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Properties of Neutrosophic $\varkappa$- bi-ideals in near-subtraction semigroups

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Abstract. This paper presents the concept of neutrosophic $\varkappa$-bi-ideals within the framework of near-subtraction semigroups and explores the associated findings. The concept of neutrosophic intersections is explored. In addition, we establish some results regarding the homomorphic preimage of a neutrosophic $\varkappa$-bi-ideal in near-subtraction semigroups.

Keywords: Subtraction semigroup, ideals, neutrosophic structures, $\varkappa-$bi-ideals, homomorphism.

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1. **Introduction**

In [15], Schein conducted a study on functions of the form $(\Sigma, \circ, \setminus)$, where Σ represents a collection of systems that is closed under function composition $\circ$ (making $(\Sigma, \circ)$ a function semigroup) and set-theoretic subtraction $\setminus$ (making $(\Sigma, \setminus)$ a subtraction algebra). Zelinka conducted a study on Schein's multiplication structure and successfully resolved a problem related to atomic subtraction algebras, as documented in [18]. In subtraction algebras [8], Jun et al. explored the notion of ideals by examining their characterization. Jun et al. [7] investigated the ideals generated by a set and their corresponding consequences. Dheena et al. [2] proposed the concepts of near-subtraction semigroups along with their strongly regular variants. The authors discovered a strongly regular equivalent assertion for a near-subtraction semigroup.

According to Zadeh [17], a fuzzy subset $\varkappa$ of a set L can be characterized as a function assigning to each element of L a value within the range $[0, 1]$. Subsequently, this notion has been efficiently employed in various domains, such as image manipulation, system regulation, engineering, robotics, industrial automation, and optimisation. Molodtsov's soft set theory, developed in 1999 as an extension of fuzzy set theory, has demonstrated its effectiveness in various domains. The idea of fuzzy bi-ideals in near-subtraction semigroups was introduced by Chinnadurai et al. in their paper [1]. The authors also provided several characterizations of these fuzzy bi-ideals.

In response to the persistent uncertainty, Smarandache devised neutrosophic sets. Both fuzzy sets and intuitionistic fuzzy sets exhibit an increased scope. The three attributes, namely falsity (F), indeterminacy (I), and truth (T), can be employed to characterise neutrosophic sets. In order to address problems arising from vague information, these collections can be utilised in various manners. A neutrosophic set can be used to differentiate between absolute and relative membership functions. Smarandache utilised these collections for unorthodox analyses, including sports outcomes (loss, draw, and victory), decision-making theory, and similar subjects. Khan et al. studied ϵ-neutrosophic $\varkappa$-subsemigroup and a semigroup in [9]. Elavarasan et al. [3] investigated the concept of neutrosophic $\varkappa$-ideals in semigroups. Elavarasan et al. analysed the properties of neutrosophic bi-filters and filters in semigroups [4]. In [12], Muhiuddin et al. defined and described neutrosophic $\varkappa$-interior ideals and neutrosophic $\varkappa$-ideals in ordered semigroups, respectively. This area has been explored by several authors (See [4–6, 13, 14, 16]).

In [11], Muhiuddin et al. examined neutrosophic $\mathfrak{N}-$ideals and discovered several analogous findings in near-subtraction semigroups. In addition, they showcased the idea of a neutrosophic $\varkappa-$ intersection. Furthermore, the researchers explored the concept of a homomorphism in a near-subtraction semigroup with a neutrosophic $\varkappa-$ structure. They derived several results based on the preimage of a neutrosophic $\varkappa-$ left (or right) ideal in a homomorphic neutrosophic $\varkappa-$ structure.

As an extension of these ideas, this paper introduces the idea of neutrosophic $\varkappa-$ bi-ideals in near-subtraction semigroups and makes some statements that go with it. Here, we present an example that illustrates the fact that not all neutrosophic $\varkappa-$ bi-ideals can be classified as neutrosophic $\varkappa-$ ideals. Moreover, we establish the definition of the preimage of a neutrosophic $\varkappa-$ bi-ideal in near-subtraction semigroups.

2. **Preliminaries**

We will provide some fundamental definitions of near-subtraction semigroups and hybrid structures. The power set of a set J is represented by $\mathfrak{P}(J)$.

Definition 2.1. [15] Subtraction algebra is defined as a set $\mathbb{N}(\neq \emptyset)$ with the binary operation "$-$" that satisfies the conditions:

(i) $h_1 - (w_1 - h_1) = h_1$,

(ii) $h_1 - (h_1 - w_1) = w_1 - (w_1 - h_1)$,

(iii) $(h_1 - w_1) - s_1 = (h_1 - s_1) - w_1 \ \forall h_1, w_1, s_1 \in \mathbb{N}$.

Definition 2.2. [2] A right (resp., left) near-subtraction semigroup refers to a non-empty set $\mathbb{N}$ equipped with two binary operations, denoted by "$-$" and "$\cdot$", which satisfy the following specific conditions:

(a) $(\mathbb{N}, -)$ is a subtraction algebra.

(b) $(\mathbb{N}, \cdot)$ is a semigroup.

(c) $(j_0 - j_1)j_2 = j_0j_2 - j_1j_2$ (resp., $j_0(j_1 - j_2) = j_0j_1 - j_0j_2$) $\forall j_0, j_1, j_2 \in \mathbb{N}$.

It is obvious that $0l_0 = 0 \ \forall l_0 \in \mathbb{N}$.

Unless otherwise specified, the term $\mathbb{N}$ refers to a right near-subtraction semigroup (briefly, NSS).

Definition 2.3. [2] For a NSS $\mathbb{N}$,

(i) $\mathbb{N}$ is called a zero-symmetric if $k_10 = 0 \ \forall k_1 \in \mathbb{N}$.

(ii) $J \subseteq \mathbb{N}\backslash\{\emptyset\}$ is defined as a near-subtraction subsemigroup of $\mathbb{N}$ if $y_0 - y_1, y_0y_1 \in J$ whenever $y_0, y_1 \in J$.

Definition 2.4. [10] For $C, D \in \mathfrak{P}(\mathbb{N})$, the product and $*$ product are described as below:

$$CD = \{c_1d_1 \mid c_1 \in C \ and \ d_1 \in D\}.$$

$$C * D = \{c_1d_1 - c_1(c_1' - d_1) \mid c_1, c_1' \in C \ and \ d_1 \in D\}.$$

Definition 2.5. [2] A subset J of a subtraction algebra $\mathbb{N}$ is considered a subalgebra of $\mathbb{N}$ if, for any elements h_0 and a_1 in J, the difference $h_0 - a_1$ also belongs to J.

Definition 2.6. [2] $F \in \mathfrak{P}(\mathbb{N})\backslash\{\emptyset\}$ is termed as

(i) a left ideal if F is a subalgebra of $(\mathbb{N}, -)$ & $fs_0 - f(v - s_0) \in F \ \forall f, v \in \mathbb{N}; s_0 \in F$.

(ii) a right ideal if F is a subalgebra of $(\mathbb{N}, -)$ & $F\mathbb{N} \subseteq F$.

(iii) an ideal if F is both a left & a right ideal.

Definition 2.7. [10] A subalgebra $\mathbb{W}$ of $\mathbb{N}$ is termed as a bi-ideal if $\mathbb{W}\mathbb{N}\mathbb{W} \cap \mathbb{W}\mathbb{N} * \mathbb{W} \subseteq \mathbb{W}$.

3. Basics of Neutrosophic $\varkappa$- structures

This section introduces the fundamental concepts of neutrosophic $\varkappa$-structures of $\mathbb{N}$, which are required for the sequel. For $\mathbb{N}(\neq \emptyset)$, $\mathcal{F}(\mathbb{N}, \mathbb{I}^-)$ refers to the gathering of negative-valued functions from $\mathbb{N}$ to $\mathbb{I}^-$, where $\mathbb{I}^- = [-1, 0]$. An element $k_1 \in \mathcal{F}(\mathbb{N}, \mathbb{I}^-)$ is called as a $\varkappa$-function on $\mathbb{N}$, & $\varkappa$-structure represents $(\mathbb{N}, k_1)$ of $\mathbb{N}$."

Definition 3.1. [9] The neutrosophic $\varkappa$-structure of a set $\mathbb{K}(\neq \emptyset)$ is described as follows:

$$\mathbb{K}_{\mathcal{W}} := \frac{\mathbb{K}}{(T_{\mathcal{W}}, I_{\mathcal{W}}, F_{\mathcal{W}})} = \left\{ \frac{r_0}{(T_{\mathcal{W}}(r_0), I_{\mathcal{W}}(r_0), F_{\mathcal{W}}(r_0))} \ : \ r_0 \in \mathbb{K} \right\},$$

where, in $\mathbb{K}$, $T_{\mathcal{W}}$ denotes the function corresponding to the degree of negative truth, $I_{\mathcal{W}}$ signifies the function associated with the degree of negative uncertainty, and $F_{\mathcal{W}}$ indicates the function that measures the extent of negative falsity.

Clearly $\mathbb{K}_{\mathcal{W}}$ satisfies the requirement: $-3 \leq T_{\mathcal{W}}(h_1) + I_{\mathcal{W}}(h_1) + F_{\mathcal{W}}(h_1) \leq 0 \ \forall h_1 \in \mathbb{K}$.

Definition 3.2. [9] For $\mathbb{K}(\neq \emptyset)$, let $\mathbb{K}_N = \frac{\mathbb{K}}{(T_N, I_N, F_N)}$ and $\theta, \varphi, \gamma \in \mathbb{I}^-$ with $\theta+\varphi+\gamma \in [-1, 0]$. Consider the sets:

$T_N^\theta = \{a_1 \in \mathbb{K} \mid T_N(a_1) \leq \theta\}, I_N^\varphi = \{a_1 \in \mathbb{K} \mid I_N(a_1) \geq \varphi\}, F_N^\gamma = \{a_1 \in \mathbb{K} | F_N(a_1) \leq \gamma\}$.

Then $\mathbb{K}_N(\theta, \varphi, \gamma) = \{a_1 \in \mathbb{K} | T_N(a_1) \leq \theta, I_N(a_1) \geq \varphi, F_N(a_1) \leq \gamma\}$ is termed as a $(\theta, \varphi, \gamma)$-*level set* of $\mathbb{K}_N$. Clearly $\mathbb{K}_N(\theta, \varphi, \gamma) = T_N^\theta \cap I_N^\varphi \cap F_N^\gamma$.

