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# Muirhead Mean Operators in Quadripartitioned Single-Valued Neutrosophic Sets for Multi-Criteria Decision-Making with Indeterminate Information

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**Abstract**. This paper addresses the critical challenge of managing uncertainty, indeterminacy, and interdependencies in multi-criteria decision-making (MCDM) by introducing novel aggregation operators. Existing methods often fall short in effectively combining diverse and conflicting information, particularly in complex and uncertain environments. To bridge this gap, we propose the integration of the Muirhead Mean (MM) operator a versatile tool that generalizes arithmetic and geometric means-into the framework of Quaripartitioned Single-Valued Neutrosophic (QSVN) set theory, which accounts for truth, contradiction, ignorance, and falsity. The mathematical properties of the proposed operators are thoroughly analyzed, demonstrating their robustness and flexibility. A new decision-making process is developed to tackle uncertainty and interdependencies, supported by illustrative examples that highlight the practical significance of this approach. This work offers a substantial advancement in the design of dependable and adaptable decision-making frameworks for complex real-world scenarios.

## 1. Introduction

A common step in the decision-making process is picking the optimal choice from a variety of options, which can be quite difficult for academics studying fuzzy systems. Particularly when working with classical data, Multi-Attribute Decision-Making (MADM), a subset of decision-making techniques, is frequently used to determine the best choices from a range of preferences. Each alternative is traditionally given a rating value by decision-makers based on exact, binary information that is represented as 0 or 1. Real-world situations, however, are rarely as simple. Traditional methods are less successful since decision-making usually entails vague, uncertain, and imprecise information because of the environment's intrinsic complexity.

Decision-makers frequently give each choice a rating value, usually in the form of 0 or 1, based on exact data and information. But real-world situations are frequently more complicated, with vague, unclear, or imprecise information. Fuzzy set (FS) theory [1]. intuitionistic fuzzy set (IFS) theory [2], interval-valued fuzzy sets (IVFS) [3], and interval-valued intuitionistic fuzzy sets (IVIFS) [4], which take into account both membership and non-membership values, are some of the mathematical theories that have been developed to address such situations. Many methods were then put forth by researchers, including medical diagnosis [15], pattern recognition [12]- [14], similarity measurements [9]- [11], and Multi-Attribute Decision-Making (MADM) [5]- [8]. However, such data, which entails reluctance and indeterminacy, cannot be handled by these theories.

Smarandache [16] developed the Neutrosophic Set (NS) for this purpose. Indeterminacy, non-membership, and membership values within the non-standard unit interval (0, 1) are all addressed by NS theory. However, it is difficult to apply NS theory to practical problems because of this non-standard unit interval. Therefore, different types of NS were proposed to make things easier. The Single-Valued Neutrosophic set (SVNS) was proposed by Wang et al. [17], the Interval-Valued Neutrosophic set (IVNS) by Zhang et al. [18]. In MCDM situations, where the primary goal is to aggregate a group of inputs into a single number, Aggregation Operators (AOs) typically play a significant role. Ye [19] introduced the single-valued neutrosophic (SVN) weighted averaging (SVNWA) and SVN weighted geometric average (SVNWGA) operators in that approach, together with the operational rules of SVNSs. The improved operations of SVN numbers (SVNNs) and their accompanying ordered weighted average/geometric aggregation operator were defined by Peng et al. [20]. Nancy and Garg [21] used the Frank norm procedures to create the weighted average and geometric average operators. Based on Hamacher operations, Liu et al. [22] created a few generalized neutrosophic aggregation operators.

The aggregation operations under the Interval Neutrosophic Set (INS) environment were introduced by Zhang et al. [24], and some of its generalized operators were suggested by Aiwu et al. [25]. A nonlinear optimization model was created by Garg and Nancy [23] to address the

MCDM problem in the context of the INS. Based to an analysis of the AOs stated above, all of these studies make the assumption that there is no association between the argument values because all of the input arguments used during aggregate are independent of one another. Nonetheless, there is always a suitable link between them in real-world issues. For example, there is a correlation between a home's price and its location if someone want to buy one. It is obvious that the two components interact and are dependent on one another. Neutrosophic systems have significantly enhanced the ability to manage uncertainty in decision-making processes. Building on these breakthroughs, this paper introduces an innovative approach that integrates the Muirhead Mean operator with QSVN sets, addressing the pressing need for reliable and flexible aggregation methods in MCDM scenarios

The Bonferroni Mean (BM) [26], Maclaurin symmetric mean (MSM) [27], Heronian Mean (HM) [28], and other helpful aggregation functions are used to take into account the interdependence of the input arguments. The primary feature of BM, as stated by Yager [29], is its capacity to record the interaction between the input arguments. BM aggregation operators were introduced by Garg and Arora [30] in the intuitionistic fuzzy soft set setting. While some of these functions can capture more than two relationships, BM can only record the relationship between two arguments. Liu and Wang [31] extended the BM to a neutrosophic environment and introduced the SVN normalized weighted Bonferroni mean (SVNNWBM) operator, utilizing the benefits of these functions in a neutrosophic domain. The MSM aggregation operators were suggested by Wang et al. [33] in order to capture the correlation between the aggregated arguments. HM operators were introduced by Li et al. [32] to address the MCDM issues in an SVNS setting. Prioritized AOs were introduced by Garg and Nancy [34] in the linguistic SVNS context to address the decision-making issues. Some prioritized weighted averaging and geometric aggregation procedures for SVNNs were created by Wu et al. [35]. Using the Frank operations, Ji et al. [36] developed the single-valued prioritized BM operator. The Muirhead mean (MM) [37], a potent and practical aggregation method, is an alternative to these aggregations. The main benefit of the MM is its ability to take into account the connections between all of the arguments, which makes it more potent and thorough than BM, MSM, and HM. Furthermore, MM has a parameter vector that can increase the flexibility of the aggregation process. We might infer from the aforementioned analysis that real-world decision-making challenges are become increasingly complex. A more advantageous method of expressing the uncertain information is required in order to choose the best alternative or alternatives for the MCDM challenges. Furthermore, it is crucial to address the issue of how to take into account the connections among input arguments. In order to integrate the prioritized aggregation and MM and propose the prioritized MM (PMM) operator, we take into account the benefits of the SVNS while keeping all these aspects in mind. While there have been several distance measurements for SVN sets proposed recently, there are currently few references. A real-world case was examined by Chatterjee et al. [38] in order to gain a better understanding of the QSVNS environment, and the results showed that such situations are common. They proved QSVNS's skills by resolving a pattern recognition decision-making task. The paper [39] examines a multi-criteria decision-making model that uses trigonometric aggregation operations of single-valued neutrosophic credibility numbers.

