delta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}}\right)C_{yxN}\right] \quad (5)$$

$$Bias(\hat{Y}_{US2rN}) = \delta_N \bar{Y}_N \left[\left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}}\right)C_{yxN}\right]$$

$$MSE(\hat{Y}_{US2rN}) = \delta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}}\right)C_{yxN}\right] \quad (6)$$

Where $\delta_N \in [\delta_L,\delta_U], \bar{Y}_N \in [\bar{Y}_L,\bar{Y}_U]$, $\bar{X}_N \in [\bar{X}_L,\bar{X}_U]$, $n_N \in [n_L,n_U]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$, $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$.

[26] suggested an additional neutrosophic estimator in which the coefficient of kurtosis is considered as an auxiliary variable.

$$\hat{Y}_{SKrN} = \bar{y}_N\left(\frac{\bar{X}_N+\beta_{2(x)N}}{\bar{x}_N+\beta_{2(x)N}}\right)$$

$$\hat{Y}_{SKrN} = (\bar{Y}_N+\Delta_{yN})\left(1+\frac{\Delta_{xN}}{\bar{X}_N+\beta_{2(x)N}}\right)^{-1}$$

Where $\hat{Y}_{SKrN} \in [\hat{Y}_{SKrL}, \hat{Y}_{SKrU}], \bar{y}_N \in [\bar{y}_L,\bar{y}_U]$, $\bar{Y}_N \in [\bar{Y}_L,\bar{Y}_U]$, $\bar{x}_N \in [\bar{x}_L,\bar{x}_U]$, $\bar{X}_N \in [\bar{X}_L,\bar{X}_U]$, $\Delta_{yN} \in [\Delta_{yL},\Delta_{yU}]$, $\Delta_{xN} \in [\Delta_{xL},\Delta_{xU}]$, $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$

For the Bias and MSE of $\hat{Y}_{SKrN}$ up to the first order of approximation we get,

$$Bias(\hat{Y}_{SKrN}) = \delta_N \bar{Y}_N \left[\left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}}\right)^2 C_{xN}^2 - \left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}}\right)C_{yxN}\right]$$

$$MSE(\hat{Y}_{SKrN}) = \delta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}}\right)C_{yxN}\right] \quad (7)$$

Where $\delta_N \in [\delta_L,\delta_U]$, $n_N \in [n_L,n_U]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$.

[26] proposed a generalized exponential-type neutrosophic estimator to calculate a finite population mean:

$$\hat{Y}_{KNN} = \bar{y}_N exp\left[\alpha\left(\frac{\bar{X}_N^{\frac{1}{h}} - \bar{x}_N^{\frac{1}{h}}}{\bar{X}_N^{\frac{1}{h}} + (a-1)\bar{x}_N^{\frac{1}{h}}}\right)\right]$$

$$\hat{Y}_{KNN} = (\bar{Y}_N+\Delta_{yN}) exp\left[-\frac{\alpha\Delta_{xN}}{ah\bar{X}_N}\left(1+\frac{\Delta_{xN}}{h\bar{X}_N}-\frac{\Delta_{xN}}{ah\bar{X}_N}\right)^{-1}\right]$$

Where $\hat{Y}_{KNN} \in [\hat{Y}_{KNL}, \hat{Y}_{KNU}], \bar{y}_N \in [\bar{y}_L,\bar{y}_U]$, $\bar{Y}_N \in [\bar{Y}_L,\bar{Y}_U]$, $\bar{x}_N \in [\bar{x}_L,\bar{x}_U]$, $\bar{X}_N \in [\bar{X}_L,\bar{X}_U]$, $\Delta_{yN} \in [\Delta_{yL},\Delta_{yU}]$, $\Delta_{xN} \in [\Delta_{xL},\Delta_{xU}]$, $(-\infty < \alpha < \infty)$ and $h(h>0)$ are real constant, $a(a \neq 0)$ is another constant.

For the Bias and MSE of $\hat{Y}_{SKrN}$ up to the first order of approximation we get,

$$Bias(\hat{Y}_{KNN}) = \delta_N \bar{Y}_N \left[\frac{\alpha C_{xN}^2}{ah^2} - \frac{\alpha C_{xN}^2}{a^2h^2} + \frac{\alpha^2 C_{xN}^2}{2a^2h^2} - \frac{\alpha C_{yxN}}{ah}\right]$$

$$MSE(\hat{Y}_{KNN}) = \delta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{\alpha^2 C_{xN}^2}{a^2h^2} - \frac{2\alpha C_{yxN}}{ah}\right] \quad (8)$$

Where $\delta_N \in [\delta_L,\delta_U]$, $n_N \in [n_L,n_U]$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $C_{yxN} \in [C_{yxL}, C_{yxU}]$.

