Decomposition of Neutrosophic Fuzzy Matrices Using Some Alpha – Cuts

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Abstract

The aim of this paper is to investigate the properties of various types of $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ (Alpha) - cuts on Neutrosophic Fuzzy Matrices (NFM). We introduce different kinds of Alpha cuts on Neutrosophic Fuzzy Sets (NFS) and discuss their properties with other existing operators on NFM. Finally, we provide representations and decompositions of NFM using these Alpha cuts. To support and clarify the findings, counterexamples are included in the discussion of the decomposition of NFM.

Keywords: Neutrosophic fuzzy sets; Neutrosophic fuzzy matrix; Neutrosophic fuzzy value; Reflexive matrix; Cut matrix; Irreflexive; Symmetric.

Abbreviations and notations

FM	Fuzzy Matrices
IFM	Intuitionistic Fuzzy Matrices
IFSs	Intuitionistic Fuzzy Sets

NFSs	Neutrosophic fuzzy Sets
NFM	Neutrosophic fuzzy matrices.
GI	Generalized Inverse
$M^2 = M$	M is idempotent NFM
$M \geq I_n$	M is reflexive NFM
$M \wedge I_n = 0$	M is irreflexive NFM
$M^T = M$	M is symmetric NFM
$M \wedge M^T \leq I_n$	M is antisymmetric NFM

1. **Introduction**

The study of fuzzy sets, first introduced by Zadeh [32], marked a significant milestone in the field of mathematical modeling and uncertainty management. Fuzzy sets laid the groundwork for diverse generalizations and extensions, including intuitionistic fuzzy sets (Atanassov [1][2]), which expanded the scope of fuzzy systems by incorporating degrees of membership and non-membership. Building upon this, neutrosophic sets introduced by Smarandache [3] further generalized intuitionistic fuzzy sets, offering a powerful framework to handle indeterminate and inconsistent information.

Matrix theory has played a pivotal role in extending these fuzzy and neutrosophic concepts to address real-world problems systematically. Initial efforts focused on fuzzy matrices and their properties, such as decomposition and convergence (Thomson [28]; Kim and Roush [14]; Hashimoto [7]; Mishref and Emam [35]). Intuitionistic fuzzy matrices (IFMs), introduced by Pal, Khan, and Shyamala [23], have since become a key area of research, enabling applications in decision-making, clustering, and relational equations (Xu and Yager [30]; Meenakshi and Gandhimathi [18]; Xu [29]). Theoretical advancements have been accompanied by the development of various operators and decomposition theorems for IFMs, enhancing their applicability (Bustince and Burillo [5]; Hai, Xing, and Biao [6]; Jose and Kuriakose [13]). Recent studies have introduced novel concepts such as the adjoint and determinant of square IFMs (Im, Lee, and Park [9][10]), as well as canonical forms (Jeong and Lee [11]). These innovations have paved the way for complex structures like quasi-cuts and their relevance to intuitionistic fuzzy and neutrosophic matrices (Barbhuiya [4]; Huang [8]).

The introduction of neutrosophic matrices, particularly neutrosophic fuzzy matrices, has further enriched this domain. Notable contributions include studies on generalized symmetric neutrosophic fuzzy matrices and their variants (Anandhkumar et al. [15]; Anandhkumar, Punithavalli, and Janaki [16]), as well as interval-valued and k-column symmetric neutrosophic fuzzy matrices (Anandhkumar et al. [24]). These frameworks address complex systems by incorporating degrees of truth, indeterminacy, and falsity, thus providing a nuanced approach to uncertainty modeling.

Research has also explored computational aspects, such as the equivalence, invertibility, and eigenvalues of intuitionistic fuzzy and neutrosophic matrices (Mondal and Pal [19]; Pradhan and Pal [25]). Furthermore, new operators and aggregation methods have been proposed for multi-attribute decision-making and clustering (Zhang [34]; Zhang and Zheng [33]). Despite these advancements, several challenges remain, including the efficient computation of neutrosophic matrix operations, extensions of classical properties such as transitivity and decomposition (Hashimoto [36]; Murugadas and Lalitha [21]), and the generalization of existing theories to accommodate emerging frameworks like Fermatean neutrosophic fuzzy matrices (Anandhkumar et al. [38]). Addressing these challenges requires integrating foundational principles with novel approaches to ensure broader applicability and scalability.

2. Motivation and Contribution of the Study

Motivation:

Neutrosophic Fuzzy Matrices (NFM) serve as a powerful tool for modeling uncertainty, indeterminacy, and imprecision in complex systems. While Neutrosophic Fuzzy Sets (NFS) have been extensively studied, particularly in decision-making and information fusion, the concept of Alpha (α)–cuts—a well-established tool in classical and fuzzy set theory—has not been thoroughly explored in the context of NFM. The lack of a systematic approach to defining and analyzing Alpha-cuts for Neutrosophic Fuzzy Matrices limits the deeper mathematical understanding and practical applications of NFMs. This gap in the literature highlights the need to develop new methodologies to enhance the interpretability, decomposition, and manipulation of NFMs.

Contribution:

This study makes the following key contributions:

(i) **Novel Definitions:** We introduce and formally define various types of α-cuts tailored specifically for Neutrosophic Fuzzy Sets and extend these concepts to Neutrosophic Fuzzy Matrices.

(ii) **Theoretical Insights:** We investigate the algebraic and structural properties of these α-cuts and establish their relationships with existing operators in the framework of NFMs.

(iii) **Decomposition Framework:** A comprehensive method for the decomposition and representation of NFMs using α-cuts is proposed, enabling better interpretability and analysis of complex data structures.

(iv) **Counterexamples and Clarifications:** To support the theoretical claims and clarify the limitations of certain decomposition scenarios, we provide well-constructed counterexamples, offering deeper insights into the behavior of NFMs under various α-cut operations.

(v) **Foundation for Further Research:** The results of this study lay the groundwork for future explorations into optimization, pattern recognition, and machine learning applications involving Neutrosophic Fuzzy Matrices.

3. Literature Review

The foundation of fuzzy set theory was established by Zadeh [32], providing a framework for modeling uncertainty and imprecision. Building on this, Atanassov [1][2] introduced intuitionistic fuzzy sets, which expanded fuzzy sets to include membership and non-membership degrees. Smarandache [3] further generalized these concepts with neutrosophic sets, enabling the representation of truth, indeterminacy, and falsity degrees, thereby addressing more complex uncertainties.

Matrix-based approaches to fuzzy and intuitionistic fuzzy systems have gained significant attention. Early works explored fuzzy matrices, such as decomposition (Hashimoto [7]) and convergence properties (Thomson [28]). Pal et al. [23] introduced intuitionistic fuzzy matrices (IFMs), facilitating advanced studies on aggregation, clustering, and relational equations (Xu and Yager [30]; Meenakshi and Gandhimathi [18]). Operators like determinants and adjoints for square IFMs were developed by Im et al. [9][10], while Jeong and Lee [11] proposed canonical forms, enhancing the structural understanding of IFMs. The decomposition of intuitionistic fuzzy sets and matrices has been extensively studied, with contributions from Hai et al. [6], Jose and Kuriakose [13], and Murugadas and Lalitha [21]. Barbhuiya [4] and Huang [8] advanced quasi-cut concepts, which have implications for both intuitionistic fuzzy and neutrosophic matrices. Mondal and Pal [19] addressed properties like similarity, invertibility, and eigenvalues of IFMs, while Pradhan and Pal [25] explored generalized inverses of these matrices.

The advent of neutrosophic matrices introduced further dimensions to this domain. Anandhkumar et al. [15] studied generalized symmetric neutrosophic fuzzy matrices, while Anandhkumar et al. [16][24] explored interval-valued and k-column symmetric matrices. These frameworks accommodate a more granular representation of uncertainty, truth, and falsity. Hashimoto [36] and Murugadas and Lalitha [21] extended transitivity and decomposition theories to neutrosophic contexts.Advanced aggregation and decision-making techniques have also been proposed, with Zhang [34] introducing a ranking method for intuitionistic fuzzy values. Zhang and Zheng [33] proposed new operators for fuzzy matrices, while Xu [29] contributed to clustering methodologies. Recent work by Anandhkumar et al. [38] examined Fermatean neutrosophic fuzzy matrices, emphasizing the need for continued innovation in this rapidly evolving field. J, J & S, R [39] have studied Some Operations on Neutrosophic Hypersoft Matrices and Their Applications. Ranulfo Paiva Barbosa (Sobrinho), & Smarandache [40] have presented Pura Vida Neutrosophic Algebra. In recent years, the study of neutrosophic fuzzy matrices has gained significant attention due to their ability to handle indeterminate and inconsistent information, which is common in real-world applications. Several researchers have made remarkable contributions in this domain, particularly in the development of Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) and their applications in decision-making problems. Anandhkumar et al. [41] introduced the determinant theory for QPNFMs and demonstrated its effectiveness in multi-criteria decision-making problems. Further advancements were made by Radhika et al. [42], who developed the concept of Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices to enhance decision-making processes.

