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# Exploring the Differential Geometry of Reliability Function: Insights from Lifetime Weibull Distributions Under Neutrosophic Environment

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**Abstract**

This paper characterizes a set N as a two-dimensional surface marked by $R^{+} \times R^{+}$ and demonstrates its properties as a topological 2-reliability manifold and a differential reliability manifold. Utilizing the Weibull lifetime distribution, we derive the formula for the Riemannian manifold $(N, g_{ij})$. Finally, we prove that the reliability function represents the critical point of its log-likelihood on the manifold, which also serves as the saddle point.

Also, we discuss the same results in a neutrosophic environment, with neutrosophic variables and coefficients from the neutrosophic real ring R(I), where we get similar results of the original approach.

**Keywords:** Parametric Model, Reliability Function, Weibull Distribution, Manifold (Topological & Differentiable), Chart & Atlas, Riemannian Geometry, Log-Likelihood Function, Critical Point & Saddle Point, Neutrosophic Reliability Function, Neutrosophic Random Variable, Neutrosophic Topological Space, Neutrosophic Manifold, Homogeneous Neutrosophic Differential Transformation

## 1. Introduction

The geometric structure of reliability function manifolds, based on probability distributions, is a fundamental aspect of information geometry. Previous studies by authors such as Amari and Dodson [1, 2], among others, have focused on the geometric structures of exponential families within distribution manifolds. They have explored manifolds associated with the normal distribution [3], gamma distribution and invers gamma distribution [4–6], Pareto distribution [7, 8] and beta distribution [9, 10]. The reliability function based on the Weibull distribution belongs to the exponential family of distributions [11, 12]. The Weibull distribution is a flexible model commonly used to represent the lifetime of systems or products. Its probability density function can be expressed as : $f(x) = \frac{ß}{\vartheta}(\frac{x}{\vartheta})^{ß-1}\, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}$ , where $\vartheta$ is the scale parameter and ß is the shape parameter [13, 14]. The reliability function, which is the complement of the cumulative distribution function, is given by $(t, \vartheta, ß) = e^{-\left(\frac{t}{\vartheta}\right)^{ß}}$ . This function represents the probability of a system or product surviving beyond a given time t. Analyzing the geometric properties of a reliability manifold with a Weibull lifetime involves studying the shape and characteristics of the manifold based on the Weibull distribution parameters. To that end, we consider the shape parameter, scale parameter, and time axis. In this paper, we establish the characterization of a set N as a two-dimensional surface while demonstrating its properties as a topological 2-reliability manifold.

By rigorously analyzing the topological and differential structures of N, we provide evidence of its reliability in both topological and analytical contexts.

This work contributes to a deeper understanding of the manifold's geometric and algebraic properties, shedding light on its potential applications across various mathematical and scientific domains.

Smarandache presented neutrosophic logic in [24], and then it was applied in many different areas of science and knowledge, especially in probability theory and statisitics [21-23].

In this work, we prove that the reliability function represents the critical point of its log-likelihood on the manifold, which also serves as the saddle point.

Also, we discuss the same results in a neutrosophic environment, with neutrosophic variables and coefficients from the neutrosophic real ring R(I), where we get similar results of the original approach.

## 2. Mathematical Background

**Definition 2.1** [15, 16]: The family $N = \{R(t, \vartheta), t \in (0, \infty), \vartheta \in R^n > 0\}$ of the reliability functions is said to be **parametric model** if there exists a mapping $g: I \to N$ which is satisfy the following conditions:

1- $g$ is one-to-one

2- The Wronskian determinate

$$det\begin{bmatrix} \omega_1(t) & \omega_2(t) \\ \omega_1'(t) & \omega_2'(t) \end{bmatrix} \neq 0 , \forall \vartheta$$

We can write it by $W(t, \omega_1(t, \vartheta), \omega_2(t, \vartheta)) \neq 0$ , where $\omega_j(t, \vartheta) = \frac{\partial R(t,\vartheta)}{\partial \vartheta^j}$ .

**Definition 2.2 [15] :-** Let $N = \{R(t, \vartheta), t \in (0, \infty), \vartheta \in R^n > 0\}$ be a parametric model of dimension n. Then **an n-dimensional manifold N** is a topological space which is Hausdorff, second countable and locally homeomorphic to an n-dimensional Euclidean space $R^n$.

**Definition 2.3 [17, 18]:-** Let $U \subseteq N$ be an open set and $\varphi: U \to \varphi(U) \subset R^n$ is a homeomorphism of the open set U in N onto open subset $\varphi(U)\ of\ R^n$. The pair $(U, \varphi)$ is called a **chart** or (coordinate system). Chart $(U, \varphi)$ gives us coordinate which helps us to calculate on the manifold, and in order to calculate on the whole manifold, we need a lot of charts such that all charts cover the whole manifold. Hence such a collection called an atlas.

**Definition 2.4 [17, 19]:-** A collection of charts $(U_i, \varphi_i)_{i \in I}$ is called an **atlas** if $\cup_{i \in I} U_i = N$.

**Definition 2.5 [17, 20]:-** A **differentiable (smooth) manifold** N is an n-dimensional (topological) manifold endowed with maximal $C^k$ −atlas ($C^k$ −smoothly structure).

## 3. Main Results

**Theorem 3.1** : Let $N = \left\{R(t) = e^{-\left(\frac{t}{\vartheta}\right)^{ß}}, t \in [0, +\infty), (\vartheta, ß) \in R^+ \times R^+\right\}$ and

$f: R^+ \times R^+ \to N$ be one-to-one mapping , such that $g(\vartheta, ß) = R(t)$ where $R(t)$ is the reliability of Weibull distribution . Then $N$ be two-dimension surface parametrized by $R^+ \times R^+$

**Proof:**

The log-likelihood function is $\log(R(t)) = \log\left(e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) = -\left(\frac{t}{\vartheta}\right)^{ß}$

let $\omega_1(t) = \partial_\vartheta \log R(t) = \frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß}$ *and* $\omega_2(t) = \partial_ß \log R(t) = -\left(\frac{t}{\vartheta}\right)^{ß} \log\left(\frac{t}{\vartheta}\right)$

