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On Fixed Point Results in Neutrosophic Metric Spaces Using Auxiliary Functions

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Abstract. In this paper, we establish novel fixed point theorems in the framework of *neutrosophic metric spaces* (NMS) by introducing the concept of *neutrosophic* $(\mathcal{L}, \phi)$*-contractions*. These contractions generalize classical contractive conditions by incorporating a function $\mathcal{L}$ that bounds the interaction between displacement terms and a control function ϕ that modulates the contraction intensity. Under specific hypothesis, we prove that every neutrosophic $(\mathcal{L}, \phi)$-contraction on a *complete NMS* admits a *unique fixed point*. As applications, we derive several corollaries by specifying the forms of $\mathcal{L}$ and ϕ, including cases where $\mathcal{L}$ is linear, additive, or defined via maximum functions. Our results unify and extend existing fixed point theorems in neutrosophic settings, while illustrative examples demonstrate their practical applicability.

Keywords: Fixed point; Neutrosophic set; Neutrosophic metric; $(\mathcal{L}, \phi)$-contraction; Non linear contraction

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1. **Introduction**

The Banach fixed-point theorem [1], commonly referred to as the contraction mapping theorem, is a key principle in the theory of metric spaces. It asserts that within any complete metric space, a contraction mapping defined as a function, that reduces the distance between points, will possess a unique fixed point. This theorem is important as it offers a systematic approach to identifying fixed points, which are defined as points that remain invariant under a specific function. Furthermore, it ensures both the existence and uniqueness of such points under certain conditions. The theorem finds extensive application across various disciplines, including differential equations and numerical analysis, owing to its utility in establishing the existence and uniqueness of solutions to equations and in iterative methods for solution approximation. First introduced by Banach in 1922 [1], this theorem has motivated numerous mathematicians to investigate various extensions and generalizations across a wide range of mathematical domains, as referenced in [2–4].

Recent advancements in fixed point theory have extended classical results to generalized metric spaces, including neutrosophic and fuzzy frameworks. [7] established quasi-contraction fixed point theorems in neutrosophic fuzzy metric spaces, while [8] explored nonlinear contractions in the same setting. In complete neutrosophic metric spaces, [5] introduced ψ-quasi-contractions. Auxiliary functions have played a key role in generalizations, as seen in b-metric spaces [6] and generalized metric spaces [10]. Further contributions include Geraghty-type contractions under ωt-distance [9] and integral-type contractions in neutrosophic spaces [11]. Additionally, [12] investigated fixed points for Geraghty mappings using equivalent distances, highlighting the versatility of neutrosophic structures in unifying discontinuous and imprecise data scenarios. [13] in their work establishing new fixed point theorems. This direction was further advanced by [14], who investigated rational $(\alpha, \beta)\phi$-mω contractions within complete quasi metric spaces, providing important extensions to existing theory. These contributions collectively demonstrate the ongoing evolution of fixed point analysis through sophisticated distance function modifications and contraction mappings. [15] established new fixed point theorems for NF-L contractions in complete neutrosophic fuzzy metric spaces, while [16] developed parallel results for T-distance spaces in complete b-metric spaces. These works demonstrate the ongoing expansion of fixed point theory into increasingly abstract distance structures.

Zadeh [17] has made a significant contribution to numerous scientific disciplines through the introduction of fuzzy sets, which have extensive research and application potential, as noted in [18–21] and related references. Although this setting is highly relevant to practical applications, it is not always provided effective solutions to many challenges over the years. Consequently, there has been a renewed emphasis on research aimed at addressing these issues. In this regard, Atanassov [22] presented Intuitionistic Fuzzy Sets (IFSs) as a method to confront these challenges.

Smarandache [23] developed the concept of the Neutrosophic Set (NS), serves as a sophisticated extension of traditional set theory. This concept also exhibits a broad spectrum of applications across different domains. For instance, The authors in [24] examined the scale invariance characteristic of the stable Pareto distribution. They provided an overview of Mathematica code, particularly highlighting the application of Neutrosophic logic in guiding risk management principles. Additionally, The authors in [25] investigated the resolution of first-order differential equations by employing trapezoidal neutrosophic numbers as initial conditions. They analyzed different forms based on the relationships among truth, indeterminacy, and falsity. For a more thorough exploration of the applications of NS and its uses, the readers should refer to [28–35] and the associated references therein.

Recent developments in neutrosophic mathematics have expanded decision-making frameworks and algebraic structures. [36,37] introduced algorithmic approaches using similarity measures in interval-valued neutrosophic soft settings, while [38] developed novel Q-neutrosophic soft interval matrices with practical applications. Earlier foundational work by [39, 40] established soft expert on complex multi-fuzzy classes, and more recently, [41, 42] presented significant generalizations of interval-valued Q-neutrosophic soft matrices with broad applicability. These contributions collectively advance the theoretical foundations and practical implementations of neutrosophic systems.

2. **Preliminary**

Triangular norms (TNs), introduced by Menger [43], extending the triangle inequality in metric spaces. Triangular conorms (CNs), as duals of TNs, model the union of fuzzy sets. Both TNs and CNs are essential in fuzzy logic, particularly for intersections and unions.

In this document, we define the sets as follows: $\mathbb{R}^+ = [0, \infty)$ and $\mathcal{I} = [0, 1]$.

Definition 2.1. Let $\bullet : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be an operation. Then, $\bullet$ is called as a continuous t-norm (TNs) if it satisfies the following properties for all $\sigma, \sigma', s, s' \in \mathcal{I}$.

(1) $\sigma \bullet 1 = \sigma$,
(2) If $\sigma \leq \sigma'$ and $s \leq s'$, than $\sigma \bullet s \leq \sigma' \bullet s'$,
(3) $\bullet$ is continuous,
(4) $\bullet$ is commutative and associate.

Definition 2.2. Let $\diamond : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be an operation. Then, $\diamond$ is called to as a continuous t-conorm (CNs) if it satisfies the following conditions for all elements $\sigma, \sigma', s, s' \in \mathcal{I}$:

(1) $\sigma \diamond 0 = \sigma$,
(2) If $\sigma \leq \sigma'$ and $s \leq s'$, than $\sigma \diamond s \leq \sigma' \diamond s'$,
(3) $\diamond$ is continuous,
(4) $\diamond$ is commutative and associate.

The notion of neutrosophic metric space (NMS) was initially presented by Kirisci and Simsek. This framework has been utilized to investigate various fixed point theorems. The definition of neutrosophic metric spaces is outlined as follows.

Definition 2.3. [44] A 6-tuple $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ is defined as a Neutrosophic Metric Space (NMS) when the set $\mathcal{W}$ is non-empty. In this setting, $\bullet$ represents a continuous t-norm, and $\diamond$ denotes a continuous t-conorm. Moreover, the elements $\mathcal{N}, \mathcal{U}$, and $\mathcal{O}$ are fuzzy sets defined on $\mathcal{W}^2 \times (0,\infty)$. These components must meet the following conditions for all $\zeta, \varrho, c \in \mathcal{W}$, and all $\tau, \rho > 0$.

(1) $0 \leq \mathcal{N}(\zeta,\varrho,\tau) \leq 1$, $0 \leq \mathcal{U}(\zeta,\varrho,\tau) \leq 1$, $0 \leq \mathcal{O}(\zeta,\varrho,\tau) \leq 1$,
(2) $0 \leq \mathcal{N}(\zeta,\varrho,\tau) + \mathcal{U}(\zeta,\varrho,\tau) + \mathcal{O}(\zeta,\varrho,\tau) \leq 3$,
(3) $\mathcal{N}(\zeta,\varrho,\tau) = 1$, for $\tau > 0$ iff $\zeta = \varrho$
(4) $\mathcal{N}(\zeta,\varrho,\tau) = H(\varrho,\zeta,\tau)$, for $\tau > 0$
(5) $\mathcal{N}(\zeta,\varrho,\tau) \bullet \mathcal{N}(\varrho,c,\rho) \leq \mathcal{N}(\zeta,c,\tau+\rho)$
(6) $\mathcal{N}(\zeta,\varrho,\cdot) : \mathbb{R}^+ \to \mathcal{I}$ is continuous
(7) $\lim_{\tau\to\infty} \mathcal{N}(\zeta,\varrho,\tau) = 1$
(8) $\mathcal{U}(\zeta,\varrho,\tau) = 0$ iff $\zeta = \varrho$
(9) $\mathcal{U}(\zeta,\varrho,\tau) = \mathcal{U}(\varrho,\zeta,\tau)$,
(10) $\mathcal{U}(\zeta,\varrho,\tau) \diamond \mathcal{U}(\varrho,c,\rho) \geq \mathcal{U}(\zeta,c,\tau+\rho)$,
(11) $\mathcal{U}(\zeta,\varrho,\cdot) : \mathbb{R}^+ \to \mathcal{I}$ is continuous
(12) $\lim_{\tau\to\infty} \mathcal{U}(\zeta,\varrho,\tau) = 0$
(13) $\mathcal{O}(\zeta,\varrho,\tau) = 0$, for $\tau > 0$ iff $\zeta = \varrho$
(14) $\mathcal{O}(\zeta,\varrho,\tau) = \mathcal{O}(\varrho,\zeta,\tau)$,
(15) $\mathcal{O}(\zeta,\varrho,\tau) \diamond \mathcal{O}(\varrho,c,\rho) \geq \mathcal{O}(\zeta,c,\tau+\rho)$,
(16) $\mathcal{O}(\zeta,\varrho,\cdot) : \mathbb{R} \to \mathcal{I}$ is continuous
(17) $\lim_{\tau\to\infty} \mathcal{O}(\zeta,\varrho,\tau) = 0$
(18) If $\tau \leq 0$, then $\mathcal{N}(\zeta,\varrho,\tau) = 0, \mathcal{U}(\zeta,\varrho,\tau) = \mathcal{O}(\zeta,\varrho,\tau) = 1$

The functions $\mathcal{N}(\zeta,\varrho,\tau)$, $\mathcal{U}(\zeta,\varrho,\tau)$, and $\mathcal{O}(\zeta,\varrho,\tau)$ represent the degrees of nearness, neutralness, and non-nearness between ζ and ϱ in relation to the parameter τ, respectively.

The convergence, Cauchy-ness, completeness in NMS are given as follows.

Definition 2.4. [44] Let (ζ_n) be a sequence in a NMS $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$. Then

(1) (ζ_n) converges to $\zeta \in \mathcal{W}$ if for a given $\epsilon \in (0,1)$, $\tau > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathcal{N}(\zeta_n,\zeta,\tau) > 1-\epsilon,\ \mathcal{U}(\zeta_n,\zeta,\tau) < \epsilon,\ \mathcal{O}(\zeta_n,\zeta,\tau) < \epsilon$$

i.e.,

$$\lim_{n\to\infty} \mathcal{N}(\zeta_n,\zeta,\tau) = 1,\ \lim_{n\to\infty} \mathcal{U}(\zeta_n,\zeta,\tau) = 0,\ \lim_{n\to\infty} \mathcal{O}(\zeta_n,\zeta,\tau) = 0$$

(2) (ζ_n) is called Cauchy if for a given $\epsilon \in (0,1)$, $\tau > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$

$$\mathcal{N}(\zeta_n,\zeta_m,\tau) > 1-\epsilon,\ \mathcal{U}(\zeta_n,\zeta_m,\tau) < \epsilon,\ \mathcal{O}(\zeta_n,\zeta_m,\tau) < \epsilon$$

i.e.,

$$\lim_{n.m\to\infty}\mathcal{N}(\zeta_n,\zeta_m,\tau)=1,\ \lim_{n,m\to\infty}\mathcal{U}(\zeta_n,\zeta_m,\tau)=0,\ \lim_{n,m\to\infty}\mathcal{O}(\zeta_n,\zeta_m,\tau)=0$$

(3) $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ is said to be complete if every Cauchy sequence converges to an element in $\mathcal{W}$.

In the work of Simsek and Kirisci [45], NC-contractions were introduced within the context of neutrosophic metric spaces, demonstrating that each NC-contraction possesses a unique fixed point under specific conditions.

Definition 2.5. [45] Let $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ be a NMS. A mapping $f:\mathcal{W}\to\mathcal{W}$ is called neutrosophic contraction if there is $k\in(0,1)$ such that for each $\zeta,\varrho\in\mathcal{W}$ and $\tau>0$, we have

$$\frac{1}{\mathcal{N}(f\zeta,f\varrho,\tau)}-1\le k\left(\frac{1}{\mathcal{N}(\zeta,\varrho,\tau)}-1\right),$$

$$\frac{1}{\mathcal{U}(f\zeta,f\varrho,\tau)}-1\ge k\left(\frac{1}{\mathcal{U}(\zeta,\varrho,\tau)}-1\right),$$

and

$$\frac{1}{\mathcal{O}(f\zeta,f\varrho,\tau)}-1\ge k\left(\frac{1}{\mathcal{O}(\zeta,\varrho,\tau)}-1\right).$$

The following lemma presented by Bataihah and Hazaymeh [46]

Lemma 2.6. *[46] Let $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ be a NMS. Then*

(1) $\mathcal{N}(\zeta,\varrho,\cdot):\mathbb{R}\to\mathbb{R}$ *is non-decreasing*
(2) $\mathcal{U}(\zeta,\varrho,\cdot):\mathbb{R}\to\mathbb{R}$ *is non-increasing*
(3) $\mathcal{O}(\zeta,\varrho,\cdot):\mathbb{R}\to\mathbb{R}$ *is non-increasing*

3. Main Result

In the subsequent sections, we will first introduce a fundamental lemmas that forms the basis of our study. Following that, we will outline our contractions in the setting of complete NMS, employing an auxiliary function. Then, we will establish that these contractions possess a unique fixed point under specific conditions and investigate the significant results that arise from our main discoveries.

We commence with the subsequent essential lemma that are required to derive our principal result.

Lemma 3.1. *Let $\mathcal{V}:\mathcal{D}\to\mathcal{B}$ be a bounded real valued function, where $\mathcal{D}$, $\mathcal{B}$ are subsets of $\mathbb{R}$. Then*

(1) *If $\mathcal{V}$ is non increasing, then there is $c>0$ such that for all $\alpha\in\mathcal{D}$*

$$\mathcal{V}\left(\frac{\alpha}{2}\right)\le c\ \mathcal{V}(\alpha)$$

(2) *If $\mathcal{V}$ is non decreasing, then there is $\tau > 0$ such that for all $\alpha \in \mathcal{D}$*

$$\mathcal{V}\left(\frac{\alpha}{2}\right) \geq \tau\ \mathcal{V}(\alpha)$$

Proof. (1)Let $\alpha \in \mathcal{D}$ be arbitrary. Since $\frac{\alpha}{2} < \alpha$ and $\mathcal{V}$ is non increasing, then

$$\mathcal{V}\left(\frac{\alpha}{2}\right) \geq \mathcal{V}(\alpha).$$

So, there is $\xi_\alpha \geq 0$ such that

$$\mathcal{V}\left(\frac{\alpha}{2}\right) = \mathcal{V}(\alpha) + \xi_\alpha.$$

Thus,

$$\mathcal{V}\left(\frac{\alpha}{2}\right) \leq 2\max\{\mathcal{V}(\alpha), \xi_\alpha\}.$$

Now,

$$\mathcal{V}(\alpha).\xi_\alpha = c_\alpha \Longrightarrow c_\alpha.\mathcal{V}(\alpha) = \xi_\alpha.$$

So,

$$\mathcal{V}\left(\frac{\alpha}{2}\right) \leq 2\max\{\mathcal{V}(\alpha), c_\alpha.\mathcal{V}(\alpha)\}$$
$$\mathcal{V}\left(\frac{\alpha}{2}\right) \leq \max\{2, 2c_\alpha\}\mathcal{V}(\alpha).$$

Let $C = \max\{2, \max_{\alpha\in\mathcal{D}}\{2c_\alpha\}\}$. Then we get the result.
The proof of (2) is identical to that of (1).□

Remark 3.2. According to Lemma 2.6, and Lemma 3.1, there are $C_\mathcal{N}, C_\mathcal{U}, C_\mathcal{O}$ such that

$$\mathcal{N}(\zeta, \varrho, \frac{\alpha}{2}) \geq C_\mathcal{N}\ \mathcal{N}(\zeta, \varrho, \alpha),$$
$$\mathcal{U}(\zeta, \varrho, \frac{\alpha}{2}) \leq C_\mathcal{U}\ \mathcal{U}(\zeta, \varrho, \alpha),$$
$$\mathcal{O}(\zeta, \varrho, \frac{\alpha}{2}) \leq C_\mathcal{O}\ \mathcal{O}(\zeta, \varrho, \alpha),$$

The following definition is due to Bataihah and Hazaymeh [46]

Definition 3.3. [46]

In this framework, we characterize a real-valued function of three variables defined on the domain $\mathcal{W}^2 \times (0, \infty)$, where $\mathcal{W}$ represents any non-empty set. We denote this function as $\mathcal{H}$ and assert that it exhibits the property (UC) if, for any sequences (ζ_n) and (ω_n) within $\mathcal{W}$, the subsequent equality is satisfied.

$$\lim_{\tau\to\tau_0}\lim_{n\to\infty} \mathcal{H}(\zeta_n, \omega_n, \tau) = \lim_{n\to\infty}\lim_{\tau\to\tau_0} \mathcal{H}(\zeta_n, \omega_n, \tau).$$

whenever the two limits are exist.

In the subsequent sections of this study, we will operate under the assumption that each of the fuzzy sets $\mathcal{N}$, $\mathcal{U}$, $\mathcal{O}$ possesses the UC property.

Through this context we need the following class of functions:

Definition 3.4. Let $\mathscr{L}$ denote the collection of all functions $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the condition

$$\mathcal{L}(a, b) \leq a + b.$$

We will now present the definition of a neutrosophic $(\mathcal{L}, \phi)$-contraction.

Definition 3.5. Let $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ be a NMS. A mapping $f : \mathcal{W} \to \mathcal{W}$ is called neutrosophic $(\mathcal{L}, \phi)$-contraction if for each $\zeta, \varrho \in \mathcal{W}$ and each $\tau > 0$, we have

$$\frac{1}{\mathcal{N}(f\zeta, f\varrho, \tau)} - 1 \leq \phi\left[\frac{1}{\mathcal{N}(\zeta, \varrho, \tau)} - 1 + \mathcal{L}\left(\frac{1}{\mathcal{N}(\zeta, f\zeta, \tau)} - 1, \frac{1}{\mathcal{N}(\varrho, f\varrho, \tau)} - 1\right)\right],$$

$$\mathcal{U}(f\zeta, f\varrho, \tau) \leq \phi\left[\mathcal{U}(\zeta, \varrho, \tau) + \mathcal{L}\left(\mathcal{U}(\zeta, f\zeta, \tau), \mathcal{U}(\varrho, f\varrho, \tau)\right)\right],$$

and

$$\mathcal{O}(f\zeta, f\varrho, \tau) \leq \phi\left[\mathcal{O}(\zeta, \varrho, \tau) + \mathcal{L}\left(\mathcal{O}(\zeta, f\zeta, \tau), \mathcal{O}(\varrho, f\varrho, \tau)\right)\right],$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\phi(t) < \frac{1}{3}t, \ \forall t > 0$.

Lemma 3.6. *Let $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ be a complete NMS, Suppose that $f : \mathcal{W} \to \mathcal{W}$ is neutrosophic $(\mathcal{L}, \phi)$-contraction. Consequently, if f has a fixed point then it is unique.*

Proof. Assume that $\mu, \nu \in \mathcal{W}$ such that $f\mu = \mu$ and $f\nu = \nu$. Then, Definition 3.5 implies

$$\frac{1}{\mathcal{N}(\mu, \nu, \tau)} - 1 = \frac{1}{\mathcal{N}(f\mu, f\nu, \tau)} - 1 \leq \phi\left[\frac{1}{\mathcal{N}(\mu, \nu, \tau)} - 1 + \mathcal{L}\left(\frac{1}{\mathcal{N}(\mu, \mu, \tau)} - 1, \frac{1}{\mathcal{N}(\nu, \nu, \tau)} - 1\right)\right],$$

$$\mathcal{U}(\mu, \nu, \tau) = \mathcal{U}(f\mu, f\nu, \tau) \leq \phi\left[\mathcal{U}(\mu, \nu, \tau) + \mathcal{L}\left(\mathcal{U}(\mu, \mu, \tau), \mathcal{U}(\nu, \nu, \tau)\right)\right],$$

and

$$\mathcal{O}(\mu, \nu, \tau) = \mathcal{O}(f\mu, f\nu, \tau) \leq \phi\left[\mathcal{O}(\mu, \nu, \tau) + \mathcal{L}\left(\mathcal{O}(\mu, \mu, \tau), \mathcal{O}(\nu, \nu, \tau)\right)\right].$$

Hence, be using properties of ϕ and $\mathcal{L}$, we get

$$\frac{1}{\mathcal{N}(\mu, \nu, \tau)} - 1 \leq \frac{1}{3}\left[\frac{1}{\mathcal{N}(\mu, \nu, \tau)} - 1\right],$$

$$\mathcal{U}(\mu, \nu, \tau) \leq \frac{1}{3}\left[\mathcal{U}(\mu, \nu, \tau)\right],$$

and

$$\mathcal{O}(\mu, \nu, \tau) \leq \frac{1}{3}\left[\mathcal{O}(\mu, \nu, \tau)\right].$$

So, $\mathcal{N}(\mu,\nu,\tau)=1$, $\mathcal{U}(\mu,\nu,\tau)=0$, and $\mathcal{O}(\mu,\nu,\tau)=0$, and hence $\mu=\nu$. □

Lemma 3.7. *Let $(\mathcal{W},\mathcal{N},\mathcal{B},\mathcal{U},\mathcal{O},\diamond,\bullet)$ be a NFMS, and let (ζ_n) be a sequence such that for $\tau>0$*

$$\begin{aligned}\mathcal{N}(\zeta_p,\zeta_q,\tau)&\geq\mathcal{N}(\zeta_{p-1},\zeta_{q-1},\tau)\\ \mathcal{U}(\zeta_p,\zeta_q,\tau)&\leq\mathcal{U}(\zeta_{p-1},\zeta_{q-1},\tau)\\ \mathcal{O}(\zeta_p,\zeta_q,\tau)&\leq\mathcal{O}(\zeta_{p-1},\zeta_{q-1},\tau)\end{aligned}\tag{1}$$

and

$$\begin{aligned}\lim_{n\to\infty}\mathcal{N}(\zeta_n,\zeta_{n+1},\tau)&=1,\\ \lim_{n\to\infty}\mathcal{U}(\zeta_n,\zeta_{n+1},\tau)&=0,\\ \lim_{n\to\infty}\mathcal{O}(\zeta_n,\zeta_{n+1},\tau)&=0.\end{aligned}\tag{2}$$

If (ζ_n) is not Cauchy, then there exist an $1>\epsilon>0$ and $\tau>0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) such that one at least of the following holds.

$$\begin{aligned}\lim_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)&=1-\epsilon,\\ \lim_{k\to\infty}\mathcal{U}(\zeta_{n_k},\zeta_{m_k},\tau)&=\epsilon,\\ \lim_{k\to\infty}\mathcal{O}(\zeta_{n_k},\zeta_{m_k},\tau)&=\epsilon.\end{aligned}$$

Proof. If (ζ_n) is not Cauchy, then for each $\tau>0$

$$\begin{aligned}\lim_{n,m\to\infty}\mathcal{N}(\zeta_n,\zeta_m,\tau)&\neq1,\\ \lim_{n,m\to\infty}\mathcal{U}(\zeta_n,\zeta_m,\tau)&\neq0,\end{aligned}$$

or

$$\lim_{n,m\to\infty}\mathcal{O}(\zeta_n,\zeta_m,\tau)\neq0.$$

Case 1: If $\lim_{n,m\to\infty}\mathcal{N}(\zeta_n,\zeta_m,\tau)\neq1$, then there are $\tau>0$, and $\epsilon>0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) is selected as the smallest index satisfying the condition.

$$\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)\leq1-\epsilon,\quad m_k>n_k>k.\tag{3}$$

This implies that

$$\mathcal{N}(\zeta_{n_k},\zeta_{m_k-1},\tau)>1-\epsilon.\tag{4}$$

chose $\delta>0$. Then

$$\begin{aligned}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau+\delta) &\geq \mathcal{N}(\zeta_{n_k},\zeta_{m_k-1},\tau)\diamond\mathcal{N}(\zeta_{m_k-1},\zeta_{m_k},\delta)\\ &> (1-\epsilon)\diamond\mathcal{N}(\zeta_{m_k-1},\zeta_{m_k},\delta)\end{aligned}$$

Using Equation 2, we get

$$\liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau+\delta)\geq(1-\epsilon).$$

Also,

$$\begin{aligned}(1-\epsilon) &\leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau+\delta)\\ &= \liminf_{k\to\infty}\lim_{\delta\to 0^+}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau+\delta)\\ &= \liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau).\end{aligned}$$

Also, from 3, it follows

$$\limsup_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)\leq(1-\epsilon).$$

So, we get

$$\lim_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)=(1-\epsilon).$$

Again, we have

$$\begin{aligned}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau+\delta) &\geq \mathcal{N}(\zeta_{n_k-1},\zeta_{n_k},\delta)\diamond\mathcal{N}(\zeta_{n_k},\zeta_{m_k-1},\tau)\\ &> \mathcal{N}(\zeta_{n_k-1},\zeta_{n_k},\delta)\diamond(1-\epsilon).\end{aligned}$$

Using Equation 2, we get $\liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau+\delta)\geq(1-\epsilon)$.
Also,

$$\begin{aligned}(1-\epsilon) &\leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau+\delta)\\ &= \liminf_{k\to\infty}\lim_{\delta\to 0^+}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau+\delta)\\ &= \liminf_{k\to\infty}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau).\end{aligned}$$

From Eq 3, we get

$$\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau)\leq\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)\leq(1-\epsilon).$$

So,

$$\limsup_{k\to\infty}\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau)\leq(1-\epsilon).$$

Hence,

$$\lim_{k\to\infty} \mathcal{N}(\zeta_{n_k-1}, \zeta_{m_k-1}, \tau) = (1-\epsilon).$$

The demonstration for the remaining cases is Similar to that of Case (1). □

Our methodology combines neutrosophic set theory with fixed-point techniques through three key innovations: first, we develop $(\mathcal{L}, \phi)$-contractions that simultaneously preserve the neutrosophic structure of truth ($\mathcal{N}$), indeterminacy ($\mathcal{U}$), and falsity ($\mathcal{O}$) memberships; second, we establish the critical bounding condition $\phi(t) < \frac{1}{3}t$ to ensure proper contraction across all membership dimensions; and third, we introduce the constraint $\frac{1}{3}C_{\mathcal{O}} < 1$ to control the falsity component's influence while maintaining mathematical tractability. This integrated approach enables the extension of Banach-type fixed-point theory to neutrosophic metric spaces while preserving their essential three-valued logic structure.

Theorem 3.8. *Let $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ be a complete NMS, Suppose that $f : \mathcal{W} \to \mathcal{W}$ is neutrosophic $(\mathcal{L}, \phi)$-contraction where $\frac{1}{3}C_{\mathcal{O}} < 1$. Therefore, the function f has exactly one fixed point*

Proof. Let $\zeta_0 \in \mathcal{W}$ represent an arbitrary point. We examine the Picard sequence (ζ_n) characterized by the relation $\zeta_{n+1} = f^(\zeta_0)$ for all $n \geq 0$. From Definition 3.5, we get for each $n \in \mathbb{N}$

$$\frac{1}{\mathcal{N}(\zeta_n, \zeta_{n+1}, \tau)} - 1 \leq \phi\left[\frac{1}{\mathcal{N}(\zeta_{n-1}, \zeta_n, \tau)} - 1 + \mathcal{L}\left(\frac{1}{\mathcal{N}(\zeta_{n-1}, \zeta_n, \tau)} - 1, \frac{1}{\mathcal{N}(\zeta_n, \zeta_{n+1}, \tau)} - 1\right)\right], \tag{5}$$

$$\mathcal{U}(\zeta_n, \zeta_{n+1}, \tau) \leq \phi\left[\mathcal{U}(\zeta_{n-1}, \zeta_n, \tau) + \mathcal{L}\left(\mathcal{U}(\zeta_{n-1}, \zeta_n, \tau), \mathcal{U}(\zeta_n, \zeta_{n+1}, \tau)\right)\right], \tag{6}$$

and

$$\mathcal{O}(\zeta_n, \zeta_{n+1}, \tau) \leq \phi\left[\mathcal{O}(\zeta_{n-1}, \zeta_n, \tau) + \mathcal{L}\left(\mathcal{O}(\zeta_{n-1}, \zeta_n, \tau), \mathcal{O}(\zeta_n, \zeta_{n+1}, \tau)\right)\right]. \tag{7}$$

From the definition of $\mathcal{L}$, and ϕ we get

$$\frac{1}{\mathcal{N}(\zeta_n, \zeta_{n+1}, \tau)} - 1 < \left(\frac{1}{\mathcal{N}(\zeta_{n-1}, \zeta_n, \tau)} - 1\right),$$

$$\mathcal{U}(\zeta_n, \zeta_{n+1}, \tau) < \left(\mathcal{U}(\zeta_{n-1}, \zeta_n, \tau)\right),$$

and

$$\mathcal{O}(\zeta_n, \zeta_{n+1}, \tau) < \left(\mathcal{O}(\zeta_{n-1}, \zeta_n, \tau)\right).$$

So, we get that the sequences

$$(\frac{1}{\mathcal{N}(\zeta_n,\zeta_{n+1},\tau)}-1:n\in\mathbb{N}),$$

$$(\mathcal{U}(\zeta_n,\zeta_{n+1},\tau):n\in\mathbb{N}),$$

and

$$(\mathcal{O}(\zeta_n,\zeta_{n+1},\tau):n\in\mathbb{N})$$

are nonincreasing in $[0,\infty)$. Then, there are $l_1,l_2,l_3\geq 0$ such that

$$\frac{1}{\mathcal{N}(\zeta_n,\zeta_{n+1},\tau)}-1\to l_1,$$

$$\mathcal{U}(\zeta_n,\zeta_{n+1},\tau)\to l_2,$$

and

$$\mathcal{O}(\zeta_n,\zeta_{n+1},\tau)\to l_3.$$

Assume to the contrary that one of l_1,l_2,l_3 is greater than 0. By taking the limit in 5, 6, and 7 and using the properties of $\mathcal{L},\phi$, we get that

$l_1<l_1$, $l_2<l_2$, or $l_3<l_3$. which is a contradiction in each case. Hence $l_1=l_2=l_3=0$.

So, we have

$$\begin{aligned}\lim_{n\to\infty}\mathcal{N}(\zeta_n,\zeta_{n+1},\tau)&=1,\\ \lim_{n\to\infty}\mathcal{U}(\zeta_n,\zeta_{n+1},\tau)&=0,\\ \lim_{n\to\infty}\mathcal{O}(\zeta_n,\zeta_{n+1},\tau)&=0.\end{aligned}\tag{8}$$

Now, we calim that (ζ_n) is Cauchy sequence. Assume the contrary. Then, according to Lemma 3.7, there exist an $1>\epsilon>0$ and $\tau>0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) such that one at least of the following holds.

$$\begin{aligned}\lim_{k\to\infty}\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)&=1-\epsilon,\\ \lim_{k\to\infty}\mathcal{U}(\zeta_{n_k},\zeta_{m_k},\tau)&=\epsilon,\\ \lim_{k\to\infty}\mathcal{O}(\zeta_{n_k},\zeta_{m_k},\tau)&=\epsilon.\end{aligned}$$

Definition 3.5 implies that

$$\frac{1}{\mathcal{N}(\zeta_{n_k},\zeta_{m_k},\tau)}-1\leq\phi\left[\frac{1}{\mathcal{N}(\zeta_{n_k-1},\zeta_{m_k-1},\tau)}-1+\mathcal{L}\left(\frac{1}{\mathcal{N}(\zeta_{n_k-1},\zeta_{n_k},\tau)}-1,\frac{1}{\mathcal{N}(\zeta_{m_k-1},\zeta_{m_k},\tau)}-1\right)\right],$$

$$\mathcal{U}(\zeta_{n_k},\zeta_{m_k},\tau)\leq\phi\left[\mathcal{U}(\zeta_{n_k-1},\zeta_{m_k-1},\tau)+\mathcal{L}\left(\mathcal{U}(\zeta_{n_k-1},\zeta_{n_k},\tau),\mathcal{U}(\zeta_{m_k-1},\zeta_{m_k},\tau)\right)\right],$$

and

$$\mathcal{O}(\zeta_{n_k},\zeta_{m_k},\tau) \leq \phi\left[\mathcal{O}(\zeta_{n_k-1},\zeta_{m_k-1},\tau) + \mathcal{L}\left(\mathcal{O}(\zeta_{n_k-1},\zeta_{n_k},\tau),\mathcal{O}(\zeta_{m_k-1},\zeta_{m_k},\tau)\right)\right].$$

By taking the limit when $k \to \infty$, we get

$$\frac{\epsilon}{1-\epsilon} < \frac{\epsilon}{1-\epsilon},$$

$$\epsilon < \epsilon,$$

and

$$\epsilon < \epsilon.$$

Which is a contradiction. Hence (ζ_n) is a Cauchy sequence, thus, there is $\mu \in \mathcal{W}$ such that $\zeta_n \to \mu$.

Assume that $f\mu \neq \mu$. From Definition 3.5, we have for each $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{O}(f\mu,\zeta_{n+1},\tau) &\leq \phi\left[\mathcal{O}(\mu,\zeta_n,\tau) + \mathcal{L}\left(\mathcal{O}(\mu,f\mu,\tau),\mathcal{O}(\zeta_n,\zeta_{n+1},\tau)\right)\right]\\ &\leq \phi\left[\mathcal{O}(\mu,\zeta_n,\tau) + \mathcal{O}(\mu,f\mu,\tau) + \mathcal{O}(\zeta_n,\zeta_{n+1},\tau)\right].\end{aligned}$$

By taking the limit, we obtain

$$\lim_{n\to\infty} \mathcal{O}(f\mu,\zeta_{n+1},\tau) \leq \phi\ \mathcal{O}(\mu,f\mu,\tau) < \frac{1}{3}\mathcal{O}(\mu,f\mu,\tau). \tag{9}$$

On the other hand, using Remark 3.2, we get

$$\begin{aligned}\mathcal{O}(\mu,f\mu,\tau) &\leq \mathcal{O}(\mu,\zeta_n,\tfrac{\tau}{2}) \bullet \mathcal{O}(\zeta_n,f\mu,\tfrac{\tau}{2})\\ &\leq \mathcal{O}(\mu,\zeta_n,\tfrac{\tau}{2}) \bullet C_{\mathcal{O}}\ \mathcal{O}(\zeta_n,f\mu,\tau).\end{aligned}$$

So,

$$\mathcal{O}(\mu,f\mu,\tau) \leq C_{\mathcal{O}} \lim_{n\to\infty} \mathcal{O}(\zeta_n,f\mu,\tau). \tag{10}$$

From Eq 9, Eq 10, we get

$$\lim_{n\to\infty} \mathcal{O}(\zeta_n,f\mu,\tau) < \frac{1}{3}C_{\mathcal{O}} \lim_{n\to\infty} \mathcal{O}(\zeta_n,f\mu,\tau) < \lim_{n\to\infty} \mathcal{O}(\zeta_n,f\mu,\tau).$$

A contradiction. So, $\mu = f\mu$. The uniqueness follows from Lemma 3.6 □

Remark 3.9. The condition $\phi(t) < \frac{1}{3}t$ is crucial to ensure that the neutrosophic inequalities contract sufficiently in all three components ($\mathcal{N}$, $\mathcal{U}$, $\mathcal{O}$). Relaxing this to $\phi(t) < t$ may not guarantee the result.

Example 3.10. Consider the complete neutrosophic metric space $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ where:

- $\mathcal{W} = [0,1]$
- The t-norm $\bullet$ and t-conorm $\diamond$ are defined as $a \bullet b = \min\{a,b\}$ and $a \diamond b = \max\{a,b\}$

- The neutrosophic metrics are defined for $\tau > 0$ as:

$$\mathcal{N}(\zeta, \varrho, \tau) = \frac{\tau}{\tau + |\zeta - \varrho|}$$
$$\mathcal{U}(\zeta, \varrho, \tau) = \frac{|\zeta - \varrho|}{\tau + |\zeta - \varrho|}$$
$$\mathcal{O}(\zeta, \varrho, \tau) = \frac{|\zeta - \varrho|}{2\tau + |\zeta - \varrho|}$$

Define the mapping $f : \mathcal{W} \to \mathcal{W}$ by $f(\zeta) = \frac{\zeta}{4}$. We will verify that f satisfies all conditions of Theorem 3.8 with:

- $\mathcal{L}(a, b) = \frac{a+b}{2}$ (which satisfies $\mathcal{L}(a, b) \leq a + b$)
- $\phi(t) = \frac{t}{4}$ (which is continuous and satisfies $\phi(t) < \frac{t}{3}$ for all $t > 0$)
- $C_{\mathcal{O}} = \frac{1}{2}$ (since $\mathcal{O}(\zeta, \varrho, \tau) \leq \frac{1}{2}$)

Proof. We check each condition of Definition 3.5:

1. For the truth membership $\mathcal{N}$:

$$\frac{1}{\mathcal{N}(f\zeta, f\varrho, \tau)} - 1 = \frac{|f\zeta - f\varrho|}{\tau} = \frac{|\frac{\zeta}{4} - \frac{\varrho}{4}|}{\tau} = \frac{|\zeta - \varrho|}{4\tau}$$
$$\phi\left[\frac{1}{\mathcal{N}(\zeta, \varrho, \tau)} - 1 + \mathcal{L}\left(\frac{1}{\mathcal{N}(\zeta, f\zeta, \tau)} - 1, \frac{1}{\mathcal{N}(\varrho, f\varrho, \tau)} - 1\right)\right]$$
$$= \frac{1}{4}\left[\frac{|\zeta - \varrho|}{\tau} + \frac{1}{2}\left(\frac{|\zeta - \frac{\zeta}{4}|}{\tau} + \frac{|\varrho - \frac{\varrho}{4}|}{\tau}\right)\right]$$
$$= \frac{1}{4}\left[\frac{|\zeta - \varrho|}{\tau} + \frac{3}{8}\frac{|\zeta| + |\varrho|}{\tau}\right] \geq \frac{|\zeta - \varrho|}{4\tau}$$

2. For the indeterminacy membership $\mathcal{U}$:

$$\mathcal{U}(f\zeta, f\varrho, \tau) = \frac{|\frac{\zeta}{4} - \frac{\varrho}{4}|}{\tau + |\frac{\zeta}{4} - \frac{\varrho}{4}|} \leq \frac{|\zeta - \varrho|}{4\tau}$$
$$\phi\left[\mathcal{U}(\zeta, \varrho, \tau) + \mathcal{L}\left(\mathcal{U}(\zeta, f\zeta, \tau), \mathcal{U}(\varrho, f\varrho, \tau)\right)\right]$$
$$= \frac{1}{4}\left[\frac{|\zeta - \varrho|}{\tau + |\zeta - \varrho|} + \frac{1}{2}\left(\frac{\frac{3}{4}|\zeta|}{\tau + \frac{3}{4}|\zeta|} + \frac{\frac{3}{4}|\varrho|}{\tau + \frac{3}{4}|\varrho|}\right)\right] \geq \frac{|\zeta - \varrho|}{4\tau}$$

3. For the falsity membership $\mathcal{O}$:

$$\mathcal{O}(f\zeta, f\varrho, \tau) = \frac{|\frac{\zeta}{4} - \frac{\varrho}{4}|}{2\tau + |\frac{\zeta}{4} - \frac{\varrho}{4}|} \leq \frac{|\zeta - \varrho|}{8\tau}$$
$$\phi\left[\mathcal{O}(\zeta, \varrho, \tau) + \mathcal{L}\left(\mathcal{O}(\zeta, f\zeta, \tau), \mathcal{O}(\varrho, f\varrho, \tau)\right)\right]$$
$$= \frac{1}{4}\left[\frac{|\zeta - \varrho|}{2\tau + |\zeta - \varrho|} + \frac{1}{2}\left(\frac{\frac{3}{4}|\zeta|}{2\tau + \frac{3}{4}|\zeta|} + \frac{\frac{3}{4}|\varrho|}{2\tau + \frac{3}{4}|\varrho|}\right)\right] \geq \frac{|\zeta - \varrho|}{8\tau}$$

All conditions are satisfied, and $\frac{1}{3}C_{\mathcal{O}} = \frac{1}{6} < 1$. Therefore, by Theorem 3.8, f has a unique fixed point $\zeta^* = 0$. □

4. Consequences

Corollary 4.1. *Let $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ be a complete NMS and $f:\mathcal{W}\to\mathcal{W}$ be a self map. If there exists $0<k<\min\left\{\frac{1}{3},\frac{1}{C_{\mathcal{O}}}\right\}$ such that for all $\zeta,\varrho\in\mathcal{W}$ and $\tau>0$:*

$$\frac{1}{\mathcal{N}(f\zeta,f\varrho,\tau)}-1\leq k\left[\frac{1}{\mathcal{N}(\zeta,\varrho,\tau)}-1+\lambda\left(\frac{1}{\mathcal{N}(\zeta,f\zeta,\tau)}-1\right)+(1-\lambda)\left(\frac{1}{\mathcal{N}(\varrho,f\varrho,\tau)}-1\right)\right]$$

$$\mathcal{U}(f\zeta,f\varrho,\tau)\leq k\left[\mathcal{U}(\zeta,\varrho,\tau)+\lambda\mathcal{U}(\zeta,f\zeta,\tau)+(1-\lambda)\mathcal{U}(\varrho,f\varrho,\tau)\right]$$

$$\mathcal{O}(f\zeta,f\varrho,\tau)\leq k\left[\mathcal{O}(\zeta,\varrho,\tau)+\lambda\mathcal{O}(\zeta,f\zeta,\tau)+(1-\lambda)\mathcal{O}(\varrho,f\varrho,\tau)\right].$$

Then f has exactly one fixed point in $\mathcal{W}$.

Proof. This follows from Theorem 3.8 by taking

- $\mathcal{L}(a,b)=\lambda a+(1-\lambda)b$ which satisfies $\mathcal{L}(a,b)\leq a+b$
- $\phi(t)=kt$ with $k<\frac{1}{3}$

The given inequalities exactly match the contraction conditions with this choice of $\mathcal{L}$. □

Corollary 4.2 (Maximum Function Case)**.** *Let $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ be a complete NMS and $f:\mathcal{W}\to\mathcal{W}$ be a self map. Define $\mathcal{L}(a,b)=\max\{a,b\}$. If there exists $0<k<\min\left\{\frac{1}{3},\frac{1}{C_{\mathcal{O}}}\right\}$ such that for all $\zeta,\varrho\in\mathcal{W}$ and $\tau>0$:*

$$\frac{1}{\mathcal{N}(f\zeta,f\varrho,\tau)}-1\leq k\left[\frac{1}{\mathcal{N}(\zeta,\varrho,\tau)}-1+\max\left\{\frac{1}{\mathcal{N}(\zeta,f\zeta,\tau)}-1,\frac{1}{\mathcal{N}(\varrho,f\varrho,\tau)}-1\right\}\right]$$

$$\mathcal{U}(f\zeta,f\varrho,\tau)\leq k\left[\mathcal{U}(\zeta,\varrho,\tau)+\max\left\{\mathcal{U}(\zeta,f\zeta,\tau),\mathcal{U}(\varrho,f\varrho,\tau)\right\}\right]$$

$$\mathcal{O}(f\zeta,f\varrho,\tau)\leq k\left[\mathcal{O}(\zeta,\varrho,\tau)+\max\left\{\mathcal{O}(\zeta,f\zeta,\tau),\mathcal{O}(\varrho,f\varrho,\tau)\right\}\right].$$

Then f has a unique fixed point in $\mathcal{W}$.

Proof. This follows from Theorem 3.8 by taking

- $\mathcal{L}(a,b)=\max\{a,b\}$ which satisfies $\mathcal{L}(a,b)\leq a+b$
- $\phi(t)=kt$ with $k<\frac{1}{3}$

The given inequalities exactly match the contraction conditions with this choice of $\mathcal{L}$. □

Corollary 4.3. *Let* $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ *be a complete NMS and* $f : \mathcal{W} \to \mathcal{W}$ *be a self map. If there exists* $0 < k < \min\left\{\frac{1}{3}, \frac{1}{C_{\mathcal{O}}}\right\}$ *such that for all* $\zeta, \varrho \in \mathcal{W}$ *and* $\tau > 0$*:*

$$\frac{1}{\mathcal{N}(f\zeta, f\varrho, \tau)} - 1 \leq k\left[\frac{1}{\mathcal{N}(\zeta, \varrho, \tau)} - 1 + \frac{1}{\mathcal{N}(\zeta, f\zeta, \tau)} - 1 + \frac{1}{\mathcal{N}(\varrho, f\varrho, \tau)} - 1\right],$$

$$\mathcal{U}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{U}(\zeta, \varrho, \tau) + \mathcal{U}(\zeta, f\zeta, \tau) + \mathcal{U}(\varrho, f\varrho, \tau)\right],$$

$$\mathcal{O}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{O}(\zeta, \varrho, \tau) + \mathcal{O}(\zeta, f\zeta, \tau) + \mathcal{O}(\varrho, f\varrho, \tau)\right].$$

Then f *possesses exactly one fixed point in* $\mathcal{W}$.

Proof. This follows from Theorem 3.8 by taking: $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = a + b$, and $\phi(t) = kt, \ k < \frac{1}{3}$. □

Corollary 4.4. *Let* $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ *be a complete NMS and* $f : \mathcal{W} \to \mathcal{W}$ *be a self map. Assume that there is* $0 < k < \min\{\frac{1}{3}, \frac{1}{C_{\mathcal{O}}}\}$ *such that for each* $\zeta, \varrho \in \mathcal{W}$ *and each* $\tau > 0$, f *satisfies the following:*

$$\frac{1}{\mathcal{N}(f\zeta, f\varrho, \tau)} - 1 \leq k\left[\frac{1}{\mathcal{N}(\zeta, \varrho, \tau)} - 1\right],$$

$$\mathcal{U}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{U}(\zeta, \varrho, \tau)\right],$$

and

$$\mathcal{O}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{O}(\zeta, \varrho, \tau)\right].$$

Therefore, the function f has exactly one fixed point

Proof. This follows from Theorem 3.8 by taking: $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = 0$, and $\phi(t) = kt, \ k < \frac{1}{3}$. □

Corollary 4.5. *Let* $(\mathcal{W}, \mathcal{N}, \mathcal{U}, \mathcal{O}, \bullet, \diamond)$ *be a complete NMS and* $f : \mathcal{W} \to \mathcal{W}$ *be a self map. Assume that there is* $0 < k < \min\{\frac{1}{3}, \frac{1}{C_{\mathcal{O}}}\}$ *such that for each* $\zeta, \varrho \in \mathcal{W}$ *and each* $\tau > 0$, f *satisfies the following:*

$$\frac{1}{\mathcal{N}(f\zeta, f\varrho, \tau)} - 1 \leq k\left[\frac{1}{\mathcal{N}(\zeta, \varrho, \tau)} - 1 + \left|\frac{1}{\mathcal{N}(\zeta, f\zeta, \tau)} - \frac{1}{\mathcal{N}(\varrho, f\varrho, \tau)}\right|\right],$$

$$\mathcal{U}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{U}(\zeta, \varrho, \tau) + |\mathcal{U}(\zeta, f\zeta, \tau) - \mathcal{U}(\varrho, f\varrho, \tau)|\right],$$

and

$$\mathcal{O}(f\zeta, f\varrho, \tau) \leq k\left[\mathcal{O}(\zeta, \varrho, \tau) + |\mathcal{O}(\zeta, f\zeta, \tau) - \mathcal{O}(\varrho, f\varrho, \tau)|\right].$$

Therefore, the function f has exactly one fixed point

Proof. This follows from Theorem 3.8 by taking

- $\mathcal{L}(a,b) = |a-b|$ which satisfies $\mathcal{L}(a,b) \leq a+b$
- $\phi(t) = kt$ with $k < \frac{1}{3}$

The given inequalities exactly match the contraction conditions with this choice of $\mathcal{L}$. □

Corollary 4.6. *Let $(\mathcal{W},\mathcal{N},\mathcal{U},\mathcal{O},\bullet,\diamond)$ be a complete NMS and $f:\mathcal{W}\to\mathcal{W}$ be a self map. Assume that there is $0<k<\min\{\frac{1}{3},\frac{1}{C_{\mathcal{O}}}\}$ such that for each $\zeta,\varrho\in\mathcal{W}$ and each $\tau>0$, f satisfies the following:*

$$\frac{1}{\mathcal{N}(f\zeta,f\varrho,\tau)}-1 \leq k\left[\frac{1}{\mathcal{N}(\zeta,\varrho,\tau)}-1+2\min\left\{\frac{1}{\mathcal{N}(\zeta,f\zeta,\tau)}-1,\frac{1}{\mathcal{N}(\varrho,f\varrho,\tau)}-1\right\}\right],$$

$$\mathcal{U}(f\zeta,f\varrho,\tau) \leq k\left[\mathcal{U}(\zeta,\varrho,\tau)+2\min\left\{\mathcal{U}(\zeta,f\zeta,\tau),\mathcal{U}(\varrho,f\varrho,\tau)\right\}\right],$$

and

$$\mathcal{O}(f\zeta,f\varrho,\tau) \leq k\left[\mathcal{O}(\zeta,\varrho,\tau)+2\min\left\{\mathcal{O}(\zeta,f\zeta,\tau),\mathcal{O}(\varrho,f\varrho,\tau)\right\}\right].$$

Therefore, the function f has exactly one fixed point

Proof. This follows from Theorem 3.8 by taking

- $\mathcal{L}(a,b) = 2\min\{a,b\}$ which satisfies $\mathcal{L}(a,b) \leq a+b$
- $\phi(t) = kt$ with $k < \frac{1}{3}$

The given inequalities exactly match the contraction conditions with this choice of $\mathcal{L}$. □

Example 4.7 (Application of Corollary 4.5)**.** Consider the NMS $(\mathbb{R},\mathcal{N},\mathcal{U},\mathcal{O})$ with:

$$\mathcal{N}(\zeta,\varrho,\tau) = \frac{\tau}{\tau+|\zeta-\varrho|}$$
$$\mathcal{U}(\zeta,\varrho,\tau) = \frac{|\zeta-\varrho|}{\tau+|\zeta-\varrho|}$$
$$\mathcal{O}(\zeta,\varrho,\tau) = \frac{|\zeta-\varrho|}{2\tau+|\zeta-\varrho|}$$

Let $f(\zeta)=\frac{\zeta}{5}$. Taking $k=\frac{1}{4}$, we verify:

- $C_{\mathcal{O}} = \frac{1}{2}$ (since $\mathcal{O}(\zeta,\varrho,\tau) \leq \frac{1}{2}$)
- $k = \frac{1}{4} < \min\left\{\frac{1}{3},2\right\} = \frac{1}{3}$

All conditions of Corollary 4.5 are satisfied, with $\zeta^* = 0$ as the unique fixed point.

Problem. Does Theorem 3.8 hold if ϕ is only assumed to be upper semicontinuous (instead of continuous) with $\phi(t) < \frac{1}{3}t$?

5. Conclusions

Our investigation has established fundamental fixed-point results for $(\mathcal{L}, \phi)$-contractions in complete neutrosophic metric spaces. By introducing this new contraction class, we have extended the classical Banach contraction principle to handle the three-dimensional nature of neutrosophic logic - simultaneously addressing truth, indeterminacy, and falsity memberships. The key condition $\phi(t) < \frac{1}{3}t$ ensures sufficient contraction across all membership dimensions, while $\frac{1}{3}C_{\mathcal{O}} < 1$ maintains proper control of the falsity component's influence.

Several important research directions emerge from this work. First, extending these results to more general spaces such as bipolar neutrosophic or partial neutrosophic metric spaces could broaden the theory's applicability. The development of multivalued versions of $(\mathcal{L}, \phi)$-contractions would be particularly valuable for applications in optimization and control theory.

Second, computational aspects deserve attention. Developing efficient iterative algorithms to approximate fixed points in neutrosophic spaces could bridge the gap between theory and practical implementation. This would be especially relevant for machine learning applications dealing with uncertain data.

Finally, applied mathematical directions offer rich possibilities. Exploring applications to neutrosophic differential equations could provide new solution methods for problems with inherent uncertainty. Similarly, investigating connections to image processing and pattern recognition in ambiguous environments may yield practical benefits for computer vision systems.

These future directions would not only deepen our theoretical understanding but also expand the practical utility of neutrosophic fixed-point methods across scientific disciplines. The framework developed here provides a solid foundation for these subsequent investigations in both pure and applied mathematics.

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