On Data Envelopment Analysis Framework for Optimizing Two-Stage Systems Under Neutrosophic Environment

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Abstract: Recently, DEA became one of the evaluation method and used widely by the researchers for the analysis. It's used to evaluate units under evaluation that produce multiple outputs by consuming multiple inputs This paper presents a neutrosophic set-based, two-stage DEA framework to surmount the weak linkages within the traditional DEA model when uncertainties and imprecise values exist. Neutrosophic sets capture the uncertainty by considering membership degrees of truth, indeterminacy, and falsity. Overall, this framework decomposes overall system efficiency into stage efficiencies and provides deep insights into the system performance. We developed a two-stage DEA optimization model with neutrosophic variables and constraints. This framework can apply to any organization in order to support decision-makers to make rational decisions and allocate resources optimally, especially in the modern competitive business environment. A case study introduced to show how the model could handle neutrosophic uncertainty, and the mathematical formulation presents succinctly for practical implementation.

Keywords: Data Envelopment Analysis, Mathematical model; Neutrosophic Sets, Two-Stage Model, Efficiency Assessment, Competitive Business Environment, Uncertainty

1. Introduction

In today's complex and interlinked world, several real-life systems work just like two-stage processes, with the output of one stage normally forming the input to the other. Examples include supply chain networks, healthcare delivery systems, and manufacturing processes. It is thus clear that for optimum resource allocation and performance improvement with a view to achieving strategic objectives, efficiency analysis of such systems becomes quite indispensable. Traditional DEA has been one of the popular methods for evaluating the efficiency of DMUs involving multiple inputs and outputs. However, conventional DEA models cannot handle uncertain, incomplete, and ambiguous data, which is a common characteristic with real-world systems. This limitation has spurred the development of advanced DEA approaches, such as fuzzy DEA and neutrosophic DEA, with a view to embracing uncertainty and indeterminacy in efficiency evaluation [1]. Neutrosophic logic is the extension of fuzzy logic, in which the membership functions of truth, indeterminacy, and falsity define a very powerful tool in mathematics when dealing with uncertainty. It is therefore going to be more applicable for modeling those complex systems containing imprecise, inconsistent partial unknown data. In such context, including neutrosophic logic within the DEA will yield more realistic estimates regarding the efficiency of systems when the conditions of uncertainty arise. Some recent studies demonstrate the potential of neutrosophic DEA in various applications; however, applying it to a two-stage system is still scant. Two-stage systems are pretty special, considering that the interdependence between stages requires specialized models that can accurately capture the flow of intermediate inputs and outputs [2].

Decision science, operations management, and economics are the fields of research that need to explain the complexity of the systems. In the beginning of the 70s, the DEA as a method of efficiency evaluation was developed by Charnes, Cooper, and Rhodes [3], and by now has already been used as a powerful tool in the multidimensional efficiency measurement of the decision-making unit. But the main reason why the traditional DEA models, for the sake of DEA that can handle the uncertainty and ambiguity of real-life data, limit their application, is the assumption made by these models that the datasets used in the process are crisp. In this regard, over the last decades, fuzzy DEA was suggested to the scientific community in order to deal with data imprecision [4]. The fuzzy logic-based DEA has been experimented with in various fields [5, 6, 7, 8, 9, 10, 11, 12, 13]. Therefore, these studies emphasize uncertainty resolution and information representation effects on technology development and efficiency. Along with neutrosophic logic, another approach, called neutrosophic logic, is provided by Smarandache [14], who tries to capture the concept of indeterminacy in yet another dimension through this creative representation of complex systems that may contain uncertain or incomplete data. Meanwhile, newly originated studies were carried out in order to add neutrosophic logic in with DEA as a means to add analysis tools under conditions of (%) uncertainty. Additionally, Additionally, various methods are available to tackle diverse challenges in a neutrosophic environment [15, 16, 17, 18, 19, 20, 21].

