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An Approach to Neutrosophic Hyper Soft Rough Matrix and its Application

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Abstract. This paper introduces a novel of Neutrosophic Hyper Soft Rough Matrix (NHSRM). We discuss various operators on NHSRM such as addition, multiplication, scalar multiplication and complement. Additionally, we develop several properties, including associativity, distributivity and De Morgan's laws, utilizing operators such as union, intersection, complement, arithmetic mean, weighted arithmetic mean, geometric mean, weighted geometric mean, harmonic mean and weighted harmonic mean. Further we propose a score function as part of this work. To demonstrate the application of the proposed work, a multi-attribute decision-making problem is considered. The score matrix obtained from three different operators are compared, and the results are illustrated graphically.

Keywords: Neutrosophic hyper soft matrix; Neutrosophic hyper soft rough matrix; Weighted arithmetic mean; Weighted geometric mean; Weighted harmonic mean; Score matrix; Multi-attribute decision-making.

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1. Introduction

In a traditional matrix, each element is well-defined and unambiguous, and these matrices are widely used in various mathematical fields, including linear transformations in vector spaces, computer graphics, signal processing, and data analysis. Fuzzy matrices have been widely studied for their utility in representing uncertainty and imprecision. In a fuzzy matrix [22], each element is either a fuzzy number or a degree of membership, typically within the interval [0, 1], indicating levels of truth, membership, or uncertainty. These matrices are particularly valuable in addressing situations involving vague or imprecise information. Their applications span various domains, including artificial intelligence, control systems, and decision support systems, where traditional matrices often lack the flexibility required to handle such complex scenarios.

The soft matrix, rooted in soft set theory, provides an advanced framework by allowing multiple and context-specific parameterizations to describe uncertainty, offering greater flexibility in modeling and analysis. This approach extends the capabilities of traditional methods by incorporating parameter dependency, reducing ambiguity, broadening applicability, and enhancing versatility. Soft matrices prove particularly useful in various fields where traditional crisp values are insufficient to capture the complexities of real-world situations. The Dynamism of fuzzy soft matrices lies in their ability to incorporate parameter dependency, reduce ambiguity, and broaden applicability across diverse fields. Applications have been explored in artificial intelligence, data analysis, decision support systems, and pattern recognition, where traditional frameworks often fail to capture the complexities and subtleties of real-world scenarios. As a result, fuzzy soft matrices have emerged as a powerful tool for tackling challenges involving high levels of uncertainty and variability. Cagman and Enginoglu [5], Borah et al. [3] developed fuzzy soft matrices, which serve as a matrix representation of fuzzy soft sets, and established operations on these matrices that facilitate theoretical studies in fuzzy soft set theory.

Fuzzy Soft Rough Matrix extends the fuzzy soft matrix by integrating rough set theory, which provides mechanisms to approximate sets using lower and upper approximations. This allows the matrix to handle situations where there is incomplete or indeterminate information, distinguishing between lower approximation and upper approximations. It is widely applied in decision support systems, pattern recognition, data mining, medical diagnostics, environmental modeling, control systems, supply chain management, and risk assessment. Muthukumar and Krishnan [14] initiated generalized fuzzy soft rough matrices and their operations, which play a crucial role in advancing theoretical studies in fuzzy soft rough sets. Their work provides a foundation for applying these matrices to model and analyze complex systems.

The Fuzzy Soft Matrix is extended to the Intuitionistic Fuzzy Soft Matrix (IFSM) by incorporating the concept of intuitionistic fuzzy sets, which includes both membership and non-membership degrees along with a degree of hesitation. The IFSM provides a framework for analyzing problems with multiple sources of uncertainty, capturing more detailed and nuanced information than standard fuzzy soft matrices. Chetia et al. [6] established five distinct types of product for intuitionistic fuzzy soft matrices and explore their theoretical properties. These operations and definitions make IFSMs a functional tool for advancing studies in intuitionistic fuzzy soft set theory, with potential applications in fields that require nuanced decision-making in engineering. Neutrosophic Fuzzy Soft Matrices (NFSMs) enhance the representation of uncertainty by explicitly incorporating indeterminacy as a distinct parameter alongside truth and falsehood, providing a more detailed and versatile framework compared to the hesitation-based representation in Intuitionistic Fuzzy Soft Matrices (IFSMs). NFSMs is designed for more complex and dynamic environments, such as big data analytics, where explicit modeling of indeterminate states is essential. Sumathi and Arockiarani [20] introduced new operations on fuzzy neutrosophic soft matrices, expanding the potential applications of these matrices in dealing with uncertainty and imprecision in complex systems. Their research highlights innovative matrix operations, such as addition and multiplication, which enhance the versatility of neutrosophic models in decision-making and other computational applications. These contributions provide a solid foundation for understanding the practical implementation of fuzzy neutrosophic soft matrices in various real-world contexts. Uma et al. [21] explored Fuzzy Neutrosophic Soft Matrices of Type I and Type II, offering significant advancements in the application of neutrosophic principles to matrix operations. Their work presents novel methods for handling uncertainty and imprecision in complex systems, particularly in decision-making environments. These contributions broaden the scope of fuzzy neutrosophic models and provide a more robust framework for analyzing real-world problems involving multi-criteria decision analysis and other computational tasks.

Smarandache [16] extended the concept of soft sets to hyper soft sets by transforming the function into a multi-attribute function. This development allowed for a more comprehensive representation of data involving multiple parameters. A hyper soft matrices lies in the latter's ability to represent and analyze multi-attribute data, enabling more flexible and comprehensive modeling of complex, interdependent systems compared to the single-parameter framework of soft matrices. These matrices are particularly useful in scenarios where multiple interrelated attributes must be considered, such as in complex system modeling, risk assessment, and multi-objective optimization. They also enhance the handling of unpredictability in medical decision support, environmental modeling, and financial forecasting. In contrast, hyper soft rough matrices go a step further by introducing the lower and upper approximations of rough set theory, which allow for distinguishing between known and unknown information, particularly when dealing with higher degree of uncertainty. It is offering a more advanced mechanism for classification and decision-making in systems where data may be ambiguous or partial.

An Intuitionistic Hyper Soft Rough Matrix (IHSRM) was introduced to offer a more comprehensive and flexible framework that can handle uncertainty at different levels through membership, non-membership, and indeterminacy. This makes it particularly suitable for complex systems, multi-criteria decision-making, and fields like data mining, machine learning, and information retrieval, where precise modeling and handling of diverse uncertainties are essential. Lei Zhou et al. [12] proposed a comprehensive framework for studying relation-based intuitionistic fuzzy rough approximation operators. This framework incorporates both constructive and axiomatic approaches and establishes several fundamental properties of intuitionistic fuzzy rough approximation operators. Jafar and Saeed [8] expanded the concept of Neutrosophic hypersoft sets to Neutrosophic Hyper Soft Matrices (NHSM), providing a detailed study of these matrices along with relevant examples. Jayasudha and Raghavi [10] explored various operations on Neutrosophic Hypersoft Matrices and demonstrated their practical applications in complex decision-making scenarios. Their work provided a comprehensive framework for applying neutrosophic principles to matrix operations, expanding the scope of these models in real-world applications. This study offers valuable insights that complement the methodologies applied in the current research, particularly in addressing the specific aspects of uncertainty in data modeling and multi-criteria decision analysis.

In this paper, we extend the concept of NHSM to NHSRM. The NHSRM is a novel mathematical model designed to address these challenges by integrating several advanced concepts from neutrosophic logic, soft set theory, hyper-soft sets, and rough set theory. This hybrid model offers a powerful and flexible tool for managing complex decision-making scenarios where traditional methods fall short. This enables NHSRM to handle situations where data is not only uncertain but also partially known or ambiguous, which is particularly useful in fields like pattern recognition, decision-making under uncertainty, and machine learning, where such approximations improve decision boundaries and classification accuracy. NHSM focuses on the representation of uncertainty without incorporating the rough set-based approximations, making it suitable for less complex scenarios. In contrast, NHSRM provides a more comprehensive framework by integrating rough set-based approximations, which makes it better suited for handling more complex and nuanced problems. The scope of this work is to develop some notions and operations with various properties of NHSRM along with a suitable decision-making problem.

In this paper, the following section of the proposed work organizes as follows:

- Section 2 presents the foundational key preliminaries, establishing the essential concepts and terminologies necessary for understanding the theoretical framework.
- Section 3 defines the key concepts, operators, and various results related to NHSRM. This section also discusses the mathematical operations used in NHSRM, such as addition, multiplication, and composition, and examines the properties and results that emerge from applying these operations.
- The proposed work illustrates the method through its application to decision-making problems, comparing the weighted arithmetic mean, weighted geometric mean, and weighted harmonic mean operators with the use of a score function in section 4. This

section effectively highlights how these operators can be applied within the NHSRM framework to tackle complex decision-making tasks.

- In section 5, an in-depth evaluation of the performance, accuracy, or efficiency of the proposed method, model, or technique is typically conducted in comparison to existing methods. In the context of this work, the NHSRM can be compared with traditional decision-making models like weighted arithmetic mean, weighted geometric mean, and weighted harmonic mean operators. These operators, when applied within NHSRM, can be assessed in terms of their effectiveness in solving decision-making problems that involve multi-attribute data and uncertainty.
- Conclusion and result are discussed in section 6.

The Neutrosophic Hyper Soft Rough Matrix (NHSRM) model is constrained by the requirement for accurate parameter adjustments, challenges in interpreting intricate outcomes, dependence on expert input for weight determination, and limited flexibility in handling dynamic data changes.

2. **Preliminaries**

Definition 1 (Neutrosophic set) [17] Let U be universe set, the neutrosophic set N on the universe set U is defined as $N = [T(x), I(x), F(x)]$, where the characteristic functions $T, I, F : U \to [0, 1]$and $0 \leqslant T(x)+I(x)+F(x) \leqslant 3$, T, I, F are neutrosophic components which defines the degree of truth, the degree of indeterminacy and the degree of falsity respectively.
Definition 2 (Soft rough set) [14] Let $R = (\varphi, A)$ be a soft set over U the pair $S = (U, R)$ is called a soft approximation space. Based on the soft approximation space S, define the following two operations:

$$\underline{apr}_S(X) = \{u \in U; \exists a \in A, u \in \varphi(a) \subseteq X)$$

$$\overline{apr}_S(X) = \{u \in U; \exists a \in A, u \in \varphi(a), \varphi(a) \cap X \neq \phi)$$

Assigning to every subset $X \subseteq U$ two sets $\underline{apr}_S(X)$ and $\overline{apr}_S(X)$, which are called the soft S lower approximation and the soft S upper approximation of X respectively. In general, $\underline{apr}_S(X)$ and $\overline{apr}_S(X)$ as soft rough approximation of X with respect to S. Moreover the sets
$Pos_S(X) = \underline{apr}_S(X)$
$Neg_S(X) = U - \overline{apr}_S(X)$
$Bnd_S(X) = \overline{apr}_S(X) - \underline{apr}_S(X)$ are called the soft S - positive region, the soft S - negative region and the soft S - boundary region of X respectively.
If $\overline{apr}_S(X) = \underline{apr}_S(X)$, X is said to be S - definable, otherwise X is called a soft S - rough set.
Definition 3 (Hyper Soft Set) [9] Let U be the universe set and P(U) be the power set of U. Suppose $a_1, a_2, a_3, ..., a_n$ where $n \geqslant 1$ be n distinct attributes whose corresponding attribute

values respectively the set $A_1, A_2...., A_n$ with $A_i \cap A_j = \phi, i \neq j$ and $i, j \in \{1, 2, ..., n\}$ then the pair $(\varphi, A_1 \times A_2 \times .... \times A_n)$ is called a hyper soft set over U, where $\varphi : A_1 \times A_2 \times .... \times A_n \to P(U)$.