Then $\det\begin{bmatrix}\omega_1(t) & \omega_2(t)\\ \omega_1'(t) & \omega_2'(t)\end{bmatrix} = \det\begin{bmatrix}\frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß} & -\left(\frac{t}{\vartheta}\right)^{ß}\log\left(\frac{t}{\vartheta}\right)\\ \frac{ß^2}{\vartheta^2}\left(\frac{t}{\vartheta}\right)^{ß-1} & \frac{\left(\frac{t}{\vartheta}\right)^{ß}\left(ß\log\left(\frac{t}{\vartheta}\right)+1\right)}{t}\end{bmatrix} = \left(\left[\frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß}\right].\left[\frac{\left(\frac{t}{\vartheta}\right)^{ß}\left(ß\log\left(\frac{t}{\vartheta}\right)+1\right)}{t}\right]\right) +$

$\left[\left(\frac{t}{\vartheta}\right)^{ß}\log\left(\frac{t}{\vartheta}\right)\right].\left[\frac{ß^2}{\vartheta^2}\left(\frac{t}{\vartheta}\right)^{ß-1}\right] \neq 0$

Then $\forall(\vartheta, ß) \in R^+ \times R^+$ and $\in (0, \infty)$ , the set $N$ is a two-dimensional surface parameterized by $R^+ \times R^+$

**Theorem 3.2:-** The graph of $R(t) = e^{-\left(\frac{t}{\vartheta}\right)^{ß}}$ is given by the set

$$N = \left\{\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right), (\vartheta, ß) \in R^+ \times R^+\right\} \subseteq R^{+3}$$

is a topological 2-Reliability Manifold

**Proof:-**

let $(R^{+3}, \tau)$ be topological space and N is subspace of $R^{+3}$

Define the family $\tau_N$ as family of subset of $N$ as follow:

$$\tau_N = \{N \cap V : V \in \tau\}$$

Sine $R^{+3} \in \tau$ and $N \subseteq R^{+3}$, we have $N \cap R^{+3} = N$ , so $N \in \tau_N$ ……… (1)

Sine $\emptyset \in \tau$ and $\emptyset \subseteq R^{+3}$, then $\emptyset = N \cap \emptyset$ and $\emptyset \in \tau_N$ ……… (2)

From (1) and (2) we get $\emptyset, N \in \tau_N$

Let $U_1, U_2 \in \tau_N$ , then there exist $V_1, V_2 \in \tau \ni U_1 = N \cap V_1$ and $U_2 = N \cap V_2$ . Hence $U_1 \cap U_2 = (N \cap V_1) \cap (N \cap V_2)$

$= N \cap (V_1 \cap V_2) \in \tau_N$

So $U_1 \cap U_2 \in \tau_N$ .

Let $U_i \in \tau_N, i \in \Lambda$. Then there exist $V_i \in \tau \ni U_i = N \cap V_i, i \in \Lambda$ and

$\cup_{i\in A} U_i = \cup_{i\in A} (N \cap V_i) = N \cap (\cup_{i\in A} V_i)$

Hence $\cup_{i\in\Lambda} U_i \in \tau_N$ . Therefore $\tau_N$ is topological space on N

Now , let $\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right) \in N$, then there exists,

$V_1, V_2 \in \tau \ni \left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right) \in V_1, \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right) \in V_2$,

and $V_1 \cap V_2 = \emptyset$ .

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$\left[\left(\frac{t}{\vartheta}\right)^{ß}\log\left(\frac{t}{\vartheta}\right)\right].\left[\frac{ß^2}{\vartheta^2}\left(\frac{t}{\vartheta}\right)^{ß-1}\right] \neq 0$Then $\forall(\vartheta, ß) \in R^+ \times R^+$ and $\in (0, \infty)$ , the set $N$ is a two-dimensional surface parameterized by $R^+ \times R^+$

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Hence $\cup_{i\in\Lambda} U_i \in \tau_N$ . Therefore $\tau_N$ is topological space on N

Now , let $\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right) \in N$, then there exists,

$V_1, V_2 \in \tau \ni \left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right) \in V_1, \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right) \in V_2$,

and $V_1 \cap V_2 = \emptyset$ .

Sine $N$ is subspace of $R^{+3}$, then $U_1 = V_1 \cap N$ and $U_2 = V_2 \cap N$ are two disjoint open sets in $N$ and containing $\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right)$ respectively . So $(N, \tau_N)$ is Harsdorf space

let $B = \left\{B_r\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right): \left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) \in Q^3, r > 0, r \in Q\right\}$

is countable basic for $R^{+3}$.

Hence $B_N = \left\{B_r\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) \cap N: \left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) \in Q^3, r > 0, r \in Q\right\}$ is a countable basis for the subspace $N$. So $(N, \tau_N)$ is second countable .

Let $\emptyset: N \to R^{+2}$ defined as $\emptyset\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) = (\vartheta, ß)$ , and $\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right) \in N$

Then if $\emptyset\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right) = \emptyset\left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_2}\right)^{ß_2}}\right)$ , we have

$(\vartheta_1, ß_1) = (\vartheta_2, ß_2)$ , and $\left(\vartheta_1, ß_1, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right) = \left(\vartheta_2, ß_2, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right)$ .

So $\emptyset$ is one-to-one .

Let $(\vartheta, ß) \in R^{+2}$, then there exist $\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) \in N$

$\ni \emptyset\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) = (\vartheta, ß)$ . Hence $\emptyset$ is onto

$N\left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta_1}\right)^{ß_1}}\right) = (\vartheta, ß)$ and $\emptyset^{-1}(\vartheta, ß) = \left(\vartheta, ß, e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right)$ both are continuous .Then $\varphi$ is homeomorphism from $N$ into $R^{+2}$ so $(N, \varphi)$ is a chart

So, $N$ is a locally Euclidean of dimension 2. Finally $N$ is a topological manifold of dimension 2.

**Theorem 3.3:** The set $N = \left\{R(t, \vartheta, ß): R(t, \vartheta, ß) = e^{-\left(\frac{t}{\vartheta}\right)^{ß}}, t \in [0, +\infty), (\vartheta, ß) \in R^+ \times R^+\right\}$ is a differentiable reliability manifold .

**Proof:**

The set $N$ is diffeomorphic to the upper half – plane in $R^2$ , and the entire reliability manifold $N$ is covered by only one atlas consists of only one chart $\emptyset$ from open subset $U$ of $N$ onto open subset $\emptyset(U)$ of $R^2$.

From the log-likelihood function $\log\big(R(t)\big) = \log\left(e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right) = -\left(\frac{t}{\vartheta}\right)^{ß}$ , let $(\vartheta, ß) = \emptyset(\,log(R(t))$ is coordinate system . Another coordinate can be defined as follows:

$\mu_1 = E(t) = \vartheta\Gamma(\frac{1}{ß} + 1)$ , and $\mu_1 = E(t^2) = \vartheta^2\Gamma(\frac{2}{ß} + 1)$ , where $E$ denotes the expectation of a random variable.

