Neutrosophic Sets and Systems, Vol. 86, 2025
University of New Mexico

Min(Max) – Min(Max) – Max(Min) (∗): Compositions of Neutrosophic Fuzzy Matrices and its Application in Medical Diagnosis

T. Harikrishnan[1], M. Anandhkumar[2], S. Prathap[3], S. Subramanian[4] , D. Ramesh[5], M.Raji[6]

[1]Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram, Tamilnadu, India

mokshihari2009@gmail.com

[2]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India

anandhkumarmm@gmail.com

[3]Department of Mathematics, Panimalar Engineering College, Chennai, Tamil Nadu, India.

prathapmaths@gmail.com

[4]Department of Mathematics, R.M.D. Engineering College, Kavarapettai 601 207

sssubramanian87@gmail.com

[5]Department of Engineering Mathematics,

College of Engineering, Koneru Lakshmaiah Educational Foundation, Vaddeswaram, Andhra Pradesh-522302, India

ram.fuzzy@gmail.com

[6]Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies,Pallavaram, Chennai

rajialagumurugan@gmail.com

Abstract: In this article, we introduce two types of compositions in Neutrosophic Fuzzy Matrices (NFM): max(min) - max(min) - min(max) (∘) and min(max) – min(max) – max(min) (∗) compositions. Using these compositions, we derive several significant results. Additionally, we demonstrate how to construct idempotent NFM from any given NFM through the min(max) – min(max) – max(min) (∗) composition. To demonstrate the application of these theoretical findings, a numerical example is provided. Furthermore, we present an algorithm designed to solve decision-making (DM) problems using the min(max) – min(max) – max(min) (∗) composition. A practical example is included to highlight the success of the proposed method. Additionally, a practical problem from our local environment was identified and successfully resolved. A thorough comparative analysis of the solutions for existing problems was conducted to evaluate their effectiveness in detail. We have adopted the same example as presented in the composition method used in Mamoni Dhar [50] is max(min)-min(max)-min(max) (∗). In our analysis, we replaced the score function used in that paper with various alternative score functions in Sahin [44], Nancy, Garg [45]. Upon comparing the results, we observed variations when different score functions were applied. Furthermore, when the min(max) – min(max) – max(min) (∗) composition method (proposed method) was applied, the results remained consistent. Finally, our proposed composition method, min(max) – min(max) – max(min) (∗) is better than Mamoni Dhar [50] composition method, max(min)-min(max)-min(max) (∗).

Keywords: Neutrosophic Fuzzy Set , Neutrosophic Fuzzy Matricrs, nearly irreflexive, weakly reflexive, Idempotent matrix

1. Introduction

In recent decades, the concept of fuzziness has expanded beyond the classical framework to better model uncertainty and vagueness in complex systems. The seminal work by Zadeh[1] on fuzzy sets laid the groundwork for this field, with further contributions by Atanassov[14] introducing intuitionistic fuzzy sets (NFS) , which account for both membership and non-membership values. Intuitionistic fuzzy sets have since evolved, influencing a wide range of applications[15,16]. However, limitations in fuzzy set theory, especially in addressing complex decision-making under uncertain conditions, led researchers like Molodtsov[2] to propose soft set theory as an alternative framework, thus broadening the conceptual landscape. Building on this, Maji et al. [3,4] further extended fuzzy and soft set theories to create fuzzy soft sets, integrating elements of fuzziness into the soft set structure to enhance decision-making applications with the need for even greater flexibility in managing indeterminacy, NFS have been developed by Smarandache [6,7]. These sets generalize traditional fuzzy and NFS by adding an indeterminate component, enabling a broader range of applications that can address not only uncertain but also inconsistent information. This has spurred advancements in both theoretical and applied areas, such as decision making Maji [9], with Deli and Broumi[11] pioneering neutrosophic soft matrices and NSM-DM, a methodology that allows for handling incomplete and contradictory data Das [12].

Within the framework of fuzzy and neutrosophic theories, fuzzy matrices have also been extensively researched. Kim and Roush [27] explored generalized fuzzy matrices, while others like Ragab and Emam[30] examined specific properties, such as the determinant and adjoint of FM. The work on FM continued with investigations into transitivity and canonical forms, as proposed by researchers like Hashimoto [22] and Kim [33], who provided foundational insights into matrix operations under fuzzy conditions. Additionally, studies have explored extensions of these matrices into the intuitionistic and neutrosophic realms. Emam [41], for instance, examined consistent and weak transitive intuitionistic fuzzy matrices, highlighting their applicability in complex decision processes. The evolution of fuzzy and intuitionistic theories has also been marked by the introduction of advanced structures, such as k-idempotent neutrosophic fuzzy matrices, which allow for novel approaches in matrix characterization and generalization. Anandhkumar et.al [38] have contributed significantly to this area, focusing on partial orderings and pseudo similarity in neutrosophic fuzzy matrices Eman [39]. In parallel, efforts have been made to refine and apply these structures in various contexts, such as soft and hypersoft sets for decision-making applications, as explored by Smarandache [6] and Vellapandi and Gunasekaran [8], enabling strategies for optimal decision processes under conditions of high uncertainty.

Several investigators have contributed significantly to the advancement of fuzzy and NFM and their applications. For instance, Emam and Fndh [42] investigated various properties of the max–min and min–max compositions of bifuzzy matrices, offering valuable insights into their mathematical structures. In the context of neutrosophic sets, Singh and Bhat [43] introduced a novel score and accuracy function, demonstrating its utility in MCDM scenarios. Similarly, Sahin [44] proposed a MCDM approach based on score functions (SF) and accuracy functions (AF) within a neutrosophic framework . Garg and Nancy [45] further refined these concepts by developing an improved score function for ranking NFS, which was successfully applied to DM problems. Recent advancements in neutrosophic fuzzy matrices have expanded their applications and theoretical underpinnings. Anandhkumar et.al [46] offered the concept of interval-valued secondary k-range symmetric NFM, providing a robust framework for representing and analyzing uncertainty. Anandhkumar et.al [47] extended this work to generalized symmetric neutrosophic fuzzy matrices, enhancing the mathematical versatility of these models. Additionally, Anandhkumar [48] explored the properties of generalized symmetric Fermatean neutrosophic fuzzy matrices, further contributing to this growing field. Moreover, Anandhkumar et.al [49] proposed secondary k-column neutrosophic fuzzy matrices, introducing new dimensions to the study of neutrosophic fuzzy matrix compositions. Mamoni Dhar [50] has studied NSM and Its Application in Medical Diagnosis (AMD).

1.1 **Abbreviations**

FM:Fuzzy Matrices
NFSs: Neutrosophic fuzzy Sets
NFM: Neutrosophic fuzzy matrices.
MCDMP: Multi-Criteria Decision- Making Problem.

1.2 The structure of the article is organized as follows

The article begins with Section 2: Main Contributions of the Work, highlighting the development of compositions of Neutrosophic Fuzzy Matrices (NFMs) to enhance uncertainty modeling and address existing methodological limitations. Section 3: Research Gap identifies critical shortcomings in handling complex compositions and relationships, which the proposed framework effectively resolves. Section 4: Comparative Analysis of the NFM Model with Existing Soft Models demonstrates the superiority of the NFM model over traditional soft computing approaches in terms of accuracy, adaptability, and efficiency. Section 5: Novelty emphasizes the unique capabilities of the framework in advancing the representation of uncertain compositions. Section 6: Literature Review justifies the study by examining related works, while Section 7: Preliminaries introduces key concepts such as neutrosophic sets, fuzzy matrices, and the composition framework. Section 8: Theorems and Results presents theoretical developments, proofs, and findings, complemented by an Example section that offers a step-by-step practical illustration. Section 9: Algorithm and Application outlines a systematic algorithm and its use in real-world problems. Finally, Section 10: Conclusion and Future Work summarizes the key findings and proposes integrating the framework into dynamic systems and machine learning models in future research.

2. Main Contribution of our Work:

Introduction of New Compositions for Neutrosophic Fuzzy Matrices (NFMs): We define two novel compositions for NFMs—namely, the max(min)–max(min)-min(max) (◦) and min(max) – min(max) – max(min) (∗) compositions. These operations are designed to facilitate enhanced manipulation and analysis of NFMs, thereby broadening the theoretical framework for neutrosophic fuzzy logic.

Development of Idempotent Neutrosophic Fuzzy Matrices: We construct idempotent NFMs through the min-min-max composition. This provides a systematic approach for deriving consistent and self-reinforcing NFMs, crucial for stable applications within neutrosophic-based systems.

Theoretical Insights and Key Results: Through these newly defined compositions, we derive several important results that contribute to the foundational understanding of neutrosophic fuzzy matrices. These results offer insights into the structure and behavior of NFMs under the proposed compositions.

Algorithm for Decision-Making (DM) Applications: We present an algorithm that leverages NFMs to solve DMP. This algorithm is designed to handle the indeterminacy and uncertainty present in complex decision environments.

Practical Application and Numerical Example: To establish the efficiency and success of the proposed compositions and algorithm, we include a arithmetical illustration demonstrating their ADM context. This example showcases the practicality of NFMs in addressing real-world decision-making scenarios.

3. Research Gap

In recent years, fuzzy and neutrosophic matrix theories have made significant progress, offering valuable tools for addressing uncertainty and imprecision in mathematical modeling and decision-making contexts. However, despite advancements in fuzzy matrices and their applications, several gaps remain that limit their effectiveness and broader applicability.

For instance, Ragab and Emam's work on min-max composition in FM [21] and Mishref and Emam's study on transitivity and subinverses in FM [29] contribute foundational concepts but do not address the limitations of composition operations under higher levels of indeterminacy.

Furthermore, the existing methods often fail to provide robust handling of transitivity when extended to neutrosophic or intuitionistic fuzzy matrices.
Recent studies, such as Uma, Murugadas, and Sriram's development of Fuzzy NSM of Type I and Type II [36], have demonstrated the adaptability of NFM structures. However, the lack of a comprehensive framework for examining the relationships and properties within these matrices under different composition rules restricts their practical utility in complex, real-world decision environments.

Additionally, Emam's research on operations within intuitionistic fuzzy matrices[39], as well as consistent and weak transitivity max-min and min-max conditions [42], indicate a need for further development of transitivity frameworks that can accommodate multi-level uncertainty and provide a unified structure for matrix-based decision analysis. The work of Pradhan and Pal[40] on strongly transitive intuitionistic fuzzy matrices suggests promising avenues, yet integrating these approaches within a consistent neutrosophic or fuzzy-neutrosophic framework remains an open challenge. We identified this gap and defined new types of Neutrosophic Fuzzy Matrices, introducing the max(min)–max(min)-min(max) ($\circ$) and min(max) – min(max) – max(min) ($*$) compositions of these matrices. Through these compositions, we derived several critical results that expand the theoretical foundation of NFMs. Notably, we constructed idempotent Neutrosophic Fuzzy Matrices from any given matrix using the min-min-max composition, adding an essential tool for achieving consistency within NFM-based models. To demonstrate these concepts and their practical applications, we included a numerical example.

Table:1 An overview of the advancements in extending NFM.

References	Extension of NFM.	**Year**
E.G. Emam et al. [21]	On the min-max composition of FM	1995
E.G. Emam, et ai [42]	Some results associated with the max–min and min–max compositions of bifuzzy matrices,	2016
Madhumangal Pal, et al [40]	Transitive and strongly transitive IFM	2017
E. G. Emama [39]	An Operation on IFM	2020
E. G. Emam [41]	On Consistent and Weak Transitive IFM	2022
P. Murugadas **Proposed**	Min(Max) – Min(Max) – Max(Min) ($*$) Compositions of NFM and its AMD	2024

4. **Comparative of NFM model with the existing soft models**

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy

FSS [3]	✓	×	×	×
IVFSS [16]	✓	×	×	×
IFSS [15]	✓	✓	✓	×
IVIFSS [46]	✓	✓	✓	×
NSS [7]	✓	✓	×	✓

Aspect	**FS**	**IFS**	**NFS**
Components	Membership degree μ	$T\mu$ and $F\ \nu$	$T\mu$, I_u and $F\nu$
Range of Values	$\mu \in [0,1]$	$\mu,\nu \in [0,1]$ with $0 \le \mu$ $\mu+\nu \le 1$	$\mu,I,\nu \in [0,1]$ with $0 \le \mu+I+\nu \le 3$
Handling of Uncertainty	Can only handle partial membership	Can model uncertainty to some extent with membership and non-membership	Explicitly models uncertainty with the indeterminacy component
Representation	Single value for membership degree μ.	Pair of values for each element: (μ,ν).	Triple of values for each element: (μ,I,ν).
Information Flexibility	Limited; only considers degree of membership	Improved flexibility with membership and non-membership	Maximum flexibility; considers T, I, and F.
Applications	Simple systems with partial truth	Systems where both acceptance and rejection are relevant	Complex, uncertain systems with significant indeterminacy
Strengths	Simplicity and ease of computation	Manages dual aspects of ambiguity	Comprehensive uncertainty representation, handling of complex systems with indeterminate information

5. Novelty

The referenced works present significant advancements across fuzzy sets, soft sets, IFS, and NFM, each addressing specific limitations and expanding the theoretical and practical frameworks for handling uncertainty. Zadeh [1] introduced fuzzy set theory, laying the groundwork for modeling imprecision in various systems. Atanassov [14-15] advanced this by developing intuitionistic fuzzy sets, which introduced an added degree of hesitation. Later, Molodtsov [2] proposed soft set theory to address uncertainties without requiring traditional fuzzy membership functions, which Maji et al. [3-5] further refined by integrating fuzzy and soft set theory. Smarandache [6-7] took this forward by introducing hypersoft and neutrosophic sets, offering tools to model multi-dimensional uncertainty.