Definition 4 (Hyper soft rough set) [11] Let U be the universe set and P(U) be the power set of U. Suppose $a_1, a_2, a_3, ..., a_n$ where $n \geqslant 1$ be n distinct attributes whose corresponding attribute values $A_1, A_2...., A_n$ respectively. Let $S_j \subseteq A_j, j \in \{1, 2, ..., n\}$then $\prod_{j=1}^{n} S_j \subseteq \prod_{j=1}^{n} A_j$. The pair $(\varphi, \prod_{j=1}^{n} S_j) = P(U)$ where φ is a mapping defined by $\varphi : \prod_{j=1}^{n} S_j \to P(U)$ is called hyper soft rough set. The lower and upper neutrosophic hyper soft approximation spaces of $X \in P(U)$ with respect to $(\varphi, \prod_{j=1}^{n} S_j^k)$ are denoted by $\underline{apr}(X)$ and $\overline{apr}(X)$ respectively, defined by

$$\underline{apr}_S(X) = \{u \in U; \exists a_1, a_2, ..., a_n \in A, u \in \varphi(a_1, a_2, ..., a_n) \subseteq X)$$

$$\overline{apr}_S(X) = \{u \in U; \exists a_1, a_2, ..., a_n \in A, u \in \varphi(a_1, a_2, ..., a_n), \varphi(a_1, a_2, ..., a_n) \cap X \neq \phi)$$

If $\overline{apr}(X) \neq \underline{apr}(X)$, then X is hyper soft rough set, otherwise it is called as hyper soft rough definable set.

Definition 5 (Neutrosophic Hyper Soft Matrix) [8] Let $U = \{u_1, u_2, ..., u_\alpha\}$ and P(U) be the universal set and the power set, respectively. Consider $A_1, A_2...., A_\beta$, for $\beta \geq 1$, where β is well-defined attributes, the corresponding attributive values are $A_1^{q_1}, A_2^{q_2}...., A_\beta^{q_n}$ and their relation $A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n}$, where $q_1, q_2, q_3, ..., q_n = 1, 2, \ldots, n$; then the pair $(\varphi, A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n})$ is said to be a neutrosophic hyper soft set over U, where $\varphi : (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n}) \to P(U)$ is defined as

$$\varphi(A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n}) = \{\langle u, T_\lambda(u), I_\lambda(u), F_\lambda(u)\rangle \in U, \lambda \in (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n})\}$$

If $B_{ij} = X(u_i, A_j^k)$, where $i = i, 2, 3, ..., \alpha, j = 1, 2, 3, ..., \beta$ and $k = q_1, q_2, q_3, ..., q_n$ then a matrix is defined as $[B_{ij}]_{\alpha\times\beta} = \begin{bmatrix} B_{11} & B_{12} & ... & B_{1\beta} \\ B_{21} & B_{22} & ... & B_{2\beta} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\alpha 1} & B_{\alpha 2} & ... & B_{\alpha\beta} \end{bmatrix}$

where $B_{ij} = (T_{A_j^k}(u_i), I_{A_j^k}(u_i), F_{A_j^k}(u_i), u_i \in U, A_j^k \in (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_n})) = (T_{ij}^B, I_{ij}^B, F_{ij}^B)$

Definition 6 (Neutrosophic Hyper Soft Rough Set) [19] Let U be an non-empty universe set and P(U) be the power set of U. Let E be the set of parameters, $E = \{A_1, A_2, ...A_n\}$, where $A_i \cap A_j = \emptyset \, for \, i \neq j$. Let $S_j \subseteq A_j, j \in \{1, 2, ..., n\}$ then $\prod_{j=1}^{n} S_j^k \subseteq \prod_{j=1}^{n} A_j^k$. The pair $(\varphi, \prod_{j=1}^{n} S_j^k) = P(U)$ where φ is a mapping defined by $\varphi : \prod_{j=1}^{n} S_j^k \to P(U)$ is called neutrosophic hyper soft rough set. The triplet $(U, \varphi, \prod_{j=1}^{n} S_j^k)$ is called neutrosophic hyper soft approximation space. The lower and upper neutrosophic hyper soft approximation spaces of $X \in P(U)$ with respect to $(U, \varphi, \prod_{j=1}^{n} S_j^k)$ are denoted by $\overline{apr}(X)$ and $\underline{apr}(X)$ respectively, defined by

$\underline{apr}_N(X) = \{(\prod_{j=1}^{n} S_j, \{\langle \frac{u}{\mu_A(u), v_A(u), w_A(u)} \rangle\}, \forall u \in U\}$ and $\overline{apr}_N(X) = N(U) - \underline{apr}_N(X^c)$

If $\overline{apr}_N(X) \neq \underline{apr}_N(X)$, then X is neutrosophic hyper soft rough set, otherwise it is called as neutrosophic hyper soft rough definable set.

3. Neutrosophic Hyper Soft Rough Matrix (NHSRM)

In this section, the definition, basic operations and various properties of NHSRM are presented.

Definition 7 (NHSRM) Let $U = \{u_1, u_2, ..., u_\alpha\}$ be an non-empty universe set and P(U) be the power set of U. Let E be the set of parameters, $E = \{A_1, A_2, ...A_n\}$, where $A_i \cap A_j = \emptyset\, for\, i \neq j$. Let $S_j \subseteq A_j, j \in \{1, 2, ..., n\}$ then $\prod_{j=1}^{n} S_j^k \subseteq \prod_{j=1}^{n} A_j^k$. The pair $(\varphi, \prod_{j=1}^{n} S_j^k) = P(U)$, where φ is a mapping defined by $\varphi : \prod_{j=1}^{n} S_j^k \to P(U)$ is called neutrosophic hyper soft rough set. Each element $u \in U$ is associated with the values determined by the hyper soft set, where each parameter can take multiple values. For each element $u \in U$ related with a parameter A_j is represented by the triplet (T_{ij}, I_{ij}, F_{ij}) where T_{ij} is the truth membership function, I_{ij} is the indeterminacy membership function and F_{ij} is the falsity membership function, $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$. If $P_{ij} = \gamma(u_i, A_j^k)$, where $i = 1, 2, 3, ..., m, j = 1, 2, 3, ..., n$ and $k = q_1, q_2, q_3, ..., q_n$ then a NHSRM is defined as

$$P = [P_{ij}] = \begin{bmatrix} < \underline{P}_{11}; \bar{P}_{11} > & < \underline{P}_{12}; \bar{P}_{12} > & .... & < \underline{P}_{1n}; \bar{P}_{1n} > \\ < \underline{P}_{21}; \bar{P}_{21} > & < \underline{P}_{22}; \bar{P}_{22} > & .... & < \underline{P}_{2n}; \bar{P}_{2n} > \\ \vdots & \vdots & \ddots & \vdots \\ < \underline{P}_{m1}; \bar{P}_{m1} > & < \underline{P}_{m2}; \bar{P}_{m2} > & .... & < \underline{P}_{mn}; \bar{P}_{mn} > \end{bmatrix}$$

Lower Approximation matrix is denoted by $\underline{P}_{ij} = (\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P)$, $0 \leqslant \underline{T}_{ij}^P + \underline{I}_{ij}^P + \underline{F}_{ij}^P \leqslant 3$ and Upper Approximation matrix is denoted by $\bar{P}_{ij} = (\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P)$, $0 \leqslant \bar{T}_{ij}^P + \bar{I}_{ij}^P + \bar{F}_{ij}^P \leqslant 3$
Thus, we can represent any neutrosophic Hyper Soft Rough Set in term of neutrosophic Fuzzy Hyper Soft Rough Matrix.

Example 1
The NHSRM P of order 4×3 is written as

$$P = \begin{bmatrix} \langle(0.8, 0.3, 0.2); (0.6, 0.5, 0.1)\rangle & \langle(0.4, 0.2, 0.3); (0.9, 0.2, 0.1)\rangle & \langle(0.5, 0.4, 0.2); (0.7, 0.5, 0.1)\rangle \\ \langle(0.3, 0.4, 0.0); (0.5, 0.8, 0.7)\rangle & \langle(0.8, 0.7, 0.5); (0.9, 0.5, 0.4)\rangle & \langle(1.0, 0.5, 0.2); (0.5, 0.6, 0.0)\rangle \\ \langle(0.9, 0.1, 0.2); (0.5, 0.6, 0.0)\rangle & \langle(0.7, 0.2, 0.1); (0.6, 0.5, 0.0)\rangle & \langle(0.8, 0.2, 0.3); (0.3, 0.2, 0.5)\rangle \\ \langle(1.0, 0.5, 0.4); (0.2, 0.3, 0.5)\rangle & \langle(0.9, 0.5, 0.1); (0.7, 0.8, 1.0)\rangle & \langle(0.6, 0.4, 0.2); (0.5, 0.2, 0.5)\rangle \end{bmatrix}$$

Definition 8 (Square NHSRM) A NHSRM of order $m \times n$ is said to be square NHSRM, if $m = n$ i.e., the number of rows and numbers of columns are equal. This "square" structure signifies that the matrix represents an equal balance between the objects in the universe of discourse and the hierarchical parameters.

Example 2
The square NHSRM P of order 3×3 is written as

$$P = \begin{bmatrix} \langle(0.8,0.3,0.2);(0.6,0.5,0.1)\rangle & \langle(0.4,0.2,0.3);(0.9,0.2,0.1)\rangle & \langle(0.5,0.4,0.2);(0.7,0.5,0.1)\rangle \\ \langle(0.3,0.4,0.0);(0.5,0.8,0.7)\rangle & \langle(0.8,0.7,0.5);(0.9,0.5,0.4)\rangle & \langle(1.0,0.5,0.2);(0.5,0.6,0.0)\rangle \\ \langle(0.9,0.1,0.2);(0.5,0.6,0.0)\rangle & \langle(0.7,0.2,0.1);(0.6,0.5,0.0)\rangle & \langle(0.8,0.2,0.3);(0.3,0.2,0.5)\rangle \end{bmatrix}$$

Definition 9 (Diagonal NHSRM) A square NHSRM of order $n \times n$ is said to be a diagonal NHSRM, if all of its non-diagonal elements are (0, 0, 1) and the main diagonal elements are considered as non-zero.

Example 3

The diagonal NHSRM P of order 3×3 is written as

$$P = \begin{bmatrix} \langle(0.8,0.3,0.2);(0.6,0.5,0.1)\rangle & \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle & \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle \\ \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle & \langle(1.0,0.3,0.5);(0.4,0.2,1.0)\rangle & \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle \\ \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle & \langle(0.0,0.0,1.0);(0.0,0.0,1.0)\rangle & \langle(0.8,0.2,0.3);(0.3,0.2,0.5)\rangle \end{bmatrix}$$

Definition 10 (Symmetric NHSRM) A square NHSRM of order $n \times n$ is said to be a symmetric NHSRM, if its transpose be equal to itself. This means that for every element in the matrix, the entry at position (i, j) is identical to the entry at the position (j, i). i.e. $P = P^T$(or) $[P_{ij}] = [P_{ji}]\,\forall i,j$

Example 4

The symmetric NHSRM P of order 3×3 is written as

$$P = \begin{bmatrix} \langle(0.8,0.3,0.2);(0.6,0.5,0.1)\rangle & \langle(0.3,0.4,0.0);(0.5,0.8,0.7)\rangle & \langle(0.5,0.4,0.2);(0.7,0.5,0.1)\rangle \\ \langle(0.3,0.4,0.0);(0.5,0.8,0.7)\rangle & \langle(0.8,0.7,0.5);(0.9,0.5,0.4)\rangle & \langle(0.7,0.2,0.1);(0.6,0.5,0.0)\rangle \\ \langle(0.5,0.4,0.2);(0.7,0.5,0.1)\rangle & \langle(0.7,0.2,0.1);(0.6,0.5,0.0)\rangle & \langle(0.8,0.2,0.3);(0.3,0.2,0.5)\rangle \end{bmatrix}$$

Definition 11 (Transpose) The transpose of a square NHSRM of order $n \times n$ is obtained by switching its rows and columns, similar to the transpose operation in standard matrices. This operation allows us to examine relationships from a different perspective, effectively interchanging the roles of elements and hierarchical parameters.

$$[P_{ij}]^T = \left[\left(\underline{T}^P_{ji}, \underline{I}^P_{ji}, \underline{F}^P_{ji}\right); (\bar{T}^P_{ji}, \bar{I}^P_{ji}, \bar{F}^P_{ji})\right]$$

Example 5

Consider Example 2, the transpose of P is given by

$$P^T = \begin{bmatrix} \langle(0.8,0.3,0.2);(0.6,0.5,0.1)\rangle & \langle(0.3,0.4,0.0);(0.5,0.8,0.7)\rangle & \langle(0.9,0.1,0.2);(0.5,0.6,0.0)\rangle \\ \langle(0.4,0.2,0.3);(0.9,0.2,0.1)\rangle & \langle(0.8,0.7,0.5);(0.9,0.5,0.4)\rangle & \langle(0.7,0.2,0.1);(0.6,0.5,0.0)\rangle \\ \langle(0.5,0.4,0.2);(0.7,0.5,0.1)\rangle & \langle(1.0,0.5,0.2);(0.5,0.6,0.0)\rangle & \langle(0.8,0.2,0.3);(0.3,0.2,0.5)\rangle \end{bmatrix}$$

Definition 12 (Addition) Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]$ we define the addition in NHSRM as follows:

$$P + Q = \left[\left(\max(\underline{T}^P_{ij}, \underline{T}^Q_{ij}), \max(\underline{I}^P_{ij}, \underline{I}^Q_{ij}), \min(\underline{F}^P_{ij}, \underline{F}^Q_{ij})\right); \left(\max(\bar{T}^P_{ij}, \bar{T}^Q_{ij}), \max(\bar{I}^P_{ij}, \bar{I}^Q_{ij}), \min(\bar{F}^P_{ij}, \bar{F}^Q_{ij})\right)\right]$$

Example 6

Let P and Q be two NHSRM of order 3×3,

$$P = \begin{bmatrix} \langle(0.8,0.3,0.2);(0.6,0.5,0.1)\rangle & \langle(0.4,0.2,0.3);(0.9,0.2,0.1)\rangle & \langle(0.5,0.4,0.2);(0.7,0.5,0.1)\rangle \\ \langle(0.3,0.4,0.0);(0.5,0.8,0.7)\rangle & \langle(0.8,0.7,0.5);(0.9,0.5,0.4)\rangle & \langle(1.0,0.5,0.2);(0.5,0.6,0.0)\rangle \\ \langle(0.9,0.1,0.2);(0.5,0.6,0.0)\rangle & \langle(0.7,0.2,0.1);(0.6,0.5,0.0)\rangle & \langle(0.8,0.2,0.3);(0.3,0.2,0.5)\rangle \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \langle(0.5,0.2,0.0);(0.4,0.3,0.1)\rangle & \langle(0.6,0.8,0.2);(0.7,0.3,0.2)\rangle & \langle(0.7,0.6,0.0);(1.0,0.4,0.1)\rangle \\ \langle(0.6,0.7,0.1);(0.8,0.9,0.3)\rangle & \langle(0.4,1.0,0.1);(0.9,0.2,0.3)\rangle & \langle(1.0,0.6,0.0);(0.7,0.1,0.3)\rangle \\ \langle(0.4,0.3,0.2);(0.7,0.6,0.2)\rangle & \langle(1.0,0.5,0.7);(0.8,0.4,1.0)\rangle & \langle(0.3,0.2,0.4);(0.5,0.7,0.6)\rangle \end{bmatrix}$$

then

$$P{+}Q = \begin{bmatrix} \langle(0.8,0.3,0.0);(0.6,0.5,0.1)\rangle & \langle(0.6,0.8,0.2);(0.9,0.3,0.1)\rangle & \langle(0.7,0.6,0.0);(1.0,0.5,0.1)\rangle \\ \langle(0.6,0.7,0.0);(0.8,0.9,0.3)\rangle & \langle(0.8,1.0,0.1);(0.9,0.5,0.3)\rangle & \langle(1.0,0.6,0.0);(0.7,0.6,0.0)\rangle \\ \langle(0.9,0.3,0.2);(0.7,0.6,0.0)\rangle & \langle(1.0,0.5,0.1);(0.8,0.5,0.0)\rangle & \langle(0.8,0.2,0.3);(0.5,0.7,0.5)\rangle \end{bmatrix}$$

Definition 13 (Multiplication) Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]$ we define the multiplication in NHSRM as follows:

$$P.Q = \left[\left(\min(\underline{T}^P_{ij}, \underline{T}^Q_{ij}), \min(\underline{I}^P_{ij}, \underline{I}^Q_{ij}), \max(\underline{F}^P_{ij}, \underline{F}^Q_{ij})\right); \left(\min(\bar{T}^P_{ij}, \bar{T}^Q_{ij}), \min(\bar{I}^P_{ij}, \bar{I}^Q_{ij}), \max(\bar{F}^P_{ij}, \bar{F}^Q_{ij})\right)\right]$$

Example 7

In Example 6, the result is derived by applying Definition 13.

$$P.Q = \begin{bmatrix} \langle(0.5,0.2,0.2);(0.4,0.3,0.1)\rangle & \langle(0.4,0.2,0.3);(0.7,0.2,0.2)\rangle & \langle(0.5,0.4,0.2);(0.7,0.4,0.1)\rangle \\ \langle(0.3,0.4,0.1);(0.5,0.8,0.7)\rangle & \langle(0.4,0.7,0.5);(0.9,0.2,0.4)\rangle & \langle(1.0,0.5,0.2);(0.5,0.1,0.3)\rangle \\ \langle(0.4,0.1,0.2);(0.5,0.6,0.2)\rangle & \langle(0.7,0.2,0.7);(0.6,0.4,1.0)\rangle & \langle(0.3,0.2,0.4);(0.3,0.2,0.6)\rangle \end{bmatrix}$$

Definition 14 (Scalar Multiplication) Let $P = [P_{ij}]$ be the NHSRM of order $m \times n$, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]$ and $\alpha \in [0,1]$ is the scalar then scalar multiplication of NHSRM P is given by

$$\alpha P = [\alpha P_{ij}] = \left[\left(\alpha\underline{T}^P_{ij}, \alpha\underline{I}^P_{ij}, \alpha\underline{F}^P_{ij}\right); \left(\alpha\bar{T}^P_{ij}, \alpha\bar{I}^P_{ij}, \alpha\bar{F}^P_{ij}\right)\right]$$

Example 8

By using definition 14 in Example 2, if $\alpha = 0.1$ then the result is as follows:

$$(0.1)P = \begin{bmatrix} \left\langle\begin{matrix}(0.08,0.03,0.02);\\(0.06,0.05,0.01)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.04,0.02,0.03);\\(0.09,0.02,0.01)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.05,0.04,0.02);\\(0.07,0.05,0.01)\end{matrix}\right\rangle \\ \left\langle\begin{matrix}(0.03,0.04,0.00);\\(0.05,0.08,0.07)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.08,0.07,0.05);\\(0.09,0.05,0.04)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.10,0.05,0.02);\\(0.05,0.06,0.00)\end{matrix}\right\rangle \\ \left\langle\begin{matrix}(0.09,0.01,0.02);\\(0.05,0.06,0.00)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.07,0.02,0.01);\\(0.06,0.05,0.00)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.08,0.02,0.03);\\(0.03,0.02,0.05)\end{matrix}\right\rangle \end{bmatrix}$$

Definition 15 (Complement) Let $P = [P_{ij}]$ be the NHSRM of order $m \times n$, where $[P_{ij}] = \left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right]$ then $P^{c} = \left[\left(\underline{F}_{ij}^{P}, 1-\underline{I}_{ij}^{P}, \underline{T}_{ij}^{P}\right); \left(\bar{F}_{ij}^{P}, 1-\bar{I}_{ij}^{P}, \bar{T}_{ij}^{P}\right)\right]$ is the complement of P.

Example 9

The complement of NHSRM P in Example 2 is formulated as

$$P^{c} = \begin{bmatrix} \langle(0.2,0.7,0.8);(0.1,0.5,0.6)\rangle & \langle(0.3,0.8,0.4);(0.5,0.8,0.7)\rangle & \langle(0.2,0.6,0.5);(0.1,0.5,0.7)\rangle \\ \langle(0.0,0.6,0.3);(0.7,0.2,0.5)\rangle & \langle(0.5,0.3,0.8);(0.4,0.5,0.9)\rangle & \langle(0.2,0.5,1.0);(0.0,0.4,0.5)\rangle \\ \langle(0.2,0.9,0.9);(0.0,0.4,0.5)\rangle & \langle(0.1,0.8,0.7);(0.0,0.5,0.6)\rangle & \langle(0.3,0.8,0.8);(0.5,0.8,0.3)\rangle \end{bmatrix}$$

Definition 16 (Score Function) Let $P = [P_{ij}]$be the NHSRM of order $m \times n$ where $[P_{ij}] = \left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right]$ then the score function is defined as

$$S(TP_n) = \frac{1}{3}\left[\left(2 + \left(\frac{\underline{T}_{ij}^{P} + \bar{T}_{ij}^{P}}{2}\right) - \left(\frac{\underline{I}_{ij}^{P} + \bar{I}_{ij}^{P}}{2}\right) - \left(\frac{\underline{F}_{ij}^{P} + \bar{F}_{ij}^{P}}{2}\right)\right)\right]$$

Proposition 1

Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order, where $[P_{ij}] = \left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}_{ij}^{Q}, \underline{I}_{ij}^{Q}, \underline{F}_{ij}^{Q}\right); \left(\bar{T}_{ij}^{Q}, \bar{I}_{ij}^{Q}, \bar{F}_{ij}^{Q}\right)\right]$ for two scalars $\alpha, \beta \in [0,1]$, then

(i) $\alpha(P+Q) = \alpha P + \alpha Q$

(ii) $\alpha(\beta P) = (\alpha\beta)P$

Proof

(i) $\alpha(P+Q)$

$= \alpha\left(\left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right] + \left[\left(\underline{T}_{ij}^{Q}, \underline{I}_{ij}^{Q}, \underline{F}_{ij}^{Q}\right); \left(\bar{T}_{ij}^{Q}, \bar{I}_{ij}^{Q}, \bar{F}_{ij}^{Q}\right)\right]\right)$

$= \left[\left(\alpha\underline{T}_{ij}^{P}, \alpha\underline{I}_{ij}^{P}, \alpha\underline{F}_{ij}^{P}\right); \left(\alpha\bar{T}_{ij}^{P}, \alpha\bar{I}_{ij}^{P}, \alpha\bar{F}_{ij}^{P}\right)\right] + \left[\left(\alpha\underline{T}_{ij}^{Q}, \alpha\underline{I}_{ij}^{Q}, \alpha\underline{F}_{ij}^{Q}\right); \left(\alpha\bar{T}_{ij}^{Q}, \alpha\bar{I}_{ij}^{Q}, \alpha\bar{F}_{ij}^{Q}\right)\right]$

$= \alpha\left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right] + \alpha\left[\left(\underline{T}_{ij}^{Q}, \underline{I}_{ij}^{Q}, \underline{F}_{ij}^{Q}\right); \left(\bar{T}_{ij}^{Q}, \bar{I}_{ij}^{Q}, \bar{F}_{ij}^{Q}\right)\right]$

$= \alpha P + \alpha Q$

(ii)$\alpha(\beta P)$

$= \alpha\left[\left(\beta\underline{T}_{ij}^{P}, \beta\underline{I}_{ij}^{P}, \beta\underline{F}_{ij}^{P}\right); \left(\beta\bar{T}_{ij}^{P}, \beta\bar{I}_{ij}^{P}, \beta\bar{F}_{ij}^{P}\right)\right]$

$= \left[\left(\alpha\beta\underline{T}_{ij}^{P}, \alpha\beta\underline{I}_{ij}^{P}, \alpha\beta\underline{F}_{ij}^{P}\right); \left(\alpha\beta\bar{T}_{ij}^{P}, \alpha\beta\bar{I}_{ij}^{P}, \alpha\beta\bar{F}_{ij}^{P}\right)\right]$

$= (\alpha\beta)\left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right]$

$= (\alpha\beta)P$

3.1. *Operations on NHSRM*

In this section, we discuss various operators such as union, intersection, arithmetic mean, weighted arithmetic mean, geometric mean, weighted geometric mean, harmonic mean and weighted harmonic mean on NHSRM.

Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]$. Then

(i) Union

$P \cup Q = \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); \left(\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij}\right)\right]$,

where

$T^R_{ij} = \max(T^P_{ij}, T^Q_{ij})$,

$I^R_{ij} = \dfrac{I^P_{ij} + I^Q_{ij}}{2}$,

$F^R_{ij} = \min(F^P_{ij}, F^Q_{ij})$

(ii) Intersection

$P \cap Q = \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); \left(\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij}\right)\right]$,

where

$T^R_{ij} = \min(T^P_{ij}, T^Q_{ij})$,

$I^R_{ij} = \dfrac{I^P_{ij} + I^Q_{ij}}{2}$,

$F^R_{ij} = \max(F^P_{ij}, F^Q_{ij})$

(iii) Arithmetic Mean

$P \odot_{AM} Q = \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); \left(\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij}\right)\right]$,

where

$T^R_{ij} = \dfrac{T^P_{ij} + T^Q_{ij}}{2}$,

$I^R_{ij} = \dfrac{I^P_{ij} + I^Q_{ij}}{2}$,

$F^R_{ij} = \dfrac{F^P_{ij} + F^Q_{ij}}{2}$

(iv) Weighted Arithmetic Mean

$P \odot_{WAM} Q = \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); \left(\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij}\right)\right]$,

where

$T^R_{ij} = \dfrac{w_1 T^P_{ij} + w_2 T^Q_{ij}}{w_1 + w_2}$,

$I^R_{ij} = \dfrac{w_1 I^P_{ij} + w_2 I^Q_{ij}}{w_1 + w_2}$,

$F^R_{ij} = \dfrac{w_1 F^P_{ij} + w_2 F^Q_{ij}}{w_1 + w_2}$, $0 < w_1 + w_2 \leqslant 1$

(v) Geometric Mean

$P \odot_{GM} Q = \left[\left(\underline{T}_{ij}^{R}, \underline{I}_{ij}^{R}, \underline{F}_{ij}^{R}\right); \left(\bar{T}_{ij}^{R}, \bar{I}_{ij}^{R}, \bar{F}_{ij}^{R}\right)\right]$,

where

$T_{ij}^{R} = \sqrt{T_{ij}^{P}.T_{ij}^{Q}}$,

$I_{ij}^{R} = \sqrt{I_{ij}^{P}.I_{ij}^{Q}}$,

$F_{ij}^{R} = \sqrt{F_{ij}^{P}.F_{ij}^{Q}}$

(vi) Weighted Geometric Mean

$P \odot_{WGM} Q = \left[\left(\underline{T}_{ij}^{R}, \underline{I}_{ij}^{R}, \underline{F}_{ij}^{R}\right); \left(\bar{T}_{ij}^{R}, \bar{I}_{ij}^{R}, \bar{F}_{ij}^{R}\right)\right]$,

where

$T_{ij}^{R} = \sqrt[w_1+w_2]{\left(T_{ij}^{P}\right)^{w_1}.\left(T_{ij}^{Q}\right)^{w_2}}$,

$I_{ij}^{R} = \sqrt[w_1+w_2]{\left(I_{ij}^{P}\right)^{w_1}.\left(I_{ij}^{Q}\right)^{w_2}}$,

$F_{ij}^{R} = \sqrt[w_1+w_2]{\left(F_{ij}^{P}\right)^{w_1}.\left(F_{ij}^{Q}\right)^{w_2}}$, $0 < w_1 + w_2 \leqslant 1$

(vii) Harmonic Mean

$P \odot_{GM} Q = \left[\left(\underline{T}_{ij}^{R}, \underline{I}_{ij}^{R}, \underline{F}_{ij}^{R}\right); \left(\bar{T}_{ij}^{R}, \bar{I}_{ij}^{R}, \bar{F}_{ij}^{R}\right)\right]$,

where

$T_{ij}^{R} = \frac{2T_{ij}^{P}T_{ij}^{Q}}{T_{ij}^{P} + T_{ij}^{Q}}$,

$I_{ij}^{R} = \frac{2I_{ij}^{P}I_{ij}^{Q}}{I_{ij}^{P} + I_{ij}^{Q}}$,

$F_{ij}^{R} = \frac{2F_{ij}^{P}F_{ij}^{Q}}{F_{ij}^{P} + F_{ij}^{Q}}$

(viii) Weighted Harmonic Mean

$P \odot_{WHM} Q = \left[\left(\underline{T}_{ij}^{R}, \underline{I}_{ij}^{R}, \underline{F}_{ij}^{R}\right); \left(\bar{T}_{ij}^{R}, \bar{I}_{ij}^{R}, \bar{F}_{ij}^{R}\right)\right]$,

where

$T_{ij}^{R} = \frac{w_1 + w_2}{\left(w_1/T_{ij}^{P}\right) + \left(w_2/T_{ij}^{Q}\right)}$,

$I_{ij}^{R} = \frac{w_1 + w_2}{\left(w_1/I_{ij}^{P}\right) + \left(w_2/I_{ij}^{Q}\right)}$,

$F_{ij}^{R} = \frac{w_1 + w_2}{\left(w_1/F_{ij}^{P}\right) + \left(w_2/F_{ij}^{Q}\right)}$, $0 < w_1 + w_2 \leqslant 1$

3.2. *Algebraic properties of NHSRM*

In this section, we present the algebraic properties of NHSRM, which are transpose property, commutative property, associative property and distributive property.

Theorem 1 Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$be two NHSRMs with same order, where $[P_{ij}] = \left[\left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\bar{T}_{ij}^{P}, \bar{I}_{ij}^{P}, \bar{F}_{ij}^{P}\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}_{ij}^{Q}, \underline{I}_{ij}^{Q}, \underline{F}_{ij}^{Q}\right); \left(\bar{T}_{ij}^{Q}, \bar{I}_{ij}^{Q}, \bar{F}_{ij}^{Q}\right)\right]$ then

(i) $(P \cup Q)^T = P^T \cup Q^T$

(ii) $(P \cap Q)^T = P^T \cap Q^T$

(iii) $(P \odot_{AM} Q)^T = P^T \odot_{AM} Q^T$

(iv) $(P \odot_{WAM} Q)^T = P^T \odot_{WAM} Q^T$

(v) $(P \odot_{GM} Q)^T = P^T \odot_{GM} Q^T$

(vi) $(P \odot_{WGM} Q)^T = P^T \odot_{WGM} Q^T$

(vii) $(P \odot_{HM} Q)^T = P^T \odot_{HM} P^T$

(viii) $(P \odot_{WHM} Q)^T = P^T \odot_{WHM} Q^T$

Proof

(i) $(P \cup Q)^T$

$$= \left[\left(\max\left(\underline{T}^P_{ij}, \underline{T}^Q_{ij}\right), \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \min\left(\underline{F}^P_{ij}, \underline{F}^Q_{ij}\right)\right); \left(\max\left(\bar{T}^P_{ij}, \bar{T}^Q_{ij}\right), \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \min\left(\bar{F}^P_{ij}, \bar{F}^Q_{ij}\right)\right)\right]^T$$

$$= \left[\left(\max\left(\underline{T}^P_{ji}, \underline{T}^Q_{ji}\right), \frac{\underline{I}^P_{ji} + \underline{I}^Q_{ji}}{2}, \min\left(\underline{F}^P_{ji}, \underline{F}^Q_{ji}\right)\right); \left(\max\left(\bar{T}^P_{ji}, \bar{T}^Q_{ji}\right), \frac{\bar{I}^P_{ji} + \bar{I}^Q_{ji}}{2}, \min\left(\bar{F}^P_{ji}, \bar{F}^Q_{ji}\right)\right)\right]$$

$$= \left[\left(\underline{T}^P_{ji}, \underline{I}^P_{ji}, \underline{F}^P_{ji}\right); \left(\bar{T}^P_{ji}, \bar{I}^P_{ji}, \bar{F}^P_{ji}\right)\right] \cup \left[\left(\underline{T}^Q_{ji}, \underline{I}^Q_{ji}, \underline{F}^Q_{ji}\right); \left(\bar{T}^Q_{ji}, \bar{I}^Q_{ji}, \bar{F}^Q_{ji}\right)\right]$$

$$= \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]^T \cup \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]^T$$

$$= P^T \cup Q^T$$

Similarly the result (ii) can be proved.

(iii) $(P \odot_{AM} Q)^T$

$$= \left[\left(\frac{\underline{T}^P_{ij} + \underline{T}^Q_{ij}}{2}, \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \frac{\underline{F}^P_{ij} + \underline{F}^Q_{ij}}{2}\right); \left(\frac{\bar{T}^P_{ij} + \bar{T}^Q_{ij}}{2}, \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \frac{\bar{F}^P_{ij} + \bar{F}^Q_{ij}}{2}\right)\right]^T$$

$$= \left[\left(\frac{\underline{T}^P_{ji} + \underline{T}^Q_{ji}}{2}, \frac{\underline{I}^P_{ji} + \underline{I}^Q_{ji}}{2}, \frac{\underline{F}^P_{ji} + \underline{F}^Q_{ji}}{2}\right); \left(\frac{\bar{T}^P_{ji} + \bar{T}^Q_{ji}}{2}, \frac{\bar{I}^P_{ji} + \bar{I}^Q_{ji}}{2}, \frac{\bar{F}^P_{ji} + \bar{F}^Q_{ji}}{2}\right)\right]$$

$$= \left[\left(\underline{T}^P_{ji}, \underline{I}^P_{ji}, \underline{F}^P_{ji}\right); \left(\bar{T}^P_{ji}, \bar{I}^P_{ji}, \bar{F}^P_{ji}\right)\right] \odot_{AM} \left[\left(\underline{T}^Q_{ji}, \underline{I}^Q_{ji}, \underline{F}^Q_{ji}\right); \left(\bar{T}^Q_{ji}, \bar{I}^Q_{ji}, \bar{F}^Q_{ji}\right)\right]$$

$$= \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]^T \odot_{AM} \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]^T$$

$$= P^T \odot_{AM} Q^T$$

(iv) $(P \odot_{WAM} Q)^T$

$$= \left[\left(\frac{w_1\underline{T}^P_{ij} + w_2\underline{T}^Q_{ij}}{w_1 + w_2}, \frac{w_1\underline{I}^P_{ij} + w_2\underline{I}^Q_{ij}}{w_1 + w_2}, \frac{w_1\underline{F}^P_{ij} + w_2\underline{F}^Q_{ij}}{w_1 + w_2}\right)\right.$$

$$\left.; \left(\frac{w_1\bar{T}^P_{ij} + w_2\bar{T}^Q_{ij}}{w_1 + w_2}, \frac{w_1\bar{I}^P_{ij} + w_2\bar{I}^Q_{ij}}{w_1 + w_2}, \frac{w_1\bar{F}^P_{ij} + w_2\bar{F}^Q_{ij}}{w_1 + w_2}\right)\right]^T$$

$$= \left[\left(\frac{w_1 \underline{T}^P_{ji} + w_2 \underline{T}^Q_{ji}}{w_1 + w_2}, \frac{w_1 \underline{I}^P_{ji} + w_2 \underline{I}^Q_{ji}}{w_1 + w_2}, \frac{w_1 \underline{F}^P_{ji} + w_2 \underline{F}^Q_{ji}}{w_1 + w_2} \right) \right.$$
$$\left. ; \left(\frac{w_1 \bar{T}^P_{ji} + w_2 \bar{T}^Q_{ji}}{w_1 + w_2}, \frac{w_1 \bar{I}^P_{ji} + w_2 \bar{I}^Q_{ji}}{w_1 + w_2}, \frac{w_1 \bar{F}^P_{ji} + w_2 \bar{F}^Q_{ji}}{w_1 + w_2} \right) \right]$$
$$= \left[\left(\underline{T}^P_{ji}, \underline{I}^P_{ji}, \underline{F}^P_{ji} \right); \left(\bar{T}^P_{ji}, \bar{I}^P_{ji}, \bar{F}^P_{ji} \right) \right] \odot_{WAM} \left[\left(\underline{T}^Q_{ji}, \underline{I}^Q_{ji}, \underline{F}^Q_{ji} \right); \left(\bar{T}^Q_{ji}, \bar{I}^Q_{ji}, \bar{F}^Q_{ji} \right) \right]$$
$$= \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij} \right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij} \right) \right]^T \odot_{WAM} \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij} \right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij} \right) \right]^T$$
$$= P^T \odot_{WAM} Q^T$$

Similarly the results (v) – (viii) can be proved.

Theorem 2 Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$be two NHSRMs with same order, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij} \right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij} \right) \right]$ and $[Q_{ij}] = \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij} \right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij} \right) \right]$ then

(i) $(P \cup Q)^c = P^c \cap Q^c$
(ii) $(P \cap Q)^c = P^c \cup Q^c$
(iii) $(P \odot_{AM} Q)^c = P^c \odot_{AM} Q^c$
(iv) $(P \odot_{WAM} Q)^c = P^c \odot_{WAM} Q^c$
(v) $(P \odot_{GM} Q)^c = P^c \odot_{GM} Q^c$
(vi) $(P \odot_{WGM} Q)^c = P^c \odot_{WGM} Q^c$
(vii) $(P \odot_{HM} Q)^c = P^c \odot_{HM} Q^c$
(viii) $(P \odot_{WHM} P)^c = P^c \odot_{WHM} Q^c$

Proof

(i) $(P \cup Q)^c$

$$= \left[\left(\max\left(\underline{T}^P_{ij}, \underline{T}^Q_{ij} \right), \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \min\left(\underline{F}^P_{ij}, \underline{F}^Q_{ij} \right) \right); \left(\max\left(\bar{T}^P_{ij}, \bar{T}^Q_{ij} \right), \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \min\left(\bar{F}^P_{ij}, \bar{F}^Q_{ij} \right) \right) \right]^c$$

$$= \left[\left(\min\left(\underline{F}^P_{ij}, \underline{F}^Q_{ij} \right), 1 - \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \max\left(\underline{T}^P_{ij}, \underline{T}^Q_{ij} \right) \right); \right.$$
$$\left. \left(\min\left(\bar{F}^P_{ij}, \bar{F}^Q_{ij} \right), 1 - \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \max\left(\bar{T}^P_{ij}, \bar{T}^Q_{ij} \right) \right) \right]$$
$$= \left[\left(\underline{F}^P_{ij}, 1 - \underline{I}^P_{ij}, \underline{T}^P_{ij} \right); \left(\bar{F}^Q_{ij}, 1 - \bar{I}^Q_{ij}, \bar{T}^Q_{ij} \right) \right] \cap \left[\left(\underline{F}^P_{ij}, 1 - \underline{I}^P_{ij}, \underline{T}^P_{ij} \right); \left(\bar{F}^Q_{ij}, 1 - \bar{I}^Q_{ji}, \bar{T}^Q_{ji} \right) \right]$$
$$= \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij} \right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij} \right) \right]^c \cap \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij} \right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij} \right) \right]^c$$
$$= P^c \cap Q^c$$

In similar manner, the result (ii) can be proved.

(iii) $(P \odot_{AM} Q)^c$

$$= \left[\left(\frac{\underline{T}_{ij}^P + \underline{T}_{ij}^Q}{2}, \frac{\underline{I}_{ij}^P + \underline{I}_{ij}^Q}{2}, \frac{\underline{F}_{ij}^P + \underline{F}_{ij}^Q}{2}\right); \left(\frac{\bar{T}_{ij}^P + \bar{T}_{ij}^Q}{2}, \frac{\bar{I}_{ij}^P + \bar{I}_{ij}^Q}{2}, \frac{\bar{F}_{ij}^P + \bar{F}_{ij}^Q}{2}\right)\right]^c$$
$$= \left[\left(\frac{\underline{F}_{ij}^P + \underline{F}_{ij}^Q}{2}, 1 - \frac{\underline{I}_{ij}^P + \underline{I}_{ij}^Q}{2}, \frac{\underline{T}_{ij}^P + \underline{T}_{ij}^Q}{2}\right); \left(\frac{\bar{F}_{ij}^P + \bar{F}_{ij}^Q}{2}, 1 - \frac{\bar{I}_{ij}^P + \bar{I}_{ij}^Q}{2}, \frac{\bar{T}_{ij}^P + \bar{T}_{ij}^Q}{2}\right)\right]$$
$$= \left[\left(\underline{F}_{ij}^P, 1 - \underline{I}_{ij}^P, \underline{T}_{ij}^P\right); \left(\bar{F}_{ij}^Q, 1 - \bar{I}_{ij}^Q, \bar{T}_{ij}^Q\right)\right] \odot_{AM} \left[\left(\underline{F}_{ij}^P, 1 - \underline{I}_{ij}^P, \underline{T}_{ij}^P\right); \left(\bar{F}_{ij}^Q, 1 - \bar{I}_{ji}^Q, \bar{T}_{ji}^Q\right)\right]$$
$$= \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P\right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P\right)\right]^c \odot_{AM} \left[\left(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q\right); \left(\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q\right)\right]^c = P^c \odot_{AM} Q^c$$

Similarly the results (iv) – (viii) can be proved.

Theorem 3 Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$be two NHSRMs with same order, where $[P_{ij}] = \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P\right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P\right)\right]$ and $[Q_{ij}] = \left[\left(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q\right); \left(\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q\right)\right]$ then

(i) $P \cup Q = Q \cup P$
(ii) $P \cap Q = Q \cap P$
(iii) $P \odot_{AM} Q = Q \odot_{AM} P$
(iv) $P \odot_{WAM} Q = Q \odot_{WAM} P$
(v) $P \odot_{GM} Q = Q \odot_{GM} P$
(vi) $P \odot_{WGM} Q = Q \odot_{WGM} P$
(vii) $P \odot_{HM} Q = Q \odot_{HM} P$
(viii) $P \odot_{WHM} Q = Q \odot_{WHM} P$

Proof

(i) $P \cup Q$

$$= \left[\left(\max\left(\underline{T}_{ij}^P, \underline{T}_{ij}^Q\right), \frac{\underline{I}_{ij}^P + \underline{I}_{ij}^Q}{2}, \min\left(\underline{F}_{ij}^P, \underline{F}_{ij}^Q\right)\right); \left(\max\left(\bar{T}_{ij}^P, \bar{T}_{ij}^Q\right), \frac{\bar{I}_{ij}^P + \bar{I}_{ij}^Q}{2}, \min\left(\bar{F}_{ij}^P, \bar{F}_{ij}^Q\right)\right)\right]$$
$$= \left[\left(\max\left(\underline{T}_{ij}^Q, \underline{T}_{ij}^P\right), \frac{\underline{I}_{ij}^Q + \underline{I}_{ij}^P}{2}, \min\left(\underline{F}_{ij}^Q, \underline{F}_{ij}^P\right)\right); \left(\max\left(\bar{T}_{ij}^Q, \bar{T}_{ij}^P\right), \frac{\bar{I}_{ij}^Q + \bar{I}_{ij}^P}{2}, \min\left(\bar{F}_{ij}^Q, \bar{F}_{ij}^P\right)\right)\right]$$
$$= \left[\left(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q\right); \left(\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q\right)\right] \cup \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P\right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P\right)\right] = Q \cup P$$

In similar manner, the results (ii) – (iv) can be proved.