3. Proposed Estimator

To address the issue of data indeterminacy with neutrosophic data, several preexisting estimators were converted into neutrosophic estimators. Using the coefficient of variation as an auxiliary variable, we have created a modified version of the neutrosophic exponential ratio estimator, inspired by [4,26].

$$\hat{Y}_{prop_N(SRS)} = \alpha_N\left(\frac{\bar{y}_N}{\bar{x}_N}\right)\bar{X}_N + (1-\alpha_N)\bar{Y}_N exp\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{x}_N}\right) \quad (9)$$

$\bar{x}_N$ & $\bar{y}_N$ are the sample means and $\bar{X}_N$ & $\bar{Y}_N$ are the population means of the auxiliary and study variables, respectively.

α_N is a balancing parameter to balance the contribution of the two parts of the estimator: the ratio term and the exponential term.

Now expand and simplifying the equation (9), we get

$$\left(\hat{Y}_{prop_N(SRS)} - \bar{Y}_N\right) = \bar{Y}_N\left(-\Delta_{x_N} + \frac{3}{2}\Delta^2_{x_N} + \alpha_N\left(\Delta_{y_N} - \Delta_{y_N}\Delta_{x_N} - \frac{1}{2}\Delta_{x_N}\right)\right) \quad (10)$$

Now take the expectation on equation (10) we get

$$Bias\left(\hat{Y}_{prop_N(SRS)}\right) = \bar{Y}_N\delta_N\left(\frac{3}{2}C^2_{x_N} - \alpha_N C_{yxN}\right)$$

Now for the MSE take the square on both the sides of equation (10), we get

$$\left(\hat{Y}_{prop_N(SRS)} - \bar{Y}_N\right)^2 = \bar{Y}^2_N\left(\Delta^2_{x_N} + \alpha^2_N\Delta^2_{y_N} - 2\alpha_N\Delta_{y_N}\Delta_{x_N}\right) \quad (11)$$

Now take the expectations on both the sides of equation (11), we get

$$MSE\left(\hat{Y}_{prop_N(SRS)}\right) = \delta_N\bar{Y}^2_N\left(C^2_{x_N} + \alpha^2_N C^2_{y_N} - 2\alpha_N C_{yxN}\right)$$

Now to optimize the value of α_N take the derivation with respect to α_N of $MSE\left(\hat{Y}_{prop_N(SRS)}\right)$ we get

$$\frac{d}{d\alpha_N}MSE\left(\hat{Y}_{prop_N(SRS)}\right) = 0$$

$$\frac{d}{d\alpha_N}\left\{\bar{Y}^2_N\theta_N\left(C^2_{x_N} + \alpha^2_N C^2_{y_N} - 2\alpha_N\rho_{yx_N}C_{y_N}C_{x_N}\right)\right\} = 0$$

$$\bar{Y}^2_h\theta_N\left[0 + 2\,\alpha_N C^2_{y_N} - 2\rho_{yx_N}C_{y_N}C_{x_N}\right] = 0$$

$$2\,\alpha_N C^2_{y_N} = 2\rho_{yx_N}C_{y_N}C_{x_N}$$

$$\alpha_N = \frac{\rho_{yx_N}C_{x_N}}{C_{y_N}} = \varphi_N$$

$$MSE\left(\hat{Y}_{prop_N(SRS)}\right) = \delta_N\bar{Y}^2_N\left(C^2_{x_N} + \varphi^2_N C^2_{y_N} - 2\varphi_N C_{yxN}\right) \quad (12)$$

4. Proposed Estimator Theoretical Comparison in Stratified Random Sampling

The following observations will be made when we compare the suggested estimator $\hat{Y}_{prop(SRS)}$ with the existing estimators covered in Sect. (2).

Observation 1: from equation (1) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_0\right)$, if and only if $MSE\left(\hat{Y}_0\right) - MSE\left(\hat{Y}_{prop(SRS)}\right) > 0$,

Or if $\delta_N\bar{Y}^2_N\left(C^2_{yN}(1-\varphi^2_N) - C^2_{x_N} + 2\varphi_N C_{yxN}\right) > 0$.

Observation 2: from equation (2) and (12)

$MSE\left(\hat{Y}_{prop(SRS)}\right) < MSE\left(\hat{Y}_{RN}\right)$, if and only if $MSE\left(\hat{Y}_{RN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$,

Or if$\delta_N\bar{Y}^2_N\left[C^2_{yN}(1-\varphi^2_N) - 2C_{yxN}(1+\varphi_N)\right] > 0$.

Observation 3: from equation (3) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_{expN}\right)$, if and only if $MSE\left(\hat{Y}_{expN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$,

Or if $\delta_N\bar{Y}^2_N\left[C^2_{yN}(1-\varphi^2_N) - \frac{3}{4}C^2_{xN} - C_{yxN}(1+2\varphi_N)\right] > 0$.

Observation 4: from equation (4) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_{SDrN}\right)$, if and only if $MSE\left(\hat{Y}_{SDrN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$,

Or if $\delta_N\bar{Y}^2_N\left[C^2_{yN}(1-\varphi^2_N) + \left(\left(\frac{\bar{X}_N}{\bar{X}_N+C_{xN}}\right)^2 - 1\right)C^2_{xN} - \left(\frac{\bar{X}_N}{\bar{X}_N+C_{xN}} + 2\varphi_N\right)C_{yxN}\right] > 0$.

Observation 5: from equation (5) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_{US1rN}\right)$, if and only if $MSE\left(\hat{Y}_{US1rN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$,

Or if$\delta_N\bar{Y}^2_N\left[C^2_{yN}(1-\varphi^2_N) + \left(\left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}}\right)^2 - 1\right)C^2_{xN} - 2C_{yxN}\left(\frac{\bar{X}_N\beta_{2(x)N}}{\bar{X}_N\beta_{2(x)N}+C_{xN}} + \varphi_N\right)\right] > 0$.

Observation 6: from equation (6) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_{US2rN}\right)$, if and only if $MSE\left(\hat{Y}_{US2rN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$,

Or if$\delta_N\bar{Y}^2_N\left[C^2_{yN}(1-\varphi^2_N) + \left(\left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}}\right)^2 - 1\right)C^2_{xN} - 2C_{yxN}\left(\frac{\bar{X}_N C_{xN}}{\bar{X}_N C_{xN}+\beta_{2(x)N}} + \varphi_N\right)\right] > 0$.

Observation 7: from equation (7) and (12)

$MSE\left(\hat{Y}_{prop_N(SRS)}\right) < MSE\left(\hat{Y}_{SKrN}\right)$, if and only if $MSE\left(\hat{Y}_{SKrN}\right) - MSE\left(\hat{Y}_{prop_N(SRS)}\right) > 0$, Or if $\delta_N \bar{Y}_N^2 \left[C_{yN}^2(1-\varphi_N^2) + \left(\left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}} \right)^2 - 1 \right) C_{xN}^2 - 2C_{yxN} \left(\frac{\bar{X}_N}{\bar{X}_N+\beta_{2(x)N}} + \varphi_N \right) \right] > 0.$

Observation 8: from equation (8) and (12)

$MSE(\hat{Y}_{prop_N(SRS)}) < MSE\ (\hat{Y}_{KNN})$, if and only if $MSE\ (\hat{Y}_{KNN}) - MSE(\hat{Y}_{prop_N(SRS)}) > 0$,

Or if $\delta_N \bar{Y}_N^2 \left[C_{yN}^2(1-\varphi_N^2) + C_{xN}^2 (\frac{\alpha^2}{a^2h^2} - 1) - 2C_{yxN} \left(\frac{\alpha}{ah} + \varphi_N \right) \right] > 0.$

5. Numerical Study

5.1 Numerical Comparison for the real and simulated data set:

Table 1. Neutrosophic and Classical real data for the population parameter.

Neutrosophic Parameter	Neutrosophic Interval	Classical Parameter	Classical Value
N_N	(750, 850) countries/districts	N	800
n_N	(75, 80) regions	n	75
$\bar{X}_N$	(140000, 160000) tests	$\bar{X}$	150,000
$\bar{Y}_N$	(25000, 35000) cases	$\bar{Y}$	30,000
S_{xN}^2	(30000, 40000)	S_x^2	35,000
S_{yN}^2	(10000, 15000)	S_y^2	12,500
ρ_{yxN}	(0.5, 0.6)	ρ_{yx}	0.55
C_{xN}	(0.001237, 0.00125)	C_x	0.0012472
C_{yN}	(0.0004, 0.003499271)	C_y	0.0037
$\beta_{2(x)N}$	(3.0, 3.4)(slightly leptokurtic)	$\beta_{2(x)}$	3.2

5.2 Empirical study

To assess the effectiveness and efficiency of the suggested neutrosophic ratio-type estimators, we empirically investigated the situation using actual COVID-19 data. The number of confirmed cases in different locations as well as the overall number of COVID-19 tests performed are included in this dataset. The study variable (Y) indicates the number of verified COVID-19 cases, whereas the auxiliary variable (X) shows the number of tests performed.

5.3 Description of the Data

The COVID-19 dataset was gathered from publicly accessible and validated sources, such as Johns Hopkins University (JHU) and the World Health Organization (WHO). The information was taken out for the time frame of March 2020–December 2021. India is the region chosen for this study, which includes data from several states and union territories.

Data Sources

- **World Health Organization (WHO):** https://covid19.who.int
- **Johns Hopkins University (JHU) COVID-19 Dashboard:** https://coronavirus.jhu.edu/map.html

Table 2. MSE of Neutrosophic and Classical estimators for the real population.