## 2. Quadripartitioned Neutrosophic Muirhead Mean Operators

**Definition 2.1.** A Quadripartitioned Neutrosophic Set (QNS) $\lambda$ comprises of the four independent degrees in particular truth degree $\mathtt{A}_\lambda$, contradiction degree $\mathtt{B}_\lambda$, ignorance degree $\mathtt{C}_\lambda$ and false degree $\mathtt{D}_\lambda$ which are defined as $\lambda = \{(u, \mathtt{A}_\lambda(u), \mathtt{B}_\lambda(u), \mathtt{C}_\lambda(u), \mathtt{D}_\lambda(u)|u \in U)\}$, where $\mathtt{A}_\lambda(u), \mathtt{B}_\lambda(u), \mathtt{C}_\lambda(u), \mathtt{D}_\lambda(u)$ is the subset of the non-standard unit interval $(0^-, 1^+)$, such that $0^- \leq \mathtt{A}_\lambda(u) + \mathtt{B}_\lambda(u) + \mathtt{C}_\lambda(u) + \mathtt{D}_\lambda(u) \leq 4^+$

**Definition 2.2.** A Single - Valued Quadripartitioned Neutrosophic Set (SQNS) $\lambda$ in $U$ are defined as $\lambda = \{(u, \mathtt{A}_\lambda(u), \mathtt{B}_\lambda(u), \mathtt{C}_\lambda(u), \mathtt{D}_\lambda(u)|u \in U)\}$, where $\mathtt{A}_\lambda(u), \mathtt{B}_\lambda(u), \mathtt{C}_\lambda(u), \mathtt{D}_\lambda(u) \in [0, 1]$ such that $(0 \leq \mathtt{A}_\lambda(u) + \mathtt{B}_\lambda(u) + \mathtt{C}_\lambda(u) + \mathtt{D}_\lambda(u) \leq 4$ for all $u \in U$.

**Definition 2.3.** Let $\lambda = (\mathtt{A}_\lambda, \mathtt{B}_\lambda, \mathtt{C}_\lambda, \mathtt{D}_\lambda)$ be a SQNN. A score function $s$ of $\lambda$ is defined as

$$s(\lambda) = \frac{\mathtt{A}_\lambda + (1 - \mathtt{B}_\lambda) + (1 - \mathtt{C}_\lambda) + \mathtt{D}_\lambda}{4}$$

**Definition 2.4.** Let $\lambda = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}\}$, $\lambda_1 = \{\mathtt{A}_1, \mathtt{B}_1, \mathtt{C}_1, \mathtt{D}_1\}$ and $\lambda_2 = \{\mathtt{A}_2, \mathtt{B}_2, \mathtt{C}_2, \mathtt{D}_2\}$ be three SQNN and $\alpha > 0$ be real number. Then, we have

$$
\begin{aligned}
(I) &\quad \lambda^c = \{\mathtt{D}, \mathtt{B}, \mathtt{C}, \mathtt{A}\} \\
(II) &\quad \lambda_1 \leq \lambda_2 \quad if \quad \mathtt{A}_1 \leq \mathtt{A}_2, \mathtt{B}_1 \geq \mathtt{B}_2, \mathtt{C}_1 \geq \mathtt{C}_2 \quad and \quad \mathtt{D}_1 \geq \mathtt{D}_2 \\
(III) &\quad \lambda_1 = \lambda_2 \quad iff \quad \lambda_1 \leq \lambda_2 \text{ and } \lambda_2 \leq \lambda_1 \\
(IV) &\quad \lambda_1 \cap \lambda_2 = \{\min(\mathtt{A}_1, \mathtt{A}_2), \max(\mathtt{B}_1, \mathtt{B}_2), \max(\mathtt{C}_1, \mathtt{C}_2), \max(\mathtt{D}_1, \mathtt{D}_2)\} \\
(V) &\quad \lambda_1 \cup \lambda_2 = \{\max(\mathtt{A}_1, \mathtt{A}_2), \min(\mathtt{B}_1, \mathtt{B}_2), \min(\mathtt{C}_1, \mathtt{C}_2), \min(\mathtt{D}_1, \mathtt{D}_2)\} \\
(VI) &\quad \lambda_1 \otimes \lambda_2 = \{\mathtt{A}_1\mathtt{A}_2, \mathtt{B}_1 + \mathtt{B}_2 - \mathtt{B}_1\mathtt{B}_2, \mathtt{C}_1 + \mathtt{C}_2 - \mathtt{C}_1\mathtt{C}_2, \mathtt{D}_1 + \mathtt{D}_2 - \mathtt{D}_1\mathtt{D}_2\} \\
(VII) &\quad \lambda_1 \oplus \lambda_2 = \{\mathtt{A}_1 + \mathtt{A}_2 - \mathtt{A}_1\mathtt{A}_2, \mathtt{B}_1\mathtt{B}_2, \mathtt{C}_1\mathtt{C}_2, \mathtt{D}_1\mathtt{D}_2\} \\
(VIII) &\quad \alpha\lambda_1 = (1 - (1 - \mathtt{A}_1)^\alpha, \mathtt{B}_1^\alpha, \mathtt{C}_1^\alpha, \mathtt{D}_1^\alpha) \\
(XI) &\quad \lambda_1^\alpha = (\mathtt{A}_1^\alpha, 1 - (1 - \mathtt{B}_1)^\alpha, 1 - (1 - \mathtt{C}_1)^\alpha, 1 - (1 - \mathtt{D}_1)^\alpha).
\end{aligned}
$$