Thus, in a study by Abdel-Basset et al. [22], a neutrosophic DEA model was proposed to be used in the efficiency evaluation of a renewable energy system, which turns out to be more accurate while having the capacity of dealing with the indeterminacy that cannot be solved by traditional DEA. Similarly, Khan et al. [23] utilized neutrosophic DEA for the performance evaluation of financial institutions to direct resource allocation to these entities, a better practice in a condition of ambiguity. The results in these studies reveal that utilization of neutrosophic DEA technology can eliminate some problems with conventional and fuzzy DEA models, especially in a partial or inconsistent data defense. In a very beneficial way for both research and practice, El-Demerdash et al. [24] proposed neutrosophic input-oriented DEA model. As an extension to the work of El-Demerdash et al., Farnam et al. [25] proposed a neutrosophic DEA model in which input and output variables are considered neutrosophic intrinsic to handle uncertainties in the input and output data. An important step forward was given by Almutairi et al. [26], who proposed a general model which is able to: decide which variables have to be modeled as deterministic or neutrosophic, consider both input and output orientation, handle different types of returns to scale, such as constant and variable. Two-stage systems are of high interest where one stage's output constitutes the input of another stage. These systems have been widely used in various settings such as supply chain management, manufacturing, and service operations; therefore, they have attracted significant attention. The traditional DEA method has limited applications in the evaluation of these systems due to the fact that the interdependencies amidst the stages are ignored and their inputs/outputs are not individually modeled. In contrast, a two-stage DEA has been proposed by some researchers with a view to providing a comprehensive solution for that problem. Tabatabaee et al., [27] proposed a new model for ranking efficient units by introducing an ideal unit and calculating the distance between each efficient unit and the ideal unit.

So, this paper proposes an optimization framework for the two-stage system using Neutrosophic Data Envelopment Analysis (NDEA). The present approach surmounts some of the limitations of classical DEA by embracing neutrosophic sets to model uncertain input-output data and also developing a two-stage model which takes care of interdependency between stages. This proposed approach will not only provide efficiency at each stage but will also provide status about overall system performance, thus helping decision-makers to identify an area of inefficiency and providing targeted improvements. This study connects neutrosophic logic with two-stage DEA, contributing to the developing body of efficiency analysis under uncertainty, hence offering the practical tool for optimization of complex real-world systems.

The rest of this paper is structured as follows. Section 2 presents the preliminaries. Section 3 presents the basic two-stages DEA model. Section 4 view the developing Neutrosophic Two-Stage DEA Model. Section 5 highlights case study. Section 6 shows the conclusion and future works.

2. Preliminaries

Neutrosophic sets were defined by Florentin Smarandache [14], thus providing a unified framework for dealing with the uncertainties, indeterminacy, and inconsistencies that arise in any kind of data. Neutrosophic sets, in contrast to classical fuzzy sets, which take into consideration only the degree of membership, and intuitionistic fuzzy sets, which consider both membership and nonmembership degrees, consider membership along with non-membership degrees and an extra dimension of indeterminacy. This indeterminate nature makes neutrosophic sets suitable for the representation of several real tasks filled with complex ambiguities and scenarios wherein the information available is incomplete or even imprecise.

Definition 1 [14]: A neutrosophic set A in a universe of discourse X is defined as: $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where $T_A(x), I_A(x), F_A(x)$ represent truth-membership function, indeterminacy-membership function, and falsity-membership function respectively. Each of these functions are independent and maps elements of X to the interval [0,1].

Definition 2 [14]: Neutrosophic sets generalize classical fuzzy sets and intuitionistic fuzzy sets. When $I_A(x) = 0$, a neutrosophic set reduces to an intuitionistic fuzzy set, and when both $I_A(x) = 0$ and $F_A(x) = 1 - T_A(x)$, it reduces to a classical fuzzy set.

Definition 3 [14]: Neutrosophic sets support a variety of operations, including union, intersection, and complement, let A and B are neutrosophic set, $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle | x \in X\}$ which are defined as follows:

1. $A \cup B = \{\langle x, max\left(T_A(x), T_B(x)\right), min\left(I_A(x), I_B(x)\right), min\left(F_A(x), F_B(x)\right)\rangle | x \in X\}$

2. $A \cap B = \{\langle x, min(T_A(x), T_B(x)), max(I_A(x), I_B(x)), max(F_A(x), F_B(x))\rangle | x \in X\}$