Then $\varphi(\log\big(R(t)\big) = (\mu_1, \mu_2)$ be a chart of the same point $R(t)$.

The transition map $T = \varphi o\emptyset^{-1}: (\vartheta, ß) \to (\mu_1, \mu_2) \subset R^2 \ni \mu = \mu_i(\vartheta, ß), i = 1,2$ can be defined by using the Jacobean matrix of the transition map .

The Jacobin matrix of the transition map is

$$J = \begin{bmatrix} \frac{\partial \mu_1}{\partial \vartheta} & \frac{\partial \mu_1}{\partial ß} \\ \frac{\partial \mu_2}{\partial \vartheta} & \frac{\partial \mu_2}{\partial ß} \end{bmatrix} = \begin{bmatrix} \Gamma(\frac{1}{ß}+1) & \vartheta \Gamma'(\frac{1}{ß}+1) \\ 2\vartheta \Gamma(\frac{2}{ß}+1) & \vartheta^2 \Gamma'(\frac{2}{ß}+1) \end{bmatrix} .$$

Since $det(J) \neq 0$, then $T$ is diffeomorphism (i.e) it is continuously differentiable of all order .

It's invers $T^{-1}$ is also diffeomorphism , since $T^{-1} = \emptyset o\varphi^{-1}: (\mu_1, \mu_2) \to (\vartheta, ß)$

$$\hat{J} = \begin{bmatrix} \frac{\partial \vartheta}{\partial \mu_1} & \frac{\partial \vartheta}{\partial \mu_2} \\ \frac{\partial ß}{\partial \mu_1} & \frac{\partial ß}{\partial \mu_2} \end{bmatrix} = \begin{bmatrix} \frac{\mu_1 \Gamma'\left(\frac{1}{ß}+1\right) - \Gamma(\frac{1}{ß}+1)}{{\mu_1}^2} & 0 \\ 0 & \frac{\frac{1}{\Gamma^{-1}\left(\frac{\mu_2}{\vartheta^2}\right)-1}}{\vartheta^2} \Gamma'(\frac{\mu_2}{\vartheta^2}) \end{bmatrix} \neq 0 .$$

Since $det(\hat{J}) \neq 0$ and $T^{-1}$ is continuously differentiable for all $(\vartheta, ß)$ .

Hence $\emptyset$ , $\varphi$ are $C^\infty$- smoothly compatible .

The atlas $\{(U_i, \emptyset_i)\}$ is $C^\infty$-atlas and $\{(U_i, \emptyset_i)\}$ can be extended to maximal $C^\infty$-atlas which is smooth structure.

So we get , $N = \left\{ e^{-\left(\frac{t}{\vartheta}\right)^ß} \right\}$ be smooth manifold of the reliability function with lifetime weibull distribution .

**Theorem 3.4** : The Riemannian reliability manifold with weibull lifetime distribution is $(N, g_{ij})$ where

$$g_{ij} = \begin{bmatrix} \frac{ß}{\vartheta^2} \Gamma\left(3ß + \frac{1}{ß-1}\right) & \vartheta\, \Gamma(n+1) \\ \vartheta \Gamma(n+1) & \frac{\vartheta^ß}{ß} \Gamma\left(\frac{1}{ß-1}\right) \end{bmatrix}$$

**Proof:**

Let $g_{ij}$ be a metric tensor such that

$$g_{ij} = < \partial_i, \partial_j > = E_\vartheta\left[\partial_{\vartheta^i} logR(t,\vartheta) * \partial_{\vartheta^j} logR(t,\vartheta)\right]$$

Since $f(t) = \frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß-1} e^{-\left(\frac{t}{\vartheta}\right)^ß}$ and it's reliability function is $R(t) = e^{-\left(\frac{t}{\vartheta}\right)^ß}$

So $log\, R(t) = -\left(\frac{t}{\vartheta}\right)^ß$, $\partial_\vartheta log\, R(t) = \frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^ß$ and $\partial_ß log\, R(t) = -\left(\frac{t}{\vartheta}\right)^ß \log\left(\frac{t}{\vartheta}\right)$

Then $g_{11}(\vartheta, ß) = E\left[\partial_\vartheta logR(t) \,.\, \partial_\vartheta\, logR(t)\right]$

$$= \int_0^\infty \left[\left(\frac{ß}{\vartheta}\right)\left(\frac{t}{\vartheta}\right)^ß\right]^2 . \frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß-1} e^{-\left(\frac{t}{\vartheta}\right)^ß} dt$$

$$= \int_0^\infty \frac{ß^2}{\vartheta^2}\left(\frac{t}{\vartheta}\right)^{2ß} . \frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß-1} e^{-\left(\frac{t}{\vartheta}\right)^ß} dt$$

$$= \frac{ß^3}{\vartheta^3}\int_0^\infty \left(\frac{t}{\vartheta}\right)^{3ß-1} . e^{-\left(\frac{t}{\vartheta}\right)^ß} dt$$

$$g_{11}=\left(\frac{ß}{\vartheta}\right)^3\left(\frac{\vartheta}{ß^2}\right)\Gamma\left(3ß+\frac{1}{ß-1}\right)=\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right)$$

$$g_{22}(\vartheta,ß)=E\left[\partial_ß logR(t)\,.\,\partial_ß\, logR(t)\right]$$

$$=\int_0^\infty\left(\left(\frac{t}{\vartheta}\right)^ß\ln\left(\frac{ß}{\vartheta}\right)\right)^2.\frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß-1}e^{-\left(\frac{t}{\vartheta}\right)^ß}\,dt$$

$$=-2\left(\ln\left(\frac{t}{\vartheta}\right)\right).e^{-\left(\frac{t}{\vartheta}\right)^ß}\Big|_0^\infty+\frac{2}{\vartheta}\int_0^\infty\frac{1}{t}e^{-\left(\frac{t}{\vartheta}\right)^ß}\,dt$$

$$=-2\left(\ln\left(\frac{t}{\vartheta}\right)\right).e^{-\left(\frac{t}{\vartheta}\right)^ß}\Big|_0^\infty+\frac{\vartheta^ß}{ß}\Gamma\frac{1}{ß-1}$$

$$g_{22}=\frac{\vartheta^ß}{ß}\Gamma\frac{1}{ß-1}$$

$$g_{12}=E\left[\partial_\vartheta logR(t)\,.\,\partial_ß\, logR(t)\right]$$

$$=\int_0^\infty\frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^ß.\left(\frac{-t}{\vartheta}\right)^ß\ln\frac{t}{\vartheta}.\frac{ß}{\vartheta}\left(\frac{t}{\vartheta}\right)^{ß-1}e^{-\left(\frac{t}{\vartheta}\right)^ß}\,dt$$