**Definition 2.5.** The prioritized weighted aggregation operators are defined as

(I) SQN prioritized weighted average (SQNPWA) $\Delta W$

$$\Delta W(\lambda_1,\lambda_2,\ldots,\lambda_n)=\Big\{1-\prod_{l=1}^{p}(1-\mathtt{A}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},\prod_{l=1}^{p}(\mathtt{B}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},\prod_{l=1}^{p}(\mathtt{C}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},\prod_{l=1}^{p}(\mathtt{D}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big\}$$

(II) SQN prioritized geometric average (SQNPGA) $\Delta G$

$$\Delta G(\lambda_1,\lambda_2,\ldots,\lambda_n)=\Big\{\prod_{l=1}^{p}(\mathtt{A}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},1-\prod_{l=1}^{p}(1-\mathtt{B}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},1-\prod_{l=1}^{p}(1-\mathtt{C}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}},1-\prod_{l=1}^{p}(1-\mathtt{D}_l)^{\frac{\mathbb{H}_l}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big\}$$

where $\mathbb{H}_1=1$ and $\mathbb{H}_l=\prod_{\mathtt{k}=1}^{l-1}s(\lambda_{\mathtt{k}}),(l=2,3,\ldots,p)$

**Definition 2.6.** For a non-negative real number $r_l$ $(l=1,2,\ldots,n)$, (MM) operator over the parameter $\mathbb{Q}=\{q_1,q_2,\ldots,q_p\}\in R^p$ is defined as

$$MM^{\mathbb{Q}}(q_1,q_2,\ldots,q_p)=\Big(\frac{1}{p!}\sum_{\delta\in S_p}\prod_{l=1}^{p}r_{\delta(l)}^{q_l}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

where $\delta$ is the permutation of $(1,2,\ldots,p)$ and $S_p$ is set of all permutation of $(1,2,\ldots,p)$. Some assigning some vectors to $\mathbb{Q}$, we can obtain some special cases of the MM.

1. If $\mathbb{Q}=(1,0,\ldots,0)$ the MM is reduced to
$MM^{(1,0,\ldots,0)}(q_1,q_2,\ldots,q_p)=\frac{1}{p}\sum_{l=1}^{p}q_l$ which is the arithemetic averaging operator.

2. If $\mathbb{Q}=(\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p})$, the MM is reduced to
$MM^{\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p}}(q_1,q_2,\ldots,q_p)=\prod_{l=1}^{p}q_l^{\frac{1}{p}}$
which is the geometric operator.

3. If $\mathbb{Q}=(1,1,0,0,\ldots,0)$ then the MM reduced to
$MM^{1,1,0,0,\ldots,0}$ then the MM is reduced to
$MM^{(1,1,0,0,\ldots,0)}(q_1,q_2,\ldots,q_p)=\Big(\frac{1}{p(p+1)}\sum_{m,l=1,m\neq l}^{p}r_m r_l\Big)^{\frac{1}{2}}$
which is the BM operator [26].

4. If $Q=(\overbrace{1,1,\ldots,1}^{\mathtt{k}},\overbrace{0,0,\ldots,0}^{\mathtt{p-k}})$, then the MM is reduced to
$MM^{(\overbrace{1,1,\ldots,1}^{\mathtt{k}},\overbrace{0,0,\ldots,0}^{\mathtt{p-k}})}(r_1,r_2,\ldots,r_p)=\Big(\frac{1}{C_{\mathtt{k}}^{p}}\sum_{1\leq m_1<\cdots<m_{\mathtt{k}}\leq p}\prod_{l}^{\mathtt{k}}r_{ml}\Big)^{\frac{1}{\mathtt{k}}}$
which is the MSM operator [27]

## 3. Quadripartitioned Neutrosophic Prioritized Muirhead Mean Operators

**Definition 3.1.** For a collection of SQNNs $\lambda_l = (1, 2, \ldots, p)$ a SQNPMM operator is a mapping SQNPMM:$\Omega \to \Omega$ defined as

$$SQNPMM(\lambda_1, \lambda_2, \ldots, \lambda_p) = \Big(\frac{1}{p!} \oplus_{\delta \in S_p} \prod_{l=1}^{p} \Big(p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}\lambda_{\delta(l)}\Big)^{q_l}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}} \tag{1}$$

where $\mathbb{H}_1 = 1$, $\mathbb{H}_l = \prod_{\mathtt{k}}^{l-1} s(\lambda_{\mathtt{k}}), (l = 2, 3, \ldots, p)$, $S_p$ is the collection of all permutations of $(1, 2, \ldots, p)$ and $\mathbb{Q} = (q_1, q_2, \ldots, q_p) \in R^p$ be a vector parameters.