3. $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle | x \in X\}$

Definition 4 [14]: A Neutrosophic Set with Triangular Fuzzy Single-Valued Membership Functions (NSTFSV) is a specialized form of a neutrosophic set. Assume that $\tilde{Q}^{TN} = \langle (x, y, z); \beta_{\tilde{Q}^{TN}}, \gamma_{\tilde{Q}^{TN}}, \delta_{\tilde{Q}^{TN}} \rangle$ be NSTFSV, then the score function (SF) defined as follows: $SF(\tilde{Q}^{TN}) = \left(\frac{1}{4}(a + 2b + c)\right)\left(\frac{1}{3}(2 + \beta_{\tilde{Q}^{TN}} - \gamma_{\tilde{Q}^{TN}} - \delta_{\tilde{Q}^{TN}})\right)$, where $a, b, c \in \mathbb{R}$ such that $a \leq b \leq c$.

3. Basic Two-stage Data Envelopment Analysis Model

Two-stage DEA models investigate processes that are characterized by two distinct successive stages. Thus, in the first stage of a two-stage process, certain inputs lead to specific outputs. The two-stage DEA process that has '*n*' DMUs working along the process. It has been formulated to evaluate the efficiency of organizations with a two-stage internal structure as shown in Figure 1. Hence, each DMU can have:

Inputs → first stage → intermediate outputs → second stage → final outputs.

It is a very useful approach in evaluating processes where outputs from the first stage serve as inputs for the second stage. It proceeds to two stages or termed (subprocess – 1 or first stage, and subprocess – 2 or second stage). In it, weight λ_i is assigned to each DMU. In first stage, '*m'* number of inputs, where x_{ij} is the amount of input *j* consumed by DMU *i* (j = 1 to m), and '*k*' number of outputs were produced, where z_{ik} is the amount of output *k* generated by DMU *i* (*k* = 1 to *s*). The outputs of the first stage are regarded as intermediate measures and input to second stage. For second stage takes '*v*' independent inputs denoted as h_{it}, the amount of independent input '*t*' used by DMU *i* (t = 1 to v) and belches out '*u*' outputs represented by y_{ir}, the amount of output '*r*' produced by DMU *i* (r = 1 to u). Let $\emptyset_p^1$ be the efficiency of the p^{th} DMU in first stage, while the efficiency of the p^{th} DMU second stage, would be $\emptyset_p^2$. Together the efficiency of the p^{th} DMU can be expressed as $\emptyset_p$. The basic two-stage output-oriented variable returns to scale (VRS) DEA model of the p^{th} DMU considers model (*M-1*) for the first stage and model (*M-2*) for the second.

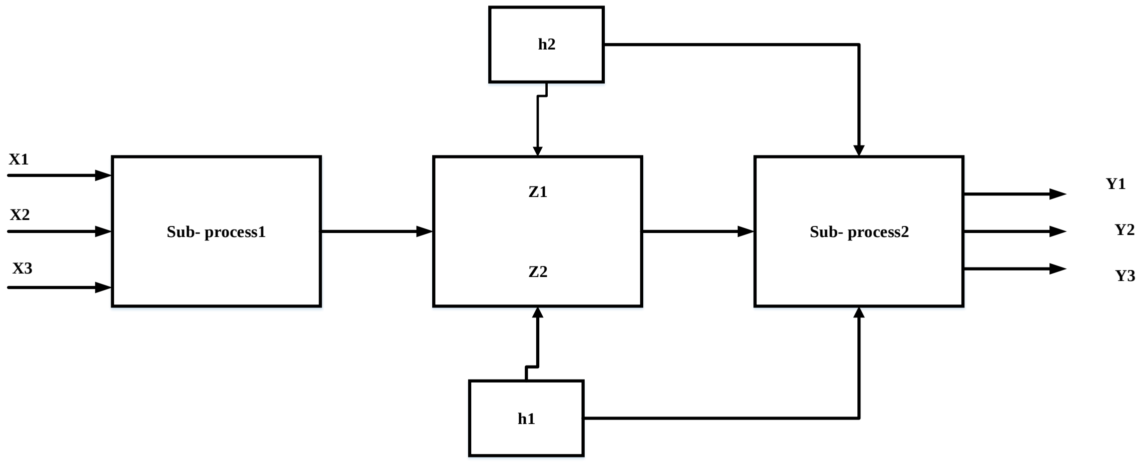

Figure 1. Two-stage DEA Model

- For 1st stage: A basic Variable Return to Scale – Output Oriented - DEA model

$$Max\ \emptyset_p^1$$

Subject to

$$\sum_{i=1}^{n} \lambda_i z_{ik} \geq \emptyset_p^1 z_{pk} \quad ,\forall k = 1 \dots s$$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq x_{pj} \quad ,\forall j = 1 \dots . m$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n) \qquad (M\text{-}1)$$