Definition 3.3. [9] For a NSS $\mathbb{N}(\neq \emptyset)$, let $\mathbb{N}_{\mathfrak{J}} := \frac{\mathbb{N}}{(T_{\mathfrak{J}}, I_{\mathfrak{J}}, F_{\mathfrak{J}})}$ and $\mathbb{N}_V := \frac{\mathbb{N}}{(T_V, I_V, F_V)}$,

(i) $\mathbb{N}_{\mathfrak{J}}$ is referred as a *neutrosophic $\varkappa$-substructure* of $\mathbb{N}_V$, denoted by $\mathbb{N}_{\mathfrak{J}} \subseteq \mathbb{N}_V$, if it meets the following conditions: $\forall a_0 \in \mathbb{N}$,

$$T_{\mathfrak{J}}(a_0) \geq T_V(a_0), I_{\mathfrak{J}}(a_0) \leq I_V(a_0), F_{\mathfrak{J}}(a_0) \geq F_V(a_0).$$

If $\mathbb{N}_{\mathfrak{J}} \subseteq \mathbb{N}_V$ & $\mathbb{N}_V \subseteq \mathbb{N}_{\mathfrak{J}}$, then $\mathbb{N}_{\mathfrak{J}} = \mathbb{N}_V$.

(ii) the intersection of $\mathbb{N}_{\mathfrak{J}}$ & $\mathbb{N}_V$ is a neutrosophic $\varkappa$-structure is termed as follows: $\mathbb{N}_{\mathfrak{J}} \cap \mathbb{N}_V = \mathbb{N}_{\mathfrak{J} \cap V} = (\mathbb{N}; T_{\mathfrak{J}\cap V}, I_{\mathfrak{J}\cap V}, F_{\mathfrak{J}\cap V})$, where, $\forall$ $a_0 \in \mathbb{N}$,

$$(T_{\mathfrak{J}} \cap T_V)(a_0) = T_{\mathfrak{J}\cap V}(a_0) = T_{\mathfrak{J}}(a_0) \vee T_V(a_0),$$

$$(I_{\mathfrak{J}} \cap I_V)(a_0) = I_{\mathfrak{J}\cap V}(a_0) = I_{\mathfrak{J}}(a_0) \wedge I_V(a_0),$$

$$(F_{\mathfrak{J}} \cap F_V)(a_0) = F_{\mathfrak{J}\cap V}(a_0) = F_{\mathfrak{J}}(a_0) \vee F_V(a_0).$$

(iii) the union of $\mathbb{N}_{\mathfrak{J}}$ & $\mathbb{N}_V$ is a neutrosophic $\varkappa$-structure is termed as follows: $\mathbb{N}_{\mathfrak{J}} \cup \mathbb{N}_V = \mathbb{N}_{\mathfrak{J} \cup V} = (\mathbb{N}; T_{\mathfrak{J}\cup V}, I_{\mathfrak{J}\cup V}, F_{\mathfrak{J}\cup V})$, where, $\forall$ $a_0 \in \mathbb{N}$,

$$(T_{\mathfrak{J}} \cup T_V)(a_0) = T_{\mathfrak{J}\cup V}(a_0) = T_{\mathfrak{J}}(a_0) \wedge T_V(a_0),$$

$$(I_{\mathfrak{J}} \cup I_V)(a_0) = I_{\mathfrak{J}\cup V}(a_0) = I_{\mathfrak{J}}(a_0) \vee I_V(a_0),$$

$$(F_{\mathfrak{J}} \cup F_V)(a_0) = F_{\mathfrak{J}\cup V}(a_0) = F_{\mathfrak{J}}(a_0) \wedge F_V(a_0).$$

(iv) the subtraction of $\mathbb{N}_{\mathfrak{J}}$ & $\mathbb{N}_V$ is a neutrosophic $\varkappa$-structure is termed as follows: $\mathbb{N}_{\mathfrak{J}} - \mathbb{N}_V = \mathbb{N}_{\mathfrak{J} - V} = (\mathbb{N}; T_{\mathfrak{J}-V}, I_{\mathfrak{J}-V}, F_{\mathfrak{J}-V})$, where, $\forall$ $a_0 \in \mathbb{N}$,

$$(T_{\mathfrak{J}} - T_V)(a_0) = T_{\mathfrak{J}-V}(a_0) = \begin{cases} \bigwedge\limits_{a_0 = f_0 - c_0} \{T_{\mathfrak{J}}(f_0) \vee T_V(c_0)\} & if\ a_0 = f_0 - c_0 \\ 0 & otherwise, \end{cases},$$

$$(I_{\mathfrak{J}} - I_V)(a_0) = I_{\mathfrak{J}-V}(a_0) = \begin{cases} \bigvee\limits_{a_0 = f_0 - c_0} \{I_{\mathfrak{J}}(f_0) \wedge I_V(c_0)\} & if\ a_0 = f_0 - c_0 \\ -1 & otherwise, \end{cases},$$

$$(F_{\mathfrak{J}} - F_V)(a_0) = F_{\mathfrak{J}-V}(a_0) = \begin{cases} \bigwedge\limits_{a_0 = f_0 - c_0} \{F_{\mathfrak{J}}(f_0) \vee F_V(c_0)\} & if\ a_0 = f_0 - c_0 \\ 0 & otherwise. \end{cases}.$$

(v) the product of $\mathbb{N}_{\mathfrak{J}}$ & $\mathbb{N}_V$ is a neutrosophic $\varkappa$-structure over $\mathbb{N}$ and is defined as follows: $\mathbb{N}_{\mathfrak{J}}\mathbb{N}_V = \mathbb{N}_{\mathfrak{J}V} = (\mathbb{N}; T_{\mathfrak{J}V}, I_{\mathfrak{J}V}, F_{\mathfrak{J}V})$, where, for any $h_0 \in \mathbb{N}$,

$$(T_{\mathfrak{J}}T_V)(h_0) = T_{\mathfrak{J}V}(h_0) = \begin{cases} \bigwedge\limits_{h_0=f_0c_0} \{T_{\mathfrak{J}}(f_0) \vee T_V(c_0)\} & if\ h_0 = f_0c_0 \\ 0 & otherwise, \end{cases},$$

$$(I_{\mathfrak{J}}I_V)(h_0) = I_{\mathfrak{J}V}(h_0) = \begin{cases} \bigvee\limits_{h_0=f_0c_0} \{I_{\mathfrak{J}}(f_0) \wedge I_V(c_0)\} & if\ h_0 = f_0c_0 \\ -1 & otherwise, \end{cases},$$

$$(F_{\mathfrak{J}}F_V)(h_0) = F_{\mathfrak{J}V}(h_0) = \begin{cases} \bigwedge\limits_{h_0=f_0c_0} \{F_{\mathfrak{J}}(f_0) \vee F_V(c_0)\} & if\ h_0 = f_0c_0 \\ 0 & otherwise. \end{cases}.$$

(vi) the *-product of $\mathbb{N}_{\mathfrak{J}}$ & $\mathbb{N}_V$ is a neutrosophic $\varkappa$-structure over $\mathbb{N}$ and is defined as follows: $\mathbb{N}_{\mathfrak{J}} * \mathbb{N}_V = \mathbb{N}_{\mathfrak{J}*V} = (\mathbb{N}; T_{\mathfrak{J}*V}, I_{\mathfrak{J}*V}, F_{\mathfrak{J}*V})$, where, for any $h_0 \in \mathbb{N}$,

$$(T_{\mathfrak{J}} * T_V)(h_0) = T_{\mathfrak{J}*V}(h_0) = \begin{cases} \bigwedge\limits_{h_0=z_0c_0-z_0(b_0-c_0)} \{T_{\mathfrak{J}}(z_0) \vee T_V(c_0)\} & if\ h_0 = z_0c_0 - z_0(b_0-c_0) \\ 0 & otherwise, \end{cases},$$

$$(I_{\mathfrak{J}} * I_V)(h_0) = I_{\mathfrak{J}*V}(h_0) = \begin{cases} \bigvee\limits_{h_0=z_0c_0-z_0(b_0-c_0)} \{I_{\mathfrak{J}}(z_0) \wedge I_V(c_0)\} & if\ h_0 = z_0c_0 - z_0(b_0-c_0) \\ -1 & otherwise, \end{cases},$$

$$(F_{\mathfrak{J}} * F_V)(h_0) = F_{\mathfrak{J}*V}(h_0) = \begin{cases} \bigwedge\limits_{h_0=z_0c_0-z_0(b_0-c_0)} \{F_{\mathfrak{J}}(z_0) \vee F_V(c_0)\} & if\ h_0 = z_0c_0 - z_0(b_0-c_0) \\ 0 & otherwise. \end{cases}.$$

Definition 3.4. For $V_0 \subseteq \mathbb{N} \neq \emptyset$, the neutrosophic $\varkappa$-structure

$$\chi_{V_0}(\mathbb{N}_D) = \frac{\mathbb{N}}{(\chi_V(T)_D, \chi_V(I)_D, \chi_V(F)_D)},$$

where

$$\chi_{V_0}(T)_D : \mathbb{N} \to \mathbb{I}^-,\ j_1 \to \begin{cases} -1 & if\ j_1 \in V_0 \\ 0 & if\ j_1 \notin V_0, \end{cases}$$

$$\chi_{V_0}(I)_D : \mathbb{N} \to \mathbb{I}^-,\ j_1 \to \begin{cases} 0 & if\ j_1 \in V_0 \\ -1 & if\ j_1 \notin V_0, \end{cases}$$

$$\chi_{V_0}(F)_D : \mathbb{N} \to \mathbb{I}^-,\ j_1 \to \begin{cases} -1 & if\ j_1 \in V_0 \\ 0 & if\ j_1 \notin V_0, \end{cases}$$

is the *characteristic neutrosophic $\varkappa$-structure* of V_0 over $\mathbb{N}$.

If $V_0 = \mathbb{N}$, then we use that $\chi_{\mathbb{N}}(\mathbb{N}_D) = \mathfrak{B}$.