**Theorem 3.2.** *For collection of SQNNs $\lambda_l = (\mathtt{A}_l, \mathtt{B}_l, \mathtt{C}_l, \mathtt{D}_l)$ $(l = 1, 2, \ldots, p)$ the aggregated value through equation 1 is again a SQNN and is given by*

$$\begin{aligned}
&QSPMM\ (\lambda_1, \lambda_2, \ldots, \lambda_n) \\
&= \Bigg(\Big(1 - \Big(\prod_{\delta \in S_p} \Big(1 - \prod_{l=1}^{p} \Big(1 - (1 - \mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}}, \\
&1 - \Big(1 - \Big(\prod_{\delta \in S_p} \Big(1 - \prod_{l=1}^{p} \Big(1 - (1 - \mathtt{B}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}}, \\
&1 - \Big(1 - \Big(\prod_{\delta \in S_p} \Big(1 - \prod_{l=1}^{p} \Big(1 - (1 - \mathtt{C}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}}, \\
&1 - \Big(1 - \Big(\prod_{\delta \in S_p} \Big(1 - \prod_{l=1}^{p} \Big(1 - (1 - \mathtt{D}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}}\Bigg).
\end{aligned} \tag{2}$$

*Proof.* For SQNN $\lambda_l (l = 1, 2, \ldots, p)$, we have

$$p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)} = \Big(1 - (1 - \mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}}, \mathtt{B}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}}, \mathtt{C}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}}, \mathtt{D}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}}\Big)$$

and

$$\left(p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}\right)^{q_l}=\Bigg((1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}})^{q_l},1-\left(1-\mathtt{B}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}})^{q_l},$$

$$1-\left(1-\mathtt{C}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}})^{q_l},1-\left(1-\mathtt{D}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}})^{q_l}\Bigg).$$

Thus,

$$\begin{aligned}
\oplus_{\delta\in S_p}\prod_{l=1}^{p}\left(p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}\right)^{q_l}&=\Bigg\{1-\prod_{\delta\in S_p}\left(1-\prod_{l=1}^{p}\left(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\right),\\
&=\prod_{\delta\in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{B}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\Bigg)\\
&=\prod_{\delta\in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{C}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\Bigg)\\
&=\prod_{\delta\in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{D}_{\delta(l)}\right)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\Bigg)\Bigg\}.
\end{aligned}$$

Now,

$$SQNPMM(\lambda_1,\lambda_2,\ldots,\lambda_p)=\left(\frac{1}{p!}\oplus_{\delta\in S_p}\prod_{l=1}^{p}\left(p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}\lambda_{\delta(l)}\right)^{q_l}\right)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

$$=\Bigg(\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}},$$

$$1-\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{B}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}},$$

$$1-\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{C}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}},$$

$$1-\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{D}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}\Bigg)$$

Thus, the equation 2 holds. Further, $0 \leq \mathtt{A}_{\delta(l)}, \mathtt{B}_{\delta(l)}, \mathtt{C}_{\delta(l)}, \mathtt{D}_{\delta(l)} \leq 1$ so we have

$$1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}} \in [0,1]$$

and

$$\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l} \in [0,1]$$

which implies that

$$1-\Big(\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big) \in [0,1]$$

Hence

$$0 \leq \Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}} \leq 1.$$

similarly, we have

$$0 \leq 1-\left(1-\prod_{\delta \in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{B}_{\delta(l)}^{\;p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\right)\right)^{\frac{1}{p!}}\right)^{\frac{1}{\sum_{l=1}^{p} q_l}} \leq 1.$$

$$0 \leq 1-\left(1-\prod_{\delta \in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{C}_{\delta(l)}^{\;p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\right)\right)^{\frac{1}{p!}}\right)^{\frac{1}{\sum_{l=1}^{p} q_l}} \leq 1.$$

$$0 \leq 1-\left(1-\prod_{\delta \in S_p}\left(1-\prod_{l=1}^{p}\left(1-\mathtt{D}_{\delta(l)}^{\;p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l}\right)\right)^{\frac{1}{p!}}\right)^{\frac{1}{\sum_{l=1}^{p} q_l}} \leq 1.$$

which complete the proof. □

**Example 3.3.** Take let $\lambda_1 = (0.4, 0.6, 0.5, 0.3)$, $\lambda_2 = (0.4, 0.7, 0.4, 0.7)$, $\lambda_3 = (0.6, 0.4, 0.3, 0.7)$ here be QSNNs and $\mathbb{Q} = (1, 0.6, 0.3)$ be the given parameter vector. By utilizing the given information and $\mathbb{H}_l = \prod_{\mathtt{k}=1}^{l-1} s(\lambda_{\mathtt{k}})$; (j=2,3), we get $\mathbb{H}_1 = 1$, $\mathbb{H}_2 = 0.4$ and $\mathbb{H}_3 = 0.16$. Therefore

$$\prod_{\delta \in S_3}(1-\prod_{l=1}^{3}\left(1-(1-\mathtt{A}_{\delta(l)})^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\right)^{q_l})$$