- For 2nd stage: A basic Variable Return to Scale – Output Oriented - DEA model

$$Max\ \emptyset_p^2$$

Subject to

$$\sum_{i=1}^{n} \lambda_i y_{ir} \geq \emptyset_p^2 y_{pr} \quad ,\forall r = 1 \dots u$$

$$\sum_{i=1}^{n} \lambda_i z_{ik} \leq z_{pk} \quad ,\forall k = 1 \dots . s$$

$$\sum_{i=1}^{n} \lambda_i h_{it} \leq h_{pt} \quad ,\forall t = 1 \dots . v$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n) \qquad (M\text{-}2)$$

- Overall Efficiency for p^{th} DMU $(\emptyset_p) = \emptyset_p^1 * \emptyset_p^2$

4. Developing a Novel Neutrosophic Two-Stage DEA Model

These two decades witness complication in the decision-making process and real uncertainties in data around the world to yield even more and highly flexible approaches regarding analytical efficiencies. An integrated neutrosophic DEA Model is here presented, which considers an output-oriented approach with two stages, during uncertain conditions with respect to its existence and flowing between such stages. The proposed model in this paper, as opposed to traditional two-stage DEA models that use deterministic data, incorporates neutrosophic sets in order to reflect the inherent vagueness and indeterminacy present in the input and output variables. This thereby provides a reasonable assessment interface for real efficiency investigation dealing with complicated systems. The formulated model takes two-stage DEA concepts further, whereby the first stage produces some intermediate outputs entering the second stage as inputs. However, it extends those principles by incorporating neutrosophic logic into handling uncertain data and also incorporates a variable return to scale assumption to account for variations in operational scale.

This dual focus on uncertainty and scalability issues renders the model especially useful in diverse areas of application in supply chain management, healthcare, and financial services, where lots of uncertainty in data and scale heterogeneity are present. The following subsections will elaborate upon the mathematical formulation of the model, key features, and application in a theoretical framework. Thus, the development of this model constitutes a significant milestone in the efficiency analysis direction, with each efficiency analysis thereby equipped by a better tool to venture the modern decision-making settings. Three separate phases are formed within our model. First, the membership function assigned to the inputs and outputs of a neutrosophic kind is defined at the first phase. Then in the second phase, defined membership functions for the neutrosophic variables to calculate the score function for them. Then the third phase concerns the computation for each DMU level of relative efficiency. Finally, the VRS output-oriented neutrosophic DEA model developed to evaluate the relative efficiency performance of the p^{th} DMU presents in the mathematical model below:

$$Max\ \emptyset_p^1$$

Subject to

$$\sum_{i=1}^{n} \lambda_i z_{ik} \geq \emptyset_p^1 z_{pk} \quad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{z}_{ik}{}^{TN} \geq \emptyset\, \tilde{z}_{pk}{}^{TN} \quad , \forall k \in K_N$$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq x_{pj} \quad , \forall j \in J_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij}{}^{TN} \leq \tilde{x}_{pj}{}^{TN} \quad , \forall j \in J_N$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n) \tag{M-3}$$

Where:

- $\tilde{z}_{ik}{}^{TN}$: output neutrosophic variable 'k' produced by DMU 'i' in first stage,
- $\tilde{x}_{ij}{}^{TN}$: input neutrosophic variable 'j' utilized by DMU 'i' in first stage,
- J_D: set of deterministic input variables for first stage,
- J_N: set of neutrosophic input variables for first stage,
- J: set of all input variables for first stage, where $J = J_D \cup J_N$
- K_D: set of deterministic output variables for first stage,
- K_N: set of neutrosophic output variables for first stage,
- *K:* set of all output variables for first stage, where $K = K_D \cup K_N$.