Estimator	MSE Neutrosophic	Estimator	MSE Classical
$\bar{y}_N$	(129.52381, 169.852941)	$\bar{y}$	151.0416667
$\bar{y}_{RN}$	(101.853441, 118.717603)	$\bar{y}_R$	112.3553024
$\bar{y}_{expN}$	(112.590958, 138.866786)	$\bar{y}_{exp}$	127.4693179
$\bar{y}_{SDrN}$	(101.853441, 118.717603)	$\bar{y}_{SDr}$	112.3553026
$\bar{y}_{US1rN}$	(101.853441, 118.717603)	$\bar{y}_{US1r}$	112.3553025

$\bar{y}_{US2rN}$	(102.117179, 119.2161305)	$\bar{y}_{US2r}$	112.7261892
$\bar{y}_{SKrN}$	(101.853768, 118.718229)	$\bar{y}_{SKr}$	112.3557668
$\bar{y}_{KNN}$	(97.1428571, 108.705882)	$\bar{y}_{KN}$	105.3515625
$\bar{y}_{PropN}$	(9.29300292, 13.8713235)	$\bar{y}_{Prop}$	11.799375

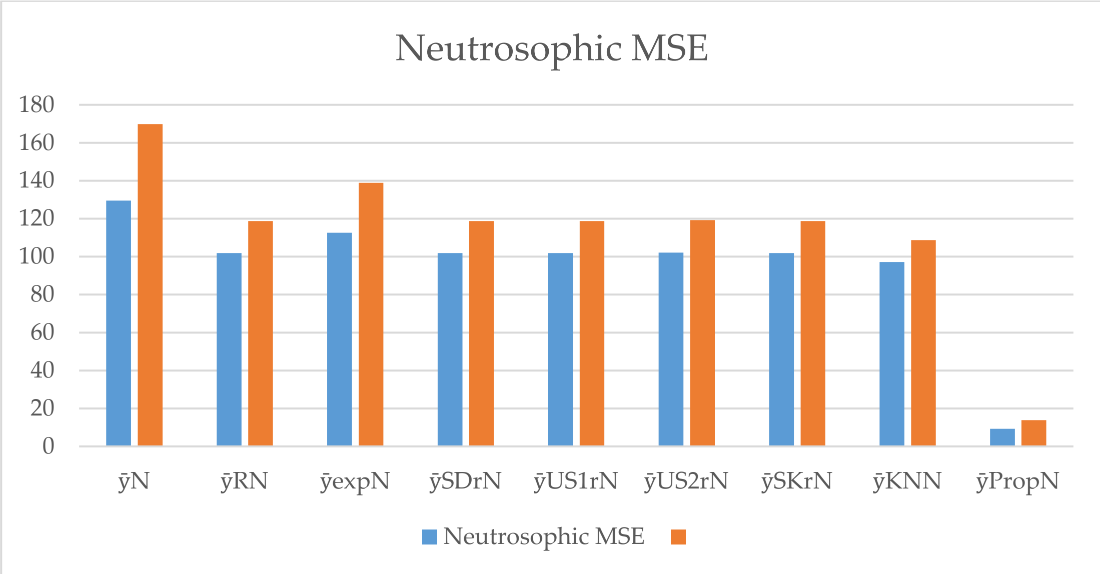

Figure 1. Neutrosophic MSE of estimators for the real population.

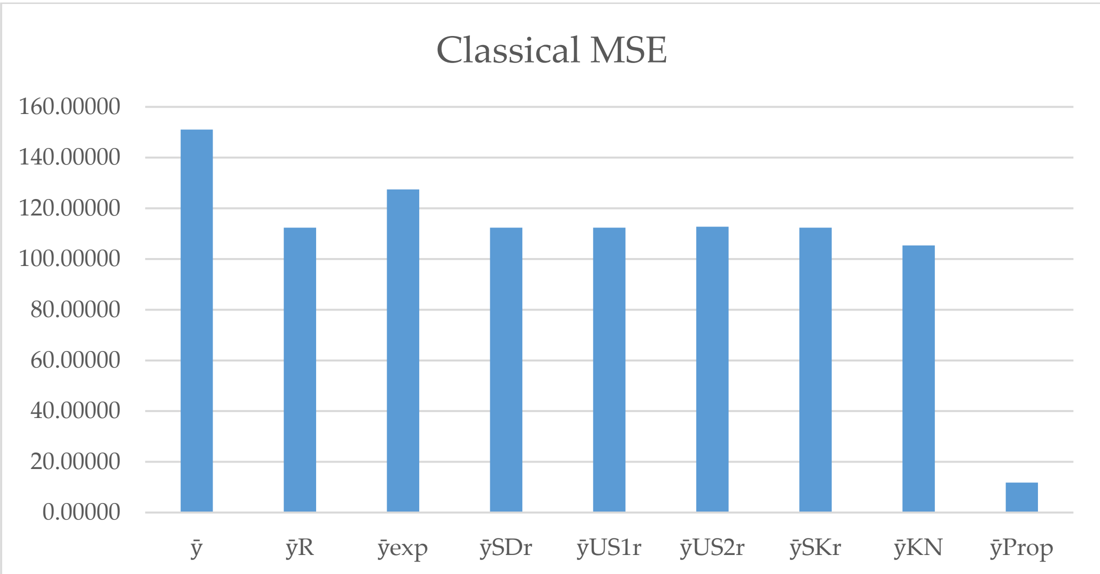

Figure 2. Classical MSE of estimators for the real population.