$$\begin{aligned}
&=\left\{1-(1-(1-0.4)^{3\times0.6410})^{1}\times(1-(1-0.4)^{3\times0.2564})^{0.6}\times(1-(1-0.6)^{3\times0.1026})^{0.3}\right\}\\
&\times\left\{1-(1-(1-0.4)^{3\times0.2564})^{1}\times(1-(1-0.4)^{3\times0.6410})^{0.6}\times(1-(1-0.6)^{3\times0.1026})^{0.3}\right\}\\
&\times\left\{1-(1-(1-0.6)^{3\times0.1026})^{1}\times(1-(1-0.4)^{3\times0.2564})^{0.6}\times(1-(1-0.4)^{3\times0.6410})^{0.3}\right\}\\
&\times\left\{1-(1-(1-0.4)^{3\times0.2564})^{1}\times(1-(1-0.6)^{3\times0.1026})^{0.6}\times(1-(1-0.4)^{3\times0.6410})^{0.3}\right\}\\
&\times\left\{1-(1-(1-0.4)^{3\times0.6410})^{1}\times(1-(1-0.6)^{3\times0.1026})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&\times\left\{1-(1-(1-0.6)^{3\times0.1026})^{1}\times(1-(1-0.4)^{3\times0.6410})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&=0.3638
\end{aligned}$$

$$\prod_{\delta \in S_3}(1-\prod_{l=1}^{3}\left(1-\mathtt{B}_{\delta(l)}^{\;3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\right)^{q_l})$$

$$= \left\{1-(1-(0.6)^{3\times 0.6410})^{1}\times(1-(0.4)^{3\times 0.2564})^{0.6}\times(1-(0.4)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.2564})^{1}\times(1-(0.6)^{3\times 0.6410})^{0.6}\times(1-(0.4)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.4)^{3\times 0.1026})^{1}\times(1-(0.7)^{3\times 0.2564})^{0.6}\times(1-(0.6)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.2564})^{1}\times(1-(0.4)^{3\times 0.1026})^{0.6}\times(1-(0.6)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.6)^{3\times 0.6410})^{1}\times(1-(0.4)^{3\times 0.1026})^{0.6}\times(1-(0.7)^{3\times 0.2564})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.4)^{3\times 0.1026})^{1}\times(1-(0.6)^{3\times 0.6410})^{0.6}\times(1-(0.7)^{3\times 0.2564})^{0.3}\right\}$$
$$= 0.3843$$

$$\prod_{\delta\in S_3}\left(1-\prod_{l=1}^{3}\left(1-\mathtt{C}_{\delta(l)}\right)^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\right)^{q_l}\Bigg)$$

$$= \left\{1-(1-(0.5)^{3\times 0.6410})^{1}\times(1-(0.4)^{3\times 0.2564})^{0.6}\times(1-(0.3)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.4)^{3\times 0.2564})^{1}\times(1-(0.5)^{3\times 0.6410})^{0.6}\times(1-(0.3)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.3)^{3\times 0.1026})^{1}\times(1-(0.4)^{3\times 0.2564})^{0.6}\times(1-(0.5)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.4)^{3\times 0.2564})^{1}\times(1-(0.3)^{3\times 0.1026})^{0.6}\times(1-(0.5)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.5)^{3\times 0.6410})^{1}\times(1-(0.3)^{3\times 0.1026})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.3)^{3\times 0.1026})^{1}\times(1-(0.5)^{3\times 0.6410})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\right\}$$
$$= 0.1607$$

and

$$\prod_{\delta\in S_3}\left(1-\prod_{l=1}^{3}\left(1-\mathtt{D}_{\delta(l)}\right)^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\right)^{q_l}\Bigg)$$

$$= \left\{1-(1-(0.3)^{3\times 0.6410})^{1}\times(1-(0.7)^{3\times 0.2564})^{0.6}\times(1-(0.7)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.2564})^{1}\times(1-(0.3)^{3\times 0.6410})^{0.6}\times(1-(0.7)^{3\times 0.1026})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.1026})^{1}\times(1-(0.7)^{3\times 0.2564})^{0.6}\times(1-(0.3)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.4)^{3\times 0.2564})^{1}\times(1-(0.7)^{3\times 0.1026})^{0.6}\times(1-(0.7)^{3\times 0.6410})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.6410})^{1}\times(1-(0.7)^{3\times 0.1026})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\right\}$$
$$\times\left\{1-(1-(0.7)^{3\times 0.1026})^{1}\times(1-(0.7)^{3\times 0.6410})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\right\}$$
$$= 0.5200$$

Hence, by using Equation(2), we get the aggregated value by SQNPMM is

$SQNPMM\ (\lambda_1, \lambda_2, \lambda_3)$

$$= \left\{(1-(0.3638)^{\frac{1}{8}})^{\frac{1}{1.9}}, 1-(1-(0.3843))^{\frac{1}{8}})^{\frac{1}{1.9}}, 1-(1-(0.1607))^{\frac{1}{8}})^{\frac{1}{1.9}}, 1-(1-(0.5200))^{\frac{1}{8}})^{\frac{1}{1.9}}\right\}$$

=(0.3260,0.6563,0.5663, 0.7378)

**Theorem 3.4.** *If $\lambda_l = (\mathtt{A}_l, \mathtt{B}_l, \mathtt{C}_l, \mathtt{D}_l)$ and $\lambda_l^{'} = (\mathtt{A}_l^{'}, \mathtt{B}_l^{'}, \mathtt{C}_l^{'}, \mathtt{D}_l^{'})$ are two SQNNs such that $\mathtt{A}_l \leq \mathtt{A}_l^{'}$, $\mathtt{B}_l \geq \mathtt{B}_l^{'}$, $\mathtt{C}_l \geq \mathtt{C}_l^{'}$ and $\mathtt{D}_l \geq \mathtt{D}_l^{'}$ for all $l$, then*

$$SQPMM(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq SQPMM(\lambda_1^{'}, \lambda_2^{'}, \ldots, \lambda_n^{'})$$