The membership functions of neutrosophic variables are assumed to be of triangular structure in this proposed model, following which a score function as defined in Definition 4 is applied to convert the VRS output-oriented triangular neutrosophic DEA model to a crisp DEA model. By this transformation, the computational function gets simplified, and the model-governing equations become easier to solve. More detail on this approach is given below.

$$Max\ \emptyset_p^1$$

Subject to

$$\sum_{i=1}^{n} \lambda_i z_{ik} \geq \emptyset_p^1 z_{pk} \quad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{z}_{ik}{}^{TN}) \geq \emptyset_p^1 \; SF(\tilde{z}_{pk}{}^{TN}) \quad , \forall k \in K_N$$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq x_{pj} \quad , \forall j \in J_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{x}_{ij}{}^{TN}) \leq SF(\tilde{x}_{pj}{}^{TN}), \forall j \in J_N$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n) \qquad (M\text{-}4)$$

Where:

- $SF(\tilde{z}_{ik}{}^{TN})$: score function for output neutrosophic variable '*k*' produced by DMU '*i*' in first satge,
- $SF(\tilde{x}_{ij}{}^{TN})$: score function for input neutrosophic variable '*j*' utilized by DMU '*i*' in first stage.

In the same way, the triangular output-oriented neutrosophic VRS DEA model has been converted into a crisp DEA model just as in the first stage, for the second stage. Since this conversion relieves the problem from the difficulties of having neutrosophotic variables, applying crisp DEA techniques becomes much easier. The function that scores the input and output of the DMU should enable a model to work under uncertainty and will remove complexities from the computation process. The very simplified version made under this approach will allow for quick evaluation of DMUs without compromise on the accuracy. The following explains how the transformation is implemented in detail.

$$Max \; \emptyset_p^2$$

Subject to

$$\sum_{i=1}^{n} \lambda_i y_{ir} \geq \emptyset_p^2 y_{pr} \quad , \forall r \in R_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{y}_{ir}{}^{TN}) \geq \emptyset_p^2 \, SF(\tilde{y}_{pr}{}^{TN}) \quad , \forall r \in R_N$$

$$\sum_{i=1}^{n} \lambda_i z_{ik} \leq z_{pk} \quad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i h_{it} \leq h_{pt} \quad , \forall t \in T_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{z}_{ik}{}^{TN}) \leq SF(\tilde{z}_{pk}{}^{TN}) \quad , \forall k \in K_N$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{h}_{it}{}^{TN}) \leq SF(\tilde{h}_{pt}{}^{TN}) \quad , \forall t \in T_N$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1, 2, \ldots, n) \qquad (M\text{-}5)$$

Where:

- $\tilde{h}_{it}{}^{TN}$: independent neutrosophic input variable '*t*' utilized by DMU '*i*' in second stage,
- $\tilde{y}_{ir}{}^{TN}$: final output neutrosophic variable '*r*' produced by DMU '*i*' in second stage,
- $SF(\tilde{h}_{it}{}^{TN})$: score function for independent neutrosophic input variable '*t*' utilized by DMU '*i*' in second stage
- $SF(\tilde{y}_{ir}{}^{TN})$: score function for final output neutrosophic variable '*r*' produced by DMU '*i*' in second stage
- R_D: set of deterministic final output variables for second stage,
- R_N: set of neutrosophic final output variables for second stage,
- *R*: set of all final output variables for second stage, where $R = R_D \cup R_N$,
- T_D: set of independent deterministic input variables for second stage,
- T_N: set of independent neutrosophic input variables for second stage,

- T: set of all input independent variables for second stage, where $T = T_D \cup T_N$.

5. Case Study

Performance evaluation of branches assumes great significance in the banking sector for decision makers. Traditional DEA could be seen almost everywhere, but in uncertain conditions due to lack of data, it is all the time being dependent on vague or imprecise data, no central tendency remains there. In this paper, we do NDEA to evaluate the efficiency of a two-stage banking system, where the first stage represents resource utilization, and the second stage includes the operations of customer service. Seven different bank branches will be analyzed using NDEA to assess efficiency by also incorporating the uncertainty regarding financial and operational aspects. The first stage consists of bank branches operating with three deterministic input variables (Number of employees, Operating expenses (in $1000s), Branch area (square meters), which are required for the banking processes) and two neutrosophic output variables (Number of processed transactions (in 1000) and Total funds managed (in $1000,00s)). In second stage, the bank branches continue to operate with two intermediate input variables (Output variables from 1st stage) as well as having an independent: deterministic input added to it (New accounts opened) and two neutrosophic final output variables (Customer satisfaction rate, and Total revenue generated ($1000,00s)). The complete case study data is shown in Tables 1, and 2. It is a problem that tries to come to terms with the relative performance of bank branches using our two-stage NDEA model.