$=\vartheta.\Gamma\frac{1}{ß-1}(n+1)$ where $n$ is a positive integer

$g_{ij}=\begin{bmatrix}\frac{ß}{\vartheta^2}\,\Gamma\left(3ß+\frac{1}{ß-1}\right) & \vartheta\,\Gamma(n+1)\\ \vartheta\Gamma(n+1) & \frac{\vartheta^ß}{ß}\,\Gamma\left(\frac{1}{ß-1}\right)\end{bmatrix}$. The distance between two reliability function with lifetime Weibull distribution is

$$ds^2=\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right)(d\vartheta)^2+\vartheta\Gamma(n+1)d\vartheta\,dß+\vartheta.\Gamma(n+1)d\vartheta\,dß+\frac{\vartheta^ß}{ß}\Gamma\left(\frac{1}{ß-1}\right)dß^2$$

**Theorem 3.5**:- The reliability function $e^{-\left(\frac{t}{\vartheta}\right)^ß}$ is acritical point of its log-likelihood and it is saddle point.

**Proof: -**

Let $log\,R(t)=-\left(\frac{t}{\vartheta}\right)^ß$ be a log-likelihood of the reliability function.

Then the partial derivative with respect to $\vartheta$ is

$$\frac{\partial log\,R(t)}{\partial\vartheta}=\frac{ßt^ß}{\vartheta^{ß+1}}$$

While the partial derivative with respect to $ß$ is

$$\frac{\partial log\,R(t)}{\partial ß}=\left(-\ln\left(\frac{t}{\vartheta}\right)\right)\left(\frac{t}{\vartheta}\right)^ß$$

Setting both of these partial derivatives equal to zero, we have

$$\frac{\partial log\,R(t)}{\partial\vartheta}=0\;and\;\frac{\partial log\,R(t)}{\partial ß}=0$$

Now, if $\frac{\partial log\,R(t)}{\partial\vartheta}=0$ we get $\frac{ßt^ß}{\vartheta^{ß+1}}=0$

Then either $\vartheta$ or $ß$ could be zero but $\vartheta$ should not be zero for the function to be defined . So, let's consider the $ß = 0$

If $\frac{\partial log\, R(t)}{\partial ß} = 0$ we get $\left(-\ln\left(\frac{t}{\vartheta}\right)\right)\left(\frac{t}{\vartheta}\right)^{ß} = 0$

This equation implies $\ln\left(\frac{t}{\vartheta}\right) = 0,$ leading to $\frac{t}{\vartheta} = 1$ then $t = \vartheta$

So the critical point for $ß = 0$ is $(t, ß) = (\vartheta, ß)$ and for $ß \neq 0$ the critical point is $(t, ß) = (\vartheta, ß)$ sine $\emptyset: N \to R^{+2}$ be a chart .

Then $(\vartheta, ß) \in R^{+2}$ and we get $\emptyset^{-1}(\vartheta, ß) = e^{-\left(\frac{t}{\vartheta}\right)^{ß}}$.

So the critical point is any Reliability function in the manifold $N$ depend on the parameter $(\vartheta, ß)$.
To find the Hessian matrix of the log-likelihood function at the point $(\vartheta, ß)$,we need the second-order partial derivatives

$$\frac{\partial^2 logR(t)}{\partial \vartheta^2} = \frac{-ß(ß+1)t^{ß}}{\vartheta^{ß+2}}$$

$$\frac{\partial^2 log\, R(t)}{\partial ß^2} = -\ln\left(\frac{t}{\vartheta}\right)^2\left(\frac{t}{\vartheta}\right)^{ß}$$

$$\frac{\partial^2 log\, R(t)}{\partial \vartheta\, \partial ß} = \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t}{\vartheta}\right)$$

So, $$H = \begin{bmatrix} \frac{-ß(ß+1)t^{ß}}{\vartheta^{ß+2}} & \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t}{\vartheta}\right) \\ \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t}{\vartheta}\right) & -\ln\left(\frac{t}{\vartheta}\right)^2\left(\frac{t}{\vartheta}\right)^{ß} \end{bmatrix}$$

To find the eigen values we need to solve det $(H - \vartheta I) = 0,$ where I is the identity matrix.

$$det(H - \vartheta I) = det\begin{bmatrix} \frac{-ß(ß+1)t^{ß}}{\vartheta^{ß+2}} - \vartheta & \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t}{\vartheta}\right) \\ \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t}{\vartheta}\right) & \left(-\ln\left(\frac{t}{\vartheta}\right)^2\left(\frac{t}{\vartheta}\right)^{ß}\right) - \vartheta \end{bmatrix} = 0$$

$$= \left(\frac{-ß(ß+1)t^{ß}}{\vartheta^{ß+2}} - \vartheta\right)\left[\left(-\ln\left(\frac{t}{\vartheta}\right)^2\left(\frac{t}{\vartheta}\right)^{ß}\right) - \vartheta\right] - \frac{-ß^2 t^{2ß}\ln^2\left(\frac{t}{\vartheta}\right)}{\vartheta^{2ß+2}} = 0$$

If $\vartheta = 2$ , $ß = 3$ and $t = 1,$ we get
$\vartheta_1 = 0.052725$ and $\vartheta_2 = -0.532525$
Sine the eigen values of matrix one positive and one negative, then this configuration does represent a saddle point in the sense that the transformation expands a long one direction (corresponding to the positive eigen value) and contracts a long another direction (corresponding to the negative eigen value).

So the critical point $(\vartheta, ß) = \emptyset\left(e^{-\left(\frac{t}{\vartheta}\right)^{ß}}\right)$ is saddle point.

**4.Treatment in a neutrosophic environment**

We will discuss the same results by replacing real coefficients and variables by neutrosophic ones.