(v) $P \odot_{GM} Q$

$$= \left[\left(\sqrt{\underline{T}_{ij}^P . \underline{T}_{ij}^Q}, \sqrt{\underline{I}_{ij}^P . \underline{I}_{ij}^Q}, \sqrt{\underline{F}_{ij}^P . \underline{F}_{ij}^Q}\right); \left(\sqrt{\bar{T}_{ij}^P . \bar{T}_{ij}^Q}, \sqrt{\bar{I}_{ij}^P . \bar{I}_{ij}^Q}, \sqrt{\bar{F}_{ij}^P . \bar{F}_{ij}^Q}\right)\right]$$
$$= \left[\left(\sqrt{\underline{T}_{ij}^Q . \underline{T}_{ij}^P}, \sqrt{\underline{I}_{ij}^Q . \underline{I}_{ij}^P}, \sqrt{\underline{F}_{ij}^Q . \underline{F}_{ij}^P}\right); \left(\sqrt{\bar{T}_{ij}^Q . \bar{T}_{ij}^P}, \sqrt{\bar{I}_{ij}^Q . \bar{I}_{ij}^P}, \sqrt{\bar{F}_{ij}^Q . \bar{F}_{ij}^P}\right)\right]$$
$$= \left[\left(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q\right); \left(\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q\right)\right] \odot_{GM} \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P\right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P\right)\right] = Q \odot_{GM} P$$

In similar manner, the results (vi) – (viii) can be proved.

Theorem 4 Let $P = [P_{ij}]$, $Q = [Q_{ij}]$and $R = [R_{ij}]$ be three NHSRMs with same order, where $[P_{ij}] = \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij}\right)\right]$, $[Q_{ij}] = \left[\left(\underline{T}^Q_{ij}, \underline{I}^Q_{ij}, \underline{F}^Q_{ij}\right); \left(\bar{T}^Q_{ij}, \bar{I}^Q_{ij}, \bar{F}^Q_{ij}\right)\right]$ and $[R_{ij}] = \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); \left(\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij}\right)\right]$ then

(i) $(P \cup Q) \cup R = P \cup (Q \cup R)$

(ii) $(P \cap Q) \cap R = P \cap (Q \cap R)$

(iii) $(P \odot_{AM} Q) \odot_{AM} R \neq P \odot_{AM} (Q \odot_{AM} R)$

(iv) $(P \odot_{GM} Q) \odot_{GM} R \neq P \odot_{GM} (Q \odot_{GM} R)$

(v) $(P \odot_{HM} Q) \odot_{HM} R \neq P \odot_{HM} (Q \odot_{HM} R)$

Proof

(i) $(P \cup Q) \cup R$

$$= \left[\left(\max\left(\underline{T}^P_{ij}, \underline{T}^Q_{ij}\right), \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \min\left(\underline{F}^P_{ij}, \underline{F}^Q_{ij}\right)\right); \left(\max\left(\bar{T}^P_{ij}, \bar{T}^Q_{ij}\right), \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \min\left(\bar{F}^P_{ij}, \bar{F}^Q_{ij}\right)\right)\right]$$

$$\cup \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); (\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij})\right]$$

$$= \left[\left(\max\left(\underline{T}^P_{ij}, \underline{T}^Q_{ij}, \underline{T}^R_{ij}\right), \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij} + \underline{I}^R_{ij}}{3}, \min\left(\underline{F}^P_{ij}, \underline{F}^Q_{ij}, \underline{F}^R_{ij}\right)\right);\right.$$

$$\left.\left(\max\left(\bar{T}^P_{ij}, \bar{T}^Q_{ij}, \bar{T}^R_{ij}\right), \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij} + \bar{I}^R_{ij}}{3}, \min\left(\bar{F}^P_{ij}, \bar{F}^Q_{ij}, \bar{F}^R_{ij}\right)\right)\right]$$

$$= \left[\left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); (\bar{T}^P_{ij}, \bar{I}^P_{ij}, \bar{F}^P_{ij})\right] \cup \left[\left(\max\left(\underline{T}^Q_{ij}, \underline{T}^R_{ij}\right), \frac{\underline{I}^Q_{ij} + \underline{I}^R_{ij}}{2}, \min\left(\underline{F}^Q_{ij}, \underline{F}^R_{ij}\right)\right);\right.$$

$$\left.\left(\max\left(\bar{T}^Q_{ij}, \bar{T}^R_{ij}\right), \frac{\bar{I}^Q_{ij} + \bar{I}^R_{ij}}{2}, \min\left(\bar{F}^Q_{ij}, \bar{F}^R_{ij}\right)\right)\right]$$

$= P \cup (Q \cup R)$

In similar manner, the result (ii) can be proved.

(iii) Consider LHS,

$$(P \odot_{AM} Q) \odot_{AM} R = \left[\left(\frac{\underline{T}^P_{ij} + \underline{T}^Q_{ij}}{2}, \frac{\underline{I}^P_{ij} + \underline{I}^Q_{ij}}{2}, \frac{\underline{F}^P_{ij} + \underline{F}^Q_{ij}}{2}\right); \left(\frac{\bar{T}^P_{ij} + \bar{T}^Q_{ij}}{2}, \frac{\bar{I}^P_{ij} + \bar{I}^Q_{ij}}{2}, \frac{\bar{F}^P_{ij} + \bar{F}^Q_{ij}}{2}\right)\right]$$

$$\odot_{AM} \left[\left(\underline{T}^R_{ij}, \underline{I}^R_{ij}, \underline{F}^R_{ij}\right); (\bar{T}^R_{ij}, \bar{I}^R_{ij}, \bar{F}^R_{ij})\right]$$

$$= \left[\left(\frac{\left(\underline{T}^P_{ij} + \underline{T}^Q_{ij}\right)/2 + \underline{T}^R_{ij}}{2}, \frac{\left(\underline{I}^P_{ij} + \underline{I}^Q_{ij}\right)/2 + \underline{I}^R_{ij}}{2}, \frac{\left(\underline{F}^P_{ij} + \underline{F}^Q_{ij}\right)/2 + \underline{F}^R_{ij}}{2}\right);\right.$$

$$\left.\left(\frac{\left(\bar{T}^P_{ij} + \bar{T}^Q_{ij}\right)/2 + \bar{T}^R_{ij}}{2}, \frac{\left(\bar{I}^P_{ij} + \bar{I}^Q_{ij}\right)/2 + \bar{I}^R_{ij}}{2}, \frac{\left(\bar{F}^P_{ij} + \bar{F}^Q_{ij}\right)/2 + \bar{F}^R_{ij}}{2}\right)\right] \quad (1)$$

Consider RHS of (iii)

$$P\odot_{AM}(Q\odot_{AM}R) = \left[\left(\underline{T}^{P}_{ij}, \underline{I}^{P}_{ij}, \underline{F}^{P}_{ij}\right); (\bar{T}^{P}_{ij}, \bar{I}^{P}_{ij}, \bar{F}^{P}_{ij})\right] \odot_{AM} \left[\left(\frac{\underline{T}^{Q}_{ij}+\underline{T}^{R}_{ij}}{2}, \frac{\underline{I}^{Q}_{ij}+\underline{I}^{R}_{ij}}{2}, \frac{\underline{F}^{Q}_{ij}+\underline{F}^{R}_{ij}}{2}\right); \left(\frac{\bar{T}^{Q}_{ij}+\bar{T}^{R}_{ij}}{2}, \frac{\bar{I}^{Q}_{ij}+\bar{I}^{R}_{ij}}{2}, \frac{\bar{F}^{Q}_{ij}+\bar{F}^{R}_{ij}}{2}\right)\right]$$

$$= \left[\left(\frac{\underline{T}^{P}_{ij}+\left(\underline{T}^{Q}_{ij}+\underline{T}^{R}_{ij}\right)/2}{2}, \frac{\underline{I}^{P}_{ij}+\left(\underline{I}^{Q}_{ij}+\underline{I}^{R}_{ij}\right)/2}{2}, \frac{\underline{F}^{P}_{ij}+\left(\underline{F}^{Q}_{ij}+\underline{F}^{R}_{ij}\right)/2}{2}\right); \left(\frac{\bar{T}^{P}_{ij}+\left(\bar{T}^{Q}_{ij}+\bar{T}^{R}_{ij}\right)/2}{2}, \frac{\bar{I}^{P}_{ij}+\left(\bar{I}^{Q}_{ij}+\bar{I}^{R}_{ij}\right)/2}{2}, \frac{\bar{F}^{P}_{ij}+\left(\bar{F}^{Q}_{ij}+\bar{F}^{R}_{ij}\right)/2}{2}\right)\right] \quad (2)$$

From Equation (1) & Equation (2), $(P\odot_{AM}Q)\odot_{AM}R \neq P\odot_{AM}(Q\odot_{AM}R)$

In similar manner, the results (iv) and (v) can be proved.

Theorem 5 Let $P = [P_{ij}]$, $Q = [Q_{ij}]$and $R = [R_{ij}]$ be three NHSRMs with same order, where $[P_{ij}] = \left[\left(\underline{T}^{P}_{ij}, \underline{I}^{P}_{ij}, \underline{F}^{P}_{ij}\right); \left(\bar{T}^{P}_{ij}, \bar{I}^{P}_{ij}, \bar{F}^{P}_{ij}\right)\right]$, $[Q_{ij}] = \left[\left(\underline{T}^{Q}_{ij}, \underline{I}^{Q}_{ij}, \underline{F}^{Q}_{ij}\right); \left(\bar{T}^{Q}_{ij}, \bar{I}^{Q}_{ij}, \bar{F}^{Q}_{ij}\right)\right]$ and $[R_{ij}] = \left[\left(\underline{T}^{R}_{ij}, \underline{I}^{R}_{ij}, \underline{F}^{R}_{ij}\right); \left(\bar{T}^{R}_{ij}, \bar{I}^{R}_{ij}, \bar{F}^{R}_{ij}\right)\right]$ then

(i) $P \cup (Q\odot_{AM}R) = (P \cup Q)\odot_{AM}(P \cup R)$

(ii) $P \cap (Q\odot_{AM}R) = (P \cap Q)\odot_{AM}(P \cap R)$

(iii) $(P\odot_{AM}Q) \cup R = (P \cup R)\odot_{AM}(Q \cup R)$

(iv) $(P\odot_{AM}Q) \cap R = (P \cap R)\odot_{AM}(Q \cap R)$

Proof

(i) $P \cup (Q\odot_{AM}R)$

$$= \left[\left(\underline{T}^{P}_{ij}, \underline{I}^{P}_{ij}, \underline{F}^{P}_{ij}\right); (\bar{T}^{P}_{ij}, \bar{I}^{P}_{ij}, \bar{F}^{P}_{ij})\right] \cup \left[\left(\left(\frac{\underline{T}^{Q}_{ij}+\underline{T}^{R}_{ij}}{2}\right), \left(\frac{\underline{I}^{Q}_{ij}+\underline{I}^{R}_{ij}}{2}\right), \left(\frac{\underline{F}^{Q}_{ij}+\underline{F}^{R}_{ij}}{2}\right)\right); \left(\left(\frac{\bar{T}^{Q}_{ij}+\bar{T}^{R}_{ij}}{2}\right), \left(\frac{\bar{I}^{Q}_{ij}+\bar{I}^{R}_{ij}}{2}\right), \left(\frac{\bar{F}^{Q}_{ij}+\bar{F}^{R}_{ij}}{2}\right)\right)\right]$$

$$= \left[\left(\max\left(\underline{T}^{P}_{ij}, \frac{\underline{T}^{Q}_{ij}+\underline{T}^{R}_{ij}}{2}\right), \left(\frac{\underline{I}^{P}_{ij}+(\underline{I}^{Q}_{ij}+\underline{I}^{R}_{ij})/2}{2}\right), \min\left(\underline{F}^{P}_{ij}, \frac{\underline{F}^{Q}_{ij}+\underline{F}^{R}_{ij}}{2}\right)\right); \left(\max\left(\bar{T}^{P}_{ij}, \frac{\bar{T}^{Q}_{ij}+\bar{T}^{R}_{ij}}{2}\right), \left(\frac{\bar{I}^{P}_{ij}+(\bar{I}^{Q}_{ij}+\bar{I}^{R}_{ij})/2}{2}\right), \min\left(\bar{F}^{P}_{ij}, \frac{\bar{F}^{Q}_{ij}+\bar{F}^{R}_{ij}}{2}\right)\right)\right]$$

$$= \Bigg[\Bigg(\max\Big(\frac{\underline{T}^P_{ij}+\underline{T}^Q_{ij}}{2},\frac{\underline{T}^P_{ij}+\underline{T}^R_{ij}}{2}\Big),\Big(\frac{(\underline{I}^P_{ij}+\underline{I}^Q_{ij})/2+(\underline{I}^P_{ij}+\underline{I}^R_{ij})/2}{2}\Big),\min\Big(\frac{\underline{F}^P_{ij}+\underline{F}^Q_{ij}}{2},\frac{\underline{F}^P_{ij}+\underline{F}^R_{ij}}{2}\Big)\Bigg);$$

$$\Bigg(\max\Big(\frac{\bar{T}^P_{ij}+\bar{T}^Q_{ij}}{2},\frac{\bar{T}^P_{ij}+\bar{T}^R_{ij}}{2}\Big),\Big(\frac{(\bar{I}^P_{ij}+\bar{I}^Q_{ij})/2+(\bar{I}^P_{ij}+\bar{I}^R_{ij})/2}{2}\Big),\min\Big(\frac{\bar{F}^P_{ij}+\bar{F}^Q_{ij}}{2},\frac{\bar{F}^P_{ij}+\bar{F}^R_{ij}}{2}\Big)\Bigg)\Bigg]$$

$$= \Bigg[\Bigg(\Bigg(\max\Big(\underline{T}^P_{ij},\underline{T}^Q_{ij}\Big),\frac{\underline{I}^P_{ij}+\underline{I}^Q_{ij}}{2},\min\Big(\underline{F}^P_{ij},\underline{F}^Q_{ij}\Big)\Bigg);\Bigg(\max\Big(\bar{T}^P_{ij},\bar{T}^Q_{ij}\Big),\frac{\bar{I}^P_{ij}+\bar{I}^Q_{ij}}{2},\min\Big(\bar{F}^P_{ij},\bar{F}^Q_{ij}\Big)\Bigg)\Bigg)$$

$$\odot_{AM}\Bigg(\Bigg(\max\Big(\underline{T}^P_{ij},\underline{T}^R_{ij}\Big),\frac{\underline{I}^P_{ij}+\underline{I}^R_{ij}}{2},\min\Big(\underline{F}^P_{ij},\underline{F}^R_{ij}\Big)\Bigg);\Bigg(\max\left(\bar{T}^P_{ij},\bar{T}^R_{ij}\right),\frac{\bar{I}^P_{ij}+\bar{I}^R_{ij}}{2},\min\left(\bar{F}^P_{ij},\bar{F}^R_{ij}\right)\Bigg)\Bigg)\Bigg]$$

$$= \Bigg[\Bigg(\Big(\underline{T}^P_{ji},\underline{I}^P_{ji},\underline{F}^P_{ji}\Big);\left(\bar{T}^P_{ji},\bar{I}^P_{ji},\bar{F}^P_{ji}\right)\Bigg)\cup\Bigg(\Big(\underline{T}^Q_{ji},\underline{I}^Q_{ji},\underline{F}^Q_{ji}\Big);\Big(\bar{T}^Q_{ji},\bar{I}^Q_{ji},\bar{F}^Q_{ji}\Big)\Bigg)$$

$$\odot_{AM}\Bigg(\Big(\underline{T}^P_{ji},\underline{I}^P_{ji},\underline{F}^P_{ji}\Big);\left(\bar{T}^P_{ji},\bar{I}^P_{ji},\bar{F}^P_{ji}\right)\Bigg)\cup\Bigg(\Big(\underline{T}^R_{ji},\underline{I}^R_{ji},\underline{F}^R_{ji}\Big);\left(\bar{T}^R_{ji},\bar{I}^R_{ji},\bar{F}^R_{ji}\right)\Bigg)\Bigg]$$

$$= (P\cup Q)\odot_{AM}(P\cup R)$$

In similar manner, the results (ii) – (iv) can be proved.

4. NHSRM in Multi-attribute Decision-making using Score Function

In this section, we present the application of NHSRM in decision-making using score function. This proposed work involves evaluating alternatives based on multi-attribute decision-making problem. The approach focuses on determining the most suitable alternative by analyzing and ranking them according to multiple attributes. This is achieved by integrating the NHSRM framework with a score function to ensure a systematic and reliable decision-making process.

Let us consider a multi-attribute decision-making problem with m alternatives $(Z_1, Z_2, ..., Z_m)$ and n attribute $(N_1, N_2, ..., N_n)$. The goal is to evaluate the best score of m alternatives based on the performance of n attributes, using a structured decision-making approach. Each alternatives Z_i, i=1,2,...,m combined with each attribute N_j, j=1,2,...,n, forms a relation represented by NHSRM. The corresponding to these NHSRM values are portrayed in the order $m \times n$. Further, the value matrix is calculated using one of three operators such as weighted arithmetic mean, weighted geometric mean and weighted harmonic mean. The value matrix is a real matrix that adheres to all the properties of NHSRM. The score matrix is then derived from the value matrix using the score function. This function simplifies complex data, such as those represented in neutrosophic sets, into a scalar value for straightforward comparison. By applying the score function within the NHSRM framework, the decision-making process becomes more efficient and transparent. The function aggregates the values of the attribute for each alternative, accounting for both the uncertainty and the partial truth-values inherent in neutrosophic sets. This enables a clearer comparison of alternatives, even when the data is incomplete or imprecise.

4.1. *Algorithm*

(1) Construct NHSRMs as defined in section 3.
(2) Apply weighted arithmetic mean, weighted geometric mean and weighted harmonic mean mentioned in section 3.2.
(3) Calculate the value matrix $V[P,Q]$ by using any one of the three operators in step 2.
(4) Determine the score matrix with the help of the value matrix, by the definition 16.
(5) Compute the aggregate scores matrix across criteria by using $S_{Total}(Z_m) = \sum_{i=1}^{m} S(Z_i)$
(6) Find the Rank alternative from the aggregate score.

Example 10

To illustrate the working of the decision-making problem, two physician P and Q needs to select the best treatment plan (Z_1, Z_2, Z_3) for an asthma patient, considering the patient's symptoms, response to past treatments, and uncertainty in diagnosis arising from the variable factors like environmental triggers and patient adherence.

Alternatives (Treatment plan)

(1) Z_1: Inhaled corticosteroids
(2) Z_2: Combination therapy (Inhaled corticosteroids+ long-acting beta agonists)
(3) Z_3: Biologics for severe asthma

Attribute for decision-making

(1) N_1: Effectiveness – Symptom Control and Frequency of Exacerbations
(2) N_2: Side effects – Short term and Long term
(3) N_3: Cost – High and Low

The two physicians give their valuable opinion about the asthma patient in the form of NHSRM as

$$P = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{ccc} N_1 & N_2 & N_3 \\ \left[\begin{array}{c} \langle(0.9,0.5,0.2);(0.8,0.3,0.1)\rangle \\ \langle(0.8,0.3,0.1);(0.7,0.1,0.1)\rangle \\ \langle(0.7,0.4,0.2);(0.8,0.2,0.1)\rangle \end{array}\right. & \begin{array}{c} \langle(0.8,0.4,0.3);(0.6,0.2,0.1)\rangle \\ \langle(0.7,0.2,0.1);(0.8,0.3,0.1)\rangle \\ \langle(0.7,0.3,0.2);(0.9,0.4,0.2)\rangle \end{array} & \left.\begin{array}{c} \langle(0.7,0.4,0.1);(0.6,0.3,0.2)\rangle \\ \langle(0.9,0.4,0.3);(0.6,0.3,0.1)\rangle \\ \langle(0.8,0.3,0.1);(0.7,0.3,0.1)\rangle \end{array}\right] \end{array}$$

$$Q = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{ccc} N_1 & N_2 & N_3 \\ \left[\begin{array}{c} \langle(0.5,0.2,0.1);(0.4,0.3,0.1)\rangle \\ \langle(0.7,0.6,0.1);(0.8,0.9,0.3)\rangle \\ \langle(0.4,0.3,0.2);(0.7,0.6,0.2)\rangle \end{array}\right. & \begin{array}{c} \langle(0.8,0.6,0.2);(0.7,0.3,0.2)\rangle \\ \langle(1.0,0.4,0.1);(0.9,0.2,0.3)\rangle \\ \langle(0.7,0.5,0.2);(0.8,0.4,0.1)\rangle \end{array} & \left.\begin{array}{c} \langle(0.7,0.6,0.3);(1.0,0.4,0.2)\rangle \\ \langle(0.8,0.6,0.3);(0.7,0.5,0.1)\rangle \\ \langle(0.9,0.3,0.2);(0.5,0.3,0.2)\rangle \end{array}\right] \end{array}$$

Weighted Arithmetic Mean

Using section 3.2 (iv), we obtained the value matrix

$$V[P,Q] = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{ccc} N_1 & N_2 & N_3 \\ \left[\begin{array}{c} \left\langle \begin{array}{c} (0.7, 0.35, 0.1); \\ (0.6, 0.3, 0.1) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.75, 0.45, 0.1); \\ (0.75, 0.5, 0.2) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.55, 0.35, 0.2); \\ (0.75, 0.4, 0.15) \end{array} \right\rangle \end{array} \right. & \begin{array}{c} \left\langle \begin{array}{c} (0.8, 0.5, 0.25); \\ (0.65, 0.25, 0.15) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.85, 0.3, 0.1); \\ (0.85, 0.25, 0.2) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.7, 0.4, 0.2); \\ (0.85, 0.4, 0.15) \end{array} \right\rangle \end{array} & \left. \begin{array}{c} \left\langle \begin{array}{c} (0.7, 0.5, 0.05); \\ (0.8, 0.35, 0.2) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.85, 0.5, 0.3); \\ (0.65, 0.4, 0.1) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.85, 0.3, 0.15); \\ (0.6, 0.3, 0.15) \end{array} \right\rangle \end{array} \right] \end{array}$$

By using Definition 16, the score matrix is derived from the above value matrix

$$S(Z_m) = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \\ \begin{bmatrix} 0.74 & 0.72 & 0.73 \\ 0.71 & 0.81 & 0.70 \\ 0.70 & 0.73 & 0.76 \end{bmatrix} \end{array}$$