Building on these foundations, Vellapandi and Gunasekaran [8] applied multi-soft set logic to decision-making, while Das et al. [12] explored neutrosophic soft matrices for collaborative decision-making. Emam and Mishref [29] examined transitivity and subinverses in FM, enhancing the understanding of composition rules, while Ragab and Emam [21, 30] provided insights into the min-max compositions crucial for matrix operations. Deli and Broumi [11] extended decision-making methodologies with neutrosophic soft matrices, and Emam with Fndh [42] explored advanced bifuzzy matrix compositions, offering valuable results for complex fuzzy environments.

Further studies by Padder and Murugadas [35] and Uma et al. [36] on idempotent intuitionistic and neutrosophic matrices have contributed significantly to matrix consistency and structural advancements in data analysis. Recent studies by Anandhkumar et al. [37-38] on k-idempotent and pseudo-similarity operations in neutrosophic matrices have introduced novel approaches for matrix ordering and comparative analysis. This paper builds upon these contributions by defining new Neutrosophic Fuzzy Matrix (NFM) compositions, specifically the max(min)–max(min)-min(max) and min(max) – min(max) – max(min) ($*$) operations. These new compositions aim to provide essential theoretical insights and demonstrate practical applications for decision-making, thereby addressing a critical gap in the field’s understanding of NFM structures and their consistency within model-based applications.

6. **Literature review**

In the study of uncertainty modeling, fuzzy sets have laid the foundational framework since Zadeh's pioneering work [1], which introduced the concept of partial membership to describe imprecise information. This led to further exploration by Atanassov [14-15], who developed intuitionistic fuzzy sets (IFSs) to include a degree of hesitation, thereby broadening the scope of fuzzy systems. Subsequent research aimed to address the limitations of IFSs, leading to advancements like interval-valued fuzzy sets [16], which provided extended flexibility in representing uncertainty. Molodtsov [2] introduced soft set theory, which enables more flexible solutions by allowing parameters without fixed membership degrees, providing a valuable tool for real-world decision-making. This was expanded by Maji et al. [3-4], who developed fuzzy soft sets to

combine fuzzy logic and soft set principles, enhancing their application potential in ambiguous environments.
Smarandache [6-7] advanced these concepts further with neutrosophic sets, which generalize IFSs and incorporate three components: T, I, F. This neutrosophic approach has provided a robust framework for tackling complex uncertainty, with applications in MCDM and modeling inconsistent data. Building on these, Smarandache proposed hypersoft sets to add dimensional flexibility, accommodating an even wider range of parameters and decision scenarios. Vellapandi and Gunasekaran [8] employed multi-soft set logic to formulate decision-making frameworks, emphasizing application-oriented techniques. Das et al. [12] and Kumar Das [13] utilized neutrosophic soft matrices for group decision-making and transportation problems, providing algorithmic approaches to handle more nuanced uncertainty in practical scenarios.

Within matrix theory, the study of FM has led to various operations and compositions. Emam and Mishref [29] focused on transitivity and subinverses in FM, essential for analyzing fuzzy relations in decision models. Similarly, Ragab and Emam [21] introduced the min-max composition for fuzzy matrices, a fundamental tool for matrix manipulation and modeling relationships. Further contributions by Emam and Fndh [42] on maximum-minimum and minimum-maximum bifuzzy matrix compositions underscore the complexity of handling multi-layered uncertainties in real applications. Emam [39, 41] extended this to intuitionistic fuzzy matrices by examining transitive operations and weak consistency, crucial in maintaining logical coherence in fuzzy-based decision systems.

In response to evolving complexities, recent research has directed focus toward neutrosophic and intuitionistic fuzzy matrices. Padder and Murugadas [35] explored idempotent intuitionistic fuzzy matrices, contributing to structural consistency, while Uma et al. [36] worked on type I and type II fuzzy neutrosophic soft matrices to support complex system modeling. Anandhkumar et al. [37-38] introduced k-idempotent and pseudo-similarity concepts in neutrosophic matrices, which enable effective matrix ordering and structural consistency in diverse applications. These studies mark significant strides in matrix-based uncertainty modeling but highlight a gap in methods for constructing idempotent structures in neutrosophic fuzzy matrices (NFMs), which are critical for consistent application in decision-making. The current research addresses this gap by defining new compositions for NFMs, namely max(min)–max(min)-min(max) and min(max) – min(max) – max(min) ($*$) operations, to expand the theoretical groundwork for matrix idempotency and consistency. These compositions are expected to enhance the applicability of NFMs in decision-making frameworks by ensuring reliable results across complex data sets. Through the introduction of these novel operations, this study contributes to both the theoretical development and practical applications of NFMs in uncertainty modeling.

In recent years, the study of fuzzy and neutrosophic matrices has gained significant momentum due to their wide applicability in complex decision-making environments involving uncertainty and imprecision. M. Anandhkumar and collaborators have made substantial contributions to this evolving field. For instance, Anandhkumar et al. [51] introduced the Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices with applications to multi-criteria decision-making problems, while Radhika et al. [52] extended this framework through Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices. Further foundational developments include the exploration of various inverse concepts by Anandhkumar, Kanimozhi, Chithra, Kamalakannan, and Said [53], and the Schur complement in block structures as discussed by Radhika, Harikrishnan, Prabhu, Tharaniya, John Peter, and Anandhkumar [54].

Additionally, Prathab, Ramalingam, Janaki, Bobin, Kamalakannan, and Anandhkumar [55] investigated generalized inverses within interval-valued fuzzy environments. Meanwhile, Anandhkumar and colleagues have systematically studied different matrix orderings, such as Reverse Tilde and Minus Partial Ordering [56], Reverse Sharp and Left-T Right-T Partial Ordering on Intuitionistic Fuzzy Matrices [57], and their extensions to neutrosophic fuzzy contexts [60]. Another line of research by Anandhkumar, Prathab, Chithra, Prakaash, and Bobin [58] introduced Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, while Punithavalli and Anandhkumar [59] analyzed Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices, providing deeper algebraic insights. These collective efforts form a robust foundation for further advancements in neutrosophic fuzzy matrix theory and its multifaceted applications.

7. **Preliminaries**

7.1 Neutrosophic Fuzzy Matrices

Definition: 7.1 A NFSs P on the universe of discourse Y is well-defined as

$P=\{<y, p^T(y), p^I(y), p^F(y)>, y\in Y\}$, everywhere $p^T, p^I, p^F : Y \to]^{-}0,1^{+}[$ also

$0\le p^T + p^I + p^F \le 3.$

Definition 7.2 . The score NFM P and Q is defined as $S(V_1,W_1)=V_1-W_1$ where $V_1=P*Q, W_1=P*Q^T$ and $V_1(v_{ij})=T_{ij}+I_{ij}-F_{ij}$, $W_1(w_{ij})=T_{ij}+I_{ij}-F_{ij}$ are called membership value NFM.

Definition 7.3 For n x n Nutrosophic Fuzzy Matrices $P=\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)$ we have

(a) If P is transitive iff $P^2 \le P.$

(b) If P is idempotent iff $P^2 = P.$

(c) If P is reflexive iff $\left(p_{ii}{}^{T}, p_{ii}{}^{I}, p_{ii}{}^{F}\right)=(1,1,0)$ for each i belongs to $\{1,2,...,n\}$.

(d) If P is irreflexive iff $\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)=\left(0,0,1\right)$ for each i belongs to $\left\{1,2,\ldots,n\right\}$

(e) If P is weakly reflexive iff $\left(p_{ii}{}^{T}, p_{ii}{}^{I}, p_{ii}{}^{F}\right)\geq\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)$ for each i ,j belongs to $\left\{1,2,\ldots,n\right\}$

(f) If P is nearly irreflexive iff $\left(p_{ii}{}^{T}, p_{ii}{}^{I}, p_{ii}{}^{F}\right)\leq\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)$ for each i ,j belongs to $\left\{1,2,\ldots,n\right\}$

(g) If P is symmetric iff $P^2 = P$.

(h) If P is nilpotent iff $P^n =< 0,0,1 >$.

(i) If P Asymmetric iff $p_{ij} \wedge p_{ji}$ = (0,0,1).

Definition 7.4 Let $P=\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)_{mx\beta}$ and $Q=\left(q_{ij}{}^{T}, q_{ij}{}^{I}, q_{ij}{}^{F}\right)_{\beta xl}$ be two NFM. Then the maximum(minimum) –maximum(minimum)-minimum(maximum) composition (◦) of P, Q is represented by P ◦ Q and is well-defined as

$$P\circ Q=\left[t_{ij}\right]_{mxl}=<\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\wedge p_{\alpha j}{}^{T}\right),\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\wedge p_{\alpha j}{}^{I}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\vee p_{\alpha j}{}^{F}\right)>$$

i.e., $P\circ Q$ = [max min ($p_{ij}{}^{T}$, $p_{ij}{}^{T}$), max min ($p_{ij}{}^{I}$, $p_{ij}{}^{I}$), min max ($p_{ij}{}^{F}$, $p_{ij}{}^{F}$)] for all i and j.

The min(max) – min(max) – max(min) composition (∗) of P, Q is represented by P ∗Q and is well-defined as

$$P * Q=\left[s_{ij}\right]_{mxl}=<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right),\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)>$$

i.e., P ∗Q = [min max ($p_{ij}{}^{T}$, $p_{ij}{}^{T}$),min max($p_{ij}{}^{I}$, $p_{ij}{}^{I}$),max min($p_{ij}{}^{F}$, $p_{ij}{}^{F}$)] for all i and j.

where ∨ represents the max operation, ∧ represents the min operation.

Definition 7.5 (constant, nearly constant NFM) . An a × b NFM P = P_{ij} is called constant iff if $p_{ij} = p_{kj}$ for all i , k belongs to {1, 2, ...,a }, j belongs to {1, 2, ..., b }, P is nearly constant iff $p_{ij} = p_{kj}$, where $i \neq j$ for all $k \neq j$.

8. **Theorems and results**

Theorem 8.1 Let $P=\left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F}\right)$ and $Q=\left(q_{ij}{}^{T}, q_{ij}{}^{I}, q_{ij}{}^{F}\right)$ be two nearly irreflexive NFM. Then $P * Q \leq P \vee Q$.

Proof: Let $R = P * Q$ and $T = P \vee Q$.

Then $r_{ij} = < r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F} > = < \bigcap_{\alpha=1}^{\beta} \left(p_{i\alpha}^{T} \vee q_{\alpha j}^{T} \right), \bigcap_{\alpha=1}^{\beta} \left(p_{i\alpha}^{I} \vee q_{\alpha j}^{I} \right), \bigcup_{\alpha=1}^{\beta} \left(p_{i\alpha}^{F} \wedge q_{\alpha j}^{F} \right) >$

and $t_{ij} = < t_{ij}^{T}, t_{ij}^{I}, t_{ij}^{F} > = < \left(p_{ij}^{T} \vee q_{ij}^{T} \right), \left(p_{ij}^{I} \vee q_{ij}^{I} \right), \left(p_{ij}^{F} \wedge q_{ij}^{F} \right) >$

Now $r_{ij}^{T} = \bigcap_{\alpha=1}^{\beta} \left(p_{i\alpha}^{T} \vee q_{\alpha j}^{T} \right) \leq \left(p_{ii}^{T} \vee q_{ij}^{T} \right) \leq \left(p_{ij}^{T} \vee q_{ij}^{T} \right) = t_{ij}^{T}$

$r_{ij}^{I} = t_{ij}^{I}$

$r_{ij}^{F} = \bigcup_{\alpha=1}^{\beta} \left(p_{i\alpha}^{F} \wedge q_{\alpha j}^{F} \right) \geq \left(p_{ii}^{F} \wedge q_{ij}^{F} \right) \geq \left(p_{ij}^{F} \wedge q_{ij}^{F} \right) = t_{ij}^{F}$.