The study of inverses of neutrosophic fuzzy matrices has also been explored extensively. Anandhkumar et al. [43] discussed various types of inverses, while their work on pseudo-similarity for neutrosophic fuzzy matrices provided important structural insights [44]. The concept of Schur complements within the framework of k-kernel symmetric block QPNFMs was proposed by Radhika et al. [45], further enriching the theoretical foundation of neutrosophic matrix theory. Moreover, Prathab et al. [46] extended these ideas by introducing generalized inverses for interval-valued secondary k-range symmetric fuzzy matrices. In parallel, significant progress has been made in the field of intuitionistic fuzzy matrices. The works by Anandhkumar et al. [47] on reverse tilde and minus partial orderings and by Punithavalli and Anandhkumar [48] on reverse sharp and left-T right-T partial orderings have contributed to a deeper understanding of ordering relations within intuitionistic fuzzy structures. Additional studies on secondary k-range symmetric neutrosophic fuzzy matrices [49] and the generalization of k-idempotent neutrosophic fuzzy matrices [50] have further expanded the theoretical underpinnings necessary for advanced decision-making applications. Finally, the exploration of kernel and k-kernel symmetric intuitionistic fuzzy matrices by Punithavalli and Anandhkumar [51], and the investigation of reverse sharp and left-T right-T partial ordering on neutrosophic fuzzy matrices by Anandhkumar et al. [52], illustrate the ongoing efforts to refine the algebraic structures underlying these advanced mathematical frameworks.

4. **Novelty**

The references provided offer significant contributions to the field of fuzzy and neutrosophic set theory, focusing on the generalization, decomposition, and algebraic operations of fuzzy matrices and intuitionistic fuzzy sets (IFS). Atanassov's foundational work on intuitionistic fuzzy sets in the 1980s expanded the understanding of uncertainty by introducing both membership and non-membership functions (Atanassov [1], [2]). Smarandache's introduction of neutrosophic sets further generalized intuitionistic fuzzy sets, allowing for better handling of indeterminacy and contradictions in data (Smarandache [3]). Barbhuiya's exploration of quasi-cuts in fuzzy and intuitionistic fuzzy sets provides valuable insights into their structural decomposition (Barbhuiya [4]). Key developments such as the work by Bustince and Burillo on intuitionistic fuzzy relations (Bustince & Burillo [5]), Hai, Xing, and Biao's theorems on cut sets (Hai, Xing, & Biao [6]), and Hashimoto's decomposition of fuzzy matrices (Hashimoto [7]) have expanded the theoretical foundation, offering practical tools for applications in optimization and control systems.

Further, the studies on the determinant and adjoint of intuitionistic fuzzy matrices by Im, Lee, and Park (Im, Lee, & Park [9], [10]) have enriched matrix algebra in fuzzy systems, facilitating the solution of fuzzy systems. Recent advancements by Anandhkumar and colleagues have introduced generalized symmetric neutrosophic fuzzy matrices (Anandhkumar et al. [15]) and interval-valued secondary k-range symmetric neutrosophic fuzzy matrices (Anandhkumar et al. [16]), enhancing the flexibility in dealing with uncertainty and imprecision in decision-making. Additionally, works by Pradhan, Pal, and others on generalized inverses and new operators on fuzzy matrices (Pradhan & Pal [25]; Shyamala & Pal [26]), as well as Zhang's method for ranking intuitionistic fuzzy values (Zhang [34]), have provided new techniques for multi-attribute decision-making under uncertainty. Together, these references offer novel mathematical frameworks, decomposition theorems, and algebraic operations that significantly contribute to the application of fuzzy and neutrosophic logic in various fields, including artificial intelligence, economics, and optimization.

5. **Preliminaries**

In this part, we introduce operations for NFMs. For two NFMs P and Q, we define the subsequent operations $M \vee N$, $M \wedge N$.

$$M \vee N = \left[m_{ij} \vee n_{ij} \right] = \left[\max < m_{ij}^{\ T}, n_{ij}^{\ T} >, \max < m_{ij}^{\ I}, n_{ij}^{\ I} >, \min < m_{ij}^{\ F}, n_{ij}^{\ F} > \right]$$

$$M \wedge N = \left[m_{ij} \wedge n_{ij} \right] = \left[\min < m_{ij}^{\ T}, n_{ij}^{\ T} >, \min < m_{ij}^{\ I}, n_{ij}^{\ I} >, \max < m_{ij}^{\ F}, n_{ij}^{\ F} > \right]$$

Definition: 5.1 A NFSs P on the universe of discourse Y is well-defined as

$U = \left\{ < y, m^T(y), m^I(y), m^F(y) >, y \in Y \right\}$, everywhere $m^T, m^I, m^F : Y \to]^-0,1^+[$ also

$0 \le m^T + m^I + m^F \le 3.$

Definition:5.2 A neutrosophic Fuzzy Matrices U is less than or equal to V (M and N are comparable)

That is $U \le V$ if $\left(m_{ij}^{\ T}, m_{ij}^{\ I}, m_{ij}^{\ F} \right) \le \left(n_{ij}^{\ T}, n_{ij}^{\ I}, n_{ij}^{\ F} \right)$ means $m_{ij}^{\ T} \le n_{ij}^{\ T}, m_{ij}^{\ I} \le n_{ij}^{\ I}, m_{ij}^{\ F} \ge n_{ij}^{\ F}$.

Example:5.1 Let us consider 3x3 NFM

$$U = \begin{bmatrix} <0.5,0.5,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.2> \end{bmatrix}$$

$$V = \begin{bmatrix} <0.9,0.6,0.1> & <0.7,0.5,0.2> & <0.9,0.3,0.1> \\ <0.8,0.7,0.2> & <1,0.7,0.2> & <0.9,0.8,0.2> \\ <0.8,0.4,0.1> & <1,0.6,0.1> & <0.9,0.3,0.1> \end{bmatrix}$$

Definition 5.3. A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by $N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

Definition 5.4 A square NFM is referred to as a Neutrosophic Fuzzy Permutation Matrix (NFPM) if each row and each column contains exactly one element with a value of (1,1,0) while all other entries are (0,0,1).

Definition 5.5 For identity NFM of order n x n is represented by I_n and is well-defined by $\left(\delta_{ij}^{\ T}, \delta_{ij}^{\ I}, \delta_{ij}^{\ F} \right) = \begin{cases} (1,1,0) \ if \ i = j \\ (0,0,1) \ if \ i \ne j \end{cases}$.

We now present the following operations for NFMs $M = \left(m^T, m^I, m^F\right)$ and $N = \left(n^T, n^I, n^F\right)$ us defined the binary operation

(i) $M * N = \left[m_{ij} * n_{ij} \right].$

(ii) $M \Theta N = \left[m_{ij} \Theta n_{ij} \right].$

(iii) $M \times N = \left[\bigcup_{k=1}^{n} \left(m_{ik} \wedge n_{kj} \right) \right].$

(iv) $M^{k+1} = M^k \times M, (k = 1, 2, 3, ...)$

(v) $M^T = \left[{m_{ji}}^T, {m_{ji}}^I, {m_{ji}}^F \right]$ (the transpose of M)

(vi) $M^c = \left[{m_{ij}}^F, {m_{ij}}^I, {m_{ij}}^T \right]$ (the complement of M)

(vii) $\Delta M = M \Theta M^T$

(viii) $\nabla M = M \wedge M^T$

Definition 5.6 Let $i, j, k \le n$ and let $M = \left[m_{ij} = \left({m_{ij}}^T, {m_{ij}}^I, {m_{ij}}^F \right) \right]$ be an NFM. Then M is called **Transitive iff** $M^2 = MM \le M$ $i.e., {m_{ik}}^T \wedge {m_{kj}}^T \le {m_{ij}}^T, {m_{ik}}^I \wedge {m_{kj}}^I \le {m_{ij}}^I$ and ${m_{ik}}^F \vee {m_{kj}}^F \ge {m_{ij}}^F$ for every $i, j, k \le n.$ **Nilpotent iff** $M^n = MM...M$ (n-times) = 0

6. Decomposition of a Neutrosophic Fuzzy Matrices Using Some $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$ Cuts

Definition:6.1 For any $\left(m^T, m^I, m^F\right), \left(n^T, n^I, n^F\right) \in \left(NFM\right)$, we define

$\left(m^T, m^I, m^F\right) \oplus \left(n^T, n^I, n^F\right) = \left\{ \left(m^T + n^T - 1\right) \wedge 1, \left(m^I + n^I - 1\right) \wedge 1, \left(m^F + n^F - 1\right) \vee 0 \right\}$ and

$\left(m^T, m^I, m^F\right) \square \left(n^T, n^I, n^F\right) = \left\{ \left(m^T + n^T - 1\right) \vee 0, \left(m^I + n^I - 1\right) \vee 0, \left(m^F + n^F\right) \wedge 1 \right\}.$

6.1 $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$ Cuts and Some Properties

Here we define $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$ cuts for NFMs and NFMs also we studied some of its properties.