Table 3. PRE of Neutrosophic and Classical estimators for the real population.

Estimator	PRE Neutrosophic	Estimator	PRE Classical
$\bar{y}_N$	(100, 100)	$\bar{y}$	100
$\bar{y}_{RN}$	(127.17, 143.07)	$\bar{y}_R$	134.432166
$\bar{y}_{expN}$	(115.04, 122.31)	$\bar{y}_{exp}$	118.4925668
$\bar{y}_{SDrN}$	(127.17, 143.07)	$\bar{y}_{SDr}$	134.4321658
$\bar{y}_{US1rN}$	(127.17, 143.07)	$\bar{y}_{US1r}$	134.432166
$\bar{y}_{US2rN}$	(126.84, 142.47)	$\bar{y}_{US2r}$	133.9898632

$\bar{y}_{SKrN}$	(127.17, 143.07)	$\bar{y}_{SKr}$	134.4316104
$\bar{y}_{KNN}$	(133.33, 156.25)	$\bar{y}_{KN}$	143.3691756
$\bar{y}_{PropN}$	(1224.49, 1393.78)	$\bar{y}_{Prop}$	1280.081925

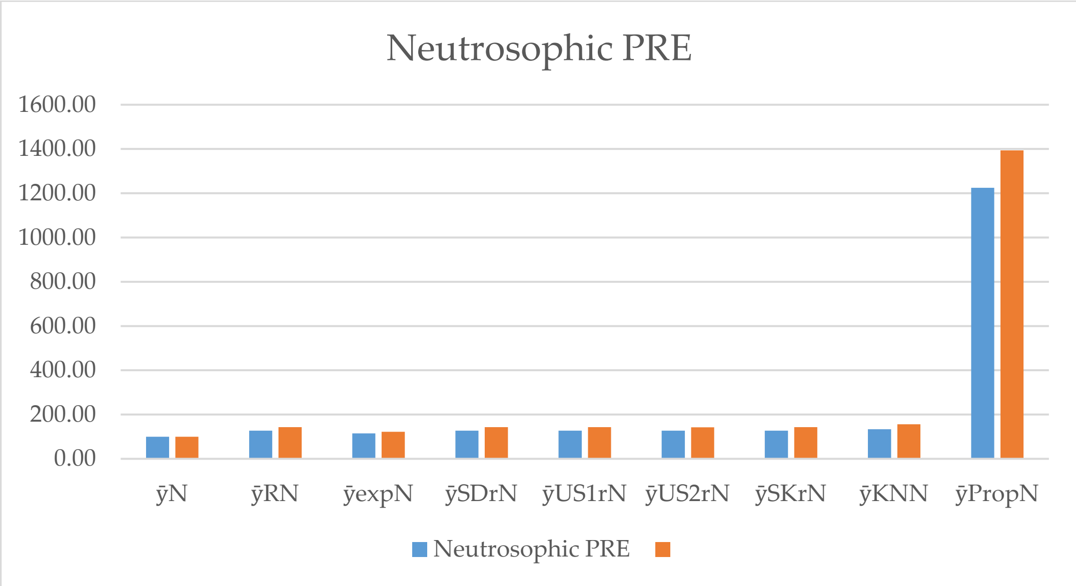

Figure 3. Neutrosophic PRE of estimators for the real population.