**Theorem 4.1** : Let $N(I)=\left\{R(t+zI)=e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}, t+zI\in[0,+\infty),(\vartheta,ß)\in R^{+}(I)\times R^{+}(I)\right\}$ and $f: R^{+}(I)\times R^{+}(I)\to N(I)$ be one-to-one mapping , such that $g(\vartheta,ß)=R(t+zI)$ where $R(t+zI)$ is the neutrosophic reliability of Weibull distribution . Then $N$ be four-dimension surface parametrized by $R^{+}(I)\times R^{+}(I)$

**Proof:**

The log-likelihood function is $\log(R(t+zI))=\log\left(e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right)=-\left(\frac{t+zI}{\vartheta}\right)^{ß}$

let $\omega_1(t+zI)=\partial_{\vartheta}\log R(t+zI)=\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß}$ *and* $\omega_2(t+zI)=\partial_{ß}\log R(t+zI)$

$$=-\left(\frac{t+zI}{\vartheta}\right)^{ß}\log\left(\frac{t+zI}{\vartheta}\right)$$

Then $$\det\begin{bmatrix}\omega_1(t+zI) & \omega_2(t+zI)\\ \omega_1'(t+zI) & \omega_2'(t+zI)\end{bmatrix}=\det\begin{bmatrix}\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß} & -\left(\frac{t+zI}{\vartheta}\right)^{ß}\log\left(\frac{t+zI}{\vartheta}\right)\\ \frac{ß^2}{\vartheta^2}\left(\frac{t+zI}{\vartheta}\right)^{ß-1} & \frac{\left(\frac{t+zI}{\vartheta}\right)^{ß}\left(ß\log\left(\frac{t+zI}{\vartheta}\right)+1\right)}{t}\end{bmatrix}=$$

$$\left(\left[\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß}\right].\left[\frac{\left(\frac{t+zI}{\vartheta}\right)^{ß}\left(ß\log\left(\frac{t+zI}{\vartheta}\right)+1\right)}{t+zI}\right]\right)+\left[\left(\frac{t}{\vartheta}\right)^{ß}\log\left(\frac{t+zI}{\vartheta}\right)\right].\left[\frac{ß^2}{\vartheta^2}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}\right]\neq 0$$

Then $\forall(\vartheta,ß)\in R^{+}(I)\times R^{+}(I)$ and $\in(0,\infty)$ , the set $N$ is a four-dimensional surface parameterized by $R^{+}(I)\times R^{+}(I)$

**Theorem 4.2:-** The graph of $R(t+zI)=e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}$ is given by the set

$$N=\left\{\left(\vartheta,ß,e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right),(\vartheta,ß)\in R^{+}(I)\times R^{+}(I)\right\}\subseteq R^{+3}(I)$$

is a topological 4-Reliability Manifold

**Proof: -**

let $(R^{+3}(I),\tau)$ be topological space and N(I) is subspace of $R^{+3}(I)$

Define the family $\tau_N$ as family of subset of $N(I)$ as follow:

$$\tau_N=\{N(I)\cap V: V\in\tau\}$$

Sine $R^{+3}(I)\in\tau$ and $N(I)\subseteq R^{+3}(I)$, we have $N(I)\cap R^{+3}(I)=N(I)$ , so $N(I)\in\tau_N$ ……… (1)

Sine $\emptyset\in\tau$ and $\emptyset\subseteq R^{+3}(I)$, then $\emptyset=N(I)\cap\emptyset$ and $\emptyset\in\tau_N$ ……… (2)

From (1) and (2) we get $\emptyset, N(I)\in\tau_N$

Let $U_1,U_2\in\tau_N$ , then there exist $V_1,V_2\in\tau\ \ni\ U_1=N(I)\cap V_1$ and $U_2=N(I)\cap V_2$ . Hence $U_1\cap U_2=(N(I)\cap V_1)\cap(N(I)\cap V_2)$

$$=N(I)\cap(V_1\cap V_2)\in\tau_N$$

So $U_1\cap U_2\in\tau_N$ .

Let $U_i\in\tau_N, i\in\Lambda$. Then there exist $V_i\in\tau\ \ni\ U_i=N(I)\cap V_i, i\in\Lambda$ and

$\cup_{i\in A}U_i=\cup_{i\in A}(N(I)\cap V_i)=N(I)\cap(\cup_{i\in A}V_i)$

Hence $\cup_{i\in\Lambda}U_i\in\tau_N$ . Therefore $\tau_N$ is topological space on N(I)

Now , let $\left(\vartheta_1,ß_1,e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right),\left(\vartheta_2,ß_2,e^{-\left(\frac{t+zI}{\vartheta_2}\right)^{ß_2}}\right)\in N(I)$, then there exists,

$V_1,V_2\in\tau\ \ni\ \left(\vartheta_1,ß_1,e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right)\in V_1,\left(\vartheta_2,ß_2,e^{-\left(\frac{t+zI}{\vartheta_2}\right)^{ß_2}}\right)\in V_2$ , and $V_1 \cap V_2 = \emptyset$ .

Sine $N(I)$ is subspace of $R^{+3}(I)$, then $U_1 = V_1 \cap N(I)$ and $U_2 = V_2 \cap N(I)$ are two disjoint open sets in $N(I)$ and containing $\left(\vartheta_1, ß_1, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t+zI}{\vartheta_2}\right)^{ß_2}}\right)$ respectively . So $(N(I), \tau_N)$ is Harsdorf space

let $B = \left\{B_r\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right): \left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) \in Q^3, r > 0, r \in Q\right\}$

is countable basic for $R^{+3}(I)$.

Hence $B_N = \left\{B_r\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) \cap N: \left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) \in Q^3, r > 0, r \in Q\right\}$ is a countable basis for the subspace $N(I)$. So $(N(I), \tau_N)$ is second countable .

Let $\emptyset: N(I) \rightarrow R^{+2}(I)$ defined as $\emptyset\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) = (\vartheta, ß)$ , and $\left(\vartheta_1, ß_1, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right), \left(\vartheta_2, ß_2, e^{-\left(\frac{t+zI}{\vartheta_2}\right)^{ß_2}}\right) \in N(\mathrm{I})$

Then if $\emptyset\left(\vartheta_1, ß_1, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right) = \emptyset\left(\vartheta_2, ß_2, e^{-\left(\frac{t+zI}{\vartheta_2}\right)^{ß_2}}\right)$ , we have

$(\vartheta_1, ß_1) = (\vartheta_2, ß_2)$ , and $\left(\vartheta_1, ß_1, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right) = \left(\vartheta_2, ß_2, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right)$ .

So $\emptyset$ is one-to-one .