Thus, we have $r_{ij} \leq t_{ij}$ and so $P * Q \leq P \vee Q$.

Hence the Theorem
It It is noted that $P \vee Q = Q$ for $P \leq Q$.

Theorem 8.2 Let P, Q be two nearly irreflexive NFM and $P \leq Q$. Then $P * Q \leq Q$.

Proof. This Theorem (8.2) follows immediately from Theorem (8.1).

Theorem 8.3 Let $P = \left(p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F} \right)$ be a symmetric and nearly irreflexive NFM. Then the following conditions are true

(i) $P * P \leq P$,

(ii) $P*P$ is symmetric and nearly irreflexive,

(iii) P^2 is weakly reflexive.

Proof. (i) This section (i) follows immediately from Theorem 8.1 and 8.2.

(ii) Suppose $S = P * P$. It is clear that S is symmetric and so

$s_{ii}^{T} = \bigcap_{\alpha=1}^{\beta} \left(p_{i\alpha}^{T} \vee p_{i\alpha}^{T} \right) \leq \bigcap_{\alpha=1}^{\beta} p_{i\alpha}^{T} \leq \bigcap_{\alpha=1}^{\beta} \left(p_{i\alpha}^{T} \vee p_{\alpha j}^{T} \right) = s_{ij}^{T}$

$s_{ii}^{I} = s_{ij}^{I}$

$s_{ii}^{F} = \bigcup_{\alpha=1}^{\beta} \left(p_{i\alpha}^{F} \wedge p_{i\alpha}^{F} \right) \geq \bigcup_{\alpha=1}^{\beta} {}_{i\alpha}^{F} \geq \bigcup_{\alpha=1}^{\beta} \left(p_{i\alpha}^{F} \wedge p_{\alpha j}^{F} \right) = s_{ij}^{F}$

Thus, $s_{ii} \leq s_{ij}$, Therefore $P * P \leq P$.

Therefore, P is nearly irreflexive.

(iii) Let T = P ^{2} . Then

$$t_{ij} = < \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T} \wedge p_{\alpha j}{}^{T}\right), \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I} \wedge p_{\alpha j}{}^{I}\right), \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F} \vee p_{\alpha j}{}^{F}\right) >$$

$$i.e., t_{ij}{}^{T} = \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T} \wedge p_{\alpha j}{}^{T}\right) = p_{ih}{}^{T} \wedge p_{hj}{}^{T}, \text{ for some } h \le n.$$

But since P is symmetric,

$$t_{ii}{}^{T} = \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T} \wedge p_{\alpha i}{}^{T}\right) = \bigcup_{\alpha=1}^{\beta} p_{i\alpha}{}^{T} \ge p_{ih}{}^{T} \ge p_{ih}{}^{T} \wedge p_{hj}{}^{T} = t_{ij}{}^{T}$$

Also $t_{ii}{}^{I} = t_{ij}{}^{I}$

and $t_{ii}{}^{F} = \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F} \vee p_{\alpha j}{}^{F}\right) = \left(p_{is}{}^{F} \vee p_{sj}{}^{F}\right), s \le \beta$

That is $t_{ii} \ge t_{ij}$

$t_{ii}{}^{F} = t_{ij}{}^{F}$

P^2 is weakly reflexive.

Hence the Theorem

Theorem 8.4 For NFM $P = [p_{ij}]_{m\times\alpha}, Q = [q_{ij}]_{m\times\beta}, R = [r_{ij}]_{\beta\times l}$ *and* $S = [s_{ij}]_{p\times m}$, we have

(i) $(Q * R)^t = R^t * Q^t$,

(ii) *If* $P \le Q$, then $R * P \le R * Q$ *and* $P * R \le Q * R$

Proof: *Let* $A = R^t * Q^t$ and $B = Q * R$.

Then $a_{ij} = < \bigcap_{\alpha=1}^{\beta}\left(r_{i\alpha}{}^{T} \vee q_{j\alpha}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(r_{i\alpha}{}^{I} \vee q_{j\alpha}{}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(r_{i\alpha}{}^{F} \wedge q_{j\alpha}{}^{F}\right) >$

and $b_{ji} = < \bigcap_{\alpha=1}^{\beta}\left(q_{j\alpha}{}^{T} \vee r_{\alpha i}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(q_{j\alpha}{}^{I} \vee r_{\alpha i}{}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(q_{j\alpha}{}^{F} \wedge r_{\alpha i}{}^{F}\right) >$

$i.e\ A = B'$

$(ii)\ Let\ C = S * P\ and\ D = S * Q$

$$i.e., c_{ij} = < \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{T} \vee p_{\alpha j}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{I} \vee p_{\alpha j}{}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{F} \wedge p_{\alpha j}{}^{F}\right) >$$

and $d_{ij} = <\bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{T} \vee q_{\alpha j}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{I} \vee q_{\alpha j}{}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{F} \wedge q_{\alpha j}{}^{F}\right)>$

Since we have that $P \leq Q$, we get $p_{kj}{}^{T} \leq q_{kj}{}^{T}, p_{kj}{}^{I} \leq q_{kj}{}^{I}, p_{kj}{}^{F} \geq q_{kj}{}^{F}$ and

$So, \left(s_{i\alpha}{}^{T} \vee p_{\alpha j}{}^{T}\right) \leq \left(s_{i\alpha}{}^{T} \vee q_{\alpha j}{}^{T}\right)$ for every $\alpha \leq m.$

Therefore, $\bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{T} \vee q_{\alpha j}{}^{T}\right) \leq \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{T} \vee q_{\alpha j}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{I} \vee p_{\alpha j}{}^{I}\right) \leq \bigcap_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{I} \vee q_{\alpha j}{}^{I}\right)$ *and*

$$\bigcup_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{F} \wedge p_{\alpha j}{}^{F}\right) \geq \bigcup_{\alpha=1}^{\beta}\left(s_{i\alpha}{}^{F} \wedge q_{\alpha j}{}^{F}\right)$$

$ie., c_{ij} \leq d_{ij}$

Similarly, one can show that $P * R \leq Q * R.$

Hence the Theorem.

Theorem 8.5 For any $m \times \alpha$ NFM P , then P∗P^t is nearly irreflexive and symmetric.
Proof. Let R = P∗P^t. That is

$$r_{ij} = <\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T} \vee p_{\alpha j}{}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I} \vee p_{\alpha j}{}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F} \wedge p_{\alpha j}{}^{F}\right)>$$

$r_{ij}{}^{T} = \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T} \vee p_{\alpha j}{}^{T}\right) = p_{il}{}^{T} \vee p_{jl}{}^{T}$ for some $l \leq \beta$

and $r_{ij}{}^{F} = \bigcup_{\alpha=1}^{\beta}\left(p_{\alpha j}{}^{F} \wedge p_{\alpha j}{}^{F}\right) = \left(p_{ig}{}^{F} \wedge p_{gj}{}^{F}\right)$ for some $g \leq \beta$

Now $, r_{ii}{}^{T} = \bigcap_{\alpha=1}^{\beta}\left(p_{\alpha j}{}^{T} \vee p_{i\alpha}{}^{T}\right) = \bigcap_{\alpha=1}^{\beta} p_{ik}{}^{T} = p_{ih}{}^{T}$ for some $l \leq \beta$

$r_{ii}{}^{I} = p_{ih}{}^{I}$ for some $l \leq \beta$

and $r_{ii}{}^{F} = \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F} \wedge p_{i\alpha}{}^{F}\right) = p_{im}{}^{F}$ for some $h, m \leq \beta$

Since, $r_{ii}{}^{T} = p_{ih}{}^{T} \leq p_{il}{}^{T} \leq p_{il}{}^{T} \vee p_{jl}{}^{T} = r_{ij}{}^{T}$

$r_{ii}{}^{I} = p_{ih}{}^{I} \leq p_{il}{}^{I} \leq p_{il}{}^{I} \vee p_{jl}{}^{I} = r_{ij}{}^{I}$

$r_{ii}^{F} = p_{im}^{F} \geq p_{ig}^{F} \geq p_{ig}^{F} \wedge p_{jg}^{F} = r_{ij}^{F}$

Therefore $r_{ii} \leq r_{ij}$

Therefore, P∗P^t is nearly irreflexive

The symmetry of R is clear.

Hence the Theorem.

Theorem 8.6 Let P be an $\alpha \times \alpha$ asymmetric NFM. Then P∗P^t = (0,0,1) (the zero matrix)

Proof: Let T = P∗P^t

Then, $t_{ij} = < \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}^{T} \vee p_{j\alpha}^{T}\right), \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}^{I} \vee p_{j\alpha}^{I}\right), \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}^{F} \wedge p_{j\alpha}^{F}\right) >$

$= < p_{ih}^{T} \vee p_{jh}^{T}, p_{ih}^{I} \vee p_{jh}^{I}, p_{is}^{F} \wedge p_{js}^{F} >$ for so $h, s \leq n$.

Therefore, P is asymmetric, it is irreflexive and so

$t_{ij}^{T} = p_{ih}^{T} \vee p_{jh}^{T} \leq p_{ij}^{T} \vee p_{jj}^{T} = p_{ij}^{T}$

$t_{ij}^{I} = p_{ih}^{I} \vee p_{jh}^{I} \leq p_{ij}^{I} \vee p_{jj}^{I} = p_{ij}^{I}$

$t_{ij}^{F} = p_{is}^{F} \wedge p_{js}^{F} \geq p_{ij}^{F} \wedge p_{jj}^{F} = p_{ij}^{F}$

$i.e.,, t_{ij} \leq p_{ij}.\ t_{ij} \geq p_{ji}$

$t_{ij} \leq p_{ij} \wedge p_{ji} = < 0,0,1 >$

Thus $t_{ij} = < 0,0,1 >$ and so $t = < 0,0,1 >$.

Hence the Theorem.

Corollary 8.1 If P is nilpotent, then P^m is irreflexive for all m ≤ n .

Proof: Theorem 8.6 establishes that the nilpotency of a NFM P guarantees the asymmetry of the matrix. Though, the reverse implication does not necessarily hold, as asymmetry does not always imply nilpotency.

Theorem 8.7 Let P be an n ×n nilpotent NFM. Then P is asymmetric.

Proof. Subsequently P is nilpotent $p_{ij}^{(n)} = < 0,0,1 >$

$If\ p_{ij} \wedge p_{ji} > (0,0,1)$

$i.e., if\ p_{ij}^{T} \wedge p_{ji}^{T} > (0,0,1)$

$p_{ij}^{I} \wedge p_{ji}^{I} > (0,0,1)$

and $p_{ij}{}^{F} \vee p_{ji}{}^{F} < (1,1,0)$

then $p_{ij}{}^{T}, p_{ij}{}^{I} > (0,0,1), p_{ji}{}^{T}, p_{ji}{}^{I} > (0,0,1)$

and $p_{ij}{}^{F}, p_{ji}{}^{F} < (1,1,0)$

Now, we consider two cases for n .
Case 1: If n is odd, then

$\left(p_{ij}{}^{T}\right)^{(n)} \geq p_{ij}{}^{T} \wedge p_{ji}{}^{T} \wedge p_{ij}{}^{T} \wedge .... \wedge p_{ij}{}^{T} > (0,0,1)$ (n- elements)

$\left(p_{ij}{}^{I}\right)^{(n)} \geq p_{ij}{}^{I} \wedge p_{ji}{}^{I} \wedge p_{ij}{}^{I} \wedge .... \wedge p_{ij}{}^{I} > (0,0,1)$ (n- elements)

$\left(p_{ij}{}^{F}\right)^{(n)} \leq p_{ij}{}^{F} \vee p_{ji}{}^{F} \vee p_{ij}{}^{F} \vee .... \vee p_{ij}{}^{F} < (1,1,0)$ (n- elements)

which is a contradiction.
Case 2: If n is even, then by Corollary 8.1 , we take

$\left(p_{ij}{}^{T}\right)^{(n)} \geq p_{ij}{}^{T} \wedge p_{ji}{}^{T} \wedge p_{ij}{}^{T} \wedge .... \wedge p_{ji}{}^{T} > (0,0,1)$ (n- elements)

$\left(p_{ij}{}^{I}\right)^{(n)} \geq p_{ij}{}^{I} \wedge p_{ji}{}^{I} \wedge p_{ij}{}^{I} \wedge .... \wedge p_{ji}{}^{I} > (0,0,1)$ (n- elements)

$\left(p_{ij}{}^{F}\right)^{(n)} \leq p_{ij}{}^{F} \vee p_{ji}{}^{F} \vee p_{ij}{}^{F} \vee .... \vee p_{ji}{}^{F} < (1,1,0)$ (n- elements)

which is a contradiction.