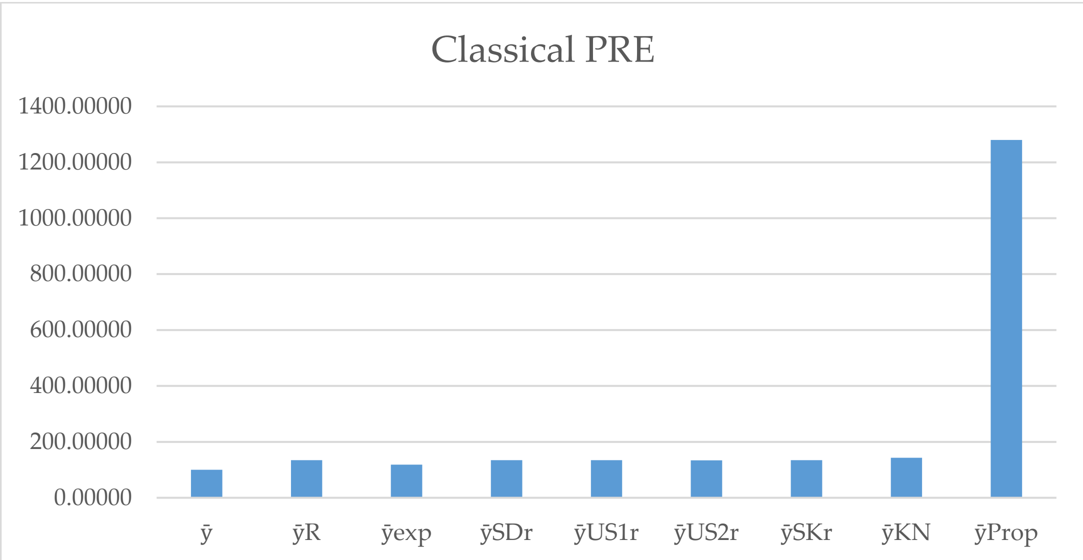

Figure 4. Classical PRE of estimators for the real population.

Table 4. Neutrosophic and Classical simulated data for the population parameter.

Neutrosophic Parameter	Neutrosophic Interval	Classical Parameter	Classical Value
N_N	(1200, 1200)	N	1200
n_N	(30, 30)	n	30
$\bar{X}_N$	(160.5, 175.3)	$\bar{X}$	169.222
$\bar{Y}_N$	(70.5, 80.7)	$\bar{Y}$	76.2299
S_{xN}^2	(37.21, 51.84)	S_x^2	41.119774
S_{yN}^2	(182.25, 327.61)	S_y^2	183.23519

ρ_{yxN}	(0.015, 0.025)	ρ_{yx}	0.0245
C_{xN}	(0.38006, 0.041072)	C_x	0..379
C_{yN}	(0.1914894, 0.2242875)	C_y	0.1776
$\beta_{2(x)N}$	(0.07, 0.21)	$\beta_{2(x)}$	0.1306

Table 5. MSE of Neutrosophic and Classical estimators for the simulated population.

Estimator	MSE Neutrosophic	Estimator	MSE Classical
$\bar{y}_N$	(5.92, 10.65)	$\bar{y}$	5.96
$\bar{y}_{RN}$	(6.12, 10.91)	$\bar{y}_R$	6.16
$\bar{y}_{expN}$	(5.964, 10.69)	$\bar{y}_{exp}$	5.99
$\bar{y}_{SDrN}$	(6.12, 10.91)	$\bar{y}_{SDr}$	6.164
$\bar{y}_{US1rN}$	(6.12, 10.91)	$\bar{y}_{US1r}$	6.16
$\bar{y}_{US2rN}$	(6.12, 10.9)	$\bar{y}_{US2r}$	6.16
$\bar{y}_{SKrN}$	(6.121, 10.906)	$\bar{y}_{SKr}$	6.164
$\bar{y}_{KNN}$	(5.922, 10.641)	$\bar{y}_{KN}$	5.95
$\bar{y}_{PropN}$	(0.23, 0.36)	$\bar{y}_{Prop}$	0.27

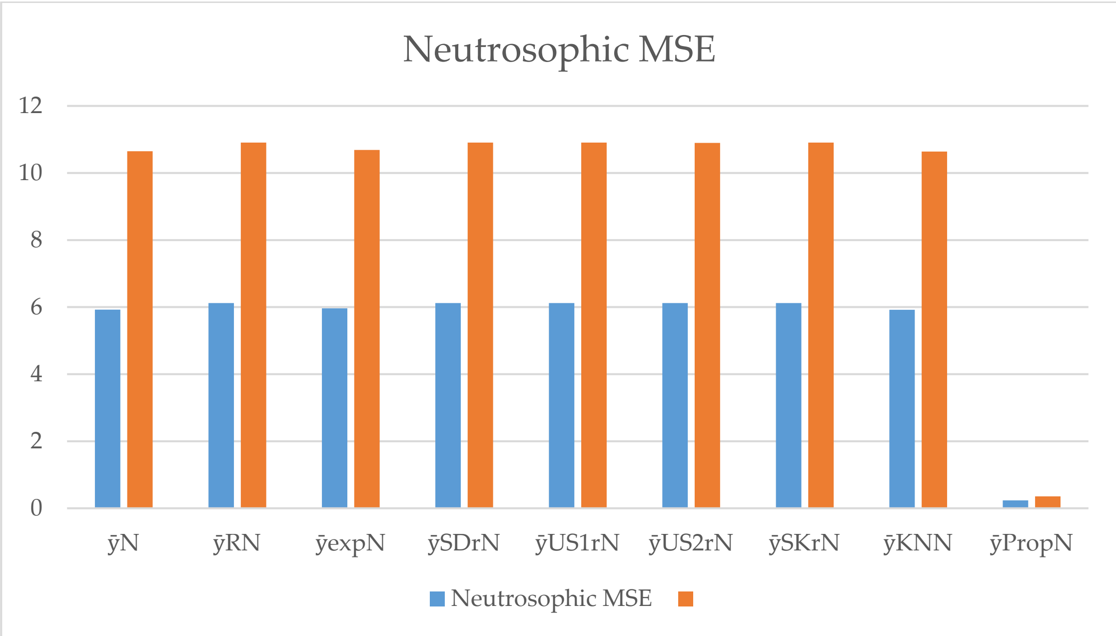

Figure 5. Neutrosophic MSE of estimators for the simulated population.