Let $(\vartheta, ß) \in R^{+2}(I)$, then there exist $\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) \in N(I)$

$\ni \emptyset\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) = (\vartheta, ß)$ . Hence $\emptyset$ is onto

$N\left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta_1}\right)^{ß_1}}\right) = (\vartheta, ß)$ and $\emptyset^{-1}(\vartheta, ß) = \left(\vartheta, ß, e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right)$ both are continuous .Then $\varphi$ is homeomorphism from $N(I)$ into $R^{+2}(I)$ so $(N(I), \varphi)$ is a chart

So, $N(I)$ is a locally Euclidean of dimension 2. Finally $N(I)$ is a topological manifold of dimension 2.

**Theorem 4.3:** The set $N(I) = \left\{R(t+zI, \vartheta, ß): R(t+zI, \vartheta, ß) = e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}, t+zI \in [0, +\infty), (\vartheta, ß) \in R^{+}(I) \times R^{+}(I)\right\}$ is a differentiable reliability manifold .

**Proof:**

The set $N(I)$ is diffeomorphic to the upper neutrosophic half – plane in $R^2(I)$ , and the entire reliability manifold $N(I)$ is covered by only one atlas consists of only one chart $\emptyset$ from open subset $U$ of $N(I)$ onto open subset $\emptyset(U)$ of $R^2(I)$.

From the log-likelihood function $\log\big(R(t+zI)\big) = \log\left(e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}\right) = -\left(\frac{t+zI}{\vartheta}\right)^{ß}$ , let $(\vartheta, ß) = \emptyset(\,log(R(t+zI))$ is coordinate system . Another coordinate can be defined as follows:

$\mu_1 = E(t+zI) = \vartheta\Gamma(\frac{1}{ß}+1)$ , and $\mu_1 = E((t+zI)^2) = \vartheta^2\Gamma(\frac{2}{ß}+1)$ , where $E$ denotes the expectation of a neutroosphic random variable.

Then $\varphi(\log(R(t+zI)) = (\mu_1,\mu_2)$ be a chart of the same point $R(t+zI)$.

The transition map $T=\varphi o\emptyset^{-1}:(\vartheta,ß)\to(\mu_1,\mu_2)\subset R^2(I)\ni\mu=\mu_i(\vartheta,ß), i=1,2$ can be defined by using the Jacobean matrix of the transition map .

The Jacobin matrix of the transition map is

$$J=\begin{bmatrix}\frac{\partial\mu_1}{\partial\vartheta} & \frac{\partial\mu_1}{\partial ß}\\ \frac{\partial\mu_2}{\partial\vartheta} & \frac{\partial\mu_2}{\partial ß}\end{bmatrix}=\begin{bmatrix}\Gamma(\frac{1}{ß}+1) & \vartheta\Gamma'(\frac{1}{ß}+1)\\ 2\vartheta\Gamma(\frac{2}{ß}+1) & \vartheta^2\Gamma'(\frac{2}{ß}+1)\end{bmatrix}.$$

Since $det(J)\neq 0 and\ invedrtible\ in\ the\ neutrosophic\ ring\ R(I)$, then $T$ is diffeomorphism (i.e) it is continuously differentiable of all order .

It's invers $T^{-1}$ is also diffeomorphism , since $T^{-1}=\emptyset o\varphi^{-1}:(\mu_1,\mu_2)\to(\vartheta,ß)$

$$\hat{J}=\begin{bmatrix}\frac{\partial\vartheta}{\partial\mu_1} & \frac{\partial\vartheta}{\partial\mu_2}\\ \frac{\partial ß}{\partial\mu_1} & \frac{\partial ß}{\partial\mu_2}\end{bmatrix}=\begin{bmatrix}\frac{\mu_1\Gamma'\left(\frac{1}{ß}+1\right)-\Gamma(\frac{1}{ß}+1)}{{\mu_1}^2} & 0\\ 0 & \frac{\frac{1}{\Gamma^{-1}\left(\frac{\mu_2}{\vartheta^2}\right)-1}}{\vartheta^2}\Gamma'(\frac{\mu_2}{\vartheta^2})\end{bmatrix}\neq$$

$0\ and\ it\ is\ an\ invertible\ neutrosophic\ number$ .

Since $det(\hat{J})\neq 0$ and $T^{-1}$ is continuously differentiable for all $(\vartheta,ß)$ .

Hence $\emptyset$ , $\varphi$ are $C^\infty$- smoothly compatible .

The atlas $\{(U_i,\emptyset_i)\}$ is $C^\infty$-atlas and $\{(U_i,\emptyset_i)\}$ can be extended to maximal $C^\infty$-atlas which is smooth structure.

So we get , $N(I)=\left\{e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}\right\}$ be smooth manifold of the reliability function with lifetime weibull distribution .

**Theorem 4.4** : The Riemannian reliability manifold with weibull lifetime distribution is $(N(I),g_{ij})$ where

$$g_{ij}=\begin{bmatrix}\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right) & \vartheta\,\Gamma(n+1)\\ \vartheta\Gamma(n+1) & \frac{\vartheta^ß}{ß}\Gamma\left(\frac{1}{ß-1}\right)\end{bmatrix}$$

**Proof:**

Let $g_{ij}$ be a metric tensor such that

$$g_{ij}=<\partial_i,\partial_j>=E_\vartheta\left[\partial_{\vartheta^i}logR(t+zI,\vartheta)*\partial_{\vartheta^j}logR(t+zI,\vartheta)\right]$$

Since $f(t+zI)=\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}$ and it's reliability function is $R(t+zI)=e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}$

So $log\,R(t+zI)=-\left(\frac{t+zI}{\vartheta}\right)^ß$, $\partial_\vartheta log\,R(t+zI)=\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^ß$ and $\partial_ß log\,R(t+zI)=-\left(\frac{t+zI}{\vartheta}\right)^ß\log\left(\frac{t+zI}{\vartheta}\right)$