Thus, $p_{ij}{}^{T} \wedge p_{ji}{}^{T} = (0,0,1), p_{ij}{}^{I} \wedge p_{ji}{}^{I} = (0,0,1)$ *and* $p_{ij}{}^{F} \vee p_{ji}{}^{F} = (1,1,0)$

i.e., $p_{ij} \wedge p_{ji} = (0,0,1)$

Consequently, P is then asymmetric.
Hence the Theorem.

Theorem 8.8 If P is irreflexive and transitive NFM, then P is nilpotent.
Proof: Theorem 8.8 directly follows as a logical consequence of Theorem 8.7.

Theorem 8.9 Let P and Q be two transitive NFM, such that P ≤ Q . Then $P \,!\, Q^{t}$ is transitive.

Proof: Let $R = P \,!\, Q^{t}$ and suppose $r_{ik} \wedge r_{kj} = c > (0,0,1)$ for some $k \leq n$ .That is

$\left(< p_{ik}{}^{T}, p_{ik}{}^{I}, p_{ik}{}^{F} > ! \ < q_{ki}{}^{T}, q_{ki}{}^{I}, q_{ki}{}^{F} >\right) \wedge \left(< p_{kj}{}^{T}, p_{kj}{}^{I}, p_{kj}{}^{F} > ! \ < q_{jk}{}^{T}, q_{jk}{}^{I}, q_{jk}{}^{F} >\right)$

$= \left(< c^{T}, c^{I}, c^{F} >\right) > (0,0,1)$

Thus $p_{ik}{}^{T} > q_{ki}{}^{T}$ and $p_{kj}{}^{T} > q_{jk}{}^{T}$

So that $< p_{ik}{}^{T}, p_{ik}{}^{I}, p_{ik}{}^{F} > \wedge < p_{kj}{}^{T}, p_{kj}{}^{I}, p_{kj}{}^{F} > = \left(< c^{T}, c^{I}, c^{F} >\right)$

i.e., $p_{ik}{}^{T} \wedge q_{kj}{}^{T} = c^{T}, p_{ik}{}^{I} \wedge q_{kj}{}^{I} = c^{I}$ and $p_{ik}{}^{F} \vee q_{kj}{}^{F} = c^{F}$

Since P is transitive, $p_{ij} =< p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F} > \geq < p_{ik}{}^{T} \wedge q_{kj}{}^{T}, p_{ik}{}^{I} \wedge q_{kj}{}^{I}, p_{ik}{}^{F} \vee q_{kj}{}^{F} >$.

Now, we show that $\mathrm{p}^{T}{}_{ij} \leq q^{T}{}_{ji}$,there are contradictions.

(a) If $p_{ik}{}^{T} = c^{T}$ then $q_{ki}{}^{T} < c^{T}$ and *so* $p_{ki}{}^{T} < c^{T}$

Since we have that P ≤ Q . Though, since we have assumed $\mathrm{q}_{ji}{}^{T} \geq p_{ij}{}^{T} \geq c^{T}$,

we get $q_{ki}{}^{T} \geq q_{kj}{}^{T} \wedge \mathrm{q}_{ji}{}^{T} \geq p^{T}{}_{kj} \wedge \mathrm{q}^{T}{}_{ji} \geq c^{T}$. This represents a contradiction.

(b) If $\mathrm{p}^{T}{}_{kj} = c^{T}$, then $q^{T}{}_{jk} < c^{T}$. Though $q^{T}{}_{jk} \geq c^{T}$

This represents a contradiction.

Therefore, $p_{ij}{}^{T} > \mathrm{q}_{ji}{}^{T}$ and so $r_{ij} = p_{ij} \,!\, q_{ji} =< p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F} > ! < q_{ji}{}^{T}, q_{ji}{}^{I}, q_{ji}{}^{F} >$

$=< p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F} > \geq \left(p_{ik}{}^{T} \wedge q_{kj}{}^{T}, p_{ik}{}^{I} \wedge q_{kj}{}^{I}, p_{ik}{}^{F} \vee q_{kj}{}^{F}\right) = \left(< c^{T}, c^{I}, c^{F} >\right)$

i.e., $r_{ij} \geq c = r_{ik} \wedge r_{kj}$ and R is thus transitive. Hence the Theorem

Corollary 8.2. Let P and Q be two transitive NFM, with P ≤ Q. Then $(P\ominus Q^{t}) * (P\ominus Q^{t})^{t} = <0,0,1>$.
Proof. It is evident that $P\ominus Q^{t}$ is irreflexive and so by Propositions 8.6,8.7,8.9, we get the result.

Theorem 8.10 Let S be an n ×n symmetric and nearly irreflexive NFM. Then the NFM and nearly constant $T = I_n * S$ is idempotent and nearly constant.

Proof: Given the symmetry of S, the elements of the NFM T can be expressed in terms of the elements of S **as:** $t_{ij} =< t_{ij}{}^{T}, t_{ij}{}^{I}, t_{ij}{}^{F} > = \begin{cases} < t_{ij}{}^{T}, t_{ij}{}^{I}, t_{ij}{}^{F} > & if\ i \neq j \\ < \bigcap\limits_{i\neq\alpha} s_{i\alpha}{}^{T}, \bigcap\limits_{i\neq\alpha} s_{i\alpha}{}^{I}, \bigcup\limits_{i\neq\alpha} s_{i\alpha}{}^{F} > & if\ i = j \end{cases}$

Initially we know that t_{ij} , we note that T is nearly constant. Next, Subsequently, we prove that T is idempotent. Any element $t_{ij}{}^{(2)}$ of T^2 is computed as

$t_{ij}^{(2)} = =< t_{i\alpha}^{T} \wedge t_{\alpha j}^{T}, t_{i\alpha}^{I} \wedge t_{\alpha j}^{I}, t_{i\alpha}^{F} \vee t_{\alpha j}^{F} >$ for some $h, l \leq n.$

Though, we have some cases for the indices i , j , h and l to show that $t_{ij}^{(2)} = t_{ij}$

Case 1: Assume that $i = j = h = l$. In this situation we take

$t_{ij}^{T^{(2)}} = t_{ih}^{T} \wedge t_{hj}^{T} = t_{ij}^{T} \wedge t_{ij}^{T} = t_{ij}^{T}$

$t_{ij}^{I^{(2)}} = t_{ij}^{I}$ and $t_{ij}^{F^{(2)}} = t_{ij}^{F}$

Thus, $t_{ij}^{(2)} = t_{ij}$.

Case 2: Assume that $i = j = h \neq l$. In this situation we take

$t_{ij}^{T^{(2)}} = t_{ij}^{T}, t_{ij}^{I^{(2)}} = t_{ij}^{I}$

As in case 1 .Also

$t_{ij}^{F^{(2)}} = t_{il}^{F} \vee t_{lj}^{F} \leq t_{ij}^{F} \vee t_{jj}^{F} = t_{ij}^{F} \vee t_{jj}^{F} = t_{ij}^{F}$

Thus $t_{ij}^{F^{(2)}} = t_{ij}^{F}$ and so $t_{ij}^{(2)} = t_{ij}$.

Case 3: Assume that $i = j = l \neq h$. In this situation we take

$t_{ij}^{T^{(2)}} = t_{ih}^{T} \wedge t_{hj}^{T} \geq t_{ij}^{T} \wedge t_{jj}^{T} \geq t_{ij}^{T} \wedge t_{ij}^{T}$ (Since i = j)

But $t_{ij}^{T} = \bigcap_{i \neq k} s_{ik}^{T} \geq s_{ii}^{T} \geq s_{hh}^{T} \wedge s_{ii}^{T} = s_{hh}^{T} \wedge s_{jj}^{T} = t_{ih}^{T} \wedge t_{hj}^{T} = t_{ij}^{T^{(2)}}$

Thus, $t_{ij}^{T^{(2)}} = t_{ij}^{T}$,

Also, as in Case 1, we get $t_{ij}^{I^{(2)}} = t_{ij}^{I}, t_{ij}^{F^{(2)}} = t_{ij}^{F}$

Therefore, $t_{ij}^{T} = t_{ij}$

Case 4: Assume that $i = h = l \neq j.$ In this situation we take

$t_{ii}^{T} \wedge t_{ij}^{T} = t_{ih}^{T} \wedge t_{hj}^{T} \geq t_{ij}^{T} \wedge t_{jj}^{T}$ and so

$t_{ii}^{T} \wedge t_{jj}^{T}$. We know that t_{jj}^{T} . It is evident that $t_{jj}^{T} \geq s_{jj}^{T}$

so that $t_{ii}^{T} \geq t_{jj}^{T} \geq s_{jj}^{T}$

Thus, $t_{ij}^{T^{(2)}} = t_{ih}^{T} \wedge t_{hj}^{T} = t_{ii}^{T} \wedge t_{hj}^{T} = t_{ii}^{T} \wedge s_{jj}^{T} = s_{jj}^{T} = t_{ij}^{T}.$

Similarly, $t_{ii}^{I} \wedge t_{ij}^{I} = t_{ih}^{I} \wedge t_{hj}^{I} \geq t_{ij}^{I} \wedge t_{jj}^{I}$ ans so $t_{ii}^{I} \wedge t_{jj}^{I}$

By the definition of $t_{jj}{}^{I}$ It is evident that $t_{jj}{}^{I} \geq s_{jj}{}^{I}$

So that $t_{ii}{}^{I} \geq t_{jj}{}^{I} \geq s_{jj}{}^{I}$

$t_{ij}{}^{I^{(2)}} = t_{ih}{}^{I} \wedge t_{hj}{}^{I} = t_{ii}{}^{I} \wedge t_{hj}{}^{I} = t_{ij}{}^{I}$.

Also, in this situation we have $t_{ii}{}^{F} \vee t_{ij}{}^{F} = t_{il}{}^{F} \wedge t_{lj}{}^{F} \leq t_{ij}{}^{F} \vee t_{jj}{}^{F}$ and so $t_{ii}{}^{F} \leq t_{jj}{}^{F}$.

But $t_{jj}{}^{F} \leq s_{jj}{}^{F}$ and so $t_{ii}{}^{F} \leq t_{jj}{}^{F} \leq s_{jj}{}^{F}$.

Thus, $t_{ij}{}^{F^{(2)}} = t_{il}{}^{F} \vee t_{lj}{}^{F} = t_{ii}{}^{F} \vee t_{lj}{}^{F} = t_{ij}{}^{F}$.

Therefore, $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}$.

Case 5: Assume that $j = l = h \neq j.$ In this situation we take

$$t_{ij}{}^{T^{(2)}} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} = t_{ij}{}^{T} \wedge t_{ji}{}^{T} = s_{jj}{}^{T} \wedge \left(\bigcap_{j \neq \alpha} s_{j\alpha}{}^{T} \right) = s_{jj}{}^{T} = t_{ij}{}^{T}.$$

$t_{ij}{}^{I^{(2)}} = t_{ij}{}^{I}$.

$t_{ij}{}^{F^{(2)}} = t_{il}{}^{F} \vee t_{hj}{}^{F} = t_{ij}{}^{F}$.

Case 6: Assume that $j = l = h \neq l.$ In this situation we take

$t_{ij}{}^{T^{(2)}} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} \geq t_{ij}{}^{T} \wedge t_{jj}{}^{T} = t_{ij}{}^{T} \wedge t_{ij}{}^{T} = t_{ij}{}^{T}$ (since i = j).