*This property is called monotonicity.*

*Proof.* For two $(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}} \geq (1-\mathtt{A}_{\delta(l)}^{'})^{p\frac{\mathbb{H}_{\delta(l)}^{'}}{\sum_{l=1}^{p}\mathbb{H}_l^{'}}}$ where $\mathbb{H}_1 = 1$, $\mathbb{H}_l = \prod_{\mathtt{k}}^{l-1} s(\lambda_{\mathtt{k}}), (l = 2, 3, \ldots, p)$, $\mathbb{H}_1^{'} = 1$, $\mathbb{H}_l^{'} = \prod_{\mathtt{k}}^{l-1} s(\lambda_{\mathtt{k}}^{'}), (l = 2, 3, \ldots, p)$

$(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}})^{q_l} \leq (1-(1-\mathtt{A}_{\delta(l)}^{'})^{p\frac{\mathbb{H}_{\delta(l)}^{'}}{\sum_{l=1}^{p}\mathbb{H}_l^{'}}})^{q_l}$ and

$$\prod_{l=1}^{p}\left(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\right)^{q_l} \leq \prod_{l=1}^{p}\left(1-(1-\mathtt{A}_{\delta(l)}^{'})^{p\frac{\mathbb{H}_{\delta(l)}^{'}}{\sum_{l=1}^{p}\mathbb{H}_l^{'}}}\right)^{q_l}$$

Further, we have $\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big) \geq \prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}'_{\delta(l)})^{p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}\Big)^{q_l}\Big)$

$\Big(\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}} \geq \Big(\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}'_{\delta(l)})^{p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}$.
Hence, we get

$$\Big(1-\Big(\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}} \geq \Big(1-\Big(\prod_{\delta\in S_p}(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{A}'_{\delta(l)})^{p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

Similarly, we have

$$1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{B}_{\delta(l)}^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}} \geq 1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{B}'^{\,p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}_{\delta(l)}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

$$1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{C}_{\delta(l)}^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}} \geq 1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{C}'^{\,p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}_{\delta(l)}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

and

$$1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{D}_{\delta(l)}^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}} \geq 1-\Big(1-\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{D}'^{\,p\frac{\mathbb{H}'_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}'_l}}_{\delta(l)}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p}q_l}}$$

. Therefore, by definition 2.4, we have

$$SQNPMM(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq SQNPMM(\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$$

□

## 4. Quadripartitioned Neutrosophic Prioritized Dual Muirhead Mean Operators

**Definition 4.1.** A SQNPDMM operator is a mapping SQNPDMM : $\Omega^p \to \Omega$ given by

$$SQNPDMM(\lambda_1, \lambda_2, \ldots, \lambda_p) = \Big(\frac{1}{\sum_{l=1}^{p} p_l}\Big(\prod_{\delta \in S_p} \bigoplus_{l=1}^{p} \Big(p_l \lambda \delta_l\Big)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}\lambda_{\delta(l)}}\Big)^{\frac{1}{p!}} \tag{3}$$

**Theorem 4.2.** *The collective value by using Equation 4.1 is still a SQNN and is given as*

SQPDMM $(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$\begin{aligned}
=&\Big(1-\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-\mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}},\\
&\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{B}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}},\\
&\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{C}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}},\\
&\Big(1-\Big(\prod_{\delta\in S_p}\Big(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{D}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Big)^{q_l}\Big)\Big)^{\frac{1}{p!}}\Big)^{\frac{1}{\sum_{l=1}^{p} q_l}}.
\end{aligned} \tag{4}$$

*Proof.* The proof of this result follows from Theorem 3.2. □

**Example 4.3.** If we have taken the data as considered in Example 3.3 to illustrate the aggregation operator as defined in Theorem 4.2 then, we have

$$\prod_{\delta\in S_3}(1-\prod_{l=1}^{3}\Big(1-\mathtt{A}_{\delta(l)})^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\Big)^{q_l}\Big)$$

$$= \Big\{1-(1-(0.4)^{3\times 0.6410})^{1}\times(1-(0.4)^{3\times 0.2564})^{0.6}\times(1-(0.6)^{3\times 0.1026})^{0.3}\Big\}$$
$$= \Big\{1-(1-(0.4)^{3\times 0.2564})^{1}\times(1-(0.4)^{3\times 0.6410})^{0.6}\times(1-(0.6)^{3\times 0.1026})^{0.3}\Big\}$$
$$= \Big\{1-(1-(0.6)^{3\times 0.1026})^{1}\times(1-(0.4)^{3\times 0.2546})^{0.6}\times(1-(0.4)^{3\times 0.6410})^{0.3}\Big\}$$
$$= \Big\{1-(1-(0.4)^{3\times 0.2564})^{1}\times(1-(0.6)^{3\times 0.1026})^{0.6}\times(1-(0.4)^{3\times 0.6410})^{0.3}\Big\}$$
$$= \Big\{1-(1-(0.4)^{3\times 0.6410})^{1}\times(1-(0.6)^{3\times 0.1026})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\Big\}$$
$$= \Big\{1-(1-(0.6)^{3\times 0.1026})^{1}\times(1-(0.4)^{3\times 0.6410})^{0.6}\times(1-(0.4)^{3\times 0.2564})^{0.3}\Big\}$$
$$= 0.2806$$

Similarly, we have

$$\prod_{\delta\in S_3}(1-\prod_{l=1}^{3}\Big(1-(1-\mathtt{B}_{\delta(l)})^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\Big)^{q_l}\Big)$$

$$= \Big\{1-(1-(1-0.6)^{3\times 0.6410})^{1}\times(1-(1-0.7)^{3\times 0.2564})^{0.6}\times(1-(1-0.4)^{3\times 0.1026})^{0.3}\Big\}$$
$$= \Big\{1-(1-(1-0.7)^{3\times 0.2564})^{1}\times(1-(1-0.6)^{3\times 0.6410})^{0.6}\times(1-(1-0.4)^{3\times 0.1026})^{0.3}\Big\}$$
$$= \Big\{1-(1-(1-0.4)^{3\times 0.1026})^{1}\times(1-(1-0.7)^{3\times 0.2564})^{0.6}\times(1-(1-0.6)^{3\times 0.6410})^{0.3}\Big\}$$
$$= \Big\{1-(1-(1-0.7)^{3\times 0.2564})^{1}\times(1-(1-0.4)^{3\times 0.1026})^{0.6}\times(1-(1-0.6)^{3\times 0.6410})^{0.3}\Big\}$$
$$= \Big\{1-(1-(1-0.6)^{3\times 0.6410})^{1}\times(1-(1-0.4)^{3\times 0.1026})^{0.6}\times(1-(1-0.7)^{3\times 0.2564})^{0.3}\Big\}$$
$$= \Big\{1-(1-(1-0.4)^{3\times 0.1026})^{1}\times(1-(1-0.6)^{3\times 0.6410})^{0.6}\times(1-(1-0.7)^{3\times 0.2564})^{0.3}\Big\}.$$
$$= 0.2327$$

$$\prod_{\delta\in S_3}(1-\prod_{l=1}^{3}\Big(1-(1-\mathtt{C}_{\delta(l)})^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\Big)^{q_l}\Big)$$

$$
\begin{aligned}
&= \left\{1-(1-(1-0.5)^{3\times0.6410})^{1}\times(1-(1-0.4)^{3\times0.2564})^{0.6}\times(1-(1-0.3)^{3\times0.1026})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.4)^{3\times0.2564})^{1}\times(1-(1-0.5)^{3\times0.6410})^{0.6}\times(1-(1-0.3)^{3\times0.1026})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.3)^{3\times0.1026})^{1}\times(1-(1-0.4)^{3\times0.2564})^{0.6}\times(1-(1-0.5)^{3\times0.6410})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.4)^{3\times0.2564})^{1}\times(1-(1-0.3)^{3\times0.1026})^{0.6}\times(1-(1-0.5)^{3\times0.6410})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.5)^{3\times0.6410})^{1}\times(1-(1-0.3)^{3\times0.1026})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.3)^{3\times0.1026})^{1}\times(1-(1-0.5)^{3\times0.6410})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&= 0.4986
\end{aligned}
$$

and

$$
\prod_{\delta\in S_3}(1-\prod_{l=1}^{3}\left(1-(1-\mathtt{D}_{\delta(l)})^{3\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{3}\mathbb{H}_l}}\right)^{q_l})
$$

$$
\begin{aligned}
&= \left\{1-(1-(1-0.3)^{3\times0.6410})^{1}\times(1-(1-0.7)^{3\times0.2564})^{0.6}\times(1-(1-0.7)^{3\times0.1026})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.7)^{3\times0.2564})^{1}\times(1-(1-0.3)^{3\times0.6410})^{0.6}\times(1-(1-0.7)^{3\times0.1026})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.7)^{3\times0.1026})^{1}\times(1-(1-0.7)^{3\times0.2564})^{0.6}\times(1-(1-0.3)^{3\times0.6410})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.4)^{3\times0.2564})^{1}\times(1-(1-0.7)^{3\times0.1026})^{0.6}\times(1-(1-0.7)^{3\times0.6410})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.7)^{3\times0.6410})^{1}\times(1-(1-0.7)^{3\times0.1026})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&= \left\{1-(1-(1-0.7)^{3\times0.1026})^{1}\times(1-(1-0.7)^{3\times0.6410})^{0.6}\times(1-(1-0.4)^{3\times0.2564})^{0.3}\right\}\\
&= 0.1986
\end{aligned}
$$

Hence, $SQNPDMM\ (\lambda_1,\lambda_2,\lambda_3)$

$$
= \left\{1-(1-(0.2806)^{\frac{1}{8}})^{\frac{1}{1.9}},(1-(0.2327))^{\frac{1}{8}})^{\frac{1}{1.9}},(1-(0.4986))^{\frac{1}{8}})^{\frac{1}{1.9}},(1-(0.1986))^{\frac{1}{8}})^{\frac{1}{1.9}}\right\}
$$

=(0.6354,0.3896,0.2706,0.4093)

## 5. Proposed Decision-Making Approach

An MCDM problem with $n$ options $R_1, R_2, \ldots, R_n$ is examined using the $p$ criterion $S_1, S_2, \ldots, S_p$. In order to do this, a specialist was asked to assess these options in a QSN setting, and the results were provided as QSNNs. For example, when we ask an expert about the alternative $R_m$ in relation to the criterion $S_l$, it corresponds to alternative $R_m$ under criterion $S_l$.

The quadripartitioned neutrosophic decion matrix $D$ whic is represented as

$$D = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1p} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2p} \\ . & . & \dots & . \\ . & . & \dots & . \\ \lambda_{np} & \lambda_{np} & \dots & \lambda_{np} \end{bmatrix}$$

Step I: If in the considered decision-making problem, there exist two kinds of criteria, namely the benefit and the cost types, then all the cost type criteria should be normalized into the benefit type by using the following equation

$$h_{ml} = \begin{cases} (\mathtt{A}_{ml}, \mathtt{B}_{ml}, \mathtt{C}_{ml}, \mathtt{D}_{ml}) & \text{for cost type criteria,} \\ (\mathtt{D}_{ml}, \mathtt{B}_{ml}, \mathtt{C}_{ml}, \mathtt{A}_{ml}) & \text{for benefit type criteria.} \end{cases}$$

Step II: Compute $\mathbb{H}_{ml}(m = 1, 2, ..., p)$ as

$$\mathbb{H}_{ml} = \begin{cases} 1 & \text{if } l = 1, \\ (l-1)\prod_{\mathtt{k}=1}^{l-1} s(h_{mk}) & \text{if } l = 2, \dots, p. \end{cases}$$

Step III: For a given parameter $Q = q_1, q_2, \dots, q_p$ utilize either SQNPMM or SQNPDMM operator to get the collective values $h_m = (\mathtt{A}_m, \mathtt{B}_m, \mathtt{C}_m, \mathtt{D}_m)$ $(m = 1, 2, \dots, n)$ for each alternative as

$h_m = SQNPMM(h_{m1}, h_{m2}, \dots, h_{mp})$

$$= \Bigg( \Bigg( 1 - \Big( \prod_{\delta \in S_p} \Big( 1 - \prod_{l=1}^{p} \Big( 1 - (1 - \mathtt{A}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}} \Big)^{q_l} \Big) \Big)^{\frac{1}{p!}} \Bigg)^{\frac{1}{\sum_{l=1}^{p} q_l}},$$

$$1 - \Bigg( 1 - \Big( \prod_{\delta \in S_p} \Big( 1 - \prod_{l=1}^{p} \Big( 1 - (1 - \mathtt{B}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}} \Big)^{q_l} \Big) \Big)^{\frac{1}{p!}} \Bigg)^{\frac{1}{\sum_{l=1}^{p} q_l}},$$

$$1 - \Bigg( 1 - \Big( \prod_{\delta \in S_p} \Big( 1 - \prod_{l=1}^{p} \Big( 1 - (1 - \mathtt{C}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}} \Big)^{q_l} \Big) \Big)^{\frac{1}{p!}} \Bigg)^{\frac{1}{\sum_{l=1}^{p} q_l}},$$

$$1 - \Bigg( 1 - \Big( \prod_{\delta \in S_p} \Big( 1 - \prod_{l=1}^{p} \Big( 1 - (1 - \mathtt{D}_{\delta(l)})^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p} \mathbb{H}_l}} \Big)^{q_l} \Big) \Big)^{\frac{1}{p!}} \Bigg)^{\frac{1}{\sum_{l=1}^{p} q_l}} \Bigg)$$

or $h_m = SQNPDMM(h_{m1}, h_{m2}, \ldots, h_{mp})$

$$
\begin{aligned}
=&\Bigg(1-\Bigg(1-\Bigg(\prod_{\delta\in S_p}\Bigg(1-\prod_{l=1}^{p}\Big(1-\mathtt{A}_{\delta(l)}\Big)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Bigg)^{q_l}\Bigg)\Bigg)^{\frac{1}{p!}}\Bigg)^{\frac{1}{\sum_{l=1}^{p}q_l}},\\
&\Bigg(1-\Bigg(\prod_{\delta\in S_p}\Bigg(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{B}_{\delta(l)}\Big)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Bigg)^{q_l}\Bigg)\Bigg)^{\frac{1}{p!}}\Bigg)^{\frac{1}{\sum_{l=1}^{p}q_l}},\\
&\Bigg(1-\Bigg(\prod_{\delta\in S_p}\Bigg(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{C}_{\delta(l)}\Big)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Bigg)^{q_l}\Bigg)\Bigg)^{\frac{1}{p!}}\Bigg)^{\frac{1}{\sum_{l=1}^{p}q_l}},\\
&\Bigg(1-\Bigg(\prod_{\delta\in S_p}\Bigg(1-\prod_{l=1}^{p}\Big(1-(1-\mathtt{D}_{\delta(l)}\Big)^{p\frac{\mathbb{H}_{\delta(l)}}{\sum_{l=1}^{p}\mathbb{H}_l}}\Bigg)^{q_l}\Bigg)\Bigg)^{\frac{1}{p!}}\Bigg)^{\frac{1}{\sum_{l=1}^{p}q_l}}
\end{aligned}
$$

Step IV: Calculate score value of the overall aggregated values $h_m = (\mathtt{A}_m, \mathtt{B}_m, \mathtt{C}_m, \mathtt{D}_m)$ $(m = 1, 2, \ldots, n)$ by using equation

$$s(h_m) = \frac{\mathtt{A}_m + (1-\mathtt{B}_m) + (1-\mathtt{C}_m) + \mathtt{D}_m}{4}$$

Step V : Rank all the feasible alternatives $\mathfrak{A}m$ $(m = 1, 2, \ldots, n)$ according to Definition 2.4 and hence select the most desirable alternative.

## Conclusion

In order to meet the requirement for sophisticated aggregation tools in multi-criteria decision-making under unpredictable and interdependent settings, this study presents a novel integration of the Muirhead Mean operator within the Quadripartitioned Single-Valued Neutrosophic set architecture. By utilizing the flexibility of the MM operator and the dependability of QSVN set theory, the suggested aggregation operators allow for the thorough assessment of choice criteria while taking truth, contradiction, ignorance, and falsity into account. While special situations demonstrate the suggested operators' practical flexibility, theoretical examination validates their mathematical resilience. Furthermore, a methodology for generating decisions based on these operators is created, showcasing its capacity to manage intricate MCDM situations with increased precision and adaptability.

## Compliance with ethical standards

**Data availability statements** Not applicable.

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