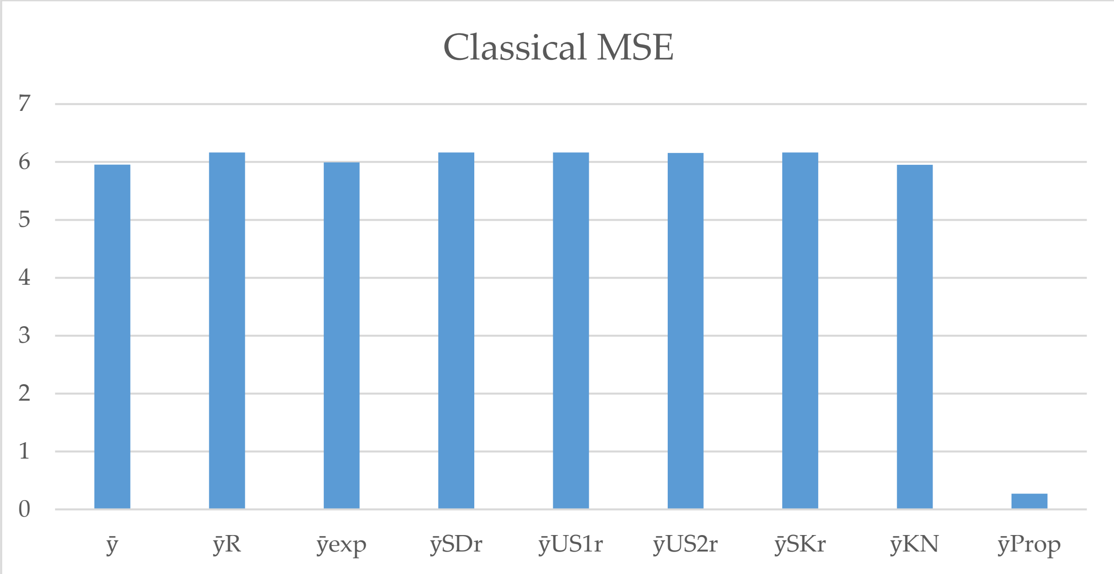

Figure 6. Classical MSE of estimators for the simulated population.

Table 6. PRE of Neutrosophic and Classical estimators for the simulated population.

Estimator	PRE Neutrosophic	Estimator	PRE Classical
$\bar{y}_N$	(100, 100)	$\bar{y}$	100
$\bar{y}_{RN}$	(96.76, 97.62)	$\bar{y}_R$	96.61
$\bar{y}_{expN}$	(99.32, 99.62)	$\bar{y}_{exp}$	99.39
$\bar{y}_{SDrN}$	(96.77, 97.62)	$\bar{y}_{SDr}$	96.61
$\bar{y}_{US1rN}$	(96.79, 99.63)	$\bar{y}_{US1r}$	96.63
$\bar{y}_{US2rN}$	(96.8, 97.69)	$\bar{y}_{US2r}$	96.76
$\bar{y}_{SKrN}$	(96.77, 97.63)	$\bar{y}_{SKr}$	96.62
$\bar{y}_{KNN}$	(100.02, 100.06)	$\bar{y}_{KN}$	100.06
$\bar{y}_{PropN}$	(2539.09, 2983.87)	$\bar{y}_{Prop}$	2197.26