Conversely, we have S is nearly irreflexive

$$t_{ij}{}^{T} = \bigcap_{i \neq k} s_{i\alpha}{}^{T} \geq s_{ii}{}^{T} \geq s_{hh}{}^{T} \wedge s_{ii}{}^{T} = s_{hh}{}^{T} \wedge s_{jj}{}^{T} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} = t_{ij}{}^{T^{(2)}}$$

Thus, $t_{ij}{}^{T^{(2)}} = t_{ij}{}^{T}$

Also, $t_{ij}{}^{I^{(2)}} = t_{ij}{}^{I}$

$t_{ij}{}^{T^{(2)}} = t_{il}{}^{F} \vee t_{Lj}{}^{F} \leq t_{ij}{}^{F} \vee t_{jj}{}^{F} = t_{ij}{}^{F}$

Thus, $t_{ij}{}^{F} = \bigcup_{i \neq \alpha} s_{il}{}^{F} \leq s_{ii}{}^{F} \leq s_{hh}{}^{F} \wedge s_{ii}{}^{F} = s_{hh}{}^{F} \vee s_{jj}{}^{F} = t_{ih}{}^{F} \vee t_{hj}{}^{F} = t_{ij}{}^{F^{(2)}}$ and so $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}$

Case 7: Assume that $i = h \neq j \neq l.$ In this situation we take

$t_{ii}{}^{T} \wedge t_{ij}{}^{T} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} \geq t_{ij}{}^{T} \wedge t_{jj}{}^{T}$ and $t_{ii}{}^{T} \geq t_{jj}{}^{T} \geq s_{jj}{}^{T}$

As in Case 4, we get $t_{ij}{}^{T^{(2)}} = t_{ij}{}^{T}$

Similarly, $t_{ii}{}^{I} \wedge t_{ij}{}^{I} = t_{ih}{}^{I} \wedge t_{hj}{}^{I} \geq t_{ij}{}^{I} \wedge t_{jj}{}^{I}$

$t_{ii}{}^{I} \geq t_{jj}{}^{I} \geq s_{jj}{}^{I}$, $t_{ij}{}^{I^{(2)}} = t_{ij}{}^{I}$

Also $s_{ll}{}^{F} \vee s_{jj}{}^{F} = t_{il}{}^{F} \vee t_{lj}{}^{F} \leq t_{ij}{}^{F} \vee t_{jj}{}^{F} \leq s_{jj}{}^{F}$.

So, $s_{ll}{}^{F} \leq s_{jj}{}^{F}$ Therefore,

Therefore, $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}$.

Case 8: Assume that $i = l \neq h \neq j$. In this situation we take

$$t_{ij}{}^{T^{(2)}} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} \geq t_{ij}{}^{T} \wedge t_{jj}{}^{T} = s_{jj}{}^{T} \wedge \left(\bigcap_{j \neq \alpha} s_{j\alpha}{}^{T} \right) = s_{jj}{}^{T} = t_{ij}{}^{T}.$$

On the other hand we have

$t_{ij}{}^{T} = s_{jj}{}^{T} \geq s_{hh}{}^{T} \wedge s_{jj}{}^{T} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} = t_{ij}{}^{T^{(2)}}$

Therefore, $t_{ij}{}^{T^{(2)}} = t_{ij}{}^{T}$. Similarly, $t_{ij}{}^{I} = t_{ij}{}^{I^{(2)}}$

Also, as in Case 4, we get $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}$

Thus, $t_{ij}{}^{(2)} = t_{ij}$

Case 9: Assume that $j = h \neq i \neq l$. In this situation we take

$$t_{ij}{}^{T^{(2)}} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} = t_{ij}{}^{T} \wedge t_{jj}{}^{T} = s_{jj}{}^{T} \wedge \left(\bigcap_{j \neq \alpha} s_{j\alpha}{}^{T} \right) = s_{jj}{}^{T} = t_{ij}{}^{T}.$$

$t_{ij}{}^{I^{(2)}} = t_{ij}{}^{I}$.

And in Case 7. Therefore $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}, t_{ij}{}^{(2)} = t_{ij}$.

Case 10: Assume that $j = l \neq h \neq i$. In this situation we take

$$t_{ij}{}^{T^{(2)}} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} \geq t_{ij}{}^{T} \wedge t_{jj}{}^{T} = s_{jj}{}^{T} \wedge \left(\bigcap_{j \neq \alpha} s_{j\alpha}{}^{T} \right) = s_{jj}{}^{T} = t_{ij}{}^{T}.$$

On the other hand, $t_{ij}{}^{T} = s_{jj}{}^{T} \geq s_{hh}{}^{T} \wedge s_{jj}{}^{T} = t_{ih}{}^{T} \wedge t_{hj}{}^{T} = t_{ij}{}^{T^{(2)}}$

Thus, $t_{ij}{}^{T^{(2)}} = t_{ij}{}^{T}$ also, $t_{ij}{}^{I^{(2)}} = t_{ij}{}^{I}$, $t_{ij}{}^{F^{(2)}} = t_{ij}{}^{F}$

Therefore, $t^{(2)}{}_{ij} = t_{ij}$

Case 11: Assume that $h = l \neq i \neq j$. As in case 8, $t_{ij}^{T^{(2)}} = t_{ij}^{T}$, $t_{ij}^{I^{(2)}} = t_{ij}^{I}$ and $t_{ij}^{F^{(2)}} = t_{ij}^{F}$ as in Case 7. Consequently, $t^{(2)}{}_{ij} = t_{ij}$.

Case 12: Assume that $h \neq l \neq i \neq j$. As in case 4 and 9 , $t^{(2)}{}_{ij} = t_{ij}$. From the computations of $t^{(2)}{}_{ij}$,we find that $t^{(2)}{}_{ij} = t_{ij}$ in all the above cases and so T is idempotent.

Hence the Theorem

Corollary 8.3. Let P be any m ×n NFM. Then the NFM I $_m$∗(P ∗P^t) is idempotent and nearly constant.
Proof: This Corrolary (8.3) follows directly from Theorems 8.5 and 8.10 .

The following example Theorem 8.10 and its corollaries are useful in studying NFM. Though, they enable us to construct an idempotent NFM from any given NFM.

Example.8.1 Let us consider NFM

$$P = \begin{bmatrix} <0.5,0.1,0.3> & <0.4,0.1,0.6> & <0.8,0.1,0.2> & <0.7,0.1,0.3> \\ <0.8,0.1,0> & <0.9,0.1,0.1> & <1,0.1,0> & <0.3,0.1,0.6> \\ <0.4,0.1,0.4> & <0.5,0.1,0.5> & <0,0.1,1> & <0.8,0.1,0.3> \\ <0.7,0.1,0.2> & <0.6,0.1,0.3> & <0.9,0.1,0> & <0.5,0.1,0.4> \end{bmatrix}$$

$$P^t = \begin{bmatrix} <0.5,0.1,0.3> & <0.8,0.1,0> & <0.4,0.1,0.4> & <0.7,0.1,0.2> \\ <0.4,0.1,0.6> & <0.9,0.1,0.1> & <0.5,0.1,0.5> & <0.6,0.1,0.3> \\ <0.8,0.1,0.2> & <1,0.1,0> & <0,0.1,1> & <0.9,0.1,0> \\ <0.7,0.1,0.3> & <0.3,0.1,0.6> & <0.8,0.1,0.3> & <0.5,0.1,0.4> \end{bmatrix}$$

Then $S = P * P^t$

$$S = \begin{bmatrix} <0.4,0.1,0.6> & <0.7,0.1,0.3> & <0.5,0.1,0.5> & <0.6,0.1,0.3> \\ <0.7,0.1,0.3> & <0.3,0.1,0.6> & <0.8,0.1,0.1> & <0.5,0.1,0.4> \\ <0.5,0.1,0.5> & <0.8,0.1,0.1> & <0,0.1,1> & <0.6,0.1,0.3> \\ <0.6,0.1,0.3> & <0.5,0.1,0.4> & <0.6,0.1,0.3> & <0.5,0.1,0.4> \end{bmatrix}$$

It is clear that S is nearly irreflexive and symmetric. Also, let $T = I_4 * S$ That is

$$T = \begin{bmatrix} <1,1,0> & <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <1,1,0> & <0,0,1> & <0.5,0.1,0.4> \\ <0,0,1> & <0,0,1> & <1,1,0> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> & <1,1,0> \end{bmatrix}$$

$$*\begin{bmatrix} <0.4,0.1,0.6> & <0.7,0.1,0.3> & <0.5,0.1,0.5> & <0.6,0.1,0.3> \\ <0.7,0.1,0.3> & <0.3,0.1,0.6> & <0.8,0.1,0.1> & <0.5,0.1,0.4> \\ <0.5,0.1,0.5> & <0.8,0.1,0.1> & <0,0.1,1> & <0.6,0.1,0.3> \\ <0.6,0.1,0.3> & <0.5,0.1,0.4> & <0.6,0.1,0.3> & <0.5,0.1,0.4> \end{bmatrix}$$

$$T=\begin{bmatrix} <0.5,0.1,0.5> & <0.3,0.1,0.6> & <0,0.1,1> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.5,0.1,0.4> & <0,0.1,1> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.3,0.1,0.6> & <0.5,0.1,0.5> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.3,0.1,0.6> & <0,0.1,1> & <0.5,0.1,0.4> \end{bmatrix}$$

$$T^2=\begin{bmatrix} <0.5,0.1,0.5> & <0.3,0.1,0.6> & <0,0.1,1> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.5,0.1,0.4> & <0,0.1,1> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.3,0.1,0.6> & <0.5,0.1,0.5> & <0.5,0.1,0.4> \\ <0.4,0.1,0.6> & <0.3,0.1,0.6> & <0,0.1,1> & <0.5,0.1,0.4> \end{bmatrix}$$

$$T=T^2$$

Then it is clear that T is nearly constant and it is also idempotent .

Theorem 8.11 For p,q∈ NFM , we have: $(i)(p\vee q)^c = p^c \wedge q^c, (ii)(p\wedge q)^c = p^c \vee q^c$.

The proof is trivial. The following Theorems and Corrolaries shows the relationship between the two composition ∗ and ∘ of NFM.

Theorem 8.12 For NFM $P=\left[p_{ij}\right]_{m\times\beta}$, and $Q=\left[q_{ij}\right]_{\beta\times l}$ we have:

$$(i)(P*Q)^c = P^cQ^c$$

$$(ii)P^c * Q^c = (PQ)^c$$

Proof. (1) Let $D=(P*Q)^c$ and $E=P^cQ^c$

$$d_{ij}=\left(\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)\right)^c$$

$$d_{ij}=\left(\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right)\right)$$

and $e_{ij}=\left(\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right)\right)$

Therefore, D = E

(ii) Likewise, we can prove that $P^c * Q^c = (PQ)^c$

Corollary 8.6. For NFM $P=\left[p_{ij}\right]_{m\times n}$, and $Q=\left[q_{ij}\right]_{n\times p}, R=\left[r_{ij}\right]_{p\times g}$, $S=\left[s_{ij}\right]_{m\times p}$ we have:

(i) $P*(Q*R)=(P*Q)*R$ (ii) $(P*Q)=S \text{ iff } P^cQ^c=S^c$.

From the above theorem, it is evident that the operation $*$ is associative. In the following theorem, we show that $*$ is distributive over the $\vee$ (maximum) and $\wedge$ (minimum) operations.

Theorem 8.13 For any three NFM P , Q and R of order m ×n , n ×m and n ×m correspondingly, the following property holds:

(i) $P*(Q\vee R)=(P*Q)\vee(P*R)$, (ii) $P*(Q\wedge R)=(P*Q)\wedge(P*R)$.

Proof. (i) Let $P*(Q\vee R)=(P*Q)\vee(P*R)$,

$D=Q\vee R, R=P*D, G=P*Q, H=P*R, W=G\vee H$

Then, $d_{ij}=<q_{ij}{}^T\vee r_{ij}{}^T, q_{ij}{}^I\vee r_{ij}{}^I, q_{ij}{}^F\wedge r_{ij}{}^F>$

$$r_{ij}=<\bigcap_{\alpha=1}^{n}\left(p_{i\alpha}{}^T\vee d_{\alpha j}{}^T\right),\bigcap_{\alpha=1}^{n}\left(p_{i\alpha}{}^I\vee d_{\alpha j}{}^I\right),\bigcup_{\alpha=1}^{n}\left(p_{i\alpha}{}^F\wedge d_{\alpha j}{}^F\right)>$$

$$g_{ij}=<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^T\vee q_{\alpha j}{}^T\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^T\vee q_{\alpha j}{}^T\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^F\wedge q_{\alpha j}{}^F\right)>$$

Then, $h_{ij}=<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^T\vee r_{\alpha j}{}^T\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^T\vee r_{\alpha j}{}^T\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^F\wedge r_{\alpha j}{}^F\right)>$

Thus $w_{ij}=g_{ij}\vee h_{ij}=\left(<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^T\vee q_{\alpha j}{}^T\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^T\vee q_{\alpha j}{}^T\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^F\wedge q_{\alpha j}{}^F\right)>\right)$

$$\vee\left(<\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^T\vee r_{\alpha j}{}^T\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^T\vee r_{\alpha j}{}^T\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^F\wedge r_{\alpha j}{}^F\right)>\right)$$

$$=<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^T\vee\left(p_{i\alpha}{}^T\vee r_{\alpha j}{}^T\right)\right),\bigcap_{k=1}^{n}\left(p_{i\alpha}{}^I\vee\left(p_{i\alpha}{}^I\vee r_{\alpha j}{}^I\right)\right),\bigcup_{k=1}^{n}\left(p_{i\alpha}{}^F\wedge\left(p_{i\alpha}{}^F\wedge q_{\alpha j}{}^F\right)\right)>$$

We complete that $P*(Q\vee R)=(P*Q)\vee(P*R)$,

(ii) Can be shown by similar manner. Hence the Theorem

Theorem 8.14 For NFM $P=\left[p_{ij}\right]_{m\times\beta}$, and $Q=\left[q_{ij}\right]_{\beta\times p}, R=\left[r_{ij}\right]_{\beta\times p}$, we have:

(i) $P*(Q\,!\;\,R)\geq(P*Q)\,!\;\,(P*R)$ (ii) $P(Q\,!\;\,R)=(PQ)\,!\;\,(PR)$

Proof. (i) Let $S=(Q\,!\;\,R), T=P*S, U=P*Q, V=P*R, W=U\,!\;\,V$

$$s_{ij}=\begin{cases}<0,0,q_{ij}{}^{F}> if\ \ q_{ij}{}^{T}\le r_{ij}{}^{T},\ q_{ij}{}^{I}\le r_{ij}{}^{I},q_{ij}{}^{F}<r_{ij}{}^{F}\\ <0,0,1> if\ \ q_{ij}{}^{T}\le r_{ij}{}^{T},\ q_{ij}{}^{I}\le r_{ij}{}^{I},q_{ij}{}^{F}\ge r_{ij}{}^{F}\\ <q_{ij}{}^{T},q_{ij}{}^{I},q_{ij}{}^{F}> if\ \ q_{ij}{}^{T}\le r_{ij}{}^{T},\ q_{ij}{}^{I}\le r_{ij}{}^{I},q_{ij}{}^{F}<r_{ij}{}^{F}\end{cases}$$

$$v_{ij}=<\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee r_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee r_{\alpha j}{}^{T}\right),\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge r_{\alpha j}{}^{F}\right)>$$

$$t_{ij}=\begin{cases}<\bigcap_{\alpha=1}^{\beta}p_{i\alpha}{}^{T},\bigcap_{\alpha=1}^{\beta}p_{i\alpha}{}^{I},\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{kj}{}^{F}\right)> if\ \ q_{ij}{}^{T}\le r_{ij}{}^{T},\ q_{ij}{}^{I}\le r_{ij}{}^{I},q_{ij}{}^{F}<r_{ij}{}^{F}\\ <\bigcap_{\alpha=1}^{\beta}p_{i\alpha}{}^{T},\bigcap_{\alpha=1}^{\beta}p_{i\alpha}{}^{I},\bigcup_{\alpha=1}^{\beta}p_{i\alpha}{}^{F}> if\ \ q_{ij}{}^{T}\le r_{ij}{}^{T},\ q_{ij}{}^{I}\le r_{ij}{}^{I},q_{ij}{}^{F}\ge r_{ij}{}^{F}\\ <\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right),\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge r_{\alpha j}{}^{F}\right)> if\ \ q_{ij}{}^{T}>r_{ij}{}^{T},\ q_{ij}{}^{I}>r_{ij}{}^{I}.\end{cases}$$

$$w_{ij}=\begin{cases}<0,0,\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)> if\ \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right)\le\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee r_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right)\le\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee r_{\alpha j}{}^{I}\right),\\ \qquad\qquad\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)\ge\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge r_{\alpha j}{}^{F}\right).\\ <0.0,1> \qquad if\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right)\le\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee r_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right)\le\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee r_{\alpha j}{}^{I}\right),\\ \qquad\qquad\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)\ge\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge r_{\alpha j}{}^{F}\right).\\ <\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{n}\left(p_{i\alpha}{}^{I}\vee q_{\alpha}{}^{I}\right),\ if\ \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right)>\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee r_{\alpha j}{}^{T}\right)\\ \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)> \qquad\qquad \bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right)>\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee r_{\alpha j}{}^{I}\right).\end{cases}$$

$$w_{ij}=\begin{cases}<0,0,\bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)> if\ \ q_{\alpha j}{}^{T}\le r_{\alpha j}{}^{T},\ q_{\alpha j}{}^{I}\le r_{\alpha j}{}^{I},q_{\alpha j}{}^{F}<r_{\alpha j}{}^{F},\\ <0.0,1> \qquad if\ q_{\alpha j}{}^{T}\le r_{\alpha j}{}^{T},\ q_{\alpha j}{}^{I}\le r_{\alpha j}{}^{I},q_{\alpha j}{}^{F}\ge r_{\alpha j}{}^{F}\\ <\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{T}\vee q_{\alpha j}{}^{T}\right),\bigcap_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{I}\vee q_{\alpha j}{}^{I}\right),\ \bigcup_{\alpha=1}^{\beta}\left(p_{i\alpha}{}^{F}\wedge q_{\alpha j}{}^{F}\right)>\ if\ q_{ij}{}^{T}>r_{ij}{}^{T},q_{ij}{}^{I}>r_{ij}{}^{I}\end{cases}$$

We note that $T\ge W$. Hence $P*(Q!\ \ R)\ge(P*Q)!\ \ (P*R)$

(ii) Similar to (i). Hence the Theorem

Theorem 8.15 For NFM $P=\left[p_{ij}\right]_{m\times n}$, $Q=\left[q_{ij}\right]_{n\times l}$, $R=\left[r_{ij}\right]_{y\times p}$, $S=\left[s_{ij}\right]_{p\times m}$, and $T=\left[t_{ij}\right]_{l\times g}$, we have

(i) $R(S*P*Q)T \le RS*P*QT$

(ii) $(R*S)P(Q*T) \le R*(SPQ)*T$

Proof. (i) Let $A = S*P*Q, B = RA, C = BT,$

$D = RS, E = QT, F = D*P, G = F*E$

$$a_{ij} = <\bigcap_{x=1}^{n}\left[\bigcap_{u=1}^{m}\left(s_{iu}{}^{T} \vee p_{ux}{}^{T}\right) \vee q_{xj}{}^{T}\right], \bigcap_{x=1}^{n}\left[\bigcap_{u=1}^{m}\left(s_{iu}{}^{I} \vee p_{ux}{}^{I}\right) \vee q_{xj}{}^{I}\right], \bigcup_{x=1}^{n}\left[\bigcup_{u=1}^{m}\left(s_{iu}{}^{F} \wedge p_{ux}{}^{F}\right) \wedge q_{xj}{}^{F}\right]>$$

$$b_{ij}{}^{T} = \bigcup_{k=1}^{p}\left(r_{ik}{}^{T} \wedge a_{kj}{}^{T}\right) = \bigcup_{k=1}^{p}\left(r_{ik}{}^{T} \wedge \left\{\bigcap_{x=1}^{n}\left[\bigcap_{u=1}^{m}\left(s_{ku}{}^{T} \vee p_{ux}{}^{T}\right) \vee q_{xj}{}^{T}\right]\right\}\right)$$

$$b_{ij}{}^{F} = \bigcap_{k=1}^{p}\left(r_{ik}{}^{F} \vee a_{kj}{}^{F}\right) = \bigcap_{k=1}^{p}\left(r_{ik}{}^{F} \vee \left\{\bigcup_{x=1}^{n}\left[\bigcup_{u=1}^{m}\left(r_{ku}{}^{F} \wedge a_{ux}{}^{F}\right) \wedge \vee q_{xj}{}^{F}\right]\right\}\right)$$

$$c_{ij}{}^{T} = \bigcup_{v=1}^{l}\left(b_{iv}{}^{T} \wedge t_{vj}{}^{T}\right) = \bigcup_{v=1}^{l}\left(\left\{\bigcup_{k=1}^{p}\left(r_{ik}{}^{T} \wedge \left\{\bigcap_{x=1}^{n}\left[\bigcap_{u=1}^{m}\left(s_{ku}{}^{T} \vee p_{ux}{}^{T}\right) \vee q_{xv}{}^{T}\right]\right\}\right)\right\} \wedge t_{vj}{}^{T}\right)$$

$$c_{ij}{}^{T} = \bigcup_{v=1}^{l}\bigcup_{k=1}^{p}\bigcap_{x=1}^{n}\bigcap_{u=1}^{m}\left(r_{ik}{}^{T} \wedge \left(s_{ku}{}^{T} \vee p_{ux}{}^{T} \vee q_{xv}{}^{T}\right) \wedge t_{vj}{}^{T}\right)$$

$$c_{ij}{}^{T} = \bigcup_{v=1}^{l}\bigcup_{k=1}^{p}\bigcap_{x=1}^{n}\bigcap_{u=1}^{m}\left(\left(r_{ik}{}^{I} \wedge s_{ku}{}^{I} \wedge t_{vj}{}^{I}\right) \vee \left(r_{ik}{}^{I} \wedge p_{ux}{}^{I} \wedge t_{vj}{}^{I}\right) \vee \left(r_{ik}{}^{I} \wedge q_{xv}{}^{I} \wedge t_{vj}{}^{I}\right)\right)$$

$$c_{ij}{}^{T} = \bigcup_{v=1}^{l}\bigcup_{k=1}^{p}\bigcap_{x=1}^{n}\bigcap_{u=1}^{m}\left(r_{ik}{}^{I} \wedge \left(s_{ku}{}^{I} \vee p_{ux}{}^{I} \vee q_{xv}{}^{I}\right) \wedge t_{vj}{}^{I}\right)$$

$$c_{ij}{}^{T} = \bigcup_{v=1}^{l}\bigcup_{k=1}^{p}\bigcap_{x=1}^{n}\bigcap_{u=1}^{m}\left(\left(r_{ik}{}^{I} \wedge s_{ku}{}^{I} \wedge t_{vj}{}^{I}\right) \vee \left(r_{ik}{}^{I} \wedge p_{ux}{}^{I} \wedge t_{vj}{}^{I}\right) \vee \left(r_{ik}{}^{I} \wedge q_{xv}{}^{I} \wedge t_{vj}{}^{I}\right)\right)$$

$$c_{ij}{}^{F} = \bigcap_{v=1}^{l}\left(b_{iv}{}^{F} \vee t_{vj}{}^{F}\right) = \bigcap_{v=1}^{l}\left(\left\{\bigcap_{k=1}^{p}\left(r_{ik}{}^{F} \vee \left\{\bigcup_{x=1}^{n}\left[\bigcup_{u=1}^{m}\left(s_{ku}{}^{F} \wedge p_{ux}{}^{F}\right)\right] \wedge q_{xv}{}^{F}\right\}\right)\right\} \vee t_{vj}{}^{I}\right)$$

$$c_{ij}{}^{F} = \bigcap_{v=1}^{l}\bigcap_{k=1}^{p}\bigcup_{x=1}^{n}\bigcup_{u=1}^{m}\left(r_{ik}{}^{I} \vee \left(s_{ku}{}^{I} \wedge p_{ux}{}^{I} \wedge q_{xv}{}^{I}\right) \vee t_{vj}{}^{I}\right)$$

$$c_{ij}{}^{F} = \bigcap_{v=1}^{l}\bigcap_{k=1}^{p}\bigcup_{x=1}^{n}\bigcup_{u=1}^{m}\left(\left(r_{ik}{}^{I} \vee s_{ku}{}^{I} \vee t_{vj}{}^{I}\right) \wedge \left(r_{ik}{}^{I} \vee p_{ux}{}^{I} \vee t_{vj}{}^{I}\right) \wedge \left(r_{ik}{}^{I} \vee q_{xv}{}^{I} \vee t_{vj}{}^{I}\right)\right)$$

Thus, $d_{ij} = <\bigcup_{k=1}^{p}\left(r_{ik}{}^{T} \wedge s_{kj}{}^{T}\right), \bigcup_{k=1}^{p}\left(r_{ik}{}^{I} \wedge s_{kj}{}^{I}\right), \bigcap_{k=1}^{p}\left(r_{ik}{}^{F} \vee s_{kj}{}^{F}\right)>$

$$f_{ij} = < \bigcap_{u=1}^{m} \left(d_{iu}{}^{T} \vee p_{uj}{}^{T} \right), \bigcap_{u=1}^{m} \left(d_{iu}{}^{I} \vee p_{uj}{}^{I} \right), \bigcup_{u=1}^{m} \left(d_{iu}{}^{F} \vee p_{uj}{}^{F} \right) >$$

$$e_{ij} = < \bigcup_{v=1}^{l} \left(q_{iv}{}^{T} \wedge t_{vj}{}^{T} \right), \bigcup_{v=1}^{l} \left(q_{iv}{}^{I} \wedge t_{vj}{}^{I} \right), \bigcap_{v=1}^{l} \left(q_{iv}{}^{F} \vee t_{vj}{}^{F} \right) >$$

Thus, $g_{ij} = \bigcap_{x=1}^{n} \left(f_{ix}{}^{T} \vee e_{xj}{}^{T} \right)$

$$c_{ij}{}^{F} = \bigcap_{x=1}^{n} \bigcap_{u=1}^{m} \bigcup_{k=1}^{p} \bigcup_{v=1}^{l} \left(r_{ik}{}^{T} \wedge s_{ku}{}^{T} \right) \vee p_{ux}{}^{T} \vee \left(q_{xv}{}^{T} \wedge t_{vj}{}^{T} \right)$$

$$= \bigcup_{v=1}^{l} \bigcup_{k=1}^{p} \bigcap_{u=1}^{m} \bigcap_{x=1}^{n} \left(r_{ik}{}^{T} \wedge s_{ku}{}^{T} \right) \vee p_{ux}{}^{T} \vee \left(q_{xv}{}^{T} \wedge t_{vj}{}^{T} \right)$$ and

$$g^{F}{}_{ij} = \bigcap_{x=1}^{n} \left(f_{ix}{}^{F} \wedge e_{xj}{}^{F} \right)$$

$$= \bigcup_{v=1}^{n} \bigcup_{k=1}^{m} \bigcap_{u=1}^{p} \bigcap_{x=1}^{l} \left(r_{ik}{}^{F} \vee s_{ku}{}^{F} \right) \wedge p_{ux}{}^{F} \wedge \left(q_{xv}{}^{F} \wedge t_{vj}{}^{F} \right)$$

$$= \bigcap_{v=1}^{l} \bigcap_{k=1}^{p} \bigcup_{u=1}^{m} \bigcup_{x=1}^{n} \left(r_{ik}{}^{F} \wedge p_{ux}{}^{F} \wedge q_{xv}{}^{F} \right) \vee \left(r_{ik}{}^{F} \wedge p_{ux}{}^{F} \wedge t_{vj}{}^{F} \right) \vee \left(s_{ku}{}^{F} \wedge p_{ux}{}^{F} \wedge q_{xv}{}^{F} \right)$$

$$\vee \left(s_{ku}{}^{F} \wedge p_{ux}{}^{F} \wedge q_{vj}{}^{F} \right)$$

We get $c_{jj}{}^{T} \leq g_{jj}{}^{T}$

Also, since $r_{ik}{}^{F} \wedge p_{ux}{}^{F} \wedge q_{xv}{}^{F} \leq r_{ik}{}^{F}, \left(s_{ku}{}^{F} \wedge p_{ux}{}^{F} \wedge q_{vj}{}^{F} \right) \leq q_{vj}{}^{F}$

$\left(r_{ik}{}^{F} \wedge p_{ux}{}^{F} \wedge t_{vj}{}^{F} \right) \leq t_{vj}{}^{F}$

We get $c_{jj}{}^{F} \geq g_{jj}{}^{F}$

Thus $c_{jj} \leq g_{jj}$ and $R \leq W$

(ii) Similar to (i). Hence the Theorem.

9. **Algorithm and Application**

In this article, utilizing the previously stated definitions (7.2) and (7.4), the aim is to develop an approximate understanding of the diseases inferred from the symptoms presented by patients to healthcare professionals. The subsequent sections include the proposed algorithm and a case study to illustrate the concept in detail.

Step 1: Construct Matrices P and Q:

Matrix P (Patients × Symptoms): Arrange patients as rows and symptoms as columns. Populate each entry with Neutrosophic fuzzy values (T,I,F) regarding the presence of a specific symptom in a particular patient.

Matrix Q (Symptoms × Diseases): Arrange symptoms as rows and diseases as columns. Populate each entry with Neutrosophic fuzzy values (T,I,F) indicating the degree of association between a symptom and a particular disease.

Step 2. Compute the Transpose of neutrosophic Fuzzy matrix Q^T.

Step 3. Compute patient symptom disease matrix $P * Q$

Step 4. Compute $V_1 = \left(p_{ij}{}^{T} + p_{ij}{}^{I} - p_{ij}{}^{F} \right),$

Step 5. Compute patient symptom non disease matrix $P * Q^T = \left(p_{ij}{}^{T}, p_{ij}{}^{I}, p_{ij}{}^{F} \right)$

Step 6. Compute $W_1 = \left(p_{ij}{}^{T} + p_{ij}{}^{I} - p_{ij}{}^{F} \right),$

Step 7. Compute score matrix $S\left(P * Q, P * Q^T \right) = V_1 - W_1.$

Step 8. Identify maximum score for the patient A_i and conclude that the patient A_i is suffering from the disease C_i.

9.1 Healthcare Diagnosis Using Multi-Criteria Decision-Making (MCDM)

In the context of healthcare, diagnosing diseases based on the symptoms observed in patients is a critical task that requires careful analysis of various factors. MCDM is a method that helps in evaluating multiple alternatives (in this case, potential diseases) by considering several criteria (symptoms).

Let us consider four patients $A = \{A_1, A_2, A_3, A_4\}$, each exhibiting a set of symptoms $B = \{B_1, B_2, B_3, B_4, B_5\}$, where: B_1 represents temperature, B_2 represents headaches, B_3 represents cough, B_4 represents stomach pain and B_5 represents body pain

The possible diseases associated with these symptoms are represented by $C = \{C_1, C_2, C_3\}$ where: C_1 corresponds to viral fever, C_2 corresponds to typhoid, and C_3 corresponds to malaria of these diseases.

In this MCDM problem, the goal is to evaluate and diagnose the most likely disease for each patient based on their symptoms. The analysis will consider various symptom values for each patient to determine which disease has the highest probability based on the observed symptoms. This requires the use of MCDM techniques, such as composition of min-min-max ($*$), score function ranking, to make informed decisions about the appropriate disease diagnosis.

Construct Matrices P and Q,

$$P=\begin{bmatrix} <0.7,0.1,0.2> & <0.6,0.1,0.3> & <0.2,0.8,0.2> & <0.6,0.1,0.2> & <0.1,0.6,0.3> \\ <0.1,0.8,0.1> & <0.4,0.4,0.5> & <0.6,0.1,0.3> & <0.1,0.7,0.2> & <0.1,0.8,0.2> \\ <0.6,0.1,0.4> & <0.8,0.1,0.2> & <0.6,0,0> & <0.5,0.3,0.3> & <0.6,0.5,0.1> \\ <0.5,0.2,0.4> & <0.5,0.4,0.1> & <0.3,0.4,0.5> & <0.7,0.2,0.1> & <0.3,0.4,0.4> \end{bmatrix}$$

$$Q=\begin{bmatrix} <0.6,0.2,0.3> & <0.6,0.2,0.2> & <0.3,0.4,0.3> \\ <0.3,0.5,0.4> & <0.2,0.6,0.3> & <0.7,0.2,0.4> \\ <0.1,0.8,0.1> & <0.5,0.4,0.3> & <0.7,0.2,0.3> \\ <0.4,0.5,0.3> & <0.7,0.2,0.1> & <0.3,0.4,0.4> \\ <0.7,0.4,0.2> & <0.1,0.8,0.2> & <0.2,0.7,0.3> \end{bmatrix}$$

$$P*Q=\begin{bmatrix} <0.2,0.2,0.2> & <0.5,0.2,0.3> & <0.2,0.2,0.3> \\ <0.4,0.5,0.4> & <0.1,0.4,0.3> & <0.2,0.2,0.4> \\ <0.5,0.2,0.3> & <0.6,0.2,0.2> & <0.5,0.2,0.3> \\ <0.3,0.2,0.3> & <0.3,0.2,0.2> & <0.3,0.4,0.3> \end{bmatrix},$$

$$V_1=\begin{bmatrix} 0.2 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0 \\ 0.4 & 0.6 & 0.4 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$$

$$P*Q^T=\begin{bmatrix} <0.2,0.2,0.3> & <0.2,0.2,0.2> & <0.3,0.2,0.3> \\ <0.2,0.5,0.3> & <0.1,0.4,0.3> & <0.3,0.2,0.5> \\ <0.8,0.2,0.4> & <0.5,0.2,0.4> & <0.5,0.2,0.3> \\ <0.3,0.2,0.4> & <0.3,0.2,0.4> & <0.3,0.4,0.5> \end{bmatrix},\ W_1=\begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0 \\ 0.6 & 0.2 & 0 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

$$S\left(P*Q,P*Q^T\right)=V_1-W_1=\begin{bmatrix} & C_1 & C_2 & C_3 \\ A_1 & 0.1 & 0.2 & -0.1 \\ A_2 & 0.1 & 0 & 0 \\ A_3 & -0.2 & 0.3 & 0 \\ A_4 & 0.1 & 0.2 & 0.2 \end{bmatrix}$$

Analysis: In the first row, the max value is 0.2, indicating that the patient is suffering from typhoid. In the second row, the max value is 0.1, indicating that the patient is suffering from viral fever. In the third row, the max value is 0.3, indicating that the patient is suffering from typhoid. In the fourth row, the max value is 0.2, suggesting that the patient may be suffering from both typhoid and malaria. Mamoni Dhar

In this study, we have adopted the same example as presented in Mamoni Dhar [50]. The results obtained are detailed below:

TABLE:1

Ai	Dhar[SF] [50]			Sahin, R.[SF][44]			Sahin,[AF][44]			Nancy, Garg,[SF][45]			Nancy, Garg,[AF][45]		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
A_1	0.2	0.4	0.2	0	0.2	0.1	0.24	0.44	0.24	0.14	0.28	0.14	0.2	0.4	0.2
A_2	0.6	0.2	0.1	-0.15	0.1	0.05	0.26	0.28	0.16	-0.165	0.2	0.085	-0.3	0.2	0.1
A_3	0.3	-0.1	0.2	0.15	-0.05	0.1	0.32	-0.08	0.28	0.195	-0.065	0.14	0.3	-0.1	0.2
A_4	0.1	0.5	0	0.05	0.25	0	0.12	0.56	0.08	0.075	0.345	0.07	0.1	0.5	0

The composition method used in Dhar [50] is max-min-min (∗). In our analysis, we replaced the score function used in that paper with various alternative score functions in Sahin [44], Nancy, Garg [45]. Upon comparing the results, we observed that Patient 2 is identified as suffering from viral fever rather than typhoid when different score functions are applied.

TABLE : 2

Ai	Dhar[SF] [50]			Sahin, R.[SF][44]			Sahin,[AF][44]			Nancy, Garg,[SF][45]			Nancy, Garg,[AF][45]		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
A_1	0.1	0.2	-0.1	0.05	0.1	-0.05	0.08	0.28	-0.12	0.055	0.2	-0.095	0.1	0.2	-0.1
A_2	0.1	0	0	0.05	0.00	0	0.25	0	-0.04	0.225	0	-0.06	0.1	0	0
A_3	-0.2	0.3	0	-0.1	0.15	0	-0.28	0.28	0	-0.12	0.165	0	-0.2	0.3	0
A_4	0.1	0.2	0.2	0.05	0.1	0.1	0.08	0.16	0.12	0.045	0.1	0.04	0.1	0.2	0.2

Furthermore, when the composition min-min-max (∗) (proposed method) is applied, the results remain consistent, confirming that Patient 2 is diagnosed with viral fever. This analysis underscores the influence of score function selection on the outcomes while showcasing the robustness of the proposed composition method. Additionally, based on the above analysis, it can be decided that Patient 4 may be affected by both typhoid and malaria.

10. Conclusion and Future Work

In this paper, we introduced new types of Neutrosophic Fuzzy Matrices (NFMs) and defined specificcompositions—namely,the-max(min)–max(min)-min(max) (∘) and min(max) – min(max) – max(min) (∗) compositions—for these matrices. Through these compositions, we derived several critical theoretical results that enhance the understanding and potential applications of NFMs. Notably, the construction of an idempotent NFM using the min-min-max composition demonstrates a significant advancement, as this approach provides a method for simplifying matrices while preserving essential neutrosophic properties. A numerical example was provided to clarify the practical implementation of these compositions, illustrating the theoretical results in a tangible form. Additionally, we proposed an algorithm that applies NFMs in the context of decision-making (DM), showcasing its effectiveness in solving DM problems that involve high degrees of uncertainty. By using the specific compositions of max(min)–max(min)-min(max)(∘)and-min(max) – min(max) – max(min) (∗), the proposed method enables flexible and reliable solutions in DM applications. This work highlights the versatility and potential of NFMs in both theoretical and applied settings, particularly in fields requiring nuanced uncertainty handling.

In this study, we revisited the example presented by Mamoni Dhar [50], which employed the max-min-min (∗) composition method. While adopting the same example, we replaced the score function used in Dhar's approach with various alternative score functions proposed by Sahin [44] and Nancy, Garg [45]. The comparative analysis revealed notable variations in the results when different score functions were applied, indicating the sensitivity of the method to the choice of score functions. Moreover, when we applied our proposed composition method, min(max)-min(max)-max(min) the results remained consistent irrespective of the score functions used. This robustness highlights the reliability of the proposed method in handling variations introduced by different score functions. Overall, our proposed min(max) – min(max) – max(min) (∗) composition method outperforms the max(min)-min(max)-min(max) (∗) method used by Mamoni Dhar [50] by ensuring consistency and reducing dependency on the choice of score functions. This improvement demonstrates the potential of our method for applications requiring robust and reliable decision-making frameworks.

Future research will focus on further exploring the theoretical properties and potential applications of NFMs, with an emphasis on expanding the range of compositions and examining their implications for broader classes of mathematical structures. Additionally, we aim to enhance the proposed DM algorithm by incorporating more complex neutrosophic operations and exploring its performance in real-world DM scenarios with higher uncertainty and larger datasets. Another promising direction involves the integration of NFMs with other mathematical frameworks, such as fuzzy graph theory and machine learning, to extend the applicability of NFMs in complex decision-making, pattern recognition, and other data-intensive fields. These efforts will contribute to the development of more sophisticated tools for uncertainty management in a variety of disciplines.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8, 338-353. Atanassov, K. (2016). Intuitionistic fuzzy sets. International journal bioautomation, 20(1), 87-96.

[2] Molodtsov, D. (1999). Soft set theory—first results. Computers & mathematics with applications, 37(4-5), 19-31.

[3]Maji, P. K. Biswas, R., & Roy, A. R. (2001). Fuzzy soft sets. Journal of fuzzy mathematics, 9(3), 589-602.

[4] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. Computers & mathematics with applications, 45(4-5), 555-562.

[5] Maji, P. K. (2009, December). More on intuitionistic fuzzy soft sets. International workshop on rough sets, fuzzy sets, data mining, and granular-soft computing (pp. 231-240). Berlin, Heidelberg: Springer.

[6] Smarandache, F. (2018). Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. Neutrosophic sets syst, 22, 168-170.

[7] Smarandache, F. (2019). Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). Journal of new theory, (29), 1-31.

[8] Vellapandi, R., & Gunasekaran, S. (2020). A new decision making approach for winning strategy based on muti soft set logic. Journal of fuzzy extension and applications, 1(2), 119-129.

[9] Maji, P. K. (2013). Neutrosophic soft sets. Annals of fuzzy mathematics and informatics, 5(1), 157-168.

[10] Maji, P. K. (2012). A neutrosophic soft set approach to a decision making problem. Annals of fuzzy mathematics and informatics, 3(2), 313-319.

[11] Deli, I., & Broumi, S. (2015). Neutrosophic soft matrices and NSM-decision making. Journal of intelligent & fuzzy systems, 28(5), 2233-2241.

[12] Das, S., Kumar, S., Kar, S., & Pal, T. (2019). Group decision making using neutrosophic soft matrix: An algorithmic approach. Journal of king saud University-computer and information sciences, 31(4), 459-468.

[13] Kumar Das, S. (2020). Application of transportation problem under pentagonal neutrosophic environment. Journal of fuzzy extension and applications, 1(1), 27-41.

[14] K. Atanassov , Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986) 87–96 .

[15] K. Atanassov , More on intuitionistic fuzzy sets, Fuzzy Sets Syst. 33 (1989) 37–45 .

[16] K. Atanassov , Opertions over interval-valued fuzzy sets, Fuzzy Sets Syst. 64 (1994) 159–174 .

[17] A.K. Adak , M. Bhowmik , M. Pal , Application of generalized in- tuitionistic fuzzy matrix in multi-criteria making problem, J. Math Compute Sci. 1 (2011) 19–31 .

[18] D. Dubois , H. Prade , An intuitionistic to bipolar representation and preference, Int. J. Intell, Syst. 23 (2008) 866–877 .
[19] U. Dudziak , B. Pekala , Equivalent bipolar fuzzy relations, Fuzzy Sets Syst. 161 (2010) 234–253 .
[20] T. Gerstenkorn , J. Manko , Bifuzzy probabilistic sets, Fuzzy Sets Syst. 71 (1995) 202–214 .
[21] M.Z. Ragab , E.G. Emam , On the min-max composition of fuzzy matrices, Fuzzy Sets Syst. 75 (1995) 83–92 .
[22] H. Hashimoto , Canonical form of a transitive matrix, Fuzzy Sets Syst. 11 (1983) 157–162 .
[23] A.K. Adak , M. Bhowmik , M. Pal , Semiring of generalized in- terval-valued intuitionistic fuzzy matrices, World Appl. Sci. J. 16 (2012) 07–16 .
[24] E.G. Emam , Some notice on nilpotent and transitive bipolar fuzzy matrices, Wulfenia J. 21 (2014) 218–232 .
[25] D. Dubois and H. Prade, Fuzzy Sets and Systems, Theory and Application (Academic Press, New York, 1980).
[26] A. Kaufmann, Introduction to the Theory of Fuzzy Subsets (Academic Press, New York, 1975).
[27] K.H. Kim and F.W. Roush, Generalized fuzzy matrices, Fuzzy Sets and Systems 4 (1980) 293-315.
[28] Li Jian-Xin, Controllable fuzzy matrices, Fuzzy Sets and Systems 45 (1992) 313-319.
[29] M.A. Mishref and E.G. Emam, Transitivity and subinverses in fuzzy matrices, Fuzzy Sets and Systems 52 (1992) 337-343.
[30] M.Z. Ragab and E.G. Emam, The determinant and adjoint of a square fuzzy matrix, Fuzzy Sets and Systems 61 (1994) 297-307.
[31] J. Verba, General decomposition problem of fuzzy relations, Fuzzy Sets and Systems 54 (1993) 69-79.
[32] J. B. Kim, "Idempotents and inverse in fuzzy matrices," Bulletin of the Malaysian Mathematical Sciences Society, vol. 6, no. 2, pp. 57-61, 1983.
[33] J. B. Kim, "On fuzzy idempotent matrices of T-Type," Information Sciences, vol. 80, no. 3-4, pp. 311-318, 1994.
[34] A. R. Meenakshi, Fuzzy Matrix: Theory and Applications. Chennai: MJP Publishers, 2008.
[35] Riyaz Ahmad Padder and P.Murugadas, On Idempotent Intuitionistic Fuzzy Matrices of T-Type International Journal of Fuzzy Logic and Intelligent Systems Vol.16,No.3, September 2016, pp.181-187
[36] R.Uma, P. Murugadas and S. Sriram, Fuzzy Neutrosophic Soft Matrices of Type I and Type II, Volume 13, Pages 211-222, 2021.
[37]M. Anandhkumar , T. Harikrishnan , S. M. Chithra , V. Kamalakannan , B. Kanimozhi, Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices, International Journal of Neutrosophic Science, 23(2) , 286-295, 2024.
[38] M. Anandhkumar , V. Kamalakannan , S. M. Chithra , Broumi Said, Pseudo Similarity of Neutrosophic Fuzzy matrices, International Journal of Neutrosophic Science, Volume 20(4) ,191-196, 2023.

[39] E. G. Emama, An Operation on Intuitionistic Fuzzy Matrices, Published by Faculty of Sciences and Mathematics, University of Nis, Serbia, Filomat 34:1 (2020), 79–88.

[40] Rajkumar Pradhan, Madhumangal Pal, Transitive and strongly transitive intuitionistic fuzzy matrices, Annals of Fuzzy Mathematics and Informatics, Volume 13, No. 4, (April 2017), pp. 485–498.

[41] E. G. Emam, On Consistent and Weak Transitive Intuitionistic Fuzzy Matrices, Fuzzy Information and Engineering ,Volume 14, 2022, Pages 16-25.

[42] E.G. Emam, M.A. Fndh, Some results associated with the max–min and min–max compositions of bifuzzy matrices, Journal of the Egyptian Mathematical Society,24(4), 515-521, 2016.

[43] Akanksha Singh, Shahid Ahmad Bhat, A novel score and accuracy function for neutrosophic sets and their real-world applications to multi-criteria decision-making process, Neutrosophic Sets and Systems, Vol. 41, 168-197, 2021.

[44] Sahin, R. (2014). Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. http://arxiv.org/abs/1412.5202. (accessed on Dec 17, 2014)

[45] Nancy, Garg, H. (2016). An improved score function for ranking neutrosophic sets and its application to decision-making process. International Journal for Uncertainty Quantification, 6, 377-385.

[46] Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi. "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 61, 1 (2024).

[47] M. Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–12.

[48]M. Anandhkumar, A. Bobin, S. M. Chithra, and V. Kamalakannan, Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices. Neu trosophic Sets and Systems 70, 1 (2024).

[49] M. Anandhkumar, G. Punithavalli, and E.Janaki, secondary k-column neutrosophic fuzzy matrices, Neutrosophic Sets and Systems 64, 2024, pp. 24-37.

[50] Mamoni Dhar, Neutrosophic Soft Matrices and Its Application in Medical Diagnosis, J. Fuzzy. Ext. Appl. Vol. 2, No. 1 (2021) 23–32.

[51].M.Anandhkumar , S. Prathap , R. Ambrose Prabhu , P.Tharaniya , K. Thirumalai , B. Kanimozhi, Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems, Neutrosophic Sets and Systems, Vol. 79, 2025.

[52]. K. Radhika, S. Senthil , N. Kavitha , R.Jegan , M.Anandhkumar, A. Bobin, Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making, Neutrosophic Sets and Systems, Vol. 78, 2025.

[53].Anandhkumar, M., Kanimozhi, B., Chithra, S.M., Kamalakannan, V., Said, B., "On various Inverse of Neutrosophic Fuzzy Matrices", International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.

[54].K. Radhika, T. Harikrishnan , R. Ambrose Prabhu, , P.Tharaniya , M.John peter, M.Anandhkumar, On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 78, 2025.

[55].H. Prathab, N. Ramalingam, E. Janaki, A. Bobin, V. Kamalakannan and M. Anandhkumar, Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses, IAENG International Journal of Computer Science, Volume 51, Issue 12, December 2024, Pages 2051-2066.

[56].M.Anandhkumar, B.Kanimozhi, S.M. Chithra, V.Kamalakannan, Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices, Mathematical Modelling of Engineering Problems, 2023, 10(4), pp. 1427–1432.

[57].G. Punithavalli1, M. Anandhkumar, Reverse Sharp and Left-T Right-T Partial Ordering on Intuitionistic Fuzzy Matrices, TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1772-1783.

[58].M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, vol. 23, no. 4, 2024, pp. 23-28.

[59]. G. Punithavalli, M. Anandhkumar, Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices,TWMS J. App. and Eng. Math. V.14, N.3, 2024, pp. 1231-1240.

[60]. Anandhkumar, M., Harikrishnan, T., Chithra, S.M., ...Kanimozhi, B., Said, B. "Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices"International Journal of Neutrosophic Science, 2023, 21(4), pp. 135–145.

Received: Nov. 7, 2024. Accepted: May 20, 2025