For any $\left(m^T, m^I, m^F\right), \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \in \left(NFM\right)$ define

(i) $$\left(m^T,m^I,m^F\right)^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\begin{cases}(1,1,0), & if\ \left(m^T,m^I,m^F\right)\geq\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right),\\(0,0,1), & \text{otherwise.}\end{cases}$$

(ii) $$\left(m^T,m^I,m^F\right)_{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\begin{cases}\left(m^T,m^I,m^F\right), & if\ \left(m^T,m^I,m^F\right)\geq\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right),\\(0,0,1), & \text{otherwise.}\end{cases}$$

(iii) $$\left(m^T,m^I,m^F\right)_{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\begin{cases}(1,1,0), & if\ \left(m^T,m^I,m^F\right)\geq\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right),\\\left(m^T,m^I,m^F\right), & \text{otherwise.}\end{cases}$$

(iv) $$\left(m^T,m^I,m^F\right)^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\begin{cases}(1,1,0), & if\ \alpha^\mu+m^T\geq 1,\alpha^\omega+m^F<1\\(0,0,1), & \text{otherwise.}\end{cases}$$

(v) $$\left(m^T,m^I,m^F\right)_{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\begin{cases}\left(m^T,m^I,m^F\right), & if\ \alpha^\mu+m^T\geq 1,\alpha^\omega+m^F<1\\(0,0,1), & \text{otherwise.}\end{cases}$$

(vi) $$\left(m^T,m^I,m^F\right)^{<\alpha^\mu,\alpha^\nu,\alpha^\omega>}=\begin{cases}(1,1,0), & if\ \alpha^\mu+m^T\geq 1,\alpha^\omega+m^F<1, or\ m^T\geq\alpha^\mu,m^F<\alpha^\omega,\\(0,0,1), & \text{otherwise.}\end{cases}$$

(vii) $$\left(m^T,m^I,m^F\right)_{<\alpha^\mu,\alpha^\nu,\alpha^\omega>}=\begin{cases}\left(m^T,m^I,m^F\right), & if\ \alpha^\mu+m^T\geq 1,\alpha^F+m^F<1, or\ m^T\geq\alpha^\mu,m^F<\alpha^\omega,\\(0,0,1), & \text{otherwise.}\end{cases}$$

(viii) $$\left(m^T,m^I,m^F\right)_{\left|\alpha^\mu,\alpha^\nu,\alpha^\omega\right|}=\begin{cases}\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right), & if\left(m^T,m^I,m^F\right)\geq\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right),\\\left(m^T,m^I,m^F\right), & if\left(m^T,m^I,m^F\right)\geq\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)\text{ and otherwise.}\end{cases}$$

Consider $M\in\left(NFM\right)_{pq},\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)\in\left(NFM\right)$.Now we extend the above definitions to NFMs as follows,

(i) $$\left[M\right]^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\left[\left(m^T,m^I,m^F\right)^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}\right].$$

(ii) $$\left[M\right]^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}=\left[\left(m^T,m^I,m^F\right)^{\left(\alpha^\mu,\alpha^\nu,\alpha^\omega\right)}\right].$$

(iii) $$\left[M\right]_{\alpha^\mu,\alpha^\nu,\alpha^\omega}=\left[\left(m^T,m^I,m^F\right)_{\alpha^\mu,\alpha^\nu,\alpha^\omega}\right].$$

(iv) $$\left[M\right]_{\left[\alpha^\mu,\alpha^\nu,\alpha^\omega\right]}=\left[\left(m^T,m^I,m^F\right)_{\left[\alpha^\mu,\alpha^\nu,\alpha^\omega\right]}\right].$$

(v) $[M]^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}=\left[\left(m^{T},m^{I},m^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\right].$

(vi) $[M]_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}=\left[\left(m^{T},m^{I},m^{F}\right)_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}\right].$

(vii) $[M]^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}=\left[\left(m^{T},m^{I},m^{F}\right)^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}\right].$

(viii) $[M]_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left[\left(m^{T},m^{I},m^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}\right].$

Proposition 6. 1. For any two NFMs $M,N\in(NFM)_{mn}$ and $M\geq N$ .We have the following results,

(i) $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\geq N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}.$

(ii) $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\geq N_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}.$

(iii) $M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\geq N_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}.$

(iv) $M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\geq N^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}.$

(v) $M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\geq N_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}.$

(vi) $M^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}\geq N^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}$

(vii) $M_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}\geq N_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}$

(viii) If all the entries of the M are comparable with $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$, $M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}\geq N_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$

Proof: (i) Consider any ijth element of $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$ as $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$

Case (1) If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$, then $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=(1,1,0)$

Sub case (1.1) If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$,

then $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=(1,1,0)$

Sub case (1.2) If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\geq\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)$,

then $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=(0,0,1)$.

Thus, $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} \geq \left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$.

Case (2) If $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$,

then $\left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right) \leq \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) \leq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$.

Hence, $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = \left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = (0,0,1)$

Case (3) If the entries of the matrix M are not comparable to $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$, then $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)} = (0,0,1)$.

Sub case (3.1) If the entries of the matrix N are also not comparable to $\left(\alpha^{T}, \alpha^{I}, \alpha^{F}\right)$ or $\left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right) \leq \left(\alpha^{T}, \alpha^{I}, \alpha^{F}\right)$, then $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)} = \left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right)^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)}$

Sub Case 3.2: If $\left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right) \geq \left(\alpha^{T}, \alpha^{I}, \alpha^{F}\right)$, then from Subcase 1.1, $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)} = \left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right)^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)} = (1,1,0)$ Suppose the entries of the matrix N are not comparable to $\left(\alpha^{T}, \alpha^{I}, \alpha^{F}\right)$. We get $M \geq N$ whenever the entries of the matrix M are comparable or not. Hence, from Case (1), (2), and (3), we have $M^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)} \geq N^{\left(\alpha^{T},\alpha^{I},\alpha^{F}\right)}$ when $M \geq N$.

(ii) Clear from (i).

(iii) Also clear from (i).

(iv) **Case 1:** If $\alpha^{\mu} + \alpha_{ij}{}^{\mu} \geq 1,\ \alpha^{\nu} + \alpha_{ij}{}^{\nu} \geq 1$ and $\alpha^{\omega} + \alpha_{ij}{}^{\omega} < 1$,

then $\left(\alpha_{ij}{}^{\mu}, \alpha_{ij}{}^{\nu}, \alpha_{ij}{}^{\omega}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} = (1,1,0)$.

Since $m_{ij} \geq n_{ij}$, $\alpha^{\mu} + n_{ij}{}^{\mu} \geq 1, \alpha^{\nu} + n_{ij}{}^{\nu} \geq 1 or\ \alpha^{\mu} + n_{ij}{}^{\mu} < 1, \alpha^{\nu} + n_{ij}{}^{\nu} < 1$.

Sub Case 1.1: If $\alpha^{\mu} + n_{ij}{}^{\mu} \geq 1, \alpha^{\nu} + n_{ij}{}^{\nu} \geq 1$ and $\alpha^{\omega} + n_{ij}{}^{\omega} < 1$,

then $\left(n^{T}, n^{I}, n^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = (1,1,0)$

Sub Case 1.2: If $\alpha^{\mu} + n_{ij}{}^{\mu} < 1, \alpha^{\nu} + n_{ij}{}^{\nu} < 1$ and $\alpha^{\omega} + n_{ij}{}^{\omega} > 1$

and $\alpha^{\mu} + n_{ij}{}^{\mu} < 1, \alpha^{\nu} + n_{ij}{}^{\nu} < 1$ and $\alpha^{\omega} + n_{ij}{}^{\omega} < 1$, then $\left(n_{ij}{}^{T}, n_{ij}{}^{I}, n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = (0,0,1)$.

In this case $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \geq \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$

Case 2: If $\alpha^{\mu} + n_{ij}^{\ \mu} \leq 1, \alpha^{\nu} + n_{ij}^{\ \nu} \leq 1$ and $\alpha^{\omega} + m_{ij}^{\ \omega} > 1$ and $\alpha^{\mu} + m_{ij}^{\ \mu} \leq 1, \alpha^{\nu} + m_{ij}^{\ \nu} \leq 1$ and $\alpha^{\omega} + m_{ij}^{\ \omega} < 1$, then $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} = (0,0,1)$.

Since $m_{ij} \geq n_{ij}$ gives $\alpha^{\mu} + n_{ij}^{\ \mu} \leq 1, \alpha^{\nu} + n_{ij}^{\ \nu} \leq 1$

$\Rightarrow \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = (0,0,1)$. In this case

$\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} = \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$.

Hence $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \geq \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$

(v) Similar to (iv)

(vi) We can write $M_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}$ and $M^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}$ in terms of $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}, M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}, M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$ and $M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$ as follows:

$M_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>} = M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} \cup M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$ and $M^{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>} = M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} \cup M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$.

Now from (i) we have $M_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>} \geq N_{<\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}>}$

(vii) From (ii) it is clear from the above.

(viii) **Case 1:** If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$,

then $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|} = \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$.

Now $\left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|} = \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$

when $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$

and $\left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|} = \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right) \leq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$

when $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \geq \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)$.

Case 2: If $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)\le\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$, then $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}^{T},n_{ij}^{I},n_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ gives $M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}\ge N_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$.

Case 3: This inequality is not valid when $\left(n_{ij}^{T},n_{ij}^{I},n_{ij}^{F}\right)$ is not comparable with $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ since the value of $\left(n_{ij}^{T},n_{ij}^{I},n_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(n_{ij}^{T},n_{ij}^{I},n_{ij}^{F}\right)$ when $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$ may be either $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ or $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)$ and in this case $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ is not comparable with $\left(n_{ij}^{T},n_{ij}^{I},n_{ij}^{F}\right)$.

Proposition 6.2. Consider any two elements $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right),\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)\in\left(NFM\right)$ such that $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\ge\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)$ and $M\in\left(NFM\right)_{mn}$. We have

(i) $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\le M^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}$

(ii) $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\le M_{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}$

(iii) $M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\le M_{\beta^{\mu},\beta^{\nu},\beta^{\omega}}$

(iv) $M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\ge M^{\left[\beta^{\mu},\beta^{\nu},\beta^{\omega}\right]}$

(v) $M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\le M_{\left[\beta^{\mu},\beta^{\nu},\beta^{\omega}\right]}$

(vi) $M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}\ge M_{\left|\beta^{\mu},\beta^{\nu},\beta^{\omega}\right|}$

Proof: (i) Consider any ijth element of $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$ as $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$.

Case1: When $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)\ge\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\ge\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)$

$\Rightarrow\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)\ge\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)$

$i.e.,\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(1,1,0\right)$

$=\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}$ when $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)\le\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)\le\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$, then

$\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(0,0,1\right)$ and $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(0,0,1\right)$ .Otherwise $\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)\le\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)\le\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ gives $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(1,1,0\right)$ .In this case, $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\le M^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}$

Case2: Suppose for some i j, $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)$ is not comparable to $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$. We have

$\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(0,0,1\right)$.

Moreover $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(0,0,1\right) or \left(1,1,0\right)$ .On the other hand, if $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)$ is not comparable to $\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)$, then

$\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(0,0,1\right)=\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}$

since $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\ge\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)$.

(ii) like(i).

(iii) like(i).

(iv) Since $\alpha^{\mu}\ge\beta^{\mu},\alpha^{\nu}\ge\beta^{\nu}$ and $\alpha^{\omega}<\beta^{\omega}$, $\alpha^{\mu}+m_{ij}^{T}\ge 1\Rightarrow\beta^{\mu}+m_{ij}^{T}\ge 1 or\ \beta^{\mu}+m_{ij}^{T}\le 1.$

$\alpha^{\nu}+m_{ij}^{I}\ge 1\Rightarrow\beta^{\nu}+m_{ij}^{I}\ge 1 or\ \beta^{\nu}+m_{ij}^{I}\le 1.$

Similarly

$\alpha^{\omega}+m_{ij}^{F}<1\Rightarrow\beta^{\omega}+m_{ij}^{F}\le 1 or\ \beta^{\omega}+m_{ij}^{F}>1$. Hence $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(1,1,0\right)$

$\Rightarrow\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(1,1,0\right)\ or \left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(0,0,1\right).$

When $\alpha^{\mu}+m_{ij}^{T}\le 1\Rightarrow\beta^{\mu}+m_{ij}^{T}\le 1,\alpha^{\nu}+m_{ij}^{I}\le 1\Rightarrow\beta^{\nu}+m_{ij}^{I}\le 1,$ it is clear

$\beta^{\omega}+m_{ij}^{F}\le 1 or\ \beta^{\omega}+m_{ij}^{F}\ge 1$.

Therefore, in this case, $\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(m_{ij}^{T},m_{ij}^{I},m_{ij}^{F}\right)^{\left(\beta^{\mu},\beta^{\nu},\beta^{\omega}\right)}=\left(0,0,1\right)$.

In general, $M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}\ge M^{\left[\beta^{\mu},\beta^{\nu},\beta^{\omega}\right]}$.

(v) From(iv) it is clear. (vi)Case 1:

If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \geq \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right)$ then

$\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|} = \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \geq \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|}$.

Case2: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$ then $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|} = \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$

and $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$

when $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$.

Also, $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$

and $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|} = \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right) < \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$

when $\left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right) < \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$.

From above we have $M_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|} \geq M_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|}$.

Case (3): If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ is not comparable with $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$ for any I,j then $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$.

Sub Case 3.1: If $m_{ij}^{T} < \alpha^{\mu}, m_{ij}^{I} < \alpha^{\nu}$ and $m_{ij}^{F} < \alpha^{\varpi}$ .Now $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|} = \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right)$ when $m_{ij}^{T} \geq \beta^{\mu}, m_{ij}^{I} \geq \beta^{\nu}$ and $m_{ij}^{F} < \beta^{\varpi}$ and $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ when $m_{ij}^{T} < \beta^{\mu}, m_{ij}^{I} < \beta^{\nu}$ and $m_{ij}^{F} < \beta^{\varpi}$ since $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \geq \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right)$.

Sub Case (3.2) If $m_{ij}^{T} \geq \alpha^{\mu}, m_{ij}^{I} \geq \alpha^{\nu}$ and $m_{ij}^{F} \geq \alpha^{\varpi}$, then $m_{ij}^{T} \geq \beta^{\mu}, m_{ij}^{I} \geq \beta^{\nu}$ and

$m_{ij}^{F} \geq \beta^{\varpi}\ or < \beta^{\varpi}$

. Therefore,

$\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right|} = \left(\beta^{\mu}, \beta^{\nu}, \beta^{\varpi}\right) or \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \leq \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left|\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right|}$.

Similarly,

we can prove the inequality when $\left(\beta^{\mu},\beta^{\nu},\beta^{\varpi}\right)$ is not comparable with $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$.

Hence, from the above three cases we have $M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right|}\geq M_{\left|\beta^{\mu},\beta^{\nu},\beta^{\varpi}\right|}$.

Proposition 6.3. For any two NFMs $M,N\in\left(NFM\right)_{mn}$, we have the following inequalities

(i) $\left(M\oplus N\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\geq M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\oplus N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$.

(ii) $\left(M\oplus N\right)_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\geq M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\oplus N_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$.

(iii) $\left(M\oplus N\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}\geq M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}\oplus N_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$.

(iv) $\left(M\oplus N\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\geq M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\oplus N^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}$.

(v) $\left(M\oplus N\right)_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\geq M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\oplus N_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}$.

Proof: (i) Consider the ijth element of $\left(M\oplus N\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$ is

$\left[\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\oplus\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\right]^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$ as $\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)$.

Now,

$$\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)=\left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\wedge 1,\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\wedge 1,\left(m_{ij}{}^{F}+n_{ij}{}^{F}-1\right)\vee 0\right]^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$$

$$=\begin{cases}(1,1,0) & if\left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\wedge 1,\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\wedge 1,\left(m_{ij}{}^{F}+n_{ij}{}^{F}-1\right)\vee 0\right]\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)\\(0,0,1) & if\left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\wedge 1,\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\wedge 1,\left(m_{ij}{}^{F}+n_{ij}{}^{F}-1\right)\vee 0\right]<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)\\ & otherwise\end{cases}$$

$$=\begin{cases}(1,1,0) & if\ m_{ij}{}^{T}+n_{ij}{}^{T}\geq\alpha^{\mu},m_{ij}{}^{I}+n_{ij}{}^{I}\geq\alpha^{\nu}\ and\ m_{ij}{}^{F}+n_{ij}{}^{F}-1<\alpha^{\varpi},\\(0,0,1) & if\ m_{ij}{}^{T}+n_{ij}{}^{T}<\alpha^{\mu},m_{ij}{}^{I}+n_{ij}{}^{I}<\alpha^{\nu}\ and\ m_{ij}{}^{F}+n_{ij}{}^{F}-1\geq\alpha^{\varpi},\\ & otherwise\end{cases}$$

Assume $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)$ as the ijth element of

$\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\oplus\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$ *i.e.,* $M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\oplus N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$.

Case (1) If $m_{ij}{}^{T}+n_{ij}{}^{T}\geq\alpha^{\mu}, m_{ij}{}^{I}+n_{ij}{}^{I}\geq\alpha^{\nu}$ *and* $m_{ij}{}^{F}+n_{ij}{}^{F}-1<\alpha^{\omega}$.

Sub Case 1.1: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(1,1,0)\oplus(1,1,0)$.

Sub Case 1.2: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(1,1,0)\oplus(0,0,1)=(1,1,0)$.

Sub Case 1.3: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(0,0,1)\oplus(1,1,0)=(1,1,0)$.

Sub Case 1.4: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(0,0,1)\oplus(0,0,1)=(0,0,1)$. In this case $\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)=(1,1,0)\geq\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)$.

Case 2: If $m_{ij}{}^{T}+n_{ij}{}^{T}<\alpha^{\mu}, m_{ij}{}^{I}+n_{ij}{}^{I}\geq\alpha^{\nu}$ *and* $m_{ij}{}^{F}+n_{ij}{}^{F}-1<\alpha^{\omega}$, since

$m_{ij}{}^{T}\leq\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\leq\alpha^{\mu}, n_{ij}{}^{T}\leq\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\leq\alpha^{\mu}$ and

$m_{ij}{}^{I}\leq\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\leq\alpha^{\nu}, n_{ij}{}^{I}\leq\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\leq\alpha^{\nu}$ and

$m_{ij}{}^{F}\geq m_{ij}{}^{F}+\left(n_{ij}{}^{F}-1\right)\geq\alpha^{\omega}, m_{ij}{}^{F}\geq m_{ij}{}^{F}+\left(m_{ij}{}^{T}-1\right)\geq\alpha^{\omega}, m_{ij}{}^{F}\geq m_{ij}{}^{F}+\left(m_{ij}{}^{I}-1\right)\geq\alpha^{\omega}$,

i.e., $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$,

and

$\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right), \left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(0,0,1)\oplus(0,0,1)=(0,0,1)=\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)$.

Case 3:

When $\left(m_{ij}{}^{T}+n_{ij}{}^{T}\right)\geq\alpha^{\mu},\left(m_{ij}{}^{I}+n_{ij}{}^{I}\right)\geq\alpha^{\nu}\ \left(m_{ij}{}^{F}+n_{ij}{}^{F}-1\right)\geq\alpha^{\omega},\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)=(0,0,1)$.

Sub Case 3.1: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)$ are not comparable to $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(0,0,1)$.

Sub Case 3.2: If for some i j either $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ or $\left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)$ is comparable

to $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$ and which is of the form

$\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) or \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$ then

$\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right) = (0,0,1).$

In this case there is no possibility for $m_{ij}^{T} \ge \alpha^{\mu}, m_{ij}^{I} \ge \alpha^{\nu}$ and $m_{ij}^{F} < \alpha^{\varpi}$ *or* $n_{ij}^{T} \ge \alpha^{\mu}, n_{ij}^{I} \ge \alpha^{\nu}$

and $n_{ij}^{F} < \alpha^{\varpi}$ since $m_{ij}^{F} + n_{ij}^{F} - 1 \ge \alpha^{\varpi}$ gives $m_{ij}^{F} \, and \, n_{ij}^{F} \ge \alpha^{\varpi}$.

Hence in this case $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right) = (0,0,1) = \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right).$

Case 4: Suppose $\left(m_{ij}^{T} + n_{ij}^{T}\right) \le \alpha^{\mu}, \left(m_{ij}^{I} + n_{ij}^{I}\right) \le \alpha^{\nu}$ and

$\left(m_{ij}^{F} + n_{ij}^{F} - 1\right) \le \alpha^{\varpi}$. Then $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right) = (0,0,1) = \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right),$

since both

$m_{ij}^{T} \, and \, n_{ij}^{T} \le \alpha^{\mu}, m_{ij}^{I} \, and \, n_{ij}^{I} \le \alpha^{\nu}.$

Hence, from the above four cases we can

conclude $(M \oplus N)^{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} \ge M^{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} \oplus N^{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)}$.

$$\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right) = \begin{cases} \left(m_{ij}^{T} + n_{ij}^{T}\right) \wedge 1, \left(m_{ij}^{I} + n_{ij}^{I}\right) \wedge 1, \left(m_{ij}^{F} + n_{ij}^{F} - 1\right) \vee 0 \\ if \left[\left(m_{ij}^{T} + n_{ij}^{T}\right) \wedge 1, \left(m_{ij}^{I} + n_{ij}^{I}\right) \wedge 1, \left(m_{ij}^{F} + n_{ij}^{F} - 1\right) \vee 0\right] \ge \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \\ (0,0,1) \;\; if \; m_{ij}^{T} + n_{ij}^{T} \ge 1, m_{ij}^{I} + n_{ij}^{I} \ge 1, m_{ij}^{F} + n_{ij}^{F} - 1 < 0 \\ \left(m_{ij}^{T} + n_{ij}^{T}\right), \left(m_{ij}^{I} + n_{ij}^{I}\right), \left(m_{ij}^{F} + n_{ij}^{F} - 1\right) \\ if \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \le \left[\left(m_{ij}^{T} + n_{ij}^{T}\right), \left(m_{ij}^{I} + n_{ij}^{I}\right), \left(m_{ij}^{F} + n_{ij}^{F} - 1\right)\right] \le (1,1,0) \\ (0,0,1) \; if \left[\left(m_{ij}^{T} + n_{ij}^{T}\right), \left(m_{ij}^{I} + n_{ij}^{I}\right), \left(m_{ij}^{F} + n_{ij}^{F} - 1\right)\right] < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \end{cases}$$

Case (1) If $m_{ij}^{T} + n_{ij}^{T} \ge 1, m_{ij}^{I} + n_{ij}^{I} \ge 1, m_{ij}^{F} + n_{ij}^{F} - 1 < 0$

Sub Case 1.1: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right) = (0,0,1).$

Sub Case 1.2: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \oplus (0,0,1) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$

Sub Case 1.3: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right) = \left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right).$

Sub Case 1.4: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=(0,0,1)$ and $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right)=(1,1,0) \geq \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)$.

Case (2) $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)<\left[\left(m_{ij}^{T}+n_{ij}^{T}\right),\left(m_{ij}^{I}+n_{ij}^{I}\right),\left(m_{ij}^{F}+n_{ij}^{F}-1\right)\right]<(1,1,0)$,

Sub Case 2.1: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=\left[\left(m_{ij}^{T}+n_{ij}^{T}\right),\left(m_{ij}^{I}+n_{ij}^{I}\right),0\right]$

Sub Case 2.2: $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \oplus (0,0,1)=\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$

Sub Case 2.3 $\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=\left(n_{ij}^{T}, n_{ij}^{I}, n_{ij}^{F}\right)$.

SubCase2.4

$$\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=(0,0,1)=\left[\left(m_{ij}^{T}+n_{ij}^{T}\right),\left(m_{ij}^{I}+n_{ij}^{I}\right),\left(m_{ij}^{F}+n_{ij}^{F}-1\right)\right] \geq \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right).$$

Case 3: $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right)=\left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)=(0,0,1)$.Hence, we conclude $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right) \geq \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F}\right)$, $(M \oplus N)_{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} \geq M_{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} \oplus N_{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)}$.

(iii) Proof is similar to (i) and (ii).

(iv) Assume $(M \oplus N)^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=O$ and $M^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]} \oplus N^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=P$. Now

$$M^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=\begin{cases}(1,1,0), \text{ if } \alpha^{\mu}+m_{ij}^{T} \geq 1, \alpha^{\nu}+m_{ij}^{I} \geq 1, \alpha^{\varpi}+m_{ij}^{F}<1, \\ (0,0,1), \text{ otherwise.}\end{cases}$$

$$N^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=\begin{cases}(1,1,0), \text{ if } \alpha^{\mu}+n_{ij}^{T} \geq 1, \alpha^{\nu}+n_{ij}^{I} \geq 1, \alpha^{\varpi}+m_{ij}^{F}<1, \\ (0,0,1), \text{ otherwise.}\end{cases}$$

$$(M \oplus N)^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right)^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}$$

$$=\begin{cases}(1,1,0), \text{ if } \alpha^{\mu}+m_{ij}^{T}+n_{ij}^{T} \geq 1, \alpha^{\nu}+m_{ij}^{I}+n_{ij}^{I} \geq 1, \alpha^{\varpi}+m_{ij}^{F}+n_{ij}^{F}<2, \\ (0,0,1), \text{ otherwise.}\end{cases}$$

Case 1: If $\alpha^{\mu}+m_{ij}^{T}+n_{ij}^{T} \geq 1, \alpha^{\nu}+m_{ij}^{I}+n_{ij}^{I} \geq 1, \alpha^{\varpi}+m_{ij}^{F}+n_{ij}^{F}<2$,

then $\left(o_{ij}^{T}, o_{ij}^{I}, o_{ij}^{F}\right)^{\left[\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right]}=(1,1,0)$.

Sub Case 1.1:

$\alpha^{\mu}+m_{ij}^{T} \geq 1, \alpha^{\mu}+n_{ij}^{T} \geq 1, \alpha^{\nu}+m_{ij}^{I} \geq 1, \alpha^{\nu}+n_{ij}^{I} \geq 1, \alpha^{\varpi}+m_{ij}^{F}<1, \alpha^{\varpi}+n_{ij}^{F}<1$. Now

$\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=(1,1,0)\oplus(1,1,0)=(1,1,0).$

Sub Case 1.2:

When $\alpha^{\mu}+m_{ij}{}^{T}\geq 1, \alpha^{\mu}+n_{ij}{}^{T}<1, \alpha^{\nu}+m_{ij}{}^{I}\geq 1, \alpha^{\nu}+n_{ij}{}^{I}<1, \alpha^{\varpi}+m_{ij}{}^{F}<1, \alpha^{\varpi}+n_{ij}{}^{F}>1,$

$\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=(1,1,0)\oplus(0,0,1)=(1,1,0).$

Sub Case 1.3:

When $\alpha^{\mu}+m_{ij}{}^{T}<1, \alpha^{\mu}+n_{ij}{}^{T}>1, \alpha^{\nu}+m_{ij}{}^{I}<1, \alpha^{\nu}+n_{ij}{}^{I}>1, \alpha^{\varpi}+m_{ij}{}^{F}\geq 1, \alpha^{\varpi}+n_{ij}{}^{F}<1,$

$\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=(0,0,1)\oplus(1,1,0)=(1,1,0).$

Sub Case 1.4:

$\alpha^{\mu}+m_{ij}{}^{T}<1, \alpha^{\mu}+n_{ij}{}^{T}<1, \alpha^{\nu}+m_{ij}{}^{I}<1, \alpha^{\nu}+n_{ij}{}^{I}<1, \alpha^{\varpi}+m_{ij}{}^{F}\geq 1, \alpha^{\varpi}+n_{ij}{}^{F}\geq 1,$

$\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=(0,0,1)\oplus(0,0,1)=(0,0,1).$ In this case $\left(o_{ij}{}^{T}, o_{ij}{}^{I}, o_{ij}{}^{F}\right)\geq\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right).$

Case 2: If

$\alpha^{\mu}+m_{ij}{}^{T}+n_{ij}{}^{T}\geq 1, \alpha^{\nu}+m_{ij}{}^{I}+n_{ij}{}^{I}\geq 1, \alpha^{\varpi}+m_{ij}{}^{F}+n_{ij}{}^{F}>2, or$

$\alpha^{\mu}+m_{ij}{}^{T}+n_{ij}{}^{T}<1, \alpha^{\nu}+m_{ij}{}^{I}+n_{ij}{}^{I}<1, \alpha^{\varpi}+m_{ij}{}^{F}+n_{ij}{}^{F}<2,$

$\left(o_{ij}{}^{T}, o_{ij}{}^{I}, o_{ij}{}^{F}\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}=(1,1,0).$ Since $\alpha^{\mu}+m_{ij}{}^{T}+n_{ij}{}^{T}<1\Rightarrow\alpha^{\mu}+m_{ij}{}^{T}<1$ and $\alpha^{\mu}+n_{ij}{}^{T}<1$ .Now whatever be the values of $\alpha^{\nu}+m_{ij}{}^{I}+n_{ij}{}^{I}<1\Rightarrow\alpha^{\nu}+m_{ij}{}^{I}<1$ the value of $\alpha^{\varpi}+m_{ij}{}^{F}+n_{ij}{}^{F}$ the value of $\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=(0,0,1)\oplus(0,0,1)=(0,0,1).$ In this case $\left(o_{ij}{}^{T}, o_{ij}{}^{I}, o_{ij}{}^{F}\right)=\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right).$ From the above two cases we conclude in general $(M\oplus N)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\geq M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}\oplus N^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}.$

(v) Similar to (iv).

Proposition 6.4. For any two NFMs $M, N\in(NFM)_{mn}$, we have

(i) $(M\,\square\,N)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\leq M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\,\square\,N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}.$

(ii) $(M\,\square\,N)_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\geq M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}\oplus N_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}.$

(iii) $(M\,\square\,N)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}\leq M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}\,\square\,N_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}.$

(iv) $\left(M \square N\right)^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \leq M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \square N^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$.

(v) $\left(M \square N\right)_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \leq M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \square N_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]}$.

Proof: (i) Consider the ijth element of

$$\left(M \square N\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} \left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}-1\right)\vee 0\left(m_{ij}{}^{I}+n_{ij}{}^{I}-1\right)\vee 0,\left(m_{ij}{}^{T}+n_{ij}{}^{F}\right)\wedge 1\right]^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}.$$

$$=\begin{cases}(1,1,0)\ \ if\left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}-1\right),\left(m_{ij}{}^{I}+n_{ij}{}^{I}-1\right),\left(m_{ij}{}^{F}+n_{ij}{}^{F}\right)\right]\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\\(0,0,1)\ \ if\left[\left(m_{ij}{}^{T}+n_{ij}{}^{T}-1\right),\left(m_{ij}{}^{I}+n_{ij}{}^{I}-1\right),\left(m_{ij}{}^{F}+n_{ij}{}^{F}-1\right)\right]<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\\\qquad otherwise\end{cases}$$

Case (1) If $m_{ij}{}^{T}+n_{ij}{}^{T}-1\geq\alpha^{\mu},m_{ij}{}^{I}+n_{ij}{}^{I}-1\geq\alpha^{\nu}$ $and\ m_{ij}{}^{F}+n_{ij}{}^{F}-1<\alpha^{\omega}$. then

$\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)=(1,1,0)$. $m_{ij}{}^{T}\geq m_{ij}{}^{T}+n_{ij}{}^{T}-1\geq\alpha^{\mu},m_{ij}{}^{I}\geq m_{ij}{}^{I}+n_{ij}{}^{I}-1\geq\alpha^{\nu}$ and

$m_{ij}{}^{F}<m_{ij}{}^{F}+n_{ij}{}^{F}<\alpha^{\omega}\Rightarrow\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$. Similarly,

$\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ gives

$\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\square N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=(1,1,0)\square(1,1,0)=(1,1,0)$.

Case 2: If $m_{ij}{}^{T}+n_{ij}{}^{T}-1<\alpha^{\mu},m_{ij}{}^{I}+n_{ij}{}^{I}-1<\alpha^{\nu}$ and $m_{ij}{}^{F}+n_{ij}{}^{F}>\alpha^{\omega}$ then $\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)=(1,1,0)$.

Sub Case 2.1: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then

$\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(1,1,0)\square(0,0,1)=(0,0,1)$.

Sub Case 2.2: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$

then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(0,0,1)$.

Sub Case 2.3: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $\left(n_{ij}{}^{T},n_{ij}{}^{I},n_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$

then $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)=(1,1,0)\oplus(1,1,0)=(1,1,0)$. In this case $\left(o_{ij}{}^{T},o_{ij}{}^{I},o_{ij}{}^{F}\right)\leq\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)$.

That is $\left(M \square N\right)^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\leq M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\square N^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$.

Case3: In dual way it is clear from of Proposition 6.3.

Case4: Like Proposition 6.3. Proofs of(ii) and (iii) evident from (i). In dual of Proposition 6 .3 we can easily prove (iv) and (v) of Proposition 6.4

Proposition 6.5. Let $M \in (NFM)_{mn}$, we have

(i) $(M^c)^{(\alpha^\mu,\alpha^\nu,\alpha^\omega)} \le \left[M^{(\alpha^\mu,\alpha^\nu,\alpha^\omega)}\right]^c$.

(ii) $(M^c)_{(\alpha^\mu,\alpha^\nu,\alpha^\omega)} \le \left[M_{(\alpha^\mu,\alpha^\nu,\alpha^\omega)}\right]^c$

(iii) $(M^c)_{\alpha^\mu,\alpha^\nu,\alpha^\omega} \ge \left[M_{\alpha^\mu,\alpha^\nu,\alpha^\omega}\right]^c$.

(iv) $(M^c)^{[\alpha^\mu,\alpha^\nu,\alpha^\omega]} \le \left[M^{[\alpha^\mu,\alpha^\nu,\alpha^\omega]}\right]^c$

(v) $(M^c)_{[\alpha^\mu,\alpha^\nu,\alpha^\omega]} \le \left[M_{[\alpha^\mu,\alpha^\nu,\alpha^\omega]}\right]^c$

Proof: (i) Case 1: $M = (m_{ij}{}^T, m_{ij}{}^I, m_{ij}{}^F) \Rightarrow M^c = (m_{ij}{}^F, m_{ij}{}^I, m_{ij}{}^T)$ and

$$(M^c)^{(\alpha^\mu,\alpha^\nu,\alpha^\omega)} = \begin{cases} (1,1,0) \ if \ (m_{ij}{}^F, m_{ij}{}^I, m_{ij}{}^T) \ge (\alpha^\mu,\alpha^\nu,\alpha^\omega) \\ (0,0,1) \ if \ (m_{ij}{}^F, m_{ij}{}^I, m_{ij}{}^T) < (\alpha^\mu,\alpha^\nu,\alpha^\omega) \end{cases}$$

when $m_{ij}{}^F \ge \alpha^\mu, m_{ij}{}^T < \alpha^\omega \Rightarrow m_{ij}{}^T < \alpha^\omega, m_{ij}{}^F \ge \alpha^\mu$ and

$$m_{ij}{}^F \ge \alpha^\nu, m_{ij}{}^T < \alpha^\omega \Rightarrow m_{ij}{}^I < \alpha^\omega, m_{ij}{}^F \ge \alpha^\nu$$

$$\Rightarrow (m_{ij}{}^T, m_{ij}{}^I, m_{ij}{}^F) < (\alpha^\omega,\alpha^\nu,\alpha^\mu) \Rightarrow (m_{ij}{}^T, m_{ij}{}^I, m_{ij}{}^F)^{(\alpha^\mu,\alpha^\nu,\alpha^\omega)} = (0,0,1)$$

$$\Rightarrow (m_{ij}{}^T, m_{ij}{}^I, m_{ij}{}^F)^{(\alpha^\mu,\alpha^\nu,\alpha^\omega)} = \left[M^{(\alpha^\omega,\alpha^\nu,\alpha^\mu)}\right]^c = (0,0,1)^c = (1,1,0).$$ Therefore

$$(M^c)^{(\alpha^\omega,\alpha^\nu,\alpha^\mu)} = \left(M^{(\alpha^\omega,\alpha^\nu,\alpha^\mu)}\right)^c$$ when

$$(m_{ij}{}^F, m_{ij}{}^I, m_{ij}{}^T) < (\alpha^\mu,\alpha^\nu,\alpha^\omega) \Rightarrow m_{ij}{}^F < \alpha^\mu, m_{ij}{}^F < \alpha^\nu$$ and

$$m_{ij}{}^T \ge \alpha^\omega, m_{ij}{}^I \ge \alpha^\omega \Rightarrow m_{ij}{}^T \ge \alpha^\omega, m_{ij}{}^F < \alpha^\mu, m_{ij}{}^F < \alpha^\nu$$

$$\Rightarrow \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)$$

$$\Rightarrow \left[\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right]^{c} = (1,1,0)$$

$$\Rightarrow \left(M^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right)^{c} = (0,0,1).$$

Case2: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ is no t comparable to $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$ and $\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right)$ is comparable then from case(1) we have $\left(M^{c}\right)^{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} = \left(M^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right)^{c}$. Suppose $\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right)$ is also not comparable then $\left(M^{c}\right)^{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} = (0,0,1)$. But $\left(M^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right)^{c} = (0,0,1) \, or \, (1,1,0)$.

Hence $\left(M^{c}\right)^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)} \leq \left(M^{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right)^{c}$.

(ii) $\left(M^{c}\right)_{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} = \begin{cases} \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \; if \; \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \\ (0,0,1) \; otherwise. \end{cases}$

when $\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)$

$\Rightarrow m_{ij}^{F} \geq \alpha^{\mu}, m_{ij}^{T} < \alpha^{\varpi} \Rightarrow m_{ij}^{T} < \alpha^{\varpi}, m_{ij}^{F} \geq \alpha^{\mu}$ and

$\Rightarrow \left[\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right]^{c} = (0,0,1)^{c} = (1,1,0)$ when

$\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \Rightarrow m_{ij}^{F} < \alpha^{\mu}, m_{ij}^{T} > \alpha^{\varpi}$

$\Rightarrow \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \Rightarrow \left[\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right]^{c} = \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right).$

Hence $\left(M^{c}\right)_{\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right)} \leq \left[\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\left(\alpha^{\varpi}, \alpha^{\nu}, \alpha^{\mu}\right)}\right]^{c}$. For in comparable entries the proof is similar to (i).

(iii) Case(i) $\left(M^{c}\right)_{\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}} = \begin{cases} (1,1,0) \; if \; \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \\ \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \; if \; \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\varpi}\right) \end{cases}$

when $\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$

$\Rightarrow \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\omega}, \alpha^{\nu}, \alpha^{\mu}\right)$

$\Rightarrow \left[M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\right]^{c} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)^{c} = \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right)$

When $\left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \Rightarrow \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) > \left(\alpha^{\omega}, \alpha^{\nu}, \alpha^{\mu}\right)$

$\Rightarrow \left[M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\right]^{c} = (1,1,0)^{c} = (0,0,1).$

Case 2: $\left(M^{c}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right)$ but $\left[M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\right]^{c} = (0,0,1)\, or \left(m_{ij}^{F}, m_{ij}^{I}, m_{ij}^{T}\right)$. Hence $\left(M^{c}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} \geq \left[M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\right]^{c}$. Proofs of (iv) and (v) are like (i) and (ii).

7. **M and M^T are Comparable**

Definition 7.1. When the matrices M and M^T are comparable for any NFM $M \in (NFM)_{mn}$, we define

$$\Delta_1 M = \begin{cases} (1,1,0)\, if\ \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right) \\ \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)\ if \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right), \end{cases}$$

and $\nabla_1 M = M \vee M^{T}$.

Proposition 7.1. Let $M \in (NFM)_{mn}$, we have

(i) $\Delta_1 M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left[\Delta_1 M\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$.

(ii) $\Delta_1 \Delta M = \Delta \Delta_1 M$.

(iii) $\left[M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left[M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}\right]_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} = M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}$

Proof: **(i)Case1:** If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ then $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) = (1,1,0)$ and $\left[\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$.

Sub Case1.1: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$ then $M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$ and $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$ since $(1,1,0) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$.

Sub Case1.2: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$ then

$\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ and $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$ since $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$.

Sub Case1.3: For some i jth entries of the matrix M are not comparable with $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$, we have $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ and $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$.

In this case $\left[\Delta_1 M\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \Delta_1 M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$.

The above equality is also true when $\left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right) = (0,0,1)$.

Case2: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \leq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ then $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ and $\left[\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$

$$= \begin{cases} (1,1,0) \; if \; \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \\ \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \; otherwise. \end{cases}$$

Sub Case2.1: When $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right), \Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0)$ since $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \leq \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \leq \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$.

SubCase2.2:

When $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right), \Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ since $\left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = (1,1,0) \, if \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \leq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right) \leq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ and $\left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ if $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \leq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right) \leq \left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$. In this case $\left[\Delta_1 M\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}} = \Delta_1 M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$.

Case3: When $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ and $\left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ are not comparable to $\left(\alpha^{\mu}, \alpha^{\nu}, \alpha^{\omega}\right)$. $\Delta_1\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ gives

$\left[\Delta_1 M\right]_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)} = \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)$.Also

$\left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}} = \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right) \geq \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) = \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$.

Hence $\Delta_1 M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}} = \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)$. If $\left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)$is comparable

when$\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)$is not comparable with$\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)$then the value

of$\left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$ may be either$(1,1,0)\, or \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)$ which are greater

than$\left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right) = \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$. From the above we

conclude $\Delta_1 M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}} = \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) = \left[\Delta_1 M\right]_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$.

(ii)Now we must prove$(\Delta\Delta_1)M = (\Delta_1\Delta)M$.

Case1: If $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) \geq \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)$ then

$\Delta M = \left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right)$. $\Delta\Delta_1 M = \Delta\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) = (1,1,0)$. Now $\Delta_1 M = (1,1,0)$ and

$\Delta\Delta_1 M = \Delta(1,1,0) = (1,1,0)$.

Since $(1,1,0) \geq \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)\Delta\Delta_1 = \Delta\Delta_1$.

Case2: If $\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) < \left(m_{ji}{}^{T}, m_{ji}{}^{I}, m_{ji}{}^{F}\right)$, then $\Delta M = (0,0,1)$ and $\Delta_1\Delta M = \Delta_1(0,0,1) = (0,0,1)$ and $\Delta_1\Delta M = \Delta_1(0,0,1) = (0,0,1)$ gives

$\Delta\Delta_1 M = \Delta\left(m_{ij}{}^{T}, m_{ij}{}^{I}, m_{ij}{}^{F}\right) = (0,0,1)$. From the above two cases $\Delta\Delta_1 = \Delta_1\Delta$.

Corollary7.1. For a NFM $M \in (NFM)_{mn}$, we have

(i) M is reflexive or symmetric $\Rightarrow M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}, M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}, M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}, M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}}$ and $M^{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right]}$.

(ii) M is reflexive and $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)$ is not equal to $(0,0,1) \Rightarrow M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}, M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\varpi}\right)}$

$, M_{\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}}$ are irreflexive.

(iii) U is irreflexive and $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ is not equal to $(1,1,0)$ are irreflexive.

8. Representation and Decomposition of an NFM

Using $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ cuts defined in section 7, any NFM can be represented as a linear combination of their cuts. In the same manner we can decompose a NFM using some $\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ cuts.

Proposition 8.1 For a NFM $M \in (NFM)_{mn}$ and S = {elements of M}. Then, the following results hold

(i) $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)} \vee M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$

(ii) $M = \bigcup_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\in S} \left\{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right) \wedge M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right\}.$

(iii) $M = \bigcup_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\in S} \left\{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right) \wedge M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right\}.$

(iv) $M = [\Delta_1 M] \wedge [\nabla_1 M].$

(v) $M = M_{\left[\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right]} \vee M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}.$

Proof: The proofs of (i) to (iii) are clear from their definitions.

$$\Delta_1 M = \begin{cases} (1,1,0) \, if \, \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right), \\ \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) if \, \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right) \end{cases}$$

$$\nabla_1 M = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \vee \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$$

Case 1: If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) \geq \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ then $\Delta_1 M = (1,1,0)$ and $\nabla_1 M = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$.Hence $M = [\Delta_1, M] \wedge [\nabla_1 M] = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$.

Case 2 If $\left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right) < \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$ then $\Delta_1 M = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$ and $\nabla_1 M = \left(m_{ji}^{T}, m_{ji}^{I}, m_{ji}^{F}\right)$. Hence $M = [\Delta_1 M] \wedge [\nabla_1 M] = \left(m_{ij}^{T}, m_{ij}^{I}, m_{ij}^{F}\right)$.So

$M = [\Delta_1 M] \wedge [\nabla_1 M].$

$$M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\begin{cases}\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right) if \ \left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right),\\ \left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right) if \ \left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\end{cases}$$

Case 1: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)\geq\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ then $M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ and $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$. Hence $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\vee M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$.

Case 2: If $\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)<\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)$ or both are incomparable, then

$M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$ and $M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}=\left(0,0,1\right)$. Hence

$M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\vee M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}=\left(m_{ij}{}^{T},m_{ij}{}^{I},m_{ij}{}^{F}\right)$. Therefore $M=M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\vee M_{\left|\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right|}$.

We illustrate the above by an example as follows.

8.1 Consider a NFM

$$M=\begin{bmatrix}<0.6,0.4,0.3> & <0.4,0.4,0.2> & <0,0,1>\\ <0.2,0.4,0.3> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0.2,0.4,0.5> & <0.1,0.4,0.2>\end{bmatrix}$$

$$S=\begin{Bmatrix}<0.6,0.4,0.3>,<0.4,0.4,0.2>,<0,0,1>,<0.2,0.4,0.3>,<1,0,0>,\\ <0.7,0.4,0.1>,<0.5,0.4,0.2>,<0.2,0.4,0.5>,<0.1,0.4,0.2>\end{Bmatrix}$$

$$M_{<0.6,0.4,0.3>}=\begin{bmatrix}<0.6,0.4,0.3> & <0,0,1> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0.7,0.4,0.1>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<0.4,0.4,0.2>}=\begin{bmatrix}<0,0,1> & <0.4,0.4,0.2> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<0,0,1>}=\begin{bmatrix}<0.6,0.4,0.3> & <0.4,0.4,0.2> & <0,0,1>\\ <0.2,0.4,0.3> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0.2,0.4,0.5> & <0.1,0.4,0.2>\end{bmatrix}$$

$$M_{<0.2,0.4,0.3>}=\begin{bmatrix}<0.6,0.4,0.3> & <0.4,0.4,0.2> & <0,0,1>\\ <0.2,0.4,0.3> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<1,1,0>}=\begin{bmatrix}<0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0,0,1>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<0.7,0.4,0.1>}=\begin{bmatrix}<0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0.7,0.4,0.1>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<0.5,0.4,0.2>}=\begin{bmatrix}<0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$M_{<0.2,0.4,0.5>}=\begin{bmatrix}<0.6,0.4,0.3> & <0.4,0.4,0.2> & <0,0,1>\\ <0.2,0.4,0.3> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0.2,0.4,0.5> & <0.1,0.4,0.2>\end{bmatrix}$$

$$M_{<0.1,0.4,0.2>}=\begin{bmatrix}<0,0,1> & <0.4,0.4,0.2> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0.7,0.4,0.1>\\ <0.5,0.4,0.2> & <0,0,1> & <0.1,0.4,0.2>\end{bmatrix}$$

$$<0.6,0.4,0.3>\wedge M_{<0.6,0.4,0.3>}=\begin{bmatrix}<0.6,0.4,0.3> & <0,0,1> & <0,0,1>\\ <0,0,1> & <0.6,0.4,0.3> & <0.6,0.4,0.3>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$<0.4,0.4,0.2>\wedge M_{<0.4,0.4,0.2>}=\begin{bmatrix}<0,0,1> & <0.4,0.4,0.2> & <0,0,1>\\ <0,0,1> & <0.4,0.4,0.2> & <0.4,0.4,0.2>\\ <0.4,0.4,0.2> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$<0,0,1>\wedge M_{<0,0,1>}=\begin{bmatrix}<0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$<0.2,0.4,0.3>\wedge M_{<0.2,0.4,0.3>}=\begin{bmatrix}<0.2,0.4,0.3> & <0.2,0.4,0.3> & <0,0,1>\\ <0.2,0.4,0.3> & <0.2,0.4,0.3> & <0.2,0.4,0.3>\\ <0.2,0.4,0.3> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$<1,1,0>\wedge M_{<1,1,0>}=\begin{bmatrix}<0,0,1> & <0,0,1> & <0,0,1>\\ <0,0,1> & <1,0,0> & <0,0,1>\\ <0,0,1> & <0,0,1> & <0,0,1>\end{bmatrix}$$

$$<0.7,0.4,0.1> \wedge M_{<0.7,0.4,0.1>} = \begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0.7,0.4,0.1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$$

$$<0.5,0.4,0.2> \wedge M_{<0.5,0.4,0.2>} = \begin{bmatrix} <0,0,1> & <0.5,0.4,0.2> & <0.5,0.4,0.2> \\ <0.5,0.4,0.2> & <0,0,1> & <0,0,1> \\ <0.5,0.4,0.2> & <0.5,0.4,0.2> & <0,0,1> \end{bmatrix}$$

$$<0.2,0.4,0.5> \wedge M_{<0.2,0.4,0.5>} = \begin{bmatrix} <0.2,0.4,0.5> & <0.2,0.4,0.5> & <0,0,1> \\ <0.2,0.4,0.5> & <0.2,0.4,0.5> & <0.2,0.4,0.5> \\ <0.2,0.4,0.5> & <0.2,0.4,0.5> & <0,0,1> \end{bmatrix}$$

$$<0.1,0.4,0.2> \wedge M_{<0.1,0.4,0.2>} = \begin{bmatrix} <0,0,1> & <0.1,0.4,0.2> & <0,0,1> \\ <0,0,1> & <0.1,0.4,0.2> & <0.1,0.4,0.2> \\ <0.1,0.4,0.2> & <0,0,1> & <0.1,0.4,0.2> \end{bmatrix}$$

Now

$$\bigcup_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\in S}\left[M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right] = \bigcup_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\in S}\left[\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\wedge M_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right]$$

$$= M = \begin{bmatrix} <0.6,0.4,0.3> & <0.4,0.4,0.2> & <0,0,1> \\ <0.2,0.4,0.3> & <1,0,0> & <0.7,0.4,0.1> \\ <0.5,0.4,0.2> & <0.2,0.4,0.5> & <0.1,0.4,0.2> \end{bmatrix}$$

Similarly, we can verify $M = \bigcup_{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\in S}\left[\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)\wedge M^{\left(\alpha^{\mu},\alpha^{\nu},\alpha^{\omega}\right)}\right]$

9.conclusion and Future Direction

In conclusion, this work contributes to the theoretical development of Neutrosophic Fuzzy Matrices (NFM) by introducing and examining various types of cuts on Neutrosophic Fuzzy Sets (NFS). We have explored the properties of these cuts and their interactions with other operators, offering valuable insights into their potential applications in fuzzy matrix operations. The representations and decompositions of NFM using these cuts provide new methods for addressing uncertainty and imprecision, which are essential in many real-world decision-making processes. The inclusion of counterexamples has further clarified the findings and demonstrated the practical implications of our approach.

For future work, we aim to explore the extension of these cuts to more complex forms of Neutrosophic Fuzzy Matrices, such as those involving interval-valued or intuitionistic fuzzy sets. Additionally, the integration of these techniques into multi-criteria decision-making models, optimization problems, and other real-world applications could further enhance their utility.

Investigating the computational efficiency of the proposed methods and developing algorithms for their implementation would also be an important direction for future research. Finally, a deeper analysis of the properties of higher-order decompositions and their impact on decision-making processes in uncertain environments would be valuable for advancing the field.

In previous studies, the authors only discussed the decomposition of fuzzy matrix conditions, which deal with the truth membership function only. **In this work,** we apply the decomposition of neutrosophic fuzzy matrix conditions, which consider truth, indeterminacy, and falsity values.

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