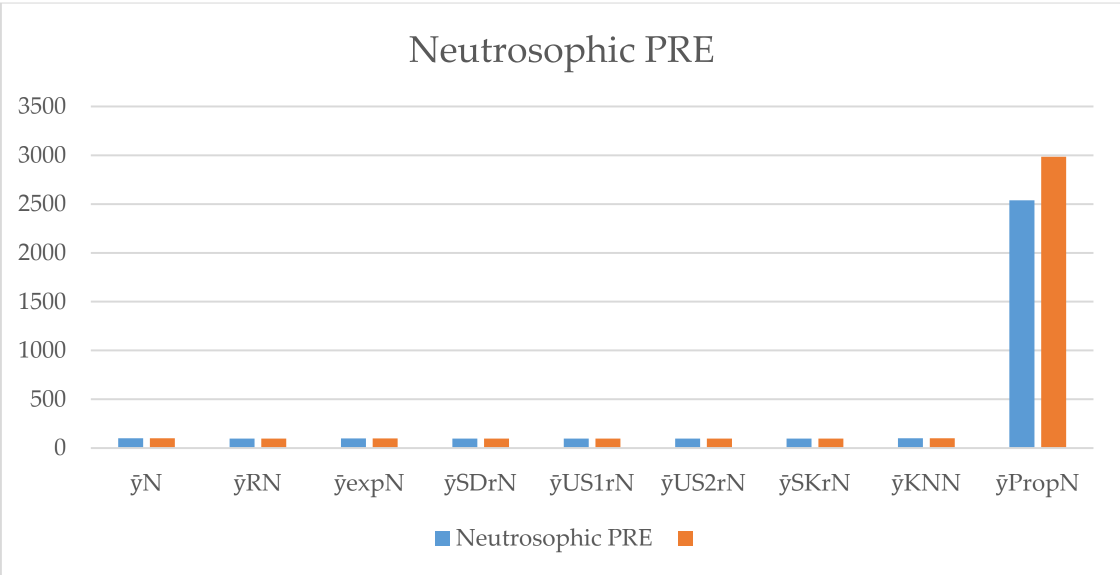

Figure 7. Neutrosophic PRE of estimators for the simulated population.

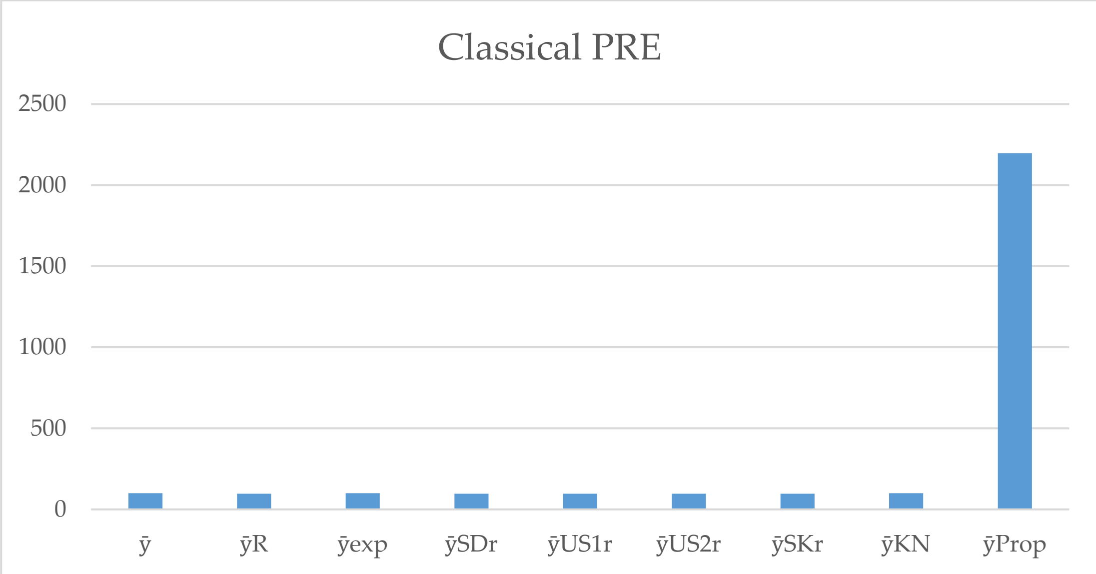

Figure 8. Classical PRE of estimators for the simulated population.

6. Result Analysis and Conclusion

Real Data Results: The findings from the COVID-19 real dataset, as summarized in Table 2, along with the Figure 1 & 2 shows that the proposed neutrosophic exponential-ratio estimator has the least MSE value when compared to the classical and alternative neutrosophic estimators. Specifically, the suggested estimator is having superior MSE intervals, giving more precision. as per shown in table 2, the range of the MSE of the suggested estimator is (9.293, 13.871), while the traditional estimator has an MSE of 11.799. Similarly, The PRE values shown in Table 3 along with the Figure 3 & 4 that is also demonstrate considerable efficiency gains. The suggested estimator gets the PRE values ranging from 1224.49% to 1393.78% when compared to the traditional estimator, demonstrating its strength and effectiveness in handling neutrosophic data.

Simulated Data Results: The simulated dataset also given in Tables 4. The suggested estimator consistently performs better than the traditional classical and alternative neutrosophic estimators in terms of MSE and PRE. As per shown in Table 5 along with Figure 5 & 6, the MSE range for the suggested estimator is (0.23, 0.36), while the standard version has an MSE of 0.27. By observing the Table 6 and Figure 7 & 8 the suggested estimator has the range of PRE values as high as 2539.09% to 2983.87%, indicating remarkable performance in contexts with varied levels of uncertainty.

7. Conclusion

This study introduces an improved neutrosophic exponential-ratio estimator for estimating population mean using simple random sampling. The data uncertainties were effectively resolved and the proposed method was found to outperform with the existing estimators in terms of MSE and PRE. Regarding types of data, neutrosophic statistics show more scope and flexibility than traditional statistics enabling them to work with complex data better. This approach must be considered in further research in connection with other designs of sampling such as stratified or cluster sampling and in other domains where uncertainty is an issue with the data.

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