Table 1 - 1st stage (resource utilization process) data for the bank branches

Branch	*Inputs*			*Intermediate Outputs*	
	Number of employees	*Operating expenses (in $1000s)*	*Branch area (square meters)*	*Processed Transactions (in 1000)*	*Total Funds Managed ($1000,00s)*
B_1	15	80	160	$\langle(4.9,5.1,5.3);0.9,0.4,0.05\rangle$	$\langle(20,22,24);0.9,0.4,0.1\rangle$
B_2	12	75	150	$\langle(4.6,4.8,5);0.9,0.15,0.1\rangle$	$\langle(18,20,22);0.9,0.7,0.1\rangle$
B_3	18	95	170	$\langle(5.3,5.5,5.7);0.9,0.1,0.04\rangle$	$\langle(23,25,27);0.9,0.4,0.1\rangle$
B_4	10	65	140	$\langle(4.1,4.3,4.5);0.8,0.5,0.12\rangle$	$\langle(16,18,20);0.8,0.5,0.1\rangle$
B_5	14	85	155	$\langle(5,5.2,5.4);0.8,0.2,0.05\rangle$	$\langle(21,23,25);0.8,0.5,0.2\rangle$
B_6	9	60	135	$\langle(3.8,4,4.2);0.9,0.5,0.15\rangle$	$\langle(14,16,18);0.9,0.5,0.2\rangle$
B_7	16	90	165	$\langle(5.1,5.3,5.5);0.9,0.4,0.03\rangle$	$\langle(22,24,26);0.9,0.4,0.1\rangle$

Table 2 - 2nd stage (customer service operations) data for the bank branches

Branch	*Independent Input*	*Final Output*	
	Number of new accounts opened	*Customer satisfaction*	*Total revenue generated ($1000,00s)*
B_1	*210*	⟨(0.8,0.87,0.94); 1.0, 0.0,0.06⟩	⟨(4.8,5.2,5.6); 1.0, 0.14,0.06⟩
B_2	*190*	⟨(0.75,0.82,0.89); 1.0,0.0,0.08⟩	⟨(4.5,4.9,5.3); 1.0,0.16,0.08⟩
B_3	*230*	⟨(0.83,0.9,0.97); 1.0,0.0,0.05⟩	⟨(5.6,6,6.4); 1.0,0.12,0.05⟩
B_4	*170*	⟨(0.68,0.75,0.82); 1.0,0.0,0.1⟩	⟨4.1,4.5,4.9; 1.0,0.18,0.1⟩
B_5	*220*	⟨(0.81,0.88,0.95); 1.0,0.0,0.06⟩	⟨(5.4,5.8,6.2); 1.0,0.13,0.06⟩
B_6	*150*	⟨(0.63,0.7,0.77); 1.0,0.0,0.15⟩	⟨(3.8,4.2,4.6); 1.0,0.2,0.15⟩
B_7	*225*	⟨(0.82,0.89,0.96); 1.0,0.0,0.05⟩	⟨(5.5,5.9,6.3); 1.0,0.12,0.05⟩

Table 3 - Bank Branches Relative Efficiency Level

Branch	*NDEA Model*			*Conventional DEA Model*		
	$\emptyset_p^1$	$\emptyset_p^2$	$\emptyset_p$	$\emptyset_p^1$	$\emptyset_p^2$	$\emptyset_p$
B_1	*1*	*0.82*	*0.82*	*0.49*	*0.58*	*0.28*
B_2	*0.65*	*1*	*0.65*	*0.4*	*1*	*0.40*
B_3	*1*	*1*	*1*	*1*	*1*	*1*
B_4	*1*	*1*	*1*	*1*	*1*	*1*
B_5	*0.48*	*0.75*	*0.36*	*0.37*	*0.86*	*0.32*
B_6	*0.29*	*0.89*	*0.26*	*0.21*	*1*	*0.21*
B_7	*1*	*0.88*	*0.88*	*0.40*	*0.50*	*0.20*