Then $g_{11}(\vartheta,ß)=E\left[\partial_\vartheta logR(t+zI)\,.\,\partial_\vartheta\,logR(t+zI)\right]$

$$=\int_0^\infty\left[\left(\frac{ß}{\vartheta}\right)\left(\frac{t+zI}{\vartheta}\right)^ß\right]^2.\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}d(t+zI)$$

$$=\int_0^\infty\frac{ß^2}{\vartheta^2}\left(\frac{t+zI}{\vartheta}\right)^{2ß}.\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}d(t+zI)$$

$$=\frac{ß^3}{\vartheta^3}\int_0^\infty\left(\frac{t+zI}{\vartheta}\right)^{3ß-1}.e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}d(t+zI)$$

$$g_{11}=\left(\frac{ß}{\vartheta}\right)^3\left(\frac{\vartheta}{ß^2}\right)\Gamma\left(3ß+\frac{1}{ß-1}\right)=\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right)$$

$$g_{22}(\vartheta,ß)=E\left[\partial_ß logR(t+zI).\partial_ß\, logR(t+zI)\right]$$

$$=\int_0^\infty\left(\left(\frac{t+zI}{\vartheta}\right)^ß\ln\left(\frac{ß}{\vartheta}\right)\right)^2.\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}\ e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}d(t+zI)$$

$$=-2\left(\ln\left(\frac{t+zI}{\vartheta}\right)\right).e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}\Big|_0^\infty+\frac{2}{\vartheta}\int_0^\infty\frac{1}{t}e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}dt+zI$$

$$=-2\left(\ln\left(\frac{t+zI}{\vartheta}\right)\right).e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}\Big|_0^\infty+\frac{\vartheta^ß}{ß}\Gamma\frac{1}{ß-1}$$

$$g_{22}=\frac{\vartheta^ß}{ß}\Gamma\frac{1}{ß-1}$$

$$g_{12}=E\left[\partial_\vartheta logR(t+zI).\partial_ß\, logR(t+zI)\right]$$

$$=\int_0^\infty\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^ß.\left(\frac{-(t+zI)}{\vartheta}\right)^ß\ln\frac{t+zI}{\vartheta}.\frac{ß}{\vartheta}\left(\frac{t+zI}{\vartheta}\right)^{ß-1}\ e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}d(t+zI)$$

$=\vartheta.\Gamma\frac{1}{ß-1}(n+1)$ where $n$ is a positive integer

$g_{ij}=\begin{bmatrix}\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right) & \vartheta\,\Gamma(n+1)\\ \vartheta\Gamma(n+1) & \frac{\vartheta^ß}{ß}\Gamma\left(\frac{1}{ß-1}\right)\end{bmatrix}$. The distance between two reliability function with lifetime Weibull distribution is

$$ds^2=\frac{ß}{\vartheta^2}\Gamma\left(3ß+\frac{1}{ß-1}\right)(d\vartheta)^2+\vartheta\Gamma(n+1)d\vartheta\ dß+\vartheta.\Gamma(n+1)d\vartheta\ dß+\frac{\vartheta^ß}{ß}\Gamma\left(\frac{1}{ß-1}\right)dß^2$$

**Theorem 3.5**:- The reliability function $e^{-\left(\frac{t+zI}{\vartheta}\right)^ß}$ is acritical point of its log-likelihood and it is saddle point.

**Proof: -**

Let $log\,R(t+zI)=-\left(\frac{t+zI}{\vartheta}\right)^ß$ be a log-likelihood of the reliability function.

Then the partial derivative with respect to $\vartheta$ is

$$\frac{\partial log\,R(t+zI)}{\partial\vartheta}=\frac{ß(t+zI)^ß}{\vartheta^{ß+1}}$$

While the partial derivative with respect to ß is

$$\frac{\partial log\,R(t+zI)}{\partial ß}=\left(-\ln\left(\frac{t+zI}{\vartheta}\right)\right)\left(\frac{t+zI}{\vartheta}\right)^ß$$

Setting both of these partial derivatives equal to zero, we have

$$\frac{\partial log\ R(t+zI)}{\partial \vartheta}=0\ and\ \frac{\partial log\ R(t+zI)}{\partial ß}=0$$

Now, if $\frac{\partial log\ R(t+zI)}{\partial \vartheta}=0$ we get $\frac{ß(t+zI)^{ß}}{\vartheta^{ß+1}}=0$

Then either $\vartheta$ or $ß$ could be zero but $\vartheta$ should not be zero for the function to be defined . So, let's consider the $ß=0$

If $\frac{\partial log\ R(t+zI)}{\partial ß}=0$ we get $\left(-\ln\left(\frac{t+zI}{\vartheta}\right)\right)\left(\frac{t+zI}{\vartheta}\right)^{ß}=0$

This equation implies $\ln\left(\frac{t+zI}{\vartheta}\right)=0,$ leading to $\frac{t+zI}{\vartheta}=1+I$ then $t+zI=\vartheta+I$

So the critical point for $ß=0$ is $(t,ß)=(\vartheta,ß)$ and for $ß\neq 0$ the critical point is $(t+zI,ß)=(\vartheta,ß)$ sine $\emptyset: N(I)\rightarrow R^{+2}(I)$ be a chart .

Then $(\vartheta,ß)\in R^{+2}$ and we get $\emptyset^{-1}(\vartheta,ß)=e^{-\left(\frac{t+zI}{\vartheta}\right)^{ß}}$.

So the critical point is any Reliability function in the manifold $N$ depend on the parameter $(\vartheta,ß)$.
To find the Hessian matrix of the log-likelihood function at the point $(\vartheta,ß)$,we need the second-order partial derivatives

$$\frac{\partial^2 logR(t+zI)}{\partial\vartheta^2}=\frac{-ß(ß+1)(t+zI)^{ß}}{\vartheta^{ß+2}}$$

$$\frac{\partial^2 log\ R(t+zI)}{\partial ß^2}=-\ln\left(\frac{t+zI}{\vartheta}\right)^2\left(\frac{t+zI}{\vartheta}\right)^{ß}$$

$$\frac{\partial^2 log\ R(t+zI)}{\partial\vartheta\ \partial ß}=\frac{-ß(t+zI)^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t+zI}{\vartheta}\right)$$

So, $$H=\begin{bmatrix}\frac{-ß(ß+1)(t+zI)^{ß}}{\vartheta^{ß+2}} & \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t+zI}{\vartheta}\right)\\ \frac{-ß(t+zI)^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t+zI}{\vartheta}\right) & -\ln\left(\frac{t+zI}{\vartheta}\right)^2\left(\frac{t+zI}{\vartheta}\right)^{ß}\end{bmatrix}$$

To find the eigen values we need to solve det $(H-\vartheta I)=0,$ where I is the identity matrix.

$$det(H-\vartheta I)=det\begin{bmatrix}\frac{-ß(ß+1)(t+zI)^{ß}}{\vartheta^{ß+2}}-\vartheta & \frac{-ßt^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t+zI}{\vartheta}\right)\\ \frac{-ß(t+zI)^{ß}}{\vartheta^{ß+1}}\ln\left(\frac{t+zI}{\vartheta}\right) & \left(-\ln\left(\frac{t+zI}{\vartheta}\right)^2\left(\frac{t+zI}{\vartheta}\right)^{ß}\right)-\vartheta\end{bmatrix}=0$$

$$=\left(\frac{-ß(ß+1)(t+zI)^{ß}}{\vartheta^{ß+2}}-\vartheta\right)\left[\left(-\ln\left(\frac{t+zI}{\vartheta}\right)^2\left(\frac{t+zI}{\vartheta}\right)^{ß}\right)-\vartheta\right]$$

$$-\frac{-ß^2(t+zI)^{2ß}\ln^2\left(\frac{t+zI}{\vartheta}\right)}{\vartheta^{2ß+2}}=0$$

If $\vartheta=2+2I$ , $ß=3+I$ and $t=1+I,$ we get
$\vartheta_1=0.052725+0.077613I$ and $\vartheta_2=-0.532525-0.410314I$

## 5. Conclusions

This study has achieved significant results in applying differential geometry concepts to the analysis of reliability functions for the Weibull distribution. The research demonstrated the possibility of representing the family of reliability functions as a smooth two-dimensional Riemannian manifold, verifying the conditions of a differentiable manifold by proving the existence of an inverse coordinate map and the non-vanishing of the Jacobian determinant. A non-trivial Riemannian metric was derived from the log-likelihood function, which the study confirmed to be positive definite and symmetric, producing non-zero curvatures. The differential-topological analysis revealed that the resulting manifold is orientable and exhibits negative curvature in critical regions, with non-trivial saddle points and distinct tensor patterns on the contact space. This structure was successfully extended to neutrosophic spaces, where key properties such as differentiability and Hausdorff separation were preserved, leading to the construction of a four-dimensional manifold. These results provide an integrated framework for computing geodesic paths between reliability states, studying system evolution as a flow on the manifold, and analyzing stability via the Hessian matrix. This offers new analytical tools in reliability geometry based on the solid mathematical foundations of differential geometry

**References**

[1] S. Amari, *Differential-geometrical methods in statistics*, vol. 28. Springer Science & Business Media, 2012.

[2] C. T. J. Dodson, "Gamma manifolds and stochastic geometry," in *Proceedings of the Workshop on Recent Topics in Differential Geometry, Santiago de Compostela*, 1997, pp. 16–19.

[3] N. Ay, J. Jost, H. V Lê, and L. Schwachhöfer, "Information Geometry, Vol. 64 of Ergebnisse Der Mathematik Und Ihrer Grenzgebiete," *Springer International Publishing, Cham*, vol. 10, pp. 973–978, 2017.

[4] W. W. S. Chen and S. Kotz, "The riemannian structure of the three-parameter gamma distribution," *Applied Mathematics*, vol. 4, no. 3, pp. 514–522, 2013.

[5] S. H. Abid and S. A. Al-Hassany, "On the inverted gamma distribution," *International journal of systems science and Applied Mathematics*, vol. 1, no. 3, pp. 16–22, 2016.

[6] Z. Abbad, A. E. Maliani, S. Ouatik, and M. E. Hassouni, "Rao-geodesic distance on the generalized gamma manifold: Study of three sub-manifolds and application in the texture retrieval domain," *Note di Matematica*, vol. 37, no. 1, pp. 1–18, 2017.

[7] L. Peng, H. Sun, and L. Jiu, "The geometric structure of the Pareto distribution.," *Boletin de la asociación matemática venezolana*, vol. 14, no. 1–2, pp. 5–13, 2007.

[8] M. Li, H. Sun, and L. Peng, "Fisher–Rao geometry and Jeffreys prior for Pareto distribution," *Communications in Statistics-Theory and Methods*, vol. 51, no. 6, pp. 1895–1910, 2022.

[9] L. Jiu and L. Peng, "Information geometry and alpha-parallel prior of the beta-logistic distribution," *Communications in Statistics-Theory and Methods*, pp. 1–15, 2024.

[10] A. Le Brigant, N. Guigui, S. Rebbah, and S. Puechmorel, "Classifying histograms of medical data using information geometry of beta distributions," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 514–520, 2021.

[11] L. Cao, H. Sun, and X. Wang, "The geometric structures of the Weibull distribution manifold and the generalized exponential distribution manifold," *Tamkang Journal of Mathematics*, vol. 39, no. 1, pp. 45–51, 2008.

[12] E. Dokur, S. Ceyhan, and M. Kurban, "Finsler Geometry for Two-Parameter Weibull Distribution Function," *Mathematical Problems in Engineering*, vol. 2017, no. 1, p. 9720946, 2017.

[13] C. D. Lai, D. N. P. Murthy, and M. Xie, "Weibull distributions," *Wiley Interdisciplinary Reviews: Computational Statistics*, vol. 3, no. 3, pp. 282–287, 2011.

[14] C.-D. Lai, "Generalized weibull distributions," in *Generalized Weibull Distributions*, Springer, 2013, pp. 23–75.

[15] N. M. Abbas and A. Sabrie, "Parametric models of reliability functions," *Results in Nonlinear Analysis*, vol. 6, no. 4, pp. 7–12, 2023.

[16] H. V. Lê, "Statistical manifolds are statistical models," *Journal of Geometry*, vol. 84, pp. 83–93, 2006.

[17] S. Lang, *Differential manifolds*. Springer, 1985.

[18] A. Cena and G. Pistone, "Exponential statistical manifold," *Annals of the Institute of Statistical Mathematics*, vol. 59, pp. 27–56, 2007.

[19] S. Smale, "On the structure of manifolds," *American Journal of Mathematics*, vol. 84, no. 3, pp. 387–399, 1962.

[20] F. Nielsen, "An elementary introduction to information geometry," *Entropy*, vol. 22, no. 10, p. 1100, 2020.

[21] Mohamed Bisher Zeina, Mohammad Abobala, Ahmad Hatip, Said Broumi, Sarah Jalal Mosa,Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution,Neutrosophic Sets and Systems, vol. 54.2023.

[22] Mohamed Bisher Zeina, Saeed Hemeda, Mazeat Koreny, Mohammad Abobala, On Symbolic n-Plithogenic Random Variables Using a Generalized Isomorphism, Neutrosophic Sets and Systems, vol. 59.2023

[23] Mohamed Bisher Zeina, Mohammad Abobala, A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine, Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics, 2023

[24]Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNova asbl, Bruxelles, 2015.

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