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Single-Valued Neutrosophic Multiple sets and its Application in Multi-Criteria Decision-Making

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Abstract. Single-valued neutrosophic set ($SVNS$) is an advanced generalization of fuzzy sets and intuitionistic fuzzy sets, developed to better represent indeterminate and inconsistent information. In contrast, multiple sets are generalized fuzzy sets that can simultaneously represent several ambiguous attributes of an item in multiple ways. In this article, we introduce the concept of the single-valued neutrosophic multiple set ($SVNMS$), which extends both multiple sets and $SVNS$, offering a more effective tool for addressing complex multi-criteria decision-making ($MCDM$) problems. Additionally, we present the theory of similarity measures between single-valued neutrosophic multiple sets and establish a multiple-attribute decision-making method based on these similarity measures. We rank the options and select the best one by comparing each alternative to the ideal alternative using similarity measures. Finally, we provide a practical example to demonstrate the application and effectiveness of $SVNMS$ in the multiple-attribute decision-making method.

Keywords: Single-valued neutrosophic set ($SVNS$), Single-valued neutrosophic multiple set ($SVNMS$), Multiple set, Similarity measure, Multicriteria decision-making

1. **Introduction**

The introduction of fuzzy sets [1], intuitionistic fuzzy sets (IFS) [2], and interval-valued intuitionistic fuzzy sets ($IVIFS$) [3] was intended to address imprecision and uncertainty. They are frequently used in machine learning, image processing, pattern recognition, information retrieval, data mining, decision-making, and other fields. