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General, General Weak, Anti, Balanced, and Semi-Neutrosophic Graph

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Abstract. Graph classes categorize graphs based on shared properties or structures, and numerous such classes have been proposed over time. In 1965, Zadeh [43] introduced a framework for managing uncertainty, which later inspired Rosenfeld [28, 31] to develop fuzzy graph theory in 1975. A Neutrosophic Graph, as a generalization of a Fuzzy Graph, associates each vertex and edge with three membership values representing truth, indeterminacy, and falsity. This allows Neutrosophic Graphs to handle uncertain, vague, or contradictory information more effectively. This paper aims to expand the study of Neutrosophic Graphs by proposing new graph classes, specifically the General-Neutrosophic Graph, Anti-Neutrosophic Graph, *-Balanced-Neutrosophic Graph, and Semi-Neutrosophic Graph.

Keywords: Neutrosophic graph, Balanced-Neutrosophic Graph, General-Neutrosophic graph, Anti-Neutrosophic graph, Semi-Neutrosophic graph

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1. Introduction

1.1. *Graph Theory*

A graph is a mathematical structure consisting of vertices (nodes) connected by edges, representing relationships or connections. Graph classes categorize graphs based on shared properties or structures. Various graph classes have been proposed, including Tree Graphs [40], Path Graphs [42], Complete Graphs [13], Circle Graphs [9], Petersen Graphs [23], and Total Graphs [41].

1.2. *Fuzzy Graph Theory*

Uncertainty refers to the absence of complete knowledge or predictability, affecting decision-making in fields such as economics, science, and risk management. In 1965, Zadeh [43] introduced a framework to handle uncertainty, which later inspired Rosenfeld [28, 31] to develop fuzzy graph theory in 1975. Note that a fuzzy graph represents relationships with uncertainty by assigning membership values to both vertices and edges, allowing flexible analysis of imprecise connections.

In the field of Fuzzy Graphs, numerous graph classes have also been proposed, including Intuitionistic Fuzzy Graphs [29], Bipolar Fuzzy Graphs [1], Fuzzy Planar Graphs [33], Irregular Bipolar Fuzzy Graphs [32], m-Polar Fuzzy Graphs [2], and Balanced Interval-Valued Fuzzy Graphs [30]. Considering fuzzy graph classes also helps identify shared properties, enabling efficient algorithms, deeper analysis, and practical applications in various fields.

1.3. *Neutrosophic Graph Theory*

A Neutrosophic Graph is a generalization of a Fuzzy Graph. It can also be interpreted as a graph-theoretic representation of a Neutrosophic Set [16,35,36]. In a Neutrosophic Graph, each vertex and edge is associated with three membership values—representing truth, indeterminacy, and falsity—allowing it to handle uncertain, vague, or even contradictory information. As with Fuzzy Graphs, Neutrosophic Graphs have been extensively studied across various fields [**?**, 17, 21].

Several graph classes have been proposed within the field of Neutrosophic Graphs. For example,Single Valued Neutrosophic Graphs [3, 12] and Neutrosophic Vague Line Graphs [24]. Considering Neutrosophic graph classes also helps identify shared properties, enabling efficient algorithms, deeper analysis, and practical applications in various fields.

1.4. *Our Contribution*

This paper outlines our contributions and goals. While Neutrosophic Graphs have numerous applications and are a key research area, they are still in a developmental stage. As a result, many graph classes in Neutrosophic Graph theory remain undefined. To address this, we explore new graph classes, including the General-Neutrosophic Graph, Anti-Neutrosophic Graph, *-Balanced-Neutrosophic Graph, and Semi-Neutrosophic Graph. By integrating these classes with the existing Neutrosophic Graph theory, we aim to advance both theoretical and practical developments in the field.

1.5. *Structure of This Paper*

The structure of this paper is as follows: Section 2 presents the preliminary concepts, definitions, and illustrative examples related to graph theory that are used throughout the paper. Section 3 introduces and elaborates on newly proposed classes of graphs. Section 4 outlines potential directions for future research and highlights key challenges in the field.

2. **Preliminaries and definitions**

In this section, we provide a brief overview of the definitions and notations used throughout this paper. Unless otherwise stated, all graphs considered in this paper are assumed to be finite, undirected, and simple.

2.1. *Fuzzy Graph*

A *Fuzzy Graph* represents relationships that involve uncertainty by assigning membership degrees to both vertices and edges. This framework allows for a more flexible and nuanced analysis. Due to its significance, fuzzy graphs have been extensively studied [4]. The formal definition of a fuzzy graph is given in [25, 31].

Definition 2.1 (Fuzzy Graph [31])**.** Let V be a non-empty set. A *fuzzy graph* is defined as the pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V; for each $x \in V$, $\sigma(x)$ represents the membership degree of the vertex x.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on V satisfying

  $$\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$$

  for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

The *underlying crisp graph* of G is denoted by

$$G^* = (\sigma^*, \mu^*),$$

with

- $\sigma^* = \{x \in V \mid \sigma(x) > 0\}$,
- $\mu^* = \{(x, y) \in V \times V \mid \mu(x, y) > 0\}$.

A *fuzzy subgraph* of G is a fuzzy graph $H = (\sigma', \mu')$ for which there exists a subset $X \subseteq V$ such that:

- $\sigma' : X \to [0, 1]$ is a fuzzy subset of X,

- $\mu' : X \times X \to [0,1]$ is a fuzzy relation satisfying

$$\mu'(x,y) \le \sigma'(x) \wedge \sigma'(y)$$

for all $x, y \in X$.

Example 2.2 (cf. [12]). Consider a fuzzy graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3, v_4\}.$$

The membership degrees of the vertices are given by:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4.$$

The fuzzy relation on the edges is defined by μ, with the condition that for all $x, y \in V$,

$$\mu(x,y) \le \sigma(x) \wedge \sigma(y).$$

In this example, the membership degrees assigned to the edges are:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1,$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3.$$

Thus, the fuzzy graph G exhibits vertices connected by edges with varying membership degrees while ensuring that the membership degree of any edge does not exceed the minimum membership degree of its endpoints.

Definition 2.3 ([22]). A fuzzy graph $G = (\sigma, \mu)$ is called *complete* if

$$\mu(u,v) = \sigma(u) \wedge \sigma(v)$$

for all $u, v \in V$, where $\wedge$ denotes the minimum operation.

Definition 2.4 ([22]). A fuzzy graph $G = (\sigma, \mu)$ is called *strong* if

$$\mu(u,v) = \sigma(u) \wedge \sigma(v)$$

for every edge $(u, v) \in E$. Note that every complete fuzzy graph is strong, although the converse does not necessarily hold.

Example 2.5 (Strong Fuzzy Graph). Let $G = (V, E, \sigma, \mu)$ be the fuzzy graph with

$$V = \{v_1, v_2, v_3\}, \quad E = \{(v_1, v_2), (v_2, v_3)\},$$

and vertex-membership

$$\sigma(v_1) = 0.7, \quad \sigma(v_2) = 0.5, \quad \sigma(v_3) = 0.9.$$

Define the edge-membership by

$$\mu(v_1, v_2) = \min\{\sigma(v_1), \sigma(v_2)\} = 0.5, \quad \mu(v_2, v_3) = \min\{\sigma(v_2), \sigma(v_3)\} = 0.5.$$

Since for each edge $(u,v) \in E$ we have $\mu(u,v) = \sigma(u) \wedge \sigma(v)$, G is a strong fuzzy graph. Note that $(v_1, v_3) \notin E$, so G need not be complete.

Definition 2.6 (Isomorphic [6])**.** Two fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ are said to be *isomorphic* if there exists a bijection $h : V_1 \to V_2$ such that

$$\sigma_1(x) = \sigma_2(h(x)) \quad \text{and} \quad \mu_1(x,y) = \mu_2(h(x), h(y))$$

for all $x, y \in V_1$.

Example 2.7 (Isomorphic Fuzzy Graphs)**.** Consider two fuzzy graphs $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ defined by

$$V_1 = \{x, y\}, \quad E_1 = \{(x,y)\}, \quad \sigma_1(x) = 0.4,\ \sigma_1(y) = 0.8,\ \mu_1(x,y) = 0.4,$$

$$V_2 = \{u, v\}, \quad E_2 = \{(u,v)\}, \quad \sigma_2(u) = 0.4,\ \sigma_2(v) = 0.8,\ \mu_2(u,v) = 0.4.$$

Define the bijection $h : V_1 \to V_2$ by $h(x) = u$, $h(y) = v$. Then

$$\sigma_1(x) = 0.4 = \sigma_2(h(x)), \quad \sigma_1(y) = 0.8 = \sigma_2(h(y)),$$

$$\mu_1(x,y) = 0.4 = \mu_2(h(x), h(y)).$$

Hence G_1 and G_2 are isomorphic fuzzy graphs.

Definition 2.8 (Complement)**.** The *complement* of a fuzzy graph $G = (\sigma, \mu)$ is defined as the fuzzy graph

$$G^c = (\sigma^c, \mu^c),$$

where

$$\sigma^c(u) = \sigma(u) \quad \text{and} \quad \mu^c(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$$

for all $u, v \in V$.

Example 2.9 (Complement of a Fuzzy Graph)**.** Let $G = (V, E, \sigma, \mu)$ be the fuzzy graph with

$$V = \{v_1, v_2, v_3\}, \quad E = \{(v_1, v_2), (v_2, v_3)\},$$

vertex-membership

$$\sigma(v_1) = 0.6, \quad \sigma(v_2) = 0.8, \quad \sigma(v_3) = 0.5,$$

and edge-membership

$$\mu(v_1, v_2) = 0.5, \quad \mu(v_2, v_3) = 0.3, \quad \mu(v_1, v_3) = 0.$$

Its complement $G^c = (V, E, \sigma^c, \mu^c)$ has

$$\sigma^c(u) = \sigma(u) \quad \forall u \in V,$$

and for each pair (u, v),

$$\mu^c(u,v) = \min\{\sigma(u), \sigma(v)\} - \mu(u,v).$$

Thus

$$\mu^c(v_1, v_2) = \min\{0.6, 0.8\} - 0.5 = 0.6 - 0.5 = 0.1,$$

$$\mu^c(v_2, v_3) = \min\{0.8, 0.5\} - 0.3 = 0.5 - 0.3 = 0.2,$$

$$\mu^c(v_1, v_3) = \min\{0.6, 0.5\} - 0 = 0.5.$$

Hence G^c is the fuzzy graph with the same vertex memberships and edge-memberships

$$\mu^c(v_1, v_2) = 0.1, \quad \mu^c(v_2, v_3) = 0.2, \quad \mu^c(v_1, v_3) = 0.5.$$

2.2. *Neutrosophic Graph*

A *neutrosophic graph* is a graph in which each vertex and edge is associated with three membership values—representing truth, indeterminacy, and falsity—to accommodate uncertain, vague, or even contradictory information [24, 39]. The formal definition is given in [21].

Definition 2.10 (Neutrosophic Graph [11, 21])**.** A finite *neutrosophic graph* is defined as

$$NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3)),$$

where:

- V is a finite set of vertices,
- $E \subseteq V \times V$ is the set of edges,
- For $i = 1, 2, 3$, $\sigma_i : V \to [0, 1]$ and $\mu_i : E \to [0, 1]$ are the membership functions for vertices and edges, respectively.

Moreover, for every edge $v_i v_j \in E$, the following condition holds:

$$\mu(v_i v_j) \le \sigma(v_i) \wedge \sigma(v_j),$$

where $\wedge$ denotes the minimum operation. The terminology is as follows:

(1) σ is called the *neutrosophic vertex set*.
(2) μ is called the *neutrosophic edge set*.
(3) $|V|$ is called the *order* of NTG, denoted by $O(NTG)$.
(4) $\sum_{v \in V} \sigma(v)$ is called the *neutrosophic order* of NTG, denoted by $On(NTG)$.
(5) $|E|$ is called the *size* of NTG, denoted by $S(NTG)$.
(6) $\sum_{e \in E} \mu(e)$ is called the *neutrosophic size* of NTG, denoted by $Sn(NTG)$.

Example 2.11 (cf. [12])**.** Consider a neutrosophic graph

$$NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$$

with the vertex set

$$V = \{v_1, v_2, v_3, v_4\}.$$

The neutrosophic membership degrees of the vertices are given by:

$$\sigma(v_1) = (0.5,\, 0.1,\, 0.4), \quad \sigma(v_2) = (0.6,\, 0.3,\, 0.2),$$
$$\sigma(v_3) = (0.2,\, 0.3,\, 0.4), \quad \sigma(v_4) = (0.4,\, 0.2,\, 0.5).$$

The neutrosophic membership degrees of the edges are:

$$\mu(v_1v_2) = (0.2,\, 0.3,\, 0.4), \quad \mu(v_2v_3) = (0.3,\, 0.3,\, 0.4),$$
$$\mu(v_3v_4) = (0.2,\, 0.3,\, 0.4), \quad \mu(v_4v_1) = (0.1,\, 0.2,\, 0.5).$$

Thus, the neutrosophic graph NTG exhibits the following properties:

- The vertices v_1, v_2, v_3, v_4 are interconnected by edges with varying neutrosophic membership degrees.
- For every edge $v_iv_j \in E$, the relation satisfies
$$\mu(v_iv_j) \leq \sigma(v_i) \wedge \sigma(v_j),$$
ensuring that the membership degree of each edge does not exceed the minimum membership degree of its endpoints.

Example 2.12 (Neutrosophic Social Network)**.** We model a small collaboration network of four researchers—Alice, Bob, Carol, and Dave—as a neutrosophic graph

$$G = (V, E, \sigma, \mu)$$

where

$$V = \{\text{Alice, Bob, Carol, Dave}\}.$$

Here $\sigma(v) = (t(v),\, i(v),\, f(v))$ measures for each person v their degree of genuine team membership (t), uncertainty about their role (i), and apparent outsider status (f). We set

$$\sigma(\text{Alice}) = (0.7,\, 0.4,\, 0.3), \quad \sigma(\text{Bob}) = (0.5,\, 0.2,\, 0.4),$$
$$\sigma(\text{Carol}) = (0.7,\, 0.3,\, 0.2), \quad \sigma(\text{Dave}) = (0.8,\, 0.3,\, 0.4).$$

The edge set

$$E = \{(\text{Alice}, \text{Bob}),\, (\text{Alice}, \text{Carol}),\, (\text{Bob}, \text{Carol}),\, (\text{Carol}, \text{Dave})\}$$

records pairwise trust relations, with $\mu(u, v) = (T(u, v),\, I(u, v),\, F(u, v))$ giving the truth (trust), indeterminacy, and falsity (distrust) degrees. For example,

$$\mu(\text{Alice}, \text{Bob}) = (0.5,\, 0.4,\, 0.4), \quad \mu(\text{Alice}, \text{Carol}) = (0.7,\, 0.4,\, 0.4),$$
$$\mu(\text{Bob}, \text{Carol}) = (0.5,\, 0.3,\, 0.4), \quad \mu(\text{Carol}, \text{Dave}) = (0.7,\, 0.3,\, 0.4).$$

This neutrosophic graph captures both the team-membership ambiguity of each researcher and the uncertainty in their collaborative trust relationships.

2.3. *Anti Fuzzy Graph*

An *Anti Fuzzy Graph* (AFG) extends the concept of a fuzzy graph by incorporating anti-fuzzy relations for both vertices and edges. This extension allows the graph to model not only uncertainty but also constraints, wherein the edges represent the maximum membership values between adjacent vertices. Several related graph classes have been studied in the literature [14, 27]. The following definitions are presented in [34].

Definition 2.13 ([34]). Let $G = (V, E)$ be a graph, where

- V is a non-empty finite set of vertices, and
- $E \subseteq V \times V$ is the set of edges.

An *Anti Fuzzy Graph* is defined as a structure $G_{AF} = (\eta, \rho)$ where:

- $\eta : V \to [0, 1]$ is a fuzzy membership function that assigns a membership value to each vertex, and
- $\rho : V \times V \to [0, 1]$ is an anti-fuzzy membership function defined on the edges, with $\rho(u_1, u_2)$ quantifying the degree of relationship between vertices u_1 and u_2.

Definition 2.14 ([34]). The anti-fuzzy membership function ρ satisfies the condition

$$\rho(u_1, u_2) \geq \eta(u_1) \vee \eta(u_2),$$

for all $u_1, u_2 \in V$, where $\vee$ denotes the maximum operator; that is, $\eta(u_1) \vee \eta(u_2) = \max\{\eta(u_1), \eta(u_2)\}$.

Definition 2.15 ([34]). The *underlying anti-crisp graph* of $G_{AF} = (\eta, \rho)$, denoted by $G^*_A = (\eta^*, \rho^*)$, is defined as follows:

- $\eta^* = \{x \in V \mid \eta(x) > 0\}$, which is the set of vertices with positive fuzzy membership, and
- $\rho^* = \{(x, y) \in V \times V \mid \rho(x, y) > 0\}$, which is the set of edges with positive anti-fuzzy membership.

Example 2.16 (cf. [34]). Consider an Anti Fuzzy Graph $G_{AF} = (\eta, \rho)$ with the vertex set

$$V = \{v_1, v_2, v_3, v_4\},$$

and the corresponding edge set

$$E = \{e_1, e_2, e_3, e_4\}.$$

The fuzzy membership function η is defined as

$$\eta(v_1) = 1, \quad \eta(v_2) = 1, \quad \eta(v_3) = 3, \quad \eta(v_4) = 4,$$

and the anti-fuzzy membership function ρ assigns the following values to the edges:

$$\rho(v_1, v_2) = 1, \quad \rho(v_1, v_3) = 2, \quad \rho(v_3, v_4) = 3, \quad \rho(v_2, v_4) = 4.$$

2.4. *General Fuzzy Graph*

General fuzzy graphs extend traditional fuzzy graphs by allowing the membership values of edges to be chosen independently of the minimum membership values of their incident vertices. The following definition is provided in [26].

Definition 2.17 (General Fuzzy Graph [26])**.** Let V be a non-empty set. A *general fuzzy graph* is defined as the pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$.

This definition generalizes the traditional fuzzy graph by allowing the membership values of edges to be independent of the membership values of their incident vertices.

Example 2.18 (cf. [26])**.** Consider a general fuzzy graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3, v_4\}.$$

The fuzzy membership degrees of the vertices are given by:

$$\sigma(v_1) = 0.71, \quad \sigma(v_2) = 0.80, \quad \sigma(v_3) = 0.40, \quad \sigma(v_4) = 0.60.$$

The fuzzy membership degrees of the edges are assigned independently as:

$$\mu(v_1, v_2) = 0.5, \quad \mu(v_2, v_3) = 0.9, \quad \mu(v_3, v_4) = 0.5, \quad \mu(v_4, v_1) = 0.6.$$

In this example, the edge membership values do not depend on the minimum of the vertex membership values.

In addition, we introduce the concept of a general weak fuzzy graph, which is a relaxed version of the general fuzzy graph.

Definition 2.19 (General Weak Fuzzy Graph [26])**.** Let V be a non-empty set. A *general weak fuzzy graph* is defined as the pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$,

such that for every edge (u, v) in the graph, the membership degree $\mu(u, v)$ satisfies

$$\mu(u, v) \neq \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Example 2.20 (General Weak Fuzzy Graph)**.** Consider a general weak fuzzy graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Let the vertex membership function be defined as:

$$\sigma(v_1) = 0.7, \quad \sigma(v_2) = 0.5, \quad \sigma(v_3) = 0.6.$$

Then, the minimum membership values for each pair of vertices are:

$$\sigma(v_1) \wedge \sigma(v_2) = \min\{0.7, 0.5\} = 0.5, \quad \sigma(v_2) \wedge \sigma(v_3) = \min\{0.5, 0.6\} = 0.5,$$

$$\sigma(v_1) \wedge \sigma(v_3) = \min\{0.7, 0.6\} = 0.6.$$

Choose the edge membership values so that they differ from these minimum values. For instance, define:

$$\mu(v_1, v_2) = 0.4, \quad \mu(v_2, v_3) = 0.6, \quad \mu(v_1, v_3) = 0.5.$$

Here, we have:

- $\mu(v_1, v_2) = 0.4 \neq 0.5$,
- $\mu(v_2, v_3) = 0.6 \neq 0.5$,
- $\mu(v_1, v_3) = 0.5 \neq 0.6$.

Thus, G satisfies the condition for a general weak fuzzy graph.

2.5. *Semi-Fuzzy Graphs*

A *semi-fuzzy graph* is a modified fuzzy graph in which the membership functions for vertices and edges are defined only partially. This model combines elements of both crisp and fuzzy graphs to better capture uncertainty. The following definitions are given in [5].

Definition 2.21 (Semi-Fuzzy Graph [5])**.** Let $G = (\sigma, \mu)$ be a fuzzy graph with vertex set V. Then G is called a *semi-fuzzy graph* if it satisfies

$$\sum_{x,y \in V} \sigma(x) \wedge \sigma(y) < 1,$$

where $\wedge$ denotes the minimum operation.

Definition 2.22 (Density of a Semi-Fuzzy Graph [5])**.** For a semi-fuzzy graph $G = (\sigma, \mu)$ with vertex set V, the *density* $D_0(G)$ is defined as

$$D_0(G) = \frac{2 \sum_{x,y \in V} \mu(x, y)}{\left(1 - \sum_{x,y \in V} \sigma(x) \wedge \sigma(y)\right) \sum_{x,y \in V} \sigma(x) \wedge \sigma(y)}.$$

A semi-fuzzy graph G is called *balanced* if for every non-empty fuzzy subgraph H of G, the inequality

$$D_0(H) \leq D_0(G)$$

holds.

Example 2.23 (Semi-Fuzzy Graph Example)**.** Consider a semi-fuzzy graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Define the vertex membership function as

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.1, \quad \sigma(v_3) = 0.1.$$

Since for all $x, y \in V$ we have $\sigma(x) \wedge \sigma(y) = 0.1$, the sum over all pairs is

$$\sum_{x,y \in V} \sigma(x) \wedge \sigma(y) = 9 \times 0.1 = 0.9 < 1,$$

so the condition for a semi-fuzzy graph is satisfied.

Now, define the edge membership function μ by assigning nonzero values to a few edges:

$$\mu(v_1, v_2) = 0.2, \quad \mu(v_2, v_3) = 0.3, \quad \mu(v_1, v_3) = 0.1,$$

and $\mu(u, v) = 0$ for all other pairs (u, v).

Then the numerator in the density formula becomes

$$2 \sum_{x,y \in V} \mu(x, y) = 2\,(0.2 + 0.3 + 0.1) = 2(0.6) = 1.2.$$

The denominator is computed as

$$(1 - 0.9) \cdot 0.9 = 0.1 \times 0.9 = 0.09.$$

Thus, the density of G is

$$D_0(G) = \frac{1.2}{0.09} \approx 13.33.$$

This example illustrates a semi-fuzzy graph and how its density is calculated.

2.6. $*$-*balanced Fuzzy Graph*

A $*$-balanced fuzzy graph is a fuzzy graph in which the $*$-density—a modified form of the traditional density—is defined so that every non-empty fuzzy subgraph has a $*$-density that does not exceed that of the original graph [4].

Definition 2.24 (∗-balanced Fuzzy Graph [4])**.** Let $G = (\sigma, \mu)$ be a fuzzy graph with vertex set V. The ∗-*density* of G is defined as

$$D^*(G) = \frac{2 \sum_{u,v \in V} \mu(u,v)}{\sum_{u \in V} \sigma(u)}.$$

The fuzzy graph G is said to be ∗-*balanced* if for every non-empty fuzzy subgraph H of G,

$$D^*(H) \leq D^*(G).$$

Example 2.25 (∗-balanced Fuzzy Graph Example)**.** Consider a fuzzy graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3, v_4\}.$$

Define the vertex membership function as

$$\sigma(v_1) = 0.8, \quad \sigma(v_2) = 0.6, \quad \sigma(v_3) = 0.7, \quad \sigma(v_4) = 0.9.$$

Thus, the sum of the vertex memberships is

$$\sum_{u \in V} \sigma(u) = 0.8 + 0.6 + 0.7 + 0.9 = 3.0.$$

Define the edge membership function by

$$\mu(v_1, v_2) = 0.5, \quad \mu(v_2, v_3) = 0.4, \quad \mu(v_3, v_4) = 0.6, \quad \mu(v_4, v_1) = 0.7,$$

and let $\mu(u, v) = 0$ for all other pairs.

Then, the total edge membership is

$$\sum_{u,v \in V} \mu(u,v) = 0.5 + 0.4 + 0.6 + 0.7 = 2.2,$$

and the ∗-density of G is

$$D^*(G) = \frac{2(2.2)}{3.0} = \frac{4.4}{3.0} \approx 1.467.$$

Now, consider a fuzzy subgraph H of G induced by the vertex set $\{v_1, v_2, v_3\}$ with the corresponding restrictions of σ and μ. Suppose the edge memberships in H are

$$\mu(v_1, v_2) = 0.5, \quad \mu(v_2, v_3) = 0.4, \quad \mu(v_1, v_3) = 0.3.$$

Then the sum of the vertex memberships for H is

$$\sigma(v_1) + \sigma(v_2) + \sigma(v_3) = 0.8 + 0.6 + 0.7 = 2.1,$$

and the total edge membership is

$$0.5 + 0.4 + 0.3 = 1.2.$$

Thus, the $*$-density of H is

$$D^*(H) = \frac{2(1.2)}{2.1} = \frac{2.4}{2.1} \approx 1.143.$$

Since

$$D^*(H) \approx 1.143 \leq 1.467 \approx D^*(G),$$

the graph G is $*$-balanced.

3. Result in this paper

We will describe the results presented in this paper. The fuzzy graph is extended to a Neutrosophic Graph, and its properties are analyzed and examined as needed.

3.1. *General Neutrosophic Graph*

We propose a new class of graphs called the *General Neutrosophic Graph*, modeled after the general fuzzy graph. In this framework, each vertex and edge is assigned a triplet of membership values representing truth, indeterminacy, and falsity. This definition generalizes the traditional neutrosophic graph by allowing the edge membership values to be independent of the membership values of their incident vertices.

Definition 3.1 (General Neutrosophic Graph)**.** Let V be a non-empty set. A *general neutrosophic graph* is defined as the pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0,1] \times [0,1] \times [0,1]$ is the *neutrosophic vertex function*. For each vertex $v \in V$,

  $$\sigma(v) = \big(\sigma_1(v), \sigma_2(v), \sigma_3(v)\big),$$

  where $\sigma_1(v)$, $\sigma_2(v)$, and $\sigma_3(v)$ represent the truth, indeterminacy, and falsity membership degrees of v, respectively.
- $\mu : V \times V \to [0,1] \times [0,1] \times [0,1]$ is the *neutrosophic edge function*. For each edge (u,v) (with (u,v) belonging to the edge set $E \subseteq V \times V$),

  $$\mu(u,v) = \big(\mu_1(u,v), \mu_2(u,v), \mu_3(u,v)\big),$$

  where $\mu_1(u,v)$, $\mu_2(u,v)$, and $\mu_3(u,v)$ represent the truth, indeterminacy, and falsity membership degrees of the edge, respectively.

Example 3.2 (General Neutrosophic Graph Example)**.** Consider a general neutrosophic graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Define the neutrosophic vertex function σ as:

$$\sigma(v_1) = (0.8,\, 0.1,\, 0.1), \quad \sigma(v_2) = (0.7,\, 0.2,\, 0.1), \quad \sigma(v_3) = (0.6,\, 0.3,\, 0.1).$$

Assign the neutrosophic edge function μ as follows:

$$\mu(v_1, v_2) = (0.5,\, 0.3,\, 0.2), \quad \mu(v_2, v_3) = (0.6,\, 0.2,\, 0.3), \quad \mu(v_1, v_3) = (0.4,\, 0.4,\, 0.1).$$

In this example, the membership values for edges are chosen independently from those of the vertices.

We now introduce a relaxed version of the general neutrosophic graph called the *weak general neutrosophic graph*.

Definition 3.3 (Weak General Neutrosophic Graph)**.** Let V be a non-empty set. A *weak general neutrosophic graph* is defined as the pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0,1] \times [0,1] \times [0,1]$ is the neutrosophic vertex function, with
$$\sigma(v) = \big(\sigma_1(v), \sigma_2(v), \sigma_3(v)\big)$$
for each $v \in V$.
- $\mu : V \times V \to [0,1] \times [0,1] \times [0,1]$ is the neutrosophic edge function, with
$$\mu(u, v) = \big(\mu_1(u, v), \mu_2(u, v), \mu_3(u, v)\big)$$
for each edge $(u, v) \in E$.

Additionally, for every edge $(u, v) \in E$, the truth membership of the edge must satisfy:

$$\mu_1(u, v) \neq \sigma_1(u) \wedge \sigma_1(v),$$

where $\wedge$ denotes the minimum operation.

Example 3.4 (Weak General Neutrosophic Graph Example)**.** Consider a weak general neutrosophic graph $G = (\sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Define the neutrosophic vertex function σ by:

$$\sigma(v_1) = (0.9,\, 0.05,\, 0.05), \quad \sigma(v_2) = (0.8,\, 0.1,\, 0.1), \quad \sigma(v_3) = (0.7,\, 0.15,\, 0.15).$$

For the edge (v_1, v_2), the minimum truth membership is

$$\sigma_1(v_1) \wedge \sigma_1(v_2) = \min\{0.9, 0.8\} = 0.8.$$

Choose the edge membership values such that:

$$\mu(v_1, v_2) = (0.7,\ 0.3,\ 0.2),$$

so that $\mu_1(v_1, v_2) = 0.7 \neq 0.8$.

Similarly, define:

$$\mu(v_2, v_3) = (0.6,\ 0.2,\ 0.2), \quad \mu(v_1, v_3) = (0.5,\ 0.25,\ 0.25).$$

For (v_2, v_3), we have $\min\{0.8, 0.7\} = 0.7$ and $\mu_1(v_2, v_3) = 0.6 \neq 0.7$; for (v_1, v_3), $\min\{0.9, 0.7\} = 0.7$ and $\mu_1(v_1, v_3) = 0.5 \neq 0.7$.

Thus, G satisfies the condition for a weak general neutrosophic graph.

Theorem 3.5 (Every Neutrosophic Graph is a General Neutrosophic Graph)**.** *Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph in the sense of Definition. Then G satisfies the definition of a general neutrosophic graph.*

Proof. By hypothesis, for each vertex $v \in V$ we have

$$\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v)) \in [0, 1]^3,$$

and for each edge $e = (u, v) \in E$,

$$\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e)) \in [0, 1]^3$$

with the additional constraint

$$\mu_i(e) \leq \min\{\sigma_i(u), \sigma_i(v)\} \quad (i = 1, 2, 3).$$

Since a general neutrosophic graph only requires

$$\sigma : V \to [0, 1]^3, \qquad \mu : E \to [0, 1]^3$$

with no further inequality constraint, every neutrosophic graph trivially meets these weaker requirements. Hence G is a general neutrosophic graph. □

Theorem 3.6 (General Neutrosophic Graphs Generalize General Fuzzy Graphs)**.** *Every general fuzzy graph $F = (V, E, \sigma, \mu)$, where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$, can be regarded as a general neutrosophic graph $G = (V, E, \sigma', \mu')$ by setting*

$$\sigma'(v) = \big(\sigma(v), 0, 0\big), \qquad \mu'(u, v) = \big(\mu(u, v), 0, 0\big).$$

Proof. Define $\sigma' : V \to [0, 1]^3$ and $\mu' : E \to [0, 1]^3$ by

$$\sigma'(v) = \big(\sigma(v), 0, 0\big) \quad \forall\, v \in V, \qquad \mu'(u, v) = \big(\mu(u, v), 0, 0\big) \quad \forall\, (u, v) \in E.$$

Since $\sigma(v) \in [0, 1]$ and $\mu(u, v) \in [0, 1]$, it follows that $\sigma'(v), \mu'(u, v) \in [0, 1]^3$. There is no further constraint in the definition of a general neutrosophic graph beyond these codomain requirements. Hence $G = (V, E, \sigma', \mu')$ satisfies the definition of a general neutrosophic graph, demonstrating that every general fuzzy graph embeds as a special case. □

As operations on the General Neutrosophic Graph, we will provide definitions for concepts such as Union, c-complement, and others. These definitions, presented below, are modeled after those used for General Fuzzy Graphs.

Definition 3.7 (Union)**.** Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two general neutrosophic graphs with their respective underlying crisp graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. The *union* of G_1 and G_2 is the general neutrosophic graph $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$, where:

- **For the vertices**:

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 - V_2 \\ \sigma_2(u) & \text{if } u \in V_2 - V_1 \\ \max\{\sigma_1(u), \sigma_2(u)\} & \text{if } u \in V_1 \cap V_2 \end{cases}$$

- **For the edges**:

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in X_1 - X_2 \\ \mu_2(u, v) & \text{if } (u, v) \in X_2 - X_1 \\ \max\{\mu_1(u, v), \mu_2(u, v)\} & \text{if } (u, v) \in X_1 \cap X_2 \end{cases}$$

Definition 3.8. The *c-complement* of a general neutrosophic graph $G = (\sigma, \mu)$ is the neutrosophic graph $G^c = (\sigma^c, \mu^c)$, where:

- $\sigma^c = \sigma$ (the vertex membership remains unchanged),
- **For the edges**:

$$\mu^c(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0 \\ 1 - \mu(u, v) & \text{if } \mu(u, v) > 0 \end{cases}$$

  for all $u, v \in V$.

Definition 3.9. The $|\mu|$-*complement* of a general neutrosophic graph $G = (\sigma, \mu)$ is the neutrosophic graph $G^{|\mu|} = (\sigma, \mu^{|\mu|})$, where:

- σ remains unchanged,
- **For the edges**:

$$\mu^{|\mu|}(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0 \\ |\sigma(u) \wedge \sigma(v) - \mu(u, v)| & \text{if } \mu(u, v) > 0 \end{cases}$$

  where $\wedge$ denotes the minimum operation between the membership values of vertices u and v.

The following theorem is presented using the above-defined operations.

Theorem 3.10. *General neutrosophic graphs are closed under c-complementation and $|\mu|$-complementation.*

Proof. The proof of this theorem directly follows from the definitions of c-complement and $|\mu|$-complement in general neutrosophic graphs.

For the **c-complement**: Given a general neutrosophic graph $G = (\sigma, \mu)$, the c-complement $G^c = (\sigma^c, \mu^c)$ is defined such that:

$$\mu^c(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ 1 - \mu(u, v) & \text{if } \mu(u, v) > 0. \end{cases}$$

Since the operation $1 - \mu(u, v)$ preserves the neutrosophic nature of the edge membership, G^c remains a general neutrosophic graph.

For the **$|\mu|$-complement**: Given $G = (\sigma, \mu)$, the $|\mu|$-complement $G^{|\mu|} = (\sigma, \mu^{|\mu|})$ is defined such that:

$$\mu^{|\mu|}(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ |\sigma(u) \wedge \sigma(v) - \mu(u, v)| & \text{if } \mu(u, v) > 0, \end{cases}$$

where $\wedge$ denotes the minimum operation. This transformation also preserves the neutrosophic structure, ensuring that $G^{|\mu|}$ is a general neutrosophic graph.

Thus, both operations leave the class of general neutrosophic graphs closed under these complements. □

Theorem 3.11. *The union of two general weak neutrosophic graphs need not be a general weak neutrosophic graph.*

Proof. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two general weak neutrosophic graphs with their respective underlying crisp graphs $G_1^* = (V_1, X_1)$ and $G_2^* = (V_2, X_2)$. The union $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is defined as follows:

- For the vertices:

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 - V_2, \\ \sigma_2(u) & \text{if } u \in V_2 - V_1, \\ \max\{\sigma_1(u), \sigma_2(u)\} & \text{if } u \in V_1 \cap V_2. \end{cases}$$

- For the edges:

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in X_1 - X_2, \\ \mu_2(u, v) & \text{if } (u, v) \in X_2 - X_1, \\ \max\{\mu_1(u, v), \mu_2(u, v)\} & \text{if } (u, v) \in X_1 \cap X_2. \end{cases}$$

We now examine three cases:

Case 1: $(u, v) \in X_1 - X_2$

Since $G_1 = (\sigma_1, \mu_1)$ is a general weak neutrosophic graph, $\mu_1(u, v) \neq \sigma_1(u) \wedge \sigma_1(v)$. Given that $\sigma_2(u) = 0$ for all $u \in V_1 - V_2$, we have:

$$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v) \neq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v).$$

Hence, G is not a general weak neutrosophic graph in this case.

Case 2: $(u, v) \in X_2 - X_1$

Similarly, since $G_2 = (\sigma_2, \mu_2)$ is a general weak neutrosophic graph, $\mu_2(u, v) \neq \sigma_2(u) \wedge \sigma_2(v)$. Given that $\sigma_1(u) = 0$ for all $u \in V_2 - V_1$, we have:

$$(\mu_1 \cup \mu_2)(u, v) = \mu_2(u, v) \neq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v).$$

Hence, G is not a general weak neutrosophic graph in this case either.

Case 3: $(u, v) \in X_1 \cap X_2$

Suppose $\sigma_2(u) > \sigma_2(v)$ and $\mu_1(u, v) = \sigma_1(u) \wedge \sigma_1(v) = \sigma_2(u) = \mu_2(u, v)$. In this case:

$$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v) = \mu_2(u, v),$$

implying that G is not a general weak neutrosophic graph in this situation.

Thus, the union of two general weak neutrosophic graphs may not result in a general weak neutrosophic graph. □

Theorem 3.12. *The $|\mu|$-complement of a general weak neutrosophic graph is a general weak neutrosophic graph.*

Proof. Let $G^{|\mu|} = (\sigma, \mu^{|\mu|})$ be the $|\mu|$-complement of a general weak neutrosophic graph $G = (\sigma, \mu)$, where $\mu^{|\mu|}(u, v)$ is defined as:

$$\mu^{|\mu|}(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ |\sigma(u) \wedge \sigma(v) - \mu(u, v)| & \text{if } \mu(u, v) > 0, \end{cases}$$

for all $u, v \in V$.

To prove that $G^{|\mu|}$ is a general weak neutrosophic graph, we must show that:

$$\mu^{|\mu|}(u, v) \neq \sigma(u) \wedge \sigma(v)$$

for all $(u, v) \in \mu^{|\mu|*}$.

Case 1: If $\mu(u, v) > 0$, then by definition:

$$\mu^{|\mu|}(u, v) = |\sigma(u) \wedge \sigma(v) - \mu(u, v)|.$$

Since $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ (as G is a general weak neutrosophic graph), it follows that $\mu^{|\mu|}(u, v) \neq \sigma(u) \wedge \sigma(v)$.

Case 2: If $\mu(u,v)=0$, then by definition:

$$\mu^{|\mu|}(u,v)=0.$$

In this case, $\mu^{|\mu|}(u,v)\neq\sigma(u)\wedge\sigma(v)$.

Thus, in both cases, $\mu^{|\mu|}(u,v)\neq\sigma(u)\wedge\sigma(v)$, proving that the $|\mu|$-complement of a general weak neutrosophic graph is indeed a general weak neutrosophic graph. □

Theorem 3.13 (c-Complement Preservation)**.** *Let $G=(V,E,\sigma,\mu)$ be a general neutrosophic graph. Its c-complement is*

$$G^c=(V,E,\sigma,\mu^c),\qquad \mu^c(e)=\begin{cases}0, & \mu(e)=0,\\ 1-\mu(e), & \mu(e)>0,\end{cases}$$

with subtraction taken componentwise in $[0,1]^3$. Then G^c is a general neutrosophic graph.

Proof. Since $0\leq\mu(e)\leq 1$ componentwise, we have $0\leq 1-\mu(e)\leq 1$, so $\mu^c(e)\in[0,1]^3$ for all $e\in E$. The vertex function remains unchanged. Therefore G^c still satisfies Definition. □

Theorem 3.14 ("Absolute-μ" Complement Preservation)**.** *Let $G=(V,E,\sigma,\mu)$ be a general neutrosophic graph. Its $|\mu|$-complement is*

$$G^{|\mu|}=(V,E,\sigma,\mu^{|\mu|}),\qquad \mu^{|\mu|}(e)=\begin{cases}0, & \mu(e)=0,\\ \big|\min\{\sigma(u),\sigma(v)\}-\mu(e)\big|, & e=(u,v)\in E,\end{cases}$$

where min *and absolute value are taken componentwise. Then $G^{|\mu|}$ is a general neutrosophic graph.*

Proof. For each edge e, $\min\{\sigma(u),\sigma(v)\}\in[0,1]^3$ and $\mu(e)\in[0,1]^3$, so their componentwise difference in absolute value also lies in $[0,1]^3$. Hence $\mu^{|\mu|}$ is well-defined and $G^{|\mu|}$ satisfies Definition. □

3.2. *Anti Neutrosophic Graph*

We define *Anti Neutrosophic Graphs* as an extension of anti-fuzzy graphs. In this model, each vertex and edge is assigned a triplet of membership values representing anti-truth, anti-indeterminacy, and anti-falsity. This framework allows us to capture the "anti" aspects of uncertainty in graph structures.

Definition 3.15 (Anti Neutrosophic Graph)**.** Let V be a non-empty set and let $E \subseteq V \times V$ be the edge set. An *anti neutrosophic graph* is defined as the triple

$$G_A = (V, \eta, \rho),$$

where:

- $\eta : V \to [0,1] \times [0,1] \times [0,1]$ is the *anti neutrosophic vertex function.* For each vertex $v \in V$,

$$\eta(v) = \big(\eta_1(v), \eta_2(v), \eta_3(v)\big),$$

  where:
  - $\eta_1(v)$ is the *anti-truth membership* of v,
  - $\eta_2(v)$ is the *anti-indeterminacy membership* of v,
  - $\eta_3(v)$ is the *anti-falsity membership* of v.
- $\rho : V \times V \to [0,1] \times [0,1] \times [0,1]$ is the *anti neutrosophic edge function.* For each edge $(u, v) \in E$,

$$\rho(u, v) = \big(\rho_1(u, v), \rho_2(u, v), \rho_3(u, v)\big),$$

  where:
  - $\rho_1(u, v)$ is the *anti-truth membership* of the edge (u, v),
  - $\rho_2(u, v)$ is the *anti-indeterminacy membership* of the edge (u, v),
  - $\rho_3(u, v)$ is the *anti-falsity membership* of the edge (u, v).

The membership functions must satisfy, for all $(u, v) \in E$,

$$\rho_1(u, v) \geq \eta_1(u) \vee \eta_1(v), \quad \rho_2(u, v) \geq \eta_2(u) \vee \eta_2(v), \quad \rho_3(u, v) \geq \eta_3(u) \vee \eta_3(v),$$

where $\vee$ denotes the maximum operation.

Theorem 3.16 (Anti–Neutrosophic Graphs Are Neutrosophic Graphs)**.** *Let $G = (V, E, \eta, \rho)$ be an anti-neutrosophic graph as in Definition, i.e.*

$$\eta : V \to [0,1]^3, \quad \rho : E \to [0,1]^3, \quad \rho_i(u, v) \geq \max\{\eta_i(u), \eta_i(v)\} \quad (i = 1, 2, 3).$$

Then G is a neutrosophic graph (Definition) by setting

$$\sigma = \eta, \qquad \mu = \rho.$$

Proof. Since η and ρ already take values in $[0,1]^3$, they satisfy the type requirements of a neutrosophic vertex-function σ and edge-function μ. Hence G meets Definition and is a neutrosophic graph. □

Theorem 3.17 (Induced Subgraphs). *Let $G = (V, E, \eta, \rho)$ be an anti-neutrosophic graph and $U \subseteq V$. The vertex-induced subgraph*

$$G[U] \;=\; \bigl(U,\; E_U,\; \eta|_U,\; \rho|_{E_U}\bigr),$$

where $E_U = \{(u, v) \in E : u, v \in U\}$, is again an anti-neutrosophic graph.

Proof. Restriction of η and ρ to U and E_U clearly preserves their codomain $[0, 1]^3$, and for every $(u, v) \in E_U$,

$$\rho_i(u, v) \;\geq\; \max\{\eta_i(u), \eta_i(v)\}$$

remains valid. Thus $G[U]$ satisfies Definition. □

Theorem 3.18 (Closure under Union). *Let*

$$G_1 = (V, E_1, \eta^1, \rho^1), \quad G_2 = (V, E_2, \eta^2, \rho^2)$$

be two anti-neutrosophic graphs on the same vertex set V. Define their union

$$G = G_1 \cup G_2 = \bigl(V,\; E_1 \cup E_2,\; \eta,\; \rho\bigr)$$

by taking the componentwise maximum

$$\eta(v) = \max\{\eta^1(v), \eta^2(v)\}, \qquad \rho(u, v) = \max\{\rho^1(u, v), \rho^2(u, v)\}.$$

Then G is an anti-neutrosophic graph.

Proof. Since max of two vectors in $[0, 1]^3$ remains in $[0, 1]^3$, the codomain requirement holds. For each $(u, v) \in E_1 \cup E_2$,

$$\rho_i(u, v) = \max\{\rho_i^1(u, v), \rho_i^2(u, v)\} \;\geq\; \max\{\eta_i^1(u), \eta_i^1(v), \eta_i^2(u), \eta_i^2(v)\} = \max\{\eta_i(u), \eta_i(v)\},$$

so Definition is satisfied. □

Theorem 3.19 (Embedding of Anti-Fuzzy Graphs). *Let $G_{af} = (V, E, \tilde{\eta}, \tilde{\rho})$ be an anti-fuzzy graph with*

$$\tilde{\eta} : V \to [0, 1], \quad \tilde{\rho} : E \to [0, 1], \quad \tilde{\rho}(u, v) \geq \max\{\tilde{\eta}(u), \tilde{\eta}(v)\}.$$

Then

$$G_{an} = (V, E, \eta, \rho), \quad \eta(v) = (\tilde{\eta}(v), 0, 0), \quad \rho(u, v) = (\tilde{\rho}(u, v), 0, 0)$$

is an anti-neutrosophic graph.

Proof. Clearly η, ρ take values in $[0, 1]^3$. Moreover

$$\rho_1(u, v) = \tilde{\rho}(u, v) \geq \max\{\tilde{\eta}(u), \tilde{\eta}(v)\} = \max\{\eta_1(u), \eta_1(v)\},$$

while $\rho_2, \rho_3, \eta_2, \eta_3$ are identically zero, so the required inequalities hold. □

Example 3.20 (Anti Neutrosophic Graph Example)**.** Consider an anti neutrosophic graph $G_A = (V, \eta, \rho)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Define the anti neutrosophic vertex function η by:

$$\eta(v_1) = (0.2,\ 0.7,\ 0.1),$$

$$\eta(v_2) = (0.3,\ 0.6,\ 0.2),$$

$$\eta(v_3) = (0.4,\ 0.5,\ 0.3).$$

For the edge (v_1, v_2), compute:

$$\eta_1(v_1) \vee \eta_1(v_2) = \max\{0.2, 0.3\} = 0.3,$$

$$\eta_2(v_1) \vee \eta_2(v_2) = \max\{0.7, 0.6\} = 0.7,$$

$$\eta_3(v_1) \vee \eta_3(v_2) = \max\{0.1, 0.2\} = 0.2.$$

A valid choice for the edge membership is:

$$\rho(v_1, v_2) = (0.4,\ 0.8,\ 0.3),$$

which satisfies

$$\rho_1(v_1, v_2) \geq 0.3$$

,

$$\rho_2(v_1, v_2) \geq 0.7$$

, and

$$\rho_3(v_1, v_2) \geq 0.2$$

.

For the edge (v_2, v_3):

$$\eta_1(v_2) \vee \eta_1(v_3) = \max\{0.3, 0.4\} = 0.4,$$

$$\eta_2(v_2) \vee \eta_2(v_3) = \max\{0.6, 0.5\} = 0.6,$$

$$\eta_3(v_2) \vee \eta_3(v_3) = \max\{0.2, 0.3\} = 0.3.$$

Choose:

$$\rho(v_2, v_3) = (0.5,\ 0.7,\ 0.4).$$

For the edge (v_1, v_3):

$$\eta_1(v_1) \vee \eta_1(v_3) = \max\{0.2, 0.4\} = 0.4,$$

$$\eta_2(v_1) \vee \eta_2(v_3) = \max\{0.7, 0.5\} = 0.7,$$

$$\eta_3(v_1) \vee \eta_3(v_3) = \max\{0.1, 0.3\} = 0.3.$$

A suitable assignment is:

$$\rho(v_1, v_3) = (0.5,\ 0.8,\ 0.4).$$

This example demonstrates an anti neutrosophic graph where each edge's membership values satisfy the condition relative to the maximum corresponding vertex memberships.

3.3. *Semi-Neutrosophic Graphs*

We define *semi-neutrosophic graphs* as an extension of semi-fuzzy graphs. In a semi-neutrosophic graph, each vertex is assigned a triplet of membership values representing its truth, indeterminacy, and falsity degrees, and each edge is similarly assigned a triplet reflecting the corresponding relationships between vertices. Formally, we have:

Definition 3.21 (Semi-Neutrosophic Graph)**.** Let V be a non-empty set. A *semi-neutrosophic graph* is defined as

$$G = (\eta, \rho),$$

where:

- $\eta = (\eta_1, \eta_2, \eta_3)$ is the *neutrosophic vertex function* with

  $$\eta_i : V \to [0, 1] \quad (i = 1, 2, 3),$$

  and for each vertex $v \in V$:
  - $\eta_1(v)$ denotes the truth membership of v,
  - $\eta_2(v)$ denotes the indeterminacy membership of v,
  - $\eta_3(v)$ denotes the falsity membership of v.
- $\rho = (\rho_1, \rho_2, \rho_3)$ is the *neutrosophic edge function* with

  $$\rho_i : V \times V \to [0, 1] \quad (i = 1, 2, 3),$$

  and for each edge $e = (u, v)$ (with $u, v \in V$):
  - $\rho_1(e)$ denotes the truth membership of e,
  - $\rho_2(e)$ denotes the indeterminacy membership of e,
  - $\rho_3(e)$ denotes the falsity membership of e.

The graph $G = (\eta, \rho)$ is called a *semi-neutrosophic graph* if it satisfies the following conditions:

- The sum of the truth memberships for all vertex pairs is less than 1:

  $$\sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y) < 1,$$

  where $\wedge$ denotes the minimum operation.

- The sum of the indeterminacy memberships for all vertex pairs is greater than 1:

$$\sum_{x,y\in V} \eta_2(x) \vee \eta_2(y) > 1,$$

where $\vee$ denotes the maximum operation, ensuring that indeterminacy is adequately represented.

- The sum of the falsity memberships for all vertex pairs is greater than 1:

$$\sum_{x,y\in V} \eta_3(x) \vee \eta_3(y) > 1,$$

ensuring that the falsity component remains within a balanced range.

Definition 3.22 (Density of a Semi-Neutrosophic Graph)**.** For a semi-neutrosophic graph $G = (\eta, \rho)$ with underlying crisp graph $G^* = (V, E)$, the *density* of G is defined as the tuple

$$D(G) = \big(D_T(G),\, D_I(G),\, D_F(G)\big),$$

where:

- The *truth density* $D_T(G)$ is given by

$$D_T(G) = \frac{2\sum_{u\in V}\sum_{v\in V} \rho_1(u,v)}{\sum_{u\in V}\sum_{v\in V}\Big(\eta_1(u) \wedge \eta_1(v)\Big)}.$$

- The *indeterminacy density* $D_I(G)$ is given by

$$D_I(G) = \frac{2\sum_{u\in V}\sum_{v\in V} \rho_2(u,v)}{\sum_{u\in V}\sum_{v\in V}\Big(\eta_2(u) \vee \eta_2(v)\Big)}.$$

- The *falsity density* $D_F(G)$ is given by

$$D_F(G) = \frac{2\sum_{u\in V}\sum_{v\in V} \rho_3(u,v)}{\sum_{u\in V}\sum_{v\in V}\Big(\eta_3(u) \vee \eta_3(v)\Big)}.$$

Definition 3.23 (Balanced and Complete Semi-Neutrosophic Graphs)**.** A semi-neutrosophic graph $G = (\eta, \rho)$ is said to be:

- *Balanced* if for every non-empty neutrosophic subgraph H of G,

$$D_T(H) \le D_T(G), \quad D_I(H) \le D_I(G), \quad D_F(H) \le D_F(G).$$

- *Complete* if for every pair of distinct vertices $u, v \in V$, the edge between them satisfies

$$\rho_1(u,v) = \eta_1(u) \wedge \eta_1(v), \quad \rho_2(u,v) = \eta_2(u) \vee \eta_2(v), \quad \rho_3(u,v) = \eta_3(u) \vee \eta_3(v).$$

Example 3.24 (Semi-Neutrosophic Graph Example)**.** Consider a semi-neutrosophic graph $G = (\eta, \rho)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

We define the neutrosophic vertex function η as follows:

$$\begin{aligned} \eta_1(v_1) &= 0.05, \quad \eta_1(v_2) = 0.04, \quad \eta_1(v_3) = 0.03, \\ \eta_2(v_1) &= 0.70, \quad \eta_2(v_2) = 0.60, \quad \eta_2(v_3) = 0.50, \\ \eta_3(v_1) &= 0.40, \quad \eta_3(v_2) = 0.30, \quad \eta_3(v_3) = 0.20. \end{aligned}$$

Note that the sum of the truth memberships for all vertex pairs is

$$\sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y) = 0.05 + 0.04 + 0.03 + 0.04 + 0.04 + 0.03 + 0.03 + 0.03 + 0.03 = 0.32 < 1.$$

For the indeterminacy and falsity components, using the maximum operator we obtain

$$\sum_{x,y \in V} \eta_2(x) \vee \eta_2(y) = 5.8 > 1, \quad \sum_{x,y \in V} \eta_3(x) \vee \eta_3(y) = 3.1 > 1.$$

Next, we define the neutrosophic edge function ρ for the undirected edges (assuming $\rho(u, v) = \rho(v, u)$ and $\rho(v, v) = 0$):

$$\begin{aligned} \rho_1(v_1, v_2) &= 0.04, \quad \rho_1(v_1, v_3) = 0.03, \quad \rho_1(v_2, v_3) = 0.03, \\ \rho_2(v_1, v_2) &= 0.80, \quad \rho_2(v_1, v_3) = 0.75, \quad \rho_2(v_2, v_3) = 0.78, \\ \rho_3(v_1, v_2) &= 0.50, \quad \rho_3(v_1, v_3) = 0.40, \quad \rho_3(v_2, v_3) = 0.45. \end{aligned}$$

Assuming symmetry, the total contributions from edges are computed over all unordered pairs (or equivalently, summing over all ordered pairs with $\rho(v, u) = \rho(u, v)$ and then multiplying by $1/2$).

Computing the Densities:

Truth Density:

$$\sum_{u,v \in V} \rho_1(u, v) = 2\Big[0.04 + 0.03 + 0.03\Big] = 0.20,$$

so

$$D_T(G) = \frac{2 \cdot 0.20}{0.32} = \frac{0.40}{0.32} \approx 1.25.$$

Indeterminacy Density:

$$\sum_{u,v \in V} \rho_2(u, v) = 2\Big[0.80 + 0.75 + 0.78\Big] = 2(2.33) = 4.66,$$

and since

$$\sum_{u,v \in V} \eta_2(u) \vee \eta_2(v) = 5.8,$$

we have

$$D_I(G) = \frac{2 \cdot 4.66}{5.8} \approx \frac{9.32}{5.8} \approx 1.61.$$

Falsity Density:

$$\sum_{u,v\in V} \rho_3(u,v) = 2\Big[0.50 + 0.40 + 0.45\Big] = 2(1.35) = 2.70,$$

and

$$\sum_{u,v\in V} \eta_3(u) \vee \eta_3(v) = 3.1,$$

so

$$D_F(G) = \frac{2 \cdot 2.70}{3.1} \approx \frac{5.40}{3.1} \approx 1.74.$$

Thus, the density of G is given by

$$D(G) \approx \big(1.25,\ 1.61,\ 1.74\big).$$

This example illustrates a semi-neutrosophic graph where the vertex and edge membership functions satisfy the stipulated conditions and demonstrates how the densities for truth, indeterminacy, and falsity are computed.

Theorem 3.25 (Semi-Neutrosophic Graphs are Neutrosophic Graphs)**.** *Let $G = (V, E, \eta, \rho)$ be a semi-neutrosophic graph as in Definition, where*

$$\eta = (\eta_1, \eta_2, \eta_3), \quad \eta_i : V \to [0, 1], \qquad \rho = (\rho_1, \rho_2, \rho_3), \quad \rho_i : E \to [0, 1],$$

and for each edge $e = (u, v) \in E$ the neutrosophic inequality $\rho_i(u, v) \le \min\{\eta_i(u), \eta_i(v)\}$ $(i = 1, 2, 3)$ holds. Then G is a neutrosophic graph.

Proof. By hypothesis, each vertex-function component η_i and each edge-function component ρ_i takes values in $[0, 1]$, so

$$\eta : V \to [0, 1]^3, \quad \rho : E \to [0, 1]^3.$$

Moreover, the condition

$$\rho_i(u, v) \le \min\{\eta_i(u), \eta_i(v)\} \quad \text{for all } (u, v) \in E,\ i = 1, 2, 3$$

is exactly the requirement for a neutrosophic graph. Therefore G satisfies all the axioms of a neutrosophic graph. □

Theorem 3.26 (Semi-Neutrosophic Graphs Generalize Semi-Fuzzy Graphs)**.** *Let (V, E, σ, μ) be a semi-fuzzy graph, so that*

$$\sigma : V \to [0, 1], \quad \mu : E \to [0, 1], \qquad \sum_{x,y\in V} \sigma(x) \wedge \sigma(y) < 1.$$

Define

$$\eta_1(v) = \sigma(v), \quad \eta_2(v) = 1, \quad \eta_3(v) = 1 \quad (\forall v \in V),$$

$$\rho_1(u, v) = \mu(u, v), \quad \rho_2(u, v) = 1, \quad \rho_3(u, v) = 1 \quad (\forall (u, v) \in E).$$

Then $G' = (V, E, \eta, \rho)$ with $\eta = (\eta_1, \eta_2, \eta_3)$, $\rho = (\rho_1, \rho_2, \rho_3)$ is a semi-neutrosophic graph.

Proof. We must verify the three semi-neutrosophic conditions:

$$\sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y) = \sum_{x,y \in V} \sigma(x) \wedge \sigma(y) \ < \ 1$$

by the semi-fuzzy hypothesis. Next,

$$\sum_{x,y \in V} \eta_2(x) \vee \eta_2(y) = \sum_{x,y \in V} 1 = |V|^2 \ > \ 1,$$

and similarly $\sum_{x,y} \eta_3(x) \vee \eta_3(y) = |V|^2 > 1$. No further constraints are imposed on ρ_i. Hence G' satisfies all requirements of Definition. □

Next, we will prove the following theorem using properties such as balanced.

Theorem 3.27. *Every complete semi-neutrosophic graph is balanced.*

Proof. Let $G = (\eta, \rho)$ be a complete semi-neutrosophic graph. According to the definition of completeness, for every pair of distinct vertices $u, v \in V$, the edge between them satisfies the following conditions:

$$\rho_1(u, v) = \eta_1(u) \wedge \eta_1(v), \quad \rho_2(u, v) = \eta_2(u) \vee \eta_2(v), \quad \rho_3(u, v) = \eta_3(u) \vee \eta_3(v),$$

where $\wedge$ represents the minimum operation, and $\vee$ represents the maximum operation.

In a complete semi-neutrosophic graph, for every edge (u, v), we have:

$$\rho_1(u, v) = \eta_1(u) \wedge \eta_1(v).$$

Thus, for the entire graph G, the total sum of the truth memberships of the edges is:

$$\sum_{x,y \in V} \rho_1(x, y) = \sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y).$$

Hence, the truth density of G simplifies to:

$$D_T(G) = \frac{2 \sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y)}{\sum_{x,y \in V} \eta_1(x) \wedge \eta_1(y)} = 2.$$

Now, consider any non-empty subgraph $H = (V_H, E_H)$ of G. To complete H, we may need to add edges to E_H, where each new edge (u, v) in the completion of H satisfies:

$$\rho_1(u, v) = \eta_1(u) \wedge \eta_1(v).$$

Let $H_0 = (V_H, E_H^0)$ be the completed subgraph, where E_H^0 is the set of all edges necessary to make H_0 a complete subgraph. The truth density of H_0 is:

$$D_T(H_0) = \frac{2 \sum_{x,y \in V_H} \eta_1(x) \wedge \eta_1(y)}{\sum_{x,y \in V_H} \eta_1(x) \wedge \eta_1(y)} = 2.$$

Since $E_H \subseteq E_H^0$, it follows that:

$$\sum_{x,y\in V_H} \rho_1(x,y) \leq \sum_{x,y\in V_H} \eta_1(x) \wedge \eta_1(y),$$

which implies that:

$$D_T(H) \leq D_T(H_0) = 2.$$

Since H_0 is a subgraph of G, we also have:

$$D_T(H_0) \leq D_T(G),$$

and therefore:

$$D_T(H) \leq D_T(H_0) \leq D_T(G).$$

Similarly, for the indeterminacy and falsity densities, the conditions for completeness imply that for each edge $(u,v) \in E$, we have:

$$\rho_2(u,v) = \eta_2(u) \vee \eta_2(v), \quad \rho_3(u,v) = \eta_3(u) \vee \eta_3(v).$$

Thus, the indeterminacy and falsity densities of G simplify as follows:

$$D_I(G) = \frac{2\sum_{x,y\in V} \eta_2(x) \vee \eta_2(y)}{\sum_{x,y\in V} \eta_2(x) \vee \eta_2(y)} = 2,$$

$$D_F(G) = \frac{2\sum_{x,y\in V} \eta_3(x) \vee \eta_3(y)}{\sum_{x,y\in V} \eta_3(x) \vee \eta_3(y)} = 2.$$

As in the truth density case, for any subgraph H, we can conclude:

$$D_I(H) \leq D_I(G), \quad D_F(H) \leq D_F(G).$$

Since for every non-empty subgraph H of G, we have:

$$D_T(H) \leq D_T(G), \quad D_I(H) \leq D_I(G), \quad D_F(H) \leq D_F(G),$$

it follows that the complete semi-neutrosophic graph G is balanced. □

Theorem 3.28. *Any complete semi-neutrosophic graph G has density of truth membership $D_T(G) > 2$.*

Proof. Let $G = (\eta, \rho)$ be a complete semi-neutrosophic graph. By definition, for every pair of vertices $u, v \in V$, the truth membership of the edge between them is given by:

$$\rho_1(u,v) = \eta_1(u) \wedge \eta_1(v),$$

where $\wedge$ denotes the minimum operation on the truth memberships of vertices u and v.

Thus, for the complete semi-neutrosophic graph G, the total sum of truth memberships of edges is:

$$\sum_{x,y\in V} \rho_1(x,y) = \sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y).$$

The truth density $D_T(G)$ is given by:

$$D_T(G) = \frac{2\sum_{x,y\in V} \rho_1(x,y)}{\sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y)}.$$

Since G is a complete graph, we know that $\rho_1(u,v) = \eta_1(u) \wedge \eta_1(v)$ for all $u, v \in V$. Therefore, the numerator simplifies to:

$$2\sum_{x,y\in V} \rho_1(x,y) = 2\sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y).$$

Let $S = \sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y)$, the truth membership sum. Now the truth density becomes:

$$D_T(G) = \frac{2S}{S} = 2.$$

However, since the sum $\sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y) < 1$ for a semi-neutrosophic graph (from the semi-neutrosophic condition $\sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y) < 1$), it follows that the denominator $1 - S$ is positive and less than 1.

Thus, the actual truth density $D_T(G)$ is:

$$D_T(G) = \frac{2}{1-S},$$

where $S = \sum_{x,y\in V} \eta_1(x) \wedge \eta_1(y) < 1$. Since $1 - S < 1$, we have:

$$\frac{1}{1-S} > 1,$$

which implies:

$$D_T(G) = 2 \times \frac{1}{1-S} > 2.$$

Therefore, the truth density $D_T(G) > 2$ for any complete semi-neutrosophic graph G. □

3.4. $*$-*balanced Neutrosophic Graph*

We define a new concept called the $*$-*balanced neutrosophic graph*, which generalizes the $*$-balanced fuzzy graph to the neutrosophic setting. In this model, each vertex v has a neutrosophic membership

$$\sigma(v) = \big(\sigma_1(v), \sigma_2(v), \sigma_3(v)\big),$$

and each edge (u,v) has a neutrosophic membership

$$\mu(u,v) = \big(\mu_1(u,v), \mu_2(u,v), \mu_3(u,v)\big),$$

where the components correspond to the truth, indeterminacy, and falsity degrees, respectively.

Definition 3.29 ($*$-density and $*$-balanced Neutrosophic Graph)**.** Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \quad \text{and} \quad \mu = (\mu_1, \mu_2, \mu_3).$$

The $*$-*density* of G is defined componentwise by

$$D_1^*(G) = \frac{2 \sum_{u,v \in V} \mu_1(u,v)}{\sum_{u \in V} \sigma_1(u)},$$

$$D_2^*(G) = \frac{2 \sum_{u,v \in V} \mu_2(u,v)}{\sum_{u \in V} \sigma_2(u)},$$

$$D_3^*(G) = \frac{2 \sum_{u,v \in V} \mu_3(u,v)}{\sum_{u \in V} \sigma_3(u)}.$$

The neutrosophic graph G is said to be $*$-*balanced* if, for every non-empty neutrosophic subgraph H of G, the following inequalities hold:

$$D_1^*(H) \leq D_1^*(G), \quad D_2^*(H) \leq D_2^*(G), \quad D_3^*(H) \leq D_3^*(G).$$

Example 3.30 ($*$-balanced Neutrosophic Graph Example)**.** Consider a neutrosophic graph $G = (V, E, \sigma, \mu)$ with the vertex set

$$V = \{v_1, v_2, v_3\}.$$

Define the neutrosophic vertex function σ by:

$$\sigma(v_1) = (0.8,\ 0.1,\ 0.1), \quad \sigma(v_2) = (0.6,\ 0.2,\ 0.2), \quad \sigma(v_3) = (0.7,\ 0.15,\ 0.15).$$

Thus, the sum of the truth memberships over all vertices is

$$\sum_{u \in V} \sigma_1(u) = 0.8 + 0.6 + 0.7 = 2.1.$$

Define the neutrosophic edge function μ on the undirected edges (assuming symmetry, i.e., $\mu(u, v) = \mu(v, u)$) as follows:

$$\mu(v_1, v_2) = (0.5,\ 0.15,\ 0.10), \quad \mu(v_2, v_3) = (0.4,\ 0.10,\ 0.15), \quad \mu(v_1, v_3) = (0.45,\ 0.12,\ 0.10).$$

The sum of the truth memberships for the edges (considered over unordered pairs) is

$$\mu_1(v_1, v_2) + \mu_1(v_2, v_3) + \mu_1(v_1, v_3) = 0.5 + 0.4 + 0.45 = 1.35.$$

Multiplying by 2 (to account for both orderings) yields

$$2\sum_{u,v\in V}\mu_1(u,v) = 2\times 1.35 = 2.70.$$

Thus, the truth $*$-density is

$$D_1^*(G) = \frac{2.70}{2.1} \approx 1.286.$$

Similarly, for the indeterminacy component, we have:

$$\sum_{u\in V}\sigma_2(u) = 0.1 + 0.2 + 0.15 = 0.45,$$

and

$$\mu_2(v_1,v_2) + \mu_2(v_2,v_3) + \mu_2(v_1,v_3) = 0.15 + 0.10 + 0.12 = 0.37.$$

Then,

$$D_2^*(G) = \frac{2\times 0.37}{0.45} \approx \frac{0.74}{0.45} \approx 1.644.$$

For the falsity component, we have:

$$\sum_{u\in V}\sigma_3(u) = 0.1 + 0.2 + 0.15 = 0.45,$$

and

$$\mu_3(v_1,v_2) + \mu_3(v_2,v_3) + \mu_3(v_1,v_3) = 0.10 + 0.15 + 0.10 = 0.35.$$

Thus,

$$D_3^*(G) = \frac{2\times 0.35}{0.45} \approx \frac{0.70}{0.45} \approx 1.556.$$

To verify the $*$-balanced property, consider the subgraph H induced by the vertex set $\{v_1, v_2\}$. In this subgraph, the sum of truth memberships is

$$\sigma_1(v_1) + \sigma_1(v_2) = 0.8 + 0.6 = 1.4,$$

and the only edge is (v_1, v_2) with $\mu_1(v_1, v_2) = 0.5$. Then,

$$D_1^*(H) = \frac{2\times 0.5}{1.4} \approx \frac{1.0}{1.4} \approx 0.714,$$

which is less than $D_1^*(G) \approx 1.286$. Similar verifications for the indeterminacy and falsity components confirm that

$$D_2^*(H) \le D_2^*(G) \quad \text{and} \quad D_3^*(H) \le D_3^*(G).$$

Hence, G is $*$-balanced.

Theorem 3.31 (*-Balanced Neutrosophic Graphs are Neutrosophic Graphs)**.** *Let* $G = (V, E, \sigma, \mu)$ *be a* $*$*-balanced neutrosophic graph as in Definition, where*

$$\sigma : V \to [0,1]^3, \quad \mu : E \to [0,1]^3,$$

and for every edge $e = (u, v) \in E$ *the neutrosophic condition*

$$\mu_i(e) \ \leq \ \min\{\sigma_i(u), \sigma_i(v)\} \quad (i = 1, 2, 3)$$

holds, together with the additional $*$*-balance inequalities for all nonempty neutrosophic subgraphs. Then* G *satisfies the axioms of a neutrosophic graph.*

Proof. By Definition, G is already a neutrosophic graph equipped with extra "$*$-balance" conditions on its densities. In particular:

- σ and μ map into $[0, 1]^3$,
- for every $e = (u, v) \in E$ and each component i, $\mu_i(e) \leq \min\{\sigma_i(u), \sigma_i(v)\}$.

These are exactly the requirements of Definition. The $*$-balance inequalities impose no further restrictions on the basic neutrosophic structure, so G indeed is a neutrosophic graph. □

Theorem 3.32 (Generalization of $*$-Balanced Fuzzy Graphs)**.** *Let* $F = (V, E, \sigma, \mu)$ *be a* $*$*-balanced fuzzy graph, i.e.*

$$\sigma : V \to [0, 1], \quad \mu : E \to [0, 1], \qquad \mu(u, v) \leq \sigma(u) \wedge \sigma(v),$$

and for every nonempty fuzzy subgraph $H \subseteq F$*,*

$$D^*(H) \ \leq \ D^*(F),$$

where

$$D^*(G) = \frac{2\sum_{(u,v)\in E_G} \mu(u, v)}{\sum_{v\in V_G} \sigma(v)}.$$

Define a neutrosophic graph

$$N = (V, E, \sigma', \mu')$$

by

$$\sigma'(v) = \big(\sigma(v), \sigma(v), \sigma(v)\big), \quad \mu'(u, v) = \big(\mu(u, v), \mu(u, v), \mu(u, v)\big).$$

Then N *is an* $*$*-balanced neutrosophic graph.*

Proof. First, σ' and μ' map into $[0, 1]^3$. Since $\mu(u, v) \leq \sigma(u)\wedge\sigma(v)$, we have for each component i:

$$\mu'_i(u, v) = \mu(u, v) \ \leq \ \min\{\sigma(u), \sigma(v)\} = \min\{\sigma'_i(u), \sigma'_i(v)\},$$

so N satisfies the neutrosophic-graph constraint.

Next, compute each component of the $*$-density of N:

$$D^*_i(N) = \frac{2\sum_{(u,v)\in E} \mu'_i(u, v)}{\sum_{v\in V} \sigma'_i(v)} = \frac{2\sum_{(u,v)\in E} \mu(u, v)}{\sum_{v\in V} \sigma(v)} = D^*(F),$$

for $i = 1, 2, 3$. Now let $H \subseteq F$ be any nonempty fuzzy subgraph, and let $N_H \subseteq N$ be the corresponding induced neutrosophic subgraph. By the same calculation,

$$D_i^*(N_H) = D^*(H) \ \leq \ D^*(F) = D_i^*(N).$$

Thus each component of the $*$-density of N_H does not exceed that of N, so N is $*$-balanced in the neutrosophic sense. □

Next, we will prove the following theorem using properties.

Theorem 3.33. *Any complete neutrosophic graph with* $|V| \geq |E|$ *has *-density* $D_1^*, D_2^*, D_3^*(G) \leq 2$.

Proof. Let $G = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a complete neutrosophic graph, where the vertex set V and edge set E satisfy $|V| \geq |E|$. We are tasked with proving that the *-density values D_1^*, D_2^*, D_3^* of the graph G are less than or equal to 2.

For a complete neutrosophic graph, by the definition of completeness:

$$\mu_1(u, v) = \sigma_1(u) \wedge \sigma_1(v) \quad \forall u, v \in V.$$

Thus, the total sum of the truth memberships over all edges is:

$$\sum_{u,v \in V} \mu_1(u, v) = \sum_{u,v \in V} \sigma_1(u) \wedge \sigma_1(v).$$

We also know that $\sigma_1(u) \wedge \sigma_1(v) \leq \sigma_1(u)$ for each pair $u, v \in V$, meaning:

$$\sum_{u,v \in V} \mu_1(u, v) \leq \sum_{u \in V} \sigma_1(u).$$

Thus, the *-density D_1^* is given by:

$$D_1^*(G) = \frac{2 \sum_{u,v \in V} \mu_1(u, v)}{\sum_{u \in V} \sigma_1(u)} \leq 2.$$

Similarly, for the indeterminacy memberships:

$$\mu_2(u, v) = \sigma_2(u) \vee \sigma_2(v) \quad \forall u, v \in V.$$

The total sum of the indeterminacy memberships over all edges is:

$$\sum_{u,v \in V} \mu_2(u, v) = \sum_{u,v \in V} \sigma_2(u) \vee \sigma_2(v).$$

Since $\sigma_2(u) \vee \sigma_2(v) \geq \sigma_2(u)$, we have:

$$\sum_{u,v \in V} \mu_2(u, v) \leq \sum_{u \in V} \sigma_2(u).$$

Thus, the *-density D_2^* is:

$$D_2^*(G) = \frac{2 \sum_{u,v \in V} \mu_2(u, v)}{\sum_{u \in V} \sigma_2(u)} \leq 2.$$

For the falsity memberships:

$$\mu_3(u,v) = \sigma_3(u) \vee \sigma_3(v) \quad \forall u, v \in V.$$

The total sum of the falsity memberships over all edges is:

$$\sum_{u,v\in V} \mu_3(u,v) = \sum_{u,v\in V} \sigma_3(u) \vee \sigma_3(v).$$

Since $\sigma_3(u) \vee \sigma_3(v) \geq \sigma_3(u)$, we have:

$$\sum_{u,v\in V} \mu_3(u,v) \leq \sum_{u\in V} \sigma_3(u).$$

Thus, the *-density D_3^* is:

$$D_3^*(G) = \frac{2\sum_{u,v\in V} \mu_3(u,v)}{\sum_{u\in V} \sigma_3(u)} \leq 2.$$

Since $D_1^*(G) \leq 2$, $D_2^*(G) \leq 2$, and $D_3^*(G) \leq 2$ for the complete neutrosophic graph G with $|V| \geq |E|$, we conclude that the *-densities of the graph satisfy:

$$D_1^*(G), D_2^*(G), D_3^*(G) \leq 2.$$

□

Theorem 3.34 (Complete Neutrosophic Graphs Are *-Balanced)**.** *Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph such that for every pair of distinct vertices $u, v \in V$,*

$$\mu_i(u,v) = \min\{\sigma_i(u), \sigma_i(v)\} \quad (i = 1, 2, 3).$$

Then for each i,

$$D_i^*(G) = \frac{2\sum_{u,v\in V} \mu_i(u,v)}{\sum_{v\in V} \sigma_i(v)} = \frac{2\sum_{u,v\in V} \min\{\sigma_i(u), \sigma_i(v)\}}{\sum_{v\in V} \sigma_i(v)} = 2,$$

*and every nonempty induced subgraph $H \subseteq G$ is also complete, so $D_i^*(H) = 2 \leq D_i^*(G)$. Hence G is *-balanced.*

Proof. Since $\mu_i(u,v) = \min\{\sigma_i(u), \sigma_i(v)\}$, one computes $\sum_{u,v} \mu_i(u,v) = \sum_{u,v} \min\{\sigma_i(u), \sigma_i(v)\}$. By symmetry this equals $\sum_v \sigma_i(v)$, giving $D_i^*(G) = 2$. Any induced subgraph of a complete graph is complete on its vertex set, so its *-density is likewise 2, yielding the required inequalities. □

Theorem 3.35 (Induced Subgraphs of *-Balanced Graphs Are *-Balanced)**.** *Let G be a *-balanced neutrosophic graph and H an induced subgraph on $U \subseteq V$. Then H is *-balanced.*

Proof. Any nonempty neutrosophic subgraph $K \subseteq H$ is also a subgraph of G. Since G is *-balanced,

$$D_i^*(K) \ \le\ D_i^*(G).$$

But $D_i^*(G)$ is unchanged when passing to the induced subgraph H (the denominators and numerators restrict to sums over U and its edges), so $D_i^*(K) \le D_i^*(H) \le D_i^*(G)$. Thus H satisfies the *-balance inequalities. □

Theorem 3.36 (Positive Scaling Invariance)**.** *Let $G = (V, E, \sigma, \mu)$ be any neutrosophic graph and $\alpha > 0$. Define $\sigma_i'(v) = \alpha\,\sigma_i(v)$ and $\mu_i'(u, v) = \alpha\,\mu_i(u, v)$. Then*

$$D_i^*(G') = \frac{2\sum_{u,v}\mu_i'(u,v)}{\sum_v \sigma_i'(v)} = \frac{2\alpha\sum_{u,v}\mu_i(u,v)}{\alpha\sum_v \sigma_i(v)} = D_i^*(G),$$

*and G is *-balanced if and only if G' is *-balanced.*

Proof. Scaling both numerator and denominator of each componentwise *-density by α cancels out. Moreover, every subgraph of G' corresponds to a scaled subgraph of G, preserving the ordering of their *-densities. Hence *-balance is invariant under positive scaling. □

4. Conclusion and Future tasks

We explored new graph classes, including the General-Neutrosophic Graph, Anti-Neutrosophic Graph, *-Balanced-Neutrosophic Graph, and Semi-Neutrosophic Graph. In considering the application of these definitions to real-world scenarios, we aim to explore whether more suitable definitions may exist. Additionally, we plan to investigate these graph classes in the context of hypergraphs [10, 15] and superhypergraphs [18–20, 37, 38], as well as their applicability to directed graphs.

We also intend to examine graph width parameters [7, 8] for these graph classes. Furthermore, our research will extend to studying graph problems and algorithms related to these graph structures.

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Data Availability

This research is entirely theoretical, without any data collection or analysis involved. We encourage future studies to explore empirical approaches to expand and validate the ideas introduced here.

Ethical Approval

As this research is exclusively theoretical in nature, it does not involve human participants or animal subjects. Therefore, ethical approval is not required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to this research or its publication.

Disclaimer

This work introduces theoretical concepts that have yet to undergo practical validation or testing. Future researchers are encouraged to apply and evaluate these ideas in empirical settings. While every effort has been made to ensure the accuracy of the findings and proper citation of references, unintentional errors or omissions may remain. Readers are advised to cross-check referenced materials independently. The opinions and conclusions expressed in this paper represent the authors' views and do not necessarily reflect those of their affiliated organizations.

References

1. Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.
2. Muhammad Akram, Muhammad Akram, and Di Cecco. *m- polar fuzzy graphs.* Springer, 2019.
3. Muhammad Akram, Sundas Shahzadi, and AB Saeid. Single-valued neutrosophic hypergraphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(1):122–135, 2018.
4. Talal Al-Hawary and Laith AlMomani. -balanced fuzzy graphs. *arXiv preprint arXiv:1804.08677*, 2018.
5. TALAL Al-Hawary and Mohammed Hashim. Semi-fuzzy graphs. *Bol. Da Socie. Paran. de Matem.*
6. K. R. Bhutani. On automorphism of fuzzy graphs. *Pattern Recognition Letters*, 9:159–162, 1989.

7. Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1–21, 1993.
8. Hans L Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical computer science*, 209(1-2):1–45, 1998.
9. André Bouchet. Circle graph obstructions. *Journal of Combinatorial Theory, Series B*, 60(1):107–144, 1994.
10. Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.
11. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.
12. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
13. Paul Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the second annual symposium on Computational geometry*, pages 169–177, 1986.
14. N Deva and A Felix. Bipolar intuitionistic anti fuzzy graphs. In *AIP Conference Proceedings*, volume 2385. AIP Publishing, 2022.
15. Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.
16. Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.
17. Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.
18. Takaaki Fujita. Superhyperbranch-width and superhypertree-width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 367, 2025.
19. Takaaki Fujita. Superhypertree-length and superhypertree-breadth in superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 41, 2025.
20. Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.
21. Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.
22. A Nagoor Gani and K Radha. On regular fuzzy graphs. 2008.
23. Derek Allan Holton and John Sheehan. *The Petersen Graph*, volume 7. Cambridge University Press, 1993.
24. S Satham Hussain, R Jahir Hussain, and Ghulam Muhiuddin. *Neutrosophic vague line graphs*, volume 36. Infinite Study, 2020.
25. John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.
26. TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.
27. Toto Nusantara, Desi Rahmadani, Yuliana Trisanti, Abdullah Bin Gani, et al. Anti fuzzy line graphs. In *Journal of Physics: Conference series*, volume 1783, page 012097. IOP Publishing, 2021.
28. Madhumangal Pal, Sovan Samanta, and Ganesh Ghorai. *Modern trends in fuzzy graph theory*. Springer, 2020.
29. R Parvathi and MG Karunambigai. Intuitionistic fuzzy graphs. In *Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer, 2006.

30. Hossein Rashmanlou and Madhumangal Pal. Balanced interval-valued fuzzy graphs. 2013.
31. Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.
32. Sovan Samanta and Madhumangal Pal. Irregular bipolar fuzzy graphs. *arXiv preprint arXiv:1209.1682*, 2012.
33. Sovan Samanta and Madhumangal Pal. Fuzzy planar graphs. *IEEE Transactions on Fuzzy Systems*, 23(6):1936–1942, 2015.
34. Azhar A Sangoora and Alyaa A Alwanb. Anti fuzzy graph. *Int. J. Adv. Appl. Math. and Mech*, 5(3):66–74, 2018.
35. Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.
36. Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
37. Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.
38. Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
39. Metawee Songsaeng and Aiyared Iampan. *Neutrosophic set theory applied to UP-algebras*. Infinite Study, 2019.
40. Ben C Stöver and Kai F Müller. Treegraph 2: combining and visualizing evidence from different phylogenetic analyses. *BMC bioinformatics*, 11:1–9, 2010.
41. Abolfazl Tehranian and Hamid Reza Maimani. A study of the total graph. 2011.
42. Yiding Yang, Xinchao Wang, Mingli Song, Junsong Yuan, and Dacheng Tao. Spagan: Shortest path graph attention network. *arXiv preprint arXiv:2101.03464*, 2021.
43. Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

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