Table 3 presents comparative analysis between NDEA model and the Conventional DEA model applied on the seven bank branches to measure their performance. The efficiency of each branch is denoted by: $\emptyset_p^1$: (First-stage efficiency-resource utilization); $\emptyset_p^2$: (Second-stage efficiency-customer service operations); and

$\emptyset_p$ (Overall efficiency). The traditional DEA models fail to account for uncertainties and imprecise data in banking operations. On the other hand, it is obvious that NDEA describes neutrosophic numbers, thus providing a more realistic and flexible view of efficiency evaluation. Both branches B_3 and B_4 scored a total efficiency level of 1 for both models in $\emptyset_p$ indicating full efficiency in terms of resource usages and customer services. The branches show more improved efficiencies on NDEA than conventional DEA, where their degrees of efficiencies were $\emptyset_{B_1}$ =0.82 and $\emptyset_{B_7}$ =0.88, compared to $\emptyset_{B_1}$ =0.28 and $\emptyset_{B_7}$ =0.20: this gives a better indication that uncertainties in transaction and customer satisfaction data were tended better within NDEA. Both branches B_5 and B_6 present low efficient scores in both models; but NDEA provides a closer mirror of the performance because it factors in uncertainty in all revenue and customers' satisfaction.

Also, again compared First-State vs. Second-State Efficiency: Branches B_1, B_3, B_4, and B_7 received a score of 1 for the first stage ($\emptyset_p^1$), singling them out as being highly efficient with regards to the branch's employees, operating expenses, and area maximization for transactions and funds managed. Branch B_2 was inefficient in the first stage ($\emptyset_p^1$=0.65) but fully efficient in the second stage ($\emptyset_p^2$=1), indicating a good service delivery and revenue generation capacity despite having a relatively lower resource utilization efficiency. Branches B_5 and B_6 were the least efficient in the first stage, at $\emptyset_p^1$=0.48 and then $\emptyset_p^1$=0.29 respectively, which points out the issue in resource allocation. Finally, to improve efficiency for inefficient branches, we can recommend Branches B_5 and B_6 should focus on improving resource utilization (optimizing staff productivity and branch expenses). Branch B_2 should enhance first-stage efficiency by increasing employee productivity and optimizing expenses. Branches B_1 and B_7 should work on enhancing customer service quality to convert operational efficiency into higher satisfaction scores and revenue.

6. Conclusions and future works

The optimization framework presented in this paper is for two-stage systems, making use of Neutrosophic Data Envelopment Analysis (NDEA) in meeting the limitations of the traditional DEA against uncertainties and imprecise data. The proposed model has made the evaluation of efficiency in very complex systems, such as the banking sector, fairly realistic because it accommodates neutrosophic numbers. The case study on bank branch performance demonstrated the effectiveness of NDEA in decomposing overall efficiency into resource utilization and customer service operations from which decision-makers could derive more profound insights. Furthermore, NDEA measured efficiency more suitably than a comparison between NDEA and ordinary DEA; particularly branches regarding which the volume of transaction, customer satisfaction level, and revenue generated are uncertain. Major results of the study include both branches B_3 and B_4 were fully efficient in both models, which proves that they optimized their resources and performed good customer service. NDEA raised efficiency scores for branches such as B_1 and B_7, undermeasured by standard DEA due to uncertainty in financial and operational data. Branches B_5 and B_6 had the least efficiency scores. Therefore, these branches have to better utilize their available resources and optimization strategy.

This study does not end at establishing the virtues of Neutrosophic Data Envelopment Analysis (NDEA) in improving the efficiency appraisals of two-stage systems but opens up a whole lot of areas yet to be covered. This becomes an opportunity for further research, which should try out a new two-stage NDEA framework once again in other industries beyond the banking sector such as healthcare, supply chain management, education, and manufacturing. Moreover, ascertaining whether it is adaptable across different industries forms the basis for validating the robustness of the methodology and improves decision-making in different environments. An attractive prospect would be combining NDEA with machine-learning techniques to add prediction capability to the NDEA. AI programs would identify patterns related to efficiency improvements, forecast future inefficiencies, and recommend automated optimization strategies for decision-makers.

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