Now compute the aggregate score matrix by adding the entires of the score matrix

$$Total\,Score = \begin{bmatrix} 2.19 \\ 2.22 \\ 2.19 \end{bmatrix}$$

Weighted Geometric Mean

Using section 3.2 (vi), we obtained the value matrix

$$V[P,Q] = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{ccc} N_1 & N_2 & N_3 \\ \left[\begin{array}{c} \left\langle \begin{array}{c} (0.82, 0.56, 0.38); \\ (0.75, 0.55, 0.32) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.87, 0.65, 0.32); \\ (0.87, 0.55, 0.42) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.73, 0.59, 0.45); \\ (0.87, 0.59, 0.38) \end{array} \right\rangle \end{array} \right. & \begin{array}{c} \left\langle \begin{array}{c} (0.89, 0.7, 0.49); \\ (0.81, 0.49, 0.38) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.91, 0.53, 0.32); \\ (0.92, 0.49, 0.42) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.84, 0.62, 0.45); \\ (0.92, 0.63, 0.38) \end{array} \right\rangle \end{array} & \left. \begin{array}{c} \left\langle \begin{array}{c} (0.84, 0.7, 0.42); \\ (0.88, 0.59, 0.45) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.92, 0.7, 0.55); \\ (0.81, 0.62, 0.32) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.92, 0.55, 0.38); \\ (0.77, 0.55, 0.38) \end{array} \right\rangle \end{array} \right] \end{array}$$

The score matrix is constructed from the above Value matrix, as per Definition 16,

$$S(Z_m) = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.61 & 0.59 \\ 0.63 & 0.68 & 0.59 \\ 0.60 & 0.61 & 0.64 \end{bmatrix} \end{array}$$

Now compute the aggregate score matrix by adding the entires of the score matrix

$$Total\,Score = \begin{bmatrix} 1.83 \\ 1.90 \\ 1.85 \end{bmatrix}$$

Weighted Harmonic Mean

Using section 3.2 (viii), we obtained the value matrix

$$V[P,Q] = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \\ \begin{bmatrix} \left\langle \begin{array}{c} (0.64, 0.29, 0.13); \\ (0.53, 0.3, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.8, 0.48, 0.24); \\ (0.65, 0.25, 0.13) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.7, 0.48, 0.15); \\ (0.75, 0.34, 0.2) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.75, 0.4, 0.1); \\ (0.75, 0.18, 0.15) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.82, 0.27, 0.1); \\ (0.85, 0.24, 0.15) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.85, 0.48, 0.3); \\ (0.65, 0.38, 0.1) \end{array} \right\rangle \\ \left\langle \begin{array}{c} (0.51, 0.35, 0.2); \\ (0.75, 0.3, 0.13) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.7, 0.38, 0.2); \\ (0.85, 0.4, 0.13) \end{array} \right\rangle & \left\langle \begin{array}{c} (0.85, 0.3, 0.13); \\ (0.58, 0.3, 0.13) \end{array} \right\rangle \end{bmatrix} \end{array}$$

By using Definition 16, the score matrix is obtained from the above value matrix

$$S(Z_m) = \begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \\ \begin{bmatrix} 0.73 & 0.73 & 0.71 \\ 0.78 & 0.82 & 0.71 \\ 0.71 & 0.74 & 0.76 \end{bmatrix} \end{array}$$

Now compute the aggregate score matrix by adding the entires of the score matrix

$$Total\,Score = \begin{bmatrix} 2.16 \\ 2.30 \\ 2.21 \end{bmatrix}$$

5. **Result and Comparative Analysis**

We proposed an algorithm for real world problems and result are compared with those of the algorithm using three operators (weighted arithmetic mean, weighted geometric mean and weighted harmonic mean) on NHSRM. The comparative study of the graphical representation of the ranking of the proposed algorithm for three operators are given in Figure 1. We accrued the result $Z_2 > Z_3 > Z_1$ and the maximum total score is obtained in the Combination therapy (Z_2), which is identified as the best treatment plan for the asthma patients, offering the highest effectiveness, controllable side effects and reasonable cost.

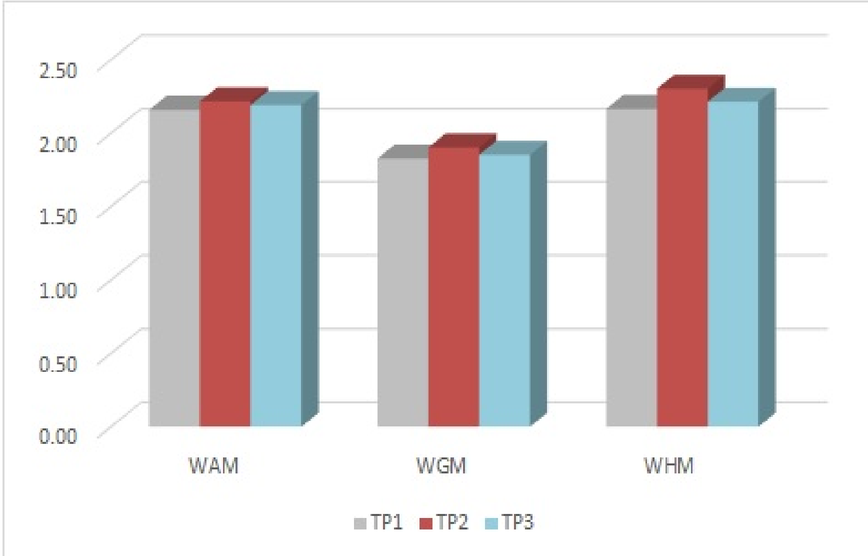

Figure 1. Comparison of score values between WAM, WGM & WHM

6. Conclusion and Future Work

In this study, a novel concept of NHSRM is introduced, incorporating the fundamental operators such as union, intersection, arithmetic mean, geometric mean and harmonic mean which are applied to explore various properties. Moreover, the concept of the score function is proposed to enhance the effectiveness and applicability of the proposed work. The NHSRM approach is also applied to multi-attribute decision-making problems, where the proposed score matrix is derived using three different operators - weighted arithmetic mean, weighted geometric mean, and weighted harmonic mean. Finally, the results were compared with outcomes from a numerical study on decision-making problems to validate the performance and robustness of the proposed NHSRM approach.

In the future, this framework can be extended and integrated with other advanced decision-making techniques, such as grey system theory, which addresses uncertainty in complex systems, machine learning algorithms that can provide adaptive and data-driven insights, and hybrid models that combine the strengths of multiple approaches. This integration could further enhance the accuracy, flexibility, and applicability of the framework, allowing it to tackle a wider range of real-world problems in dynamic and uncertain environments.

References

[1] Al-Quran, A.; Hassan, N., & Marei, E. (2019). A novel approach to neutrosophic soft rough set under uncertainty. Symmetry, 11(3), 384. https://doi.org/10.3390/sym11030384

[2] Boobalan, J., & Sriram, S. (2014). Arithmetic operations on intuitionistic fuzzy matrices. In Proceedings of the International Conference On Mathematical Sciences, Sathyabama University (pp. 484-487).

[3] Borah, M. J.; Neog, T. J., & Sut, D. K. (2012). Fuzzy soft matrix theory and its decision making. International Journal of Modern Engineering Research, 2(2), 121-127.

[4] Broumi, S.& Smarandache, F. (2013). Intuitionistic Neutrosophic soft set. Infinite Study, 130-140. https://doi.org/10.48550/arXiv.1311.3562

[5] Cagman, N., & Enginoglu, S. (2012). Fuzzy soft matrix theory and its application in decision making. Iranian Journal of Fuzzy Systems, 9(1), 109-119. https://doi.org/10.22111/ijfs.2012.229

[6] Chetia, B., & Das, P. K. (2012). Some results of intuitionistic fuzzy soft matrix theory. Advances in applied science research, 3(1), 412-423.

[7] Das, M., Mohanty, D., & Parida, K. C. (2021). On the Neutrosophic soft set with rough set theory. Soft Computing, 25(21), 13365-13376. https://doi.org/10.1007/s00500-021-06089-2

[8] Jafar, M. N., & Saeed, M. (2022). Matrix Theory for Neutrosophic Hypersoft Set and Applications in Multiattributive Multicriteria Decision-Making Problems. Journal of Mathematics, 6666408. https://doi.org/10.1155/2022/6666408

[9] Jafar, M. N., Saeed, M., Haseeb, M., & Habib, A. (2021). Matrix theory for intuitionistic fuzzy hypersoft sets and its application in multi-attributive decision-making problems. Theory and Application of Hypersoft Set, 65. https://doi.org/10.5281/zenodo.4787683

[10] Jayasudha, J., & Raghavi, S. (2024). Some Operations on Neutrosophic Hypersoft Matrices and Their Applications. Neutrosophic Systems with Applications, 21, 46-62. https://doi.org/10.61356/j.nswa.2024.21354

[11] Kamacı, H. (2021). On hybrid structures of hypersoft sets and rough sets. International Journal of Modern Science and Technology, 6(4), 69-82.

[12] Lei Zhou & Wei-Zhi Wu, (2008) On Generalized Intuitionistic Fuzzy Rough Approximation Operators, Information Sciences 178, 2448-2465. https://doi.org/10.1016/j.ins.2008.01.012

[13] Martina, D. J. S., & Deepa, G. (2023). Some algebraic properties on rough neutrosophic matrix and its application to multi-criteria decision-making. AIMS Mathematics, 8(10), 24132-24152. https://doi.org/10.3934/math.20231230.

[14] Muthukumar, P., & Krishnan, G. S. S. (2018). Generalized Fuzzy Soft Rough Matrices and Their Applications in Decision-Making Problems. International Journal of Fuzzy Systems, 20, 500-514. https://doi.org/10.1007/s40815-017-0350-x

[15] Saqlain, M., Moin, S., Jafar, M. N., Saeed, M., & Smarandache, F. (2020). Aggregate Operators of Neutrosophic Hypersoft set. Neutrosophic sets and systems, vol. 32-18. https://doi.org/10.5281/zenodo.3723155

[16] Smarandache .F. (2018). Extension of soft set to Hypersoft Set, and then Plithogenic Hypersoft Set, Neutrosophic sets and systems, 22, 168-170.

[17] Smarandache, F. (2023). New types of soft sets "Hypersoft set, Indetermsoft set, Indetermhypersoft set, and Treesoft set": an improved version. Infinite Study. Neutrosophic sets and systems, vol. 8, 35-41. https://doi.org/10.61356/j.nswa.2023.41

[18] Shanthi, S. A. (2021). Application of intuitionistic fuzzy rough matrices. Advances and Applications in Mathematical Sciences, 20(4), 501-513.

[19] Subha, V. S., & Selvakumar, R. (2024, January). An Innovative Electric Vehicle Selection with Multi-Criteria Decision-making Approach in Indian brands on a Neutrosophic Hypersoft Rough Set by using Reduct and Core. In 2024 Fourth International Conference on Advances in Electrical, Computing, Communication and Sustainable Technologies (ICAECT) (pp. 1-7). IEEE.

[20] Sumathi, I. R., & Arockiarani, I. (2014). New operations on fuzzy neutrosophic soft matrices. International journal of innovative Research and Studies, 13(3), 110-124.

[21] Uma, R., Sriram, S., & Murugadas, P. (2021). Fuzzy Neutrosophic Soft Matrices of Type I and Type II. Fuzzy Information and Engineering, 13(2), 211-222.

[22] Zadeh, L.A. (1965) Fuzzy sets. Inf. Control 8(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

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