4. Properties of neutrosophic $\varkappa$-bi-ideals

In this portion, we define the idea of neutrosophic $\varkappa$-bi-ideals of near-subtraction semigroups and build an example to show that every neutrosophic $\varkappa$-bi-ideal need not be a hybrid ideal of near-subtraction semigroups. Within near-subtraction semigroups, we define neutrosophic $\varkappa$ intersection and provide some results about neutrosophic $\varkappa$-bi-ideals.

Definition 4.1. A neutrosophic $\varkappa$-subalgebra $\mathbb{N}_J := \frac{\mathbb{N}}{(T_J,I_J,F_J)}$ of $\mathbb{N}$ is termed as a neutrosophic $\varkappa$-bi-ideal of $\mathbb{N}$ if $(\mathbb{N}_J\mathfrak{B}\mathbb{N}_J) \cap (\mathbb{N}_J\mathfrak{B} * \mathbb{N}_J) \subseteq \mathbb{N}_J$.

Definition 4.2. A neutrosophic $\varkappa$-structure $\mathbb{N}_B = \frac{\mathbb{N}}{(T_B,I_B,F_B)}$ of $\mathbb{N}$ is termed as a *neutrosophic $\varkappa$-ideal* of $\mathbb{N}$ if it fulfils the assertions:

(i) $(\forall u_1, g_1 \in \mathbb{N}) \left(\begin{array}{c} T_B(g_1 - u_1) \leq T_B(g_1) \vee T_B(u_1) \\ I_B(g_1 - u_1) \geq I_B(g_1) \wedge I_B(u_1) \\ F_B(g_1 - u_1) \leq F_B(g_1) \vee F_B(u_1) \end{array} \right)$.

(ii) $(\forall c_1, j_1, u_1 \in \mathbb{N}) \left(\begin{array}{c} T_B(c_1u_1 - c_1(j_1 - u_1)) \leq T_B(u_1) \\ I_B(c_1u_1 - c_1(j_1 - u_1)) \geq I_B(u_1) \\ F_B(c_1u_1 - c_1(j_1 - u_1)) \leq F_B(u_1) \end{array} \right)$.

(iii) $(\forall u_1, q_1 \in \mathbb{N}) \left(\begin{array}{c} T_B(u_1q_1) \leq T_B(u_1) \\ I_B(u_1q_1) \geq I_B(u_1) \\ F_B(u_1q_1) \leq F_B(u_1) \end{array} \right)$.

A subset $\mathbb{N}_B$ of $\mathbb{N}$ is termed a left hybrid ideal if conditions (i) and (ii) are satisfied. It is called a right hybrid ideal when conditions (i) and (iii) are fulfilled.

Notation 1. *For a near- subtraction semigroup $\mathbb{N}$, we use the following notations.*

(i) $\mathcal{N}_{\mathbb{L}}(\mathbb{N})$ (respectively $\mathcal{N}_{\mathbb{R}}(\mathbb{N})$) represents the gathering of all neutrosophic $\varkappa$-left(respectively right) ideals of $\mathbb{N}$.

(ii) $\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$ represents the collection of all neutrosophic $\varkappa$- bi-ideals of $\mathbb{N}$.

Example 4.3. Let $\mathbb{Q} = \{0, p, w, z\}$ in which "$-$" and "$\cdot$" are defined by:

-	0	p	w	z
0	0	0	0	0
p	p	0	p	p
w	w	w	0	w
z	z	z	z	0

$\cdot$	0	p	w	z
0	0	0	0	0
p	p	p	p	p
w	0	0	0	w
z	0	0	0	z

Then $(\mathbb{Q}, -, \cdot)$ is a NSS. Here

$$\mathbb{Q}_{\mathscr{P}}; = \left\{ \frac{0}{(-0.8,-0.1,-0.7)}, \frac{p}{(-0.6,-0.4,-0.6)}, \frac{w}{(-0.4,-0.6,-0.5)}, \frac{z}{(-0.2,-0.8,-0.4)} \right\} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}).$$

Clearly $\mathcal{N}_{\mathbb{L}}(\mathbb{N}) \cap \mathcal{N}_{\mathbb{R}}(\mathbb{N}) \subseteq \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$(see Lemma 4.5). The following example demonstrates that the reverse statement does not hold.

Example 4.4. Let $\mathbb{N} = \{0, p, w, z\}$ in which "$-$" and "$\cdot$" are defined as in Example 4.3. Then

$$\mathbb{N}_{\mathscr{P}}; = \left\{ \frac{0}{(-0.8,-0.1,-0.7)}, \frac{p}{(-0.6,-0.4,-0.6)}, \frac{w}{(-0.4,-0.6,-0.5)}, \frac{z}{(-0.6,-0.4,-0.6)} \right\} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}),$$

but $\mathbb{N}_{\mathscr{P}}$ of $\mathbb{N}$ is not a left neutrosophic $\varkappa$- ideal, since $T_{\mathscr{P}}(wz - w(0 - z)) = T_{\mathscr{P}}(w) = -0.4 \nleq -0.6 = T_{\mathscr{P}}(z)$, $I_{\mathscr{P}}(wz - w(0 - z)) = I_{\mathscr{P}}(w) = -0.6 \ngeq -0.4 = I_{\mathscr{P}}(z)$ and $F_{\mathscr{P}}(wz - w(0 - z)) = F_{\mathscr{P}}(w) = -0.5 \nleq -0.6 = F_{\mathscr{P}}(z)$.

Lemma 4.5. *Let* $\mathbb{N}_L := \frac{\mathbb{N}}{(T_L,I_L,F_L)}$. *If* $\mathbb{N}_L \in \mathcal{N}_{\mathbb{L}}(\mathbb{N})$, *then* $\mathbb{N}_L \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.

Proof. Let $a' \in \mathbb{N}$ be such that $a' = v_0y_0s_0 = z_0d_0 - z_0(q_0 - d_0)$, where $v_0, y_0, s_0, z_0, q_0, d_0 \in \mathbb{N}$. Then

$$\begin{aligned}
((T_L\mathfrak{B}T_L) \cap (T_L\mathfrak{B} * T_L))(a') &= (T_L\mathfrak{B}T_L)(a') \vee (T_L\mathfrak{B} * T_L)(a') \geq (T_L\mathfrak{B} * T_L)(a') \\
&= \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} \{(T_L\mathfrak{B})(z_0) \vee T_L(d_0)\} \\
&\geq \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} \{\mathfrak{B}(z_0) \vee T_L(z_0d_0 - z_0(q_0 - d_0))\} \\
&= \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} T_L(z_0d_0 - z_0(q_0 - d_0)) = T_L(a'), \\
((I_L\mathfrak{B}I_L) \cap (I_L\mathfrak{B} * I_L))(a') &= (I_L\mathfrak{B}I_L)(a') \wedge (I_L\mathfrak{B} * I_L)(a') \leq (I_L\mathfrak{B} * I_L)(a') \\
&= \bigvee_{a'=z_0d_0-z_0(q_0-d_0)} \{(I_L\mathfrak{B})(z_0) \wedge I_L(d_0)\} \\
&\leq \bigvee_{a'=z_0d_0-z_0(q_0-d_0)} \{(I_L\mathfrak{B})(z_0) \wedge I_L(d_0)\} \\
&\leq \bigvee_{a'=z_0d_0-z_0(q_0-d_0)} \{\mathfrak{B}(z_0) \wedge I_L(z_0d_0 - z_0(q_0 - d_0))\} \\
&= \bigvee_{a'=z_0d_0-z_0(q_0-d_0)} I_L(z_0d_0 - z_0(q_0 - d_0)) = I_L(a'), \\
((F_L\mathfrak{B}F_L) \cap (F_L\mathfrak{B} * F_L))(a') &= (F_L\mathfrak{B}F_L)(a') \cap (F_L\mathfrak{B} * F_L)(a') \geq (F_L\mathfrak{B} * F_L)(a') \\
&= \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} \{(F_L\mathfrak{B})(z_0) \vee F_L(d_0)\} \\
&\geq \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} \{\mathfrak{B}(z_0) \vee F_L(z_0d_0 - z_0(q_0 - d_0))\} \\
&= \bigwedge_{a'=z_0d_0-z_0(q_0-d_0)} F_L(z_0d_0 - z_0(q_0 - d_0)) = F_L(a').
\end{aligned}$$

If a' cannot be expressed as $a' = v_0y_0s_0 = z_0d_0 - z_0(q_0 - d_0)$, then $((T_L\mathfrak{B}T_L) \cap (T_L\mathfrak{B} * T_L))(a') = 0 \geq T_L(a')$; $((I_L\mathfrak{B}I_L) \cap (I_L\mathfrak{B} * I_L))(a') = -1 \leq I_L(a')$, $((F_L\mathfrak{B}F_L) \cap (F_L\mathfrak{B} * F_L))(a') = 0 \geq F_L(a')$. Hence $\mathbb{N}_J \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. □

The proof of the following lemma follows a similar approach to that used in the proof of Lemma 4.5. We present the proof for readers' convenience.

Lemma 4.6. *Let* $\mathbb{N}_L := \frac{\mathbb{N}}{(T_L,I_L,F_L)}$. *If* $\mathbb{N}_L \in \mathcal{N}_{\mathbb{R}}(\mathbb{N})$, *then* $\mathbb{N}_L \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.

Proof. Let $x' \in \mathbb{N} : x' = vy = xj - x(q - j)$, $v = v_1v_2$, where $v, v_1, v_2, y, x, q, j \in \mathbb{N}$. Then

$$\begin{aligned}
((T_L\mathfrak{B}T_L)\cap(T_L\mathfrak{B}*T_L))(x') &= (T_L\mathfrak{B}T_L)(x')\vee(T_L\mathfrak{B}*T_L)(x')\\
&\geq \bigwedge_{x'=vy}\{(T_L\mathfrak{B})(v)\vee T_L(y)\}\\
&= \left\{\bigwedge_{x'=vy}\left\{\bigwedge_{v=v_1v_2}\{T_L(v_1)\vee\mathfrak{B}(v_2)\}\right\}\vee T_L(y)\right\}\\
&= \left\{\bigwedge_{x'=vy}\left\{\bigwedge_{v=v_1v_2}T_L(v_1)\right\}\vee T_L(y)\right\}\\
&= T_L(v_1)\vee T_L(y)\ (\text{since }\ T_L(vy)=T_L(v_1v_2y)\leq T_L(v_1))\\
&\geq T_L(vy)=T_L(x'),\\
((I_L\mathfrak{B}I_L)\cap(I_L\mathfrak{B}*I_L))(x') &= (I_L\mathfrak{B}I_L)(x')\wedge(I_L\mathfrak{B}*I_L)(x')\\
&\leq \bigvee_{x'=vy}\{(I_L\mathfrak{B})(v)\wedge I_L(y)\}\\
&= \left\{\bigvee_{x'=vy}\left\{\bigvee_{v=v_1v_2}\{I_L(v_1)\wedge\mathfrak{B}(v_2)\}\right\}\wedge I_L(y)\right\}\\
&= \left\{\bigvee_{x'=vy}\left\{\bigvee_{v=v_1v_2}I_L(v_1)\right\}\wedge I_L(y)\right\}\\
&= I_L(v_1)\wedge I_L(y)\ (\text{since }\ I_L(vy)=I_L(v_1v_2y)\geq I_L(v_1))\\
&\leq I_L(vy)=I_L(x'),\\
((F_L\mathfrak{B}F_L)\cap(F_L\mathfrak{B}*F_L))(x') &= (F_L\mathfrak{B}F_L)(x')\vee(F_L\mathfrak{B}*F_L)(x')\\
&\geq \bigwedge_{x'=vy}\{(F_L\mathfrak{B})(v)\vee F_L(y)\}\\
&= \left\{\bigwedge_{x'=vy}\left\{\bigwedge_{v=v_1v_2}\{F_L(v_1)\vee\mathfrak{B}(v_2)\}\right\}\vee F_L(y)\right\}\\
&= \left\{\bigwedge_{x'=vy}\left\{\bigwedge_{v=v_1v_2}F_L(v_1)\right\}\vee F_L(y)\right\}\\
&= F_L(v_1)\vee F_L(y)\ (\text{since }\ F_L(vy)=F_L(v_1v_2y)\leq F_L(v_1))\\
&\geq F_L(vy)=F_L(x').
\end{aligned}$$

If x' cannot be expressed as $x'=vy=xj-x(q-j)$, then $((T_L\mathfrak{B}T_L)\cap(T_L\mathfrak{B}*T_L))(x')=0\geq T_L(x')$; $((I_L\mathfrak{B}I_L)\cap(I_L\mathfrak{B}*I_L))(x')=-1\leq I_L(x')$, $((F_L\mathfrak{B}F_L)\cap(F_L\mathfrak{B}*F_L))(x')=0\geq F_L(x')$.

Hence $\mathbb{N}_L\in\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. □

Theorem 4.7. *Let* $\mathbb{N}_{\mathfrak{K}}:=\frac{\mathbb{N}}{(T_{\mathfrak{K}},I_{\mathfrak{K}},F_{\mathfrak{K}})}$ *and* $\mathbb{N}_L:=\frac{\mathbb{N}}{(T_L,I_L,F_L)}$. *If* $\mathbb{N}_{\mathfrak{K}},\mathbb{N}_L\in\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, *then* $\mathbb{N}_{\mathfrak{K}}\cap\mathbb{N}_L\in\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.

Proof. Let $\mathbb{N}_{\mathfrak{K}}, \mathbb{N}_L \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, and let $z_0, t_0 \in \mathbb{N}$. Then

$$\begin{aligned}
(T_{\mathfrak{K}} \cap T_L)(z_0 - t_0) &= T_{\mathfrak{K}}(z_0 - t_0) \vee T_L(z_0 - t_0) \\
&\leq (T_{\mathfrak{K}}(z_0) \vee T_{\mathfrak{K}}(t_0)) \vee (T_L(z_0) \vee T_L(t_0)) \\
&= (T_{\mathfrak{K}}(z_0) \vee T_L(z_0)) \vee (T_{\mathfrak{K}}(t_0) \vee T_L(t_0)) \\
&= (T_{\mathfrak{K}} \vee T_L)(z_0) \vee (T_{\mathfrak{K}} \vee T_L)(t_0), \\
(I_{\mathfrak{K}} \cap I_L)(z_0 - t_0) &= I_{\mathfrak{K}}(z_0 - t_0) \wedge I_L(z_0 - t_0) \\
&\geq (I_{\mathfrak{K}}(z_0) \wedge I_{\mathfrak{K}}(t_0)) \wedge (I_L(z_0) \wedge I_L(t_0)) \\
&= (I_{\mathfrak{K}}(z_0) \wedge I_L(z_0)) \wedge (I_{\mathfrak{K}}(t_0) \wedge I_L(t_0)) \\
&= (I_{\mathfrak{K}} \wedge I_L)(z_0) \wedge (I_{\mathfrak{K}} \wedge I_L)(t_0), \\
(F_{\mathfrak{K}} \cap F_L)(z_0 - t_0) &= F_{\mathfrak{K}}(z_0 - t_0) \vee F_L(z_0 - t_0) \\
&\leq (F_{\mathfrak{K}}(z_0) \vee F_{\mathfrak{K}}(t_0)) \vee (F_L(z_0) \vee F_L(t_0)) \\
&= (F_{\mathfrak{K}}(z_0) \vee F_L(z_0)) \vee (F_{\mathfrak{K}}(t_0) \vee F_L(t_0)) \\
&= (F_{\mathfrak{K}} \vee F_L)(z_0) \vee (F_{\mathfrak{K}} \vee F_L)(t_0).
\end{aligned}$$

Let $\mathfrak{j}' \in \mathbb{N}$ and choose $h, w, s, \mathfrak{j}, t, a \in \mathbb{N}$ be such that $\mathfrak{j}' = hws = \mathfrak{j}a - \mathfrak{j}(t-a)$. Since $\mathbb{N}_{\mathfrak{K}}$ and $\mathbb{N}_L$ are neutrosophic $\varkappa$- bi-ideals of $\mathbb{N}$, we get

$$\begin{aligned}
\left\{\bigwedge_{\mathfrak{j}'=hws} (T_{\mathfrak{K}}(h) \vee T_{\mathfrak{K}}(s))\right\} \vee \left\{\bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (T_{\mathfrak{K}}(\mathfrak{j}) \vee T_{\mathfrak{K}}(a))\right\} &\geq T_{\mathfrak{K}}(\mathfrak{j}'), \\
\left\{\bigvee_{\mathfrak{j}'=hws} (I_{\mathfrak{K}}(h) \wedge I_{\mathfrak{K}}(s))\right\} \wedge \left\{\bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (I_{\mathfrak{K}}(\mathfrak{j}) \wedge I_{\mathfrak{K}}(a))\right\} &\leq I_{\mathfrak{K}}(\mathfrak{j}'), \\
\left\{\bigwedge_{\mathfrak{j}'=hws} (F_{\mathfrak{K}}(h) \vee F_{\mathfrak{K}}(s))\right\} \vee \left\{\bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (F_{\mathfrak{K}}(\mathfrak{j}) \vee F_{\mathfrak{K}}(a))\right\} &\geq F_{\mathfrak{K}}(\mathfrak{j}').
\end{aligned}$$

and

$$\begin{aligned}
\left\{\bigwedge_{\mathfrak{j}'=hws} (T_L(h) \vee T_L(s))\right\} \vee \left\{\bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (T_L(\mathfrak{j}) \vee T_L(a))\right\} &\geq T_L(\mathfrak{j}'), \\
\left\{\bigvee_{\mathfrak{j}'=hws} (I_L(h) \wedge I_L(s))\right\} \wedge \left\{\bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (I_L(\mathfrak{j}) \wedge I_L(a))\right\} &\leq I_L(\mathfrak{j}'), \\
\left\{\bigwedge_{\mathfrak{j}'=hws} (F_L(h) \vee F_L(s))\right\} \vee \left\{\bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (F_L(\mathfrak{j}) \vee F_L(a))\right\} &\geq F_L(\mathfrak{j}').
\end{aligned}$$

Now, $((T_{\mathfrak{K}} \cap T_L)\mathfrak{B}(T_{\mathfrak{K}} \cap T_L))(\mathfrak{j}') \vee ((T_{\mathfrak{K}} \cap T_L)\mathfrak{B} * (T_{\mathfrak{K}} \cap T_L))(\mathfrak{j}')$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} ((T_{\mathfrak{K}} \vee T_L)(h) \vee (T_{\mathfrak{K}} \vee T_L)(s)) \right\} \vee \left\{ \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} ((T_{\mathfrak{K}} \vee T_L)(\mathfrak{j}) \vee (T_{\mathfrak{K}} \vee T_L)(a)) \right\}$$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} (T_{\mathfrak{K}}(h) \vee T_L(h)) \vee (T_{\mathfrak{K}}(s) \vee T_L(s)) \right\} \vee \left\{ \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (T_{\mathfrak{K}}(\mathfrak{j}) \vee T_L(\mathfrak{j})) \vee (T_{\mathfrak{K}}(a) \vee T_L(a)) \right\}$$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} (T_{\mathfrak{K}}(h) \vee T_{\mathfrak{K}}(s)) \vee \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (T_{\mathfrak{K}}(\mathfrak{j}) \vee T_{\mathfrak{K}}(a)) \right\}$$

$$\vee \left\{ \bigwedge_{\mathfrak{j}'=hws} (T_L(h) \vee T_L(s)) \vee \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (T_L(\mathfrak{j}) \vee T_L(a)) \right\}$$

$\geq T_{\mathfrak{K}}(\mathfrak{j}') \vee T_L(\mathfrak{j}') = (T_{\mathfrak{K}} \vee T_L)(\mathfrak{j}')$, $((I_{\mathfrak{K}} \cap I_L)\mathfrak{B}(I_{\mathfrak{K}} \cap I_L))(\mathfrak{j}') \wedge ((I_{\mathfrak{K}} \cap I_L)\mathfrak{B} * (I_{\mathfrak{K}} \cap I_L))(\mathfrak{j}')$

$$= \left\{ \bigvee_{\mathfrak{j}'=hws} ((I_{\mathfrak{K}} \wedge I_L)(h) \wedge (I_{\mathfrak{K}} \wedge I_L)(s)) \right\} \wedge \left\{ \bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} ((I_{\mathfrak{K}} \wedge I_L)(\mathfrak{j}) \wedge (I_{\mathfrak{K}} \wedge I_L)(a)) \right\}$$

$$= \left\{ \bigvee_{\mathfrak{j}'=hws} (I_{\mathfrak{K}}(h) \wedge I_L(h)) \wedge (I_{\mathfrak{K}}(s) \wedge I_L(s)) \right\} \wedge \left\{ \bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (I_{\mathfrak{K}}(\mathfrak{j}) \wedge I_L(\mathfrak{j})) \wedge (I_{\mathfrak{K}}(a) \wedge I_L(a)) \right\}$$

$$= \left\{ \bigvee_{\mathfrak{j}'=hws} (I_{\mathfrak{K}}(h) \wedge I_{\mathfrak{K}}(s)) \wedge \bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (I_{\mathfrak{K}}(\mathfrak{j}) \wedge I_{\mathfrak{K}}(a)) \right\}$$

$$\wedge \left\{ \bigvee_{\mathfrak{j}'=hws} (I_L(h) \wedge I_L(s)) \wedge \bigvee_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (I_L(\mathfrak{j}) \wedge I_L(a)) \right\}$$

$\leq I_{\mathfrak{K}}(\mathfrak{j}') \wedge I_L(\mathfrak{j}') = (I_{\mathfrak{K}} \wedge I_L)(\mathfrak{j}')$,

$((F_{\mathfrak{K}} \cap F_L)\mathfrak{B}(F_{\mathfrak{K}} \cap F_L))(\mathfrak{j}') \vee ((F_{\mathfrak{K}} \cap F_L)\mathfrak{B} * (F_{\mathfrak{K}} \cap F_L))(\mathfrak{j}')$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} ((F_{\mathfrak{K}} \vee F_L)(h) \vee (F_{\mathfrak{K}} \vee F_L)(s)) \right\} \vee \left\{ \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} ((F_{\mathfrak{K}} \vee F_L)(\mathfrak{j}) \vee (F_{\mathfrak{K}} \vee F_L)(a)) \right\}$$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} (F_{\mathfrak{K}}(h) \vee F_L(h)) \vee (F_{\mathfrak{K}}(s) \vee F_L(s)) \right\} \vee \left\{ \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (F_{\mathfrak{K}}(\mathfrak{j}) \vee F_L(\mathfrak{j})) \vee (F_{\mathfrak{K}}(a) \vee F_L(a)) \right\}$$

$$= \left\{ \bigwedge_{\mathfrak{j}'=hws} (F_{\mathfrak{K}}(h) \vee F_{\mathfrak{K}}(s)) \vee \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (F_{\mathfrak{K}}(\mathfrak{j}) \vee F_{\mathfrak{K}}(a)) \right\}$$

$$\vee \left\{ \bigwedge_{\mathfrak{j}'=hws} (F_L(h) \vee F_L(s)) \vee \bigwedge_{\mathfrak{j}'=\mathfrak{j}a-\mathfrak{j}(t-a)} (F_L(\mathfrak{j}) \vee F_L(a)) \right\}$$

$\geq F_{\mathfrak{K}}(\mathfrak{j}') \vee F_L(\mathfrak{j}') = (F_{\mathfrak{K}} \vee F_L)(\mathfrak{j}')$.

Hence $\mathbb{N}_{\mathfrak{K}} \cap \mathbb{N}_L \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. □

Theorem 4.8. *If $\mathbb{N}_L := \frac{\mathbb{N}}{(T_L, I_L, F_L)} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, then the $(\eth, \varphi, \nu)$-level set $\mathbb{N}_L(\eth, \varphi, \nu)$ of $\mathbb{N}$ is a bi-ideal $\forall \eth, \varphi, \nu \in \mathbb{I}^-$.*

Proof. For $\eth, \varphi, \nu \in \mathbb{I}^-$ and $m_1, b_1 \in \mathbb{N}_L(\eth, \varphi, \nu)$, $T_L(m_1 - b_1) \leq T_L(m_1) \vee T_L(b_1) \leq \eth, I_L(m_1 - b_1) \geq I_L(m_1) \wedge I_L(b_1) \geq \varphi, F_L(m_1 - b_1) \leq F_L(m_1) \vee F_L(b_1) \leq \nu$. It follows that $m_1 - b_1 \in \mathbb{N}_L(\eth, \varphi, \nu)$.

Let $z' \in \mathbb{N}$ and $z' \in \mathbb{N}_L(\eth, \varphi, \nu)\mathfrak{B}\mathbb{N}_L(\eth, \varphi, \nu) \cap \mathbb{N}_L(\eth, \varphi, \nu)\mathfrak{B} * \mathbb{N}_L(\eth, \varphi, \nu)$. If there exist $f_1, q, u_1, c \in \mathbb{N}_L(\eth, \varphi, \nu)$ and $f_2, f, u, u_2, y \in \mathbb{N}$ such that $z' = fq = uc - u(y - c)$, $f = f_1 f_2$ and $u = u_1 u_2$, then

$$\begin{aligned}
T_L(z') \leq &((T_L\mathfrak{B}T_L) \vee (T_L\mathfrak{B} * T_L))(z') \\
= &(T_L\mathfrak{B}T_L)(z') \vee (T_L\mathfrak{B} * T_L)(z') \\
= &\left\{ \bigwedge_{z'=fq} \{(T_L\mathfrak{B})(f) \vee T_L(q)\} \right\} \vee \left\{ \bigwedge_{z'=uc-u(y-c)} \{(T_L\mathfrak{B})(u) \vee T_L(c)\} \right\} \\
= &\left\{ \bigwedge_{z'=fq} \left(\bigwedge_{f=f_1f_2} (T_L(f_1) \vee \mathfrak{B}(f_2)) \right) \vee T_L(q) \right\} \\
&\vee \left\{ \bigwedge_{z'=uc-u(y-c)} \left(\bigwedge_{u=u_1u_2} (T_L(u_1) \vee \mathfrak{B}(u_2)) \right) \vee T_L(c) \right\} \\
\leq &(T_L(f_1) \vee T_L(q) \vee T_L(u_1) \vee T_L(c)) \leq \eth, \\
I_L(z') \geq &((I_L\mathfrak{B}I_L) \wedge (I_L\mathfrak{B} * I_L))(z') \\
= &(I_L\mathfrak{B}I_L)(z') \wedge (I_L\mathfrak{B} * I_L)(z') \\
= &\left\{ \bigvee_{z'=fq} \{(I_L\mathfrak{B})(f) \wedge I_L(q)\} \right\} \wedge \left\{ \bigvee_{z'=uc-u(y-c)} \{(I_L\mathfrak{B})(u) \wedge I_L(c)\} \right\} \\
= &\left\{ \bigvee_{z'=fq} \left(\bigvee_{f=f_1f_2} (I_L(f_1) \wedge \mathfrak{B}(f_2)) \right) \wedge I_L(q) \right\} \\
&\wedge \left\{ \bigvee_{z'=uc-u(y-c)} \left(\bigvee_{u=u_1u_2} (I_L(u_1) \wedge \mathfrak{B}(u_2)) \right) \wedge I_L(c) \right\} \\
\geq &(I_L(f_1) \wedge I_L(q) \wedge I_L(u_1) \wedge I_L(c)) \geq \varphi, \\
F_L(z') \leq &((F_L\mathfrak{B}F_L) \vee (F_L\mathfrak{B} * F_L))(z') \\
= &(F_L\mathfrak{B}F_L)(z') \vee (F_L\mathfrak{B} * F_L)(z') \\
= &\left\{ \bigwedge_{z'=fq} \{(F_L\mathfrak{B})(f) \vee F_L(q)\} \right\} \vee \left\{ \bigwedge_{z'=uc-u(y-c)} \{(F_L\mathfrak{B})(u) \vee F_L(c)\} \right\}
\end{aligned}$$

$$=\left\{\bigwedge_{z'=fq}\left(\bigwedge_{f=f_1f_2}(F_L(f_1)\vee\mathfrak{B}(f_2))\right)\vee F_L(q)\right\}$$
$$\vee\left\{\bigwedge_{z'=uc-u(y-c)}\left(\bigwedge_{u=u_1u_2}(F_L(u_1)\vee\mathfrak{B}(u_2))\right)\vee F_L(c)\right\}$$
$$\leq (F_L(f_1)\vee F_L(q)\vee F_L(u_1)\vee F_L(c))\leq\nu.$$

This implies that $z'\in\mathbb{N}_L(\eth,\varphi,\nu)$. So $\mathbb{N}_L(\eth,\varphi,\nu)$ of $\mathbb{N}$ is a bi-ideal. □

Lemma 4.9. *For any subsets K, V of $\mathbb{N}$ and $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N,I_N,F_N)}$, the following statements are true:*

(i) $\chi_K(\mathbb{N}_N)\cap\chi_V(\mathbb{N}_N)=\chi_{K\cap V}(\mathbb{N}_N)$.

(ii) $\chi_K(\mathbb{N}_N)\cup\chi_V(\mathbb{N}_N)=\chi_{K\cup V}(\mathbb{N}_N)$.

(iii) $\chi_K(\mathbb{N}_N)\chi_V(\mathbb{N}_N)=\chi_{KV}(\mathbb{N}_N)$.

(iv) $\chi_K(\mathbb{N}_N)*\chi_V(\mathbb{N}_N)=\chi_{K*V}(\mathbb{N}_N)$.

(v) If $K\subseteq V$, then $\chi_K(\mathbb{N}_N)\subseteq\chi_V(\mathbb{N}_N)$.

Proof. The proofs are obvious. □

Lemma 4.10. *For a subset K of $\mathbb{N}$ and $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N,I_N,F_N)}$, the below statements are equivalent:*

(i) K of $\mathbb{N}$ is a bi-ideal,

(ii) $\chi_K(\mathbb{N}_N)\in\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.

Proof. *(i) ⇒ (ii)* For $y_1, c_1\in\mathbb{N}$, if $y_1, c_1\in K$, then $y_1-c_1\in K$ which implies $\chi_K(T_N)(y_1-c_1)=-1=\chi_K(T_N)(y_1)\vee\chi_K(T_N)(c_1)$, $\chi_K(I_N)(y_1-c_1)=0=\chi_K(I_N)(y_1)\wedge\chi_K(I_N)(c_1)$, $\chi_K(F_N)(y_1-c_1)=-1=\chi_K(F_N)(y_1)\vee\chi_K(F_N)(c_1)$. Otherwise $y_1\notin K$ or $c_1\notin K$. Then $\chi_K(T_N)(y_1-c_1)\leq 0=\chi_K(T_N)(y_1)\vee\chi_K(T_N)(c_1)$, $\chi_K(I_N)(y_1-c_1)\geq -1=\chi_K(I_N)(y_1)\wedge\chi_K(I_N)(c_1)$, $\chi_K(F_N)(y_1-c_1)\leq 0=\chi_K(F_N)(y_1)\vee\chi_K(F_N)(c_1)$. So $\chi_K(\mathbb{N}_N)$ of $\mathbb{N}$ is a neutrosophic $\varkappa$- subalgebra.

By Lemma 4.9, we have

$$\begin{aligned}\chi_K(\mathbb{N}_N)\mathfrak{B}\chi_K(\mathbb{N}_N)\cap\chi_K(\mathbb{N}_N)\mathfrak{B}*\chi_K(\mathbb{N}_N)&=\chi_{K\mathbb{N}K}(\mathbb{N}_N)\cap\chi_{K\mathbb{N}*K}(\mathbb{N}_N)\\&=\chi_{(K\mathbb{N}K\cap(K\mathbb{N}*K))}(\mathbb{N}_N)\ll\chi_K(\mathbb{N}_N).\end{aligned}$$

So, $\chi_K(\mathbb{N}_N)\in\mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.

(ii) ⇒ (i) Let $u'\in K\mathbb{N}K\cap K\mathbb{N}*K$. Then $u'=hv=uq-u(z-q)$ and $h=h_1h_2$; $u=u_1u_2$ for some $h_1, v, q, u_1\in K$ and $h_2, u, z, u_2\in\mathbb{N}$. Now,

$$\begin{aligned}\chi_K(T_N)(u')\leq&(\chi_K\mathfrak{B}\chi_K\cap\chi_K\mathfrak{B}*\chi_K)(T_N)(u')\\=&(\chi_K\mathfrak{B}\chi_K)(T_N)(u')\vee(\chi_K\mathfrak{B}*\chi_K)(T_N)(u')\\=&\bigwedge_{u'=hv}\{(\chi_K\mathfrak{B})(T_N)(h)\vee\chi_K(T_N)(v)\}\vee\bigwedge_{u'=uq-u(z-q)}\{(\chi_K\mathfrak{B})(T_N)(u)\vee\chi_K(T_N)(q)\}\end{aligned}$$

$$= \left\{ \bigwedge_{u'=hv} \left(\bigwedge_{h=h_1h_2} \chi_K(T_N)(h_1) \vee \mathfrak{B}(h_2) \right) \vee \chi_K(T_N)(v) \right\}$$
$$\vee \left\{ \bigwedge_{u'=uq-u(z-q)} \left(\bigwedge_{u=u_1u_2} \chi_K(T_N)(u_1) \vee \mathfrak{B}(u_2) \right) \vee \chi_K(T_N)(q) \right\}$$
$$\leq \chi_K(T_N)(h_1) \vee \chi_K(T_N)(v) \vee \chi_K(T_N)(u_1) \vee \chi_K(T_N)(q) = -1,$$

$$\chi_K(I_N)(u') \geq (\chi_K\mathfrak{B}\chi_K \cap \chi_K\mathfrak{B} * \chi_K)(I_N)(u')$$
$$= (\chi_K\mathfrak{B}\chi_K)(I_N)(u') \wedge (\chi_K\mathfrak{B} * \chi_K)(I_N)(u')$$
$$= \bigvee_{u'=hv} \{(\chi_K\mathfrak{B})(I_N)(h) \wedge \chi_K(I_N)(v)\} \wedge \bigvee_{u'=uq-u(z-q)} \{(\chi_K\mathfrak{B})(I_N)(u) \wedge \chi_K(I_N)(q)\}$$
$$= \left\{ \bigvee_{u'=hv} \left(\bigvee_{h=h_1h_2} \chi_K(I_N)(h_1) \wedge \mathfrak{B}(h_2) \right) \wedge \chi_K(I_N)(v) \right\}$$
$$\wedge \left\{ \bigvee_{u'=uq-u(z-q)} \left(\bigvee_{u=u_1u_2} \chi_K(I_N)(u_1) \wedge \mathfrak{B}(u_2) \right) \wedge \chi_K(I_N)(q) \right\}$$
$$\geq \chi_K(I_N)(h_1) \wedge \chi_K(I_N)(v) \wedge \chi_K(I_N)(u_1) \wedge \chi_K(I_N)(q) = 0,$$

$$\chi_K(F_N)(u') \leq (\chi_K\mathfrak{B}\chi_K \cap \chi_K\mathfrak{B} * \chi_K)(F_N)(u')$$
$$= (\chi_K\mathfrak{B}\chi_K)(F_N)(u') \vee (\chi_K\mathfrak{B} * \chi_K)(F_N)(u')$$
$$= \bigwedge_{u'=hv} \{(\chi_K\mathfrak{B})(F_N)(h) \vee \chi_K(F_N)(v)\} \vee \bigwedge_{u'=uq-u(z-q)} \{(\chi_K\mathfrak{B})(F_N)(u) \vee \chi_K(F_N)(q)\}$$
$$= \left\{ \bigwedge_{u'=hv} \left(\bigwedge_{h=h_1h_2} \chi_K(F_N)(h_1) \vee \mathfrak{B}(h_2) \right) \vee \chi_K(F_N)(v) \right\}$$
$$\vee \left\{ \bigwedge_{u'=uq-u(z-q)} \left(\bigwedge_{u=u_1u_2} \chi_K(F_N)(u_1) \vee \mathfrak{B}(u_2) \right) \vee \chi_K(F_N)(q) \right\}$$
$$\leq \chi_K(F_N)(h_1) \vee \chi_K(F_N)(v) \vee \chi_K(F_N)(u_1) \vee \chi_K(F_N)(q) = -1.$$

Thus $u' \in K$ and hence $K\mathbb{N}K \cap K\mathbb{N} * K \subseteq K$. □

Theorem 4.11. *For a neutrosophic $\varkappa$- subalgebra $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N, I_N, F_N)}$ of $\mathbb{N}$, if $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \subseteq \mathbb{N}_N$, then $\mathbb{N}_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.*

Proof. Assume $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \subseteq \mathbb{N}_N$ and let $y_1 \in \mathbb{N}$. Then

$$((T_N\mathfrak{B}T_N) \cap (T_N\mathfrak{B} * T_N))(y_1) = (T_N\mathfrak{B}T_N)(y_1) \vee (T_N\mathfrak{B} * T_N)(y_1) \geq (T_N\mathfrak{B}T_N)(y_1) \geq T_N(y_1),$$
$$((I_N\mathfrak{B}I_N) \cap (I_N\mathfrak{B} * I_N))(y_1) = (I_N\mathfrak{B}I_N)(y_1) \wedge (I_N\mathfrak{B} * I_N)(y_1) \leq (I_N\mathfrak{B}I_N)(y_1) \leq I_N(y_1),$$
$$((F_N\mathfrak{B}F_N) \cap (F_N\mathfrak{B} * F_N))(y_1) = (F_N\mathfrak{B}F_N)(y_1) \vee (F_N\mathfrak{B} * F_N)(y_1) \geq (F_N\mathfrak{B}F_N)(y_1) \geq F_N(y_1).$$

Thus $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \cap \mathbb{N}_N\mathfrak{B} * \mathbb{N}_N \subseteq \mathbb{N}_N$ and so $\mathbb{N}_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. □

Theorem 4.12. *If $\mathbb{N}$ is a zero-symmetric NSS and $\mathbb{N}_J := \frac{\mathbb{N}}{(T_J,I_J,F_J)} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, then $\mathbb{N}_J\mathfrak{B}\mathbb{N}_J \subseteq \mathbb{N}_J$.*

Proof. Let $\mathbb{N}_J \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. Then $\mathbb{N}_J\mathfrak{B}\mathbb{N}_J \cap \mathbb{N}_J\mathfrak{B} * \mathbb{N}_J \subseteq \mathbb{N}_J$. Clearly, $\mathbb{N}_J(0) \supseteq \mathbb{N}_J(d)\ \forall d \in \mathbb{N}$. Since $\mathbb{N}$ is zero-symmetric, $\mathbb{N}_J\mathfrak{B}\mathbb{N}_J \subseteq \mathbb{N}_J\mathfrak{B} * \mathbb{N}_J$. So $\mathbb{N}_J\mathfrak{B}\mathbb{N}_J \cap \mathbb{N}_J\mathfrak{B} * \mathbb{N}_J = \mathbb{N}_J\mathfrak{B}\mathbb{N}_J \subseteq \mathbb{N}_J$. Hence $\mathbb{N}_J\mathfrak{B}\mathbb{N}_J \subseteq \mathbb{N}_J$. □

Theorem 4.13. *If $\mathbb{N}$ is a zero-symmetric NSS and for a neutrosophic $\varkappa$- subalgebra $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N,I_N,F_N)}$ of $\mathbb{N}$, the below statements are equivalent:*

(a) $\mathbb{N}_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$,

(b) $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \subseteq \mathbb{N}_N$.

Proof. By Theorem 4.11 and Theorem 4.12, the proof is obvious. □

Theorem 4.14. *If $\mathbb{N}$ is a zero-symmetric NSS and $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N,I_N,F_N)} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, then $\mathbb{N}_N(q_1j_1c_1) \supseteq \mathbb{N}_N(q_1) \cap \mathbb{N}_N(c_1)\ \forall q_1, j_1, c_1 \in \mathbb{N}$.*

Proof. Suppose $\mathbb{N}_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$ of a zero-symmetric NSS. By Theorem 4.12, $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \subseteq \mathbb{N}_N$.

Let $q_1, j_1, c_1 \in \mathbb{N}$. Then

$$\begin{aligned}
T_N(q_1j_1c_1) \leq (T_N\mathfrak{B}T_N)(q_1j_1c_1) &= \bigwedge_{q_1j_1c_1=d_1m_1}\{(T_N\mathfrak{B})(d_1) \vee T_N(m_1)\} \\
&\leq (T_N\mathfrak{B})(q_1j_1) \vee T_N(c_1) \\
&\leq T_N(q_1) \vee \mathfrak{B}(j_1) \vee T_N(c_1) \\
&= T_N(q_1) \vee T_N(c_1), \\
I_N(q_1j_1c_1) \geq (I_N\mathfrak{B}I_N)(q_1j_1c_1) &= \bigvee_{q_1j_1c_1=d_1m_1}\{(I_N\mathfrak{B})(d_1) \wedge I_N(m_1)\} \\
&\geq (I_N\mathfrak{B})(q_1j_1) \wedge I_N(c_1) \\
&\geq I_N(q_1) \wedge \mathfrak{B}(j_1) \wedge I_N(c_1) \\
&= I_N(q_1) \wedge I_N(c_1), \\
F_N(q_1j_1c_1) \leq (F_N\mathfrak{B}F_N)(q_1j_1c_1) &= \bigwedge_{q_1j_1c_1=d_1m_1}\{(F_N\mathfrak{B})(d_1) \vee F_N(m_1)\} \\
&\leq (F_N\mathfrak{B})(q_1j_1) \vee F_N(c_1) \leq F_N(q_1) \vee \mathfrak{B}(j_1) \vee F_N(c_1) \\
&= F_N(q_1) \vee F_N(c_1).
\end{aligned}$$

Hence $\mathbb{N}_N(q_1j_1c_1) \supseteq \mathbb{N}_N(q_1) \cap \mathbb{N}_N(c_1)$. □

Theorem 4.15. *If $\mathbb{N}$ is a zero-symmetric NSS & $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N, I_N, F_N)} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, the below assertions are equivalent:*

(i) $\mathbb{N}_N(qjc) \supseteq \mathbb{N}_N(q) \cap \mathbb{N}_N(c)\ \forall q, j, c \in \mathbb{N}$,

(ii) $\mathbb{N}_N \mathfrak{B} \mathbb{N}_N \subseteq \mathbb{N}_N$.

Proof. $(i) \Rightarrow (ii)$ Let $v' \in \mathbb{N}$. If $\exists\ d, m, d_1, d_2 \in \mathbb{N}$ such that $v' = dm$ and $d = d_1 d_2$.

Then by hypothesis, $T_N(d_1 d_2 m) \leq T_N(d_1) \vee T_N(m), I_N(d_1 d_2 m) \geq I_N(d_1) \wedge I_N(m), F_N(d_1 d_2 m) \leq F_N(d_1) \vee F_N(m)$.

Now,

$$\begin{aligned}
(T_N \mathfrak{B} T_N)(v') &= \bigwedge_{v'=dm} (T_N \mathfrak{B})(d) \vee T_N(m) \\
&= \bigwedge_{v'=dm} \left(\bigwedge_{d=d_1 d_2} \{T_N(d_1) \vee \mathfrak{B}(d_2)\} \right) \vee T_N(m) \\
&= \bigwedge_{v'=dm} \left(\bigwedge_{d=d_1 d_2} \{T_N(d_1) \vee -1\} \right) \vee T_N(m) \\
&= \bigwedge_{v'=d_1 d_2 m} \{T_N(d_1) \vee T_N(m)\} \geq \bigwedge_{v'=d_1 d_2 m} T_N(d_1 d_2 m) = T_N(v'), \\
(I_N \mathfrak{B} I_N)(v') &= \bigvee_{v'=dm} (I_N \mathfrak{B})(d) \wedge I_N(m) \\
&= \bigvee_{v'=dm} \left(\bigvee_{d=d_1 d_2} \{I_N(d_1) \wedge \mathfrak{B}(d_2)\} \right) \wedge I_N(m) \\
&= \bigvee_{v'=dm} \left(\bigvee_{d=d_1 d_2} \{I_N(d_1) \wedge 0\} \right) \wedge I_N(m) \\
&= \bigvee_{v'=d_1 d_2 m} \{I_N(d_1) \wedge I_N(m)\} \leq \bigvee_{v'=d_1 d_2 m} I_N(d_1 d_2 m) = I_N(v'),
\end{aligned}$$

$$\begin{aligned}
(F_N \mathfrak{B} F_N)(v') &= \bigwedge_{v'=dm} (F_N \mathfrak{B})(d) \vee F_N(m) \\
&= \bigwedge_{v'=dm} \left(\bigwedge_{d=d_1 d_2} \{F_N(d_1) \vee \mathfrak{B}(d_2)\} \right) \vee F_N(m) \\
&= \bigwedge_{v'=dm} \left(\bigwedge_{d=d_1 d_2} \{F_N(d_1) \vee -1\} \right) \vee F_N(m) \\
&= \bigwedge_{v'=d_1 d_2 m} (F_N(d_1) \vee F_N(m)) \geq \bigwedge_{v'=d_1 d_2 m} F_N(d_1 d_2 m) = F_N(v').
\end{aligned}$$

Hence $\mathbb{N}_N \mathfrak{B} \mathbb{N}_N \subseteq \mathbb{N}_N$.

$(ii) \Rightarrow (i)$ Suppose that $\mathbb{N}_N \mathfrak{B} \mathbb{N}_N \subseteq \mathbb{N}_N$ and let $d, m, g, v' \in \mathbb{N}$ be such that $v' = dmg$. Then

$$\begin{aligned}
T_N(dmg) = T_N(v') \leq (T_N\mathfrak{B}T_N)(v') &= \bigwedge_{v'=fy} \{(T_N\mathfrak{B})(f) \vee T_N(y)\} \\
&= \bigwedge_{v'=fy} \left(\bigwedge_{f=f_1f_2} \{T_N(f_1) \vee \mathfrak{B}(f_2)\} \right) \vee T_N(y) \\
&\leq T_N(d) \vee \mathfrak{B}(m) \vee T_N(g) \\
&= T_N(d) \vee T_N(g), \\
I_N(dmg) = I_N(v') \geq (I_N\mathfrak{B}I_N)(v') &= \bigvee_{v'=fy} (I_N\mathfrak{B})(f) \wedge I_N(y) \\
&= \bigvee_{v'=fy} \left(\bigvee_{f=f_1f_2} I_N(f_1) \wedge \mathfrak{B}(f_2) \right) \wedge I_N(y) \\
&\geq I_N(d) \wedge \mathfrak{B}(m) \wedge I_N(g) \\
&= I_N(d) \wedge I_N(g), \\
F_N(dmg) = F_N(v') \leq (F_N\mathfrak{B}F_N)(v') &= \bigwedge_{v'=fy} \{(F_N\mathfrak{B})(f) \vee F_N(y)\} \\
&= \bigwedge_{v'=fy} \left(\bigwedge_{f=f_1f_2} \{F_N(f_1) \vee \mathfrak{B}(f_2)\} \right) \vee F_N(y) \\
&\leq F_N(d) \vee \mathfrak{B}(m) \vee F_N(g) = F_N(d) \vee F_N(g),
\end{aligned}$$

Hence $\mathbb{N}_N(qjc) \supseteq \mathbb{N}_N(q) \cap \mathbb{N}_N(c)$ $\forall q, j, c \in \mathbb{N}$. □

Theorem 4.16. *For a neutrosophic $\varkappa$- subalgebra $\mathbb{N}_N := \frac{\mathbb{N}}{(T_N, I_N, F_N)}$ of a zero-symmetric NSS, the below assertions are equivalent:*

(i) $\mathbb{N}_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$,

(ii) $\mathbb{N}_N(qjc) \supseteq \mathbb{N}_N(q) \cap \mathbb{N}_N(c)$ $\forall q, j, c \in \mathbb{N}$.

(iii) $\mathbb{N}_N\mathfrak{B}\mathbb{N}_N \subseteq \mathbb{N}_N$.

Proof. By Theorem 4.14 and Theorem 4.15, the proof is simple. □

5. Homomorphism of a neutrosophic $\varkappa$- structure

In this portion, we explore some characteristics of neutrosophic $\varkappa$- structures that are homomorphic to near-subtraction semigroups. Hereafter, $\mathbb{N}$ and $\mathbb{N}'$ denote the zero-symmetric near-subtraction semigroups.

Definition 5.1. Let $\psi : \mathbb{N} \to \mathbb{N}'$ be a mapping.

(i) ψ is a homomorphism of $\mathbb{N}$ into $\mathbb{N}'$ if $\psi(m_1 - v_1) = \psi(m_1) - \psi(v_1)$ and $\psi(m_1v_1) = \psi(m_1)\psi(v_1)$ $\forall m_1, v_1 \in \mathbb{N}$.

(ii) For $\tilde{l}_{\varsigma} \in \mathfrak{H}(\mathbb{N}^{'})$, the preimage of $\tilde{l}_{\varsigma}$ under ψ, represented as $\psi^{-1}(\tilde{l}_{\varsigma})$, is a neutrosophic $\varkappa$- structure of $\mathbb{N}$ defined by $\psi^{-1}(\tilde{l}_{\varsigma}) := (\psi^{-1}(\tilde{l}), \psi^{-1}(\varsigma))$, where $\psi^{-1}(\tilde{l})(r_1) = \tilde{l}(\psi(r_1))$ & $\psi^{-1}(\varsigma)(r_1) = \varsigma(\psi(r_1))\ \forall r_1 \in \mathbb{N}$.

Theorem 5.2. *For a homomorphism $\psi : \mathbb{N} \to \mathbb{N}^{'}$ and $\mathbb{N}^{'}_{N} := \frac{\mathbb{N}^{'}}{(T_N, I_N, F_N)}$, if $\mathbb{N}^{'}_{N} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}^{'})$, then $\psi^{-1}(\mathbb{N}^{'}_{N}) \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$.*

Proof. Suppose $\mathbb{N}^{'}_{N} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}^{'})$ and let $w_0, d_0 \in \mathbb{N}$. Then $\psi^{-1}(T_N)(w_0 - d_0) = T_N(\psi(w_0 - d_0)) = T_N(\psi(w_0) - \psi(d_0)) \leq T_N(\psi(w_0)) \vee T_N(\psi(d_0)) = \psi^{-1}(T_N)(w_0) \vee \psi^{-1}(T_N)(d_0)$, $\psi^{-1}(I_N)(w_0 - d_0) = I_N(\psi(w_0 - d_0)) = I_N(\psi(w_0) - \psi(d_0)) \geq I_N(\psi(w_0)) \wedge I_N(\psi(d_0)) = \psi^{-1}(I_N)(w_0) \wedge \psi^{-1}(I_N)(d_0)$, $\psi^{-1}(F_N)(w_0 - d_0) = F_N(\psi(w_0 - d_0)) = F_N(\psi(w_0) - \psi(d_0)) \leq F_N(\psi(w_0)) \vee F_N(\psi(d_0)) = \psi^{-1}(F_N)(w_0) \vee \psi^{-1}(F_N)(d_0)$.

By Theorem 4.16, assume that $\mathbb{N}_N \mathfrak{B} \mathbb{N}_N \subseteq \mathbb{N}_N$. Let $w, g, m, w' \in \mathbb{N}$ be such that $w' = wgm$. Then

$$\begin{aligned}
\psi^{-1}(T_N)(wgm) = T_N(\psi(w')) &\leq (T_N \mathfrak{B} T_N)(\psi(w')) \\
&= \bigwedge_{w'=fy} \{(T_N \mathfrak{B})(\psi(f)) \vee T_N(\psi(y))\} \\
&= \bigwedge_{w'=fy} \left(\bigwedge_{f=f_1 f_2} \{T_N(\psi(f_1)) \vee \mathfrak{B}(f_2)\} \right) \vee T_N(\psi(y)) \\
&\leq T_N(\psi(w)) \vee \mathfrak{B}(g) \vee T_N(\psi(m)) \\
&= T_N(\psi(w)) \vee T_N(\psi(m)) = \psi^{-1}(T_N)(w) \vee \psi^{-1}(T_N)(m), \\
\psi^{-1}(I_N)(wgm) = I_N(\psi(w')) &\geq (I_N \mathfrak{B} I_N)(\psi(w')) \\
&= \bigvee_{w'=fy} \{(I_N \mathfrak{B})(\psi(f)) \wedge I_N(\psi(y))\} \\
&= \bigvee_{w'=fy} \left(\bigvee_{f=f_1 f_2} \{I_N(\psi(f_1)) \wedge \mathfrak{B}(f_2)\} \right) \wedge I_N(\psi(y)) \\
&\geq I_N(\psi(w)) \wedge \mathfrak{B}(g) \wedge I_N(\psi(m)) \\
&= I_N(\psi(w)) \wedge I_N(\psi(m)) = \psi^{-1}(I_N)(w) \wedge \psi^{-1}(I_N)(m), \\
\psi^{-1}(F_N)(wgm) = F_N(\psi(w')) &\leq (F_N \mathfrak{B} F_N)(\psi(w')) \\
&= \bigwedge_{w'=fy} \{(F_N \mathfrak{B})(\psi(f)) \vee F_N(\psi(y))\} \\
&= \bigwedge_{w'=fy} \left(\bigwedge_{f=f_1 f_2} \{F_N(\psi(f_1)) \vee \mathfrak{B}(f_2)\} \right) \vee F_N(\psi(y)) \\
&\leq F_N(\psi(w)) \vee \mathfrak{B}(g) \vee F_N(\psi(m)) \\
&= F_N(\psi(w)) \vee F_N(\psi(m)) = \psi^{-1}(F_N)(w) \vee \psi^{-1}(F_N)(m).
\end{aligned}$$

So $\psi^{-1}(\mathbb{N}^{'}_{N}) \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$. □

Theorem 5.3. *For a onto homomorphism* $\psi : \mathbb{N} \to \mathbb{N}'$ & $\mathbb{N}'_N := \frac{\mathbb{N}'}{(T_N, I_N, F_N)}$, *if* $\psi^{-1}(\mathbb{N}'_N) \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$, *then* $\mathbb{N}'_N \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}')$.

Proof. Let $\psi^{-1}(\mathbb{N}'_N) \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N})$ & $\mathfrak{k}', z' \in \mathbb{N}'$. Then $\exists\ z, \mathfrak{k} \in \mathbb{N}$ such that $\psi(\mathfrak{k}) = \mathfrak{k}'$ and $\psi(z) = z'$. Now,

$$\begin{aligned}
T_N(\mathfrak{k}' - z') = T_N(\psi(\mathfrak{k}) - \psi(z)) &= T_N(\psi(\mathfrak{k} - z)) \\
&= \psi^{-1}(T_N)(\mathfrak{k} - z) \\
&\leq \psi^{-1}(T_N)(\mathfrak{k}) \vee \psi^{-1}(T_N)(z) \\
&= T_N(\psi(\mathfrak{k})) \vee T_N(\psi(z)) = T_N(\mathfrak{k}') \vee T_N(z'), \\
I_N(\mathfrak{k}' - z') = I_N(\psi(\mathfrak{k}) - \psi(z)) &= I_N(\psi(\mathfrak{k} - z)) \\
&= \psi^{-1}(I_N)(\mathfrak{k} - z) \\
&\geq \psi^{-1}(I_N)(\mathfrak{k}) \wedge \psi^{-1}(I_N)(z) \\
&= I_N(\psi(\mathfrak{k})) \wedge I_N(\psi(z)) = I_N(\mathfrak{k}') \wedge I_N(z'), \\
F_N(\mathfrak{k}' - z') = F_N(\psi(\mathfrak{k}) - \psi(z)) &= F_N(\psi(\mathfrak{k} - z)) \\
&= \psi^{-1}(F_N)(\mathfrak{k} - z) \\
&\leq \psi^{-1}(F_N)(\mathfrak{k}) \vee \psi^{-1}(F_N)(z) \\
&= F_N(\psi(\mathfrak{k})) \vee F_N(\psi(z)) = F_N(\mathfrak{k}') \vee F_N(z'),
\end{aligned}$$

By Theorem 4.16, assume that $\tilde{l}_\varsigma \mathfrak{B} \tilde{l}_\varsigma \ll \tilde{l}_\varsigma$. Let $\mathfrak{k}', r', m', g' \in \mathbb{N}'$. Then $\exists\ \mathfrak{k}, r, m \in \mathbb{N}$ such that $\psi(\mathfrak{k}) = \mathfrak{k}'$, $\psi(r) = r'$, $\psi(m) = m'$ and $g' = \mathfrak{k}' r' m'$. Then

$$\begin{aligned}
T_N(\mathfrak{k}' r' m') =& T_N(\psi(g')) \leq (T_N \mathfrak{B} T_N)(\psi(g')) = \psi^{-1}(T_N \mathfrak{B} T_N)(g') \\
&= \bigwedge_{g'=fy} \psi^{-1}(T_N \mathfrak{B})(f) \vee \psi^{-1}(T_N)(y) \\
&= \bigwedge_{g'=fy} \left(\bigwedge_{f=f_1 f_2} \psi^{-1}(T_N(f_1) \vee \mathfrak{B}(f_2)) \right) \vee \psi^{-1}(T_N)(y) \\
&\leq \psi^{-1}(T_N)(\mathfrak{k}) \vee \mathfrak{B}(r) \vee \psi^{-1}(T_N)(m) \\
&= \psi^{-1}(T_N)(\mathfrak{k}) \vee \psi^{-1}(T_N)(m) = T_N(\psi(\mathfrak{k})) \vee T_N(\psi(m)) = T_N(\mathfrak{k}') \vee T_N(m'), \\
I_N(\mathfrak{k}' r' m') &= I_N(\psi(g')) \geq (I_N \mathfrak{B} I_N)(\psi(g')) = \psi^{-1}(I_N \mathfrak{B} I_N)(g') \\
&= \bigvee_{g'=fy} \psi^{-1}(I_N \mathfrak{B})(f) \wedge \psi^{-1}(I_N)(y) \\
&= \bigvee_{g'=fy} \left(\bigvee_{f=f_1 f_2} \psi^{-1}(I_N(f_1) \wedge \mathfrak{B}(f_2)) \right) \wedge \psi^{-1}(I_N)(y)
\end{aligned}$$

$$
\begin{aligned}
&\geq \psi^{-1}(I_N)(\mathfrak{k}) \wedge \mathfrak{B}(r) \wedge \psi^{-1}(I_N)(m) \\
&= \psi^{-1}(I_N)(\mathfrak{k}) \wedge \psi^{-1}(I_N)(m) = I_N(\psi(\mathfrak{k})) \wedge I_N(\psi(m)) = I_N(\mathfrak{k}') \wedge I_N(m'),
\end{aligned}
$$

$$
\begin{aligned}
F_N(\mathfrak{k}'r'm') &= F_N(\psi(g')) \leq (F_N\mathfrak{B}F_N)(\psi(g')) = \psi^{-1}(F_N\mathfrak{B}F_N)(g') \\
&= \bigwedge_{g'=fy} \psi^{-1}(F_N\mathfrak{B})(f) \vee \psi^{-1}(F_N)(y) \\
&= \bigwedge_{g'=fy} \left(\bigwedge_{f=f_1f_2} \psi^{-1}(F_N(f_1) \vee \mathfrak{B}(f_2)) \right) \vee \psi^{-1}(F_N)(y) \\
&\leq \psi^{-1}(F_N)(\mathfrak{k}) \vee \mathfrak{B}(r) \vee \psi^{-1}(F_N)(m) \\
&= \psi^{-1}(F_N)(\mathfrak{k}) \vee \psi^{-1}(F_N)(m) = F_N(\psi(\mathfrak{k})) \vee F_N(\psi(m)) = F_N(\mathfrak{k}') \vee F_N(m').
\end{aligned}
$$

So $\mathbb{N}_N^{'} \in \mathcal{NB}_{\mathcal{I}}(\mathbb{N}^{'})$. □

6. Conclusions

This research examines the properties of neutrosophic $\varkappa$-bi-ideals and develops corresponding bi-ideals within the context of near-subtraction semigroups. Additionally, various aspects of the neutrosophic $\varkappa$-preimage of the neutrosophic $\varkappa$-bi-ideal of a near-subtraction semigroup are analyzed under homomorphism mapping. The findings presented in this paper aim to pave the way for defining the concept of a neutrosophic $\varkappa$-prime bi-ideal and exploring its related properties in near-subtraction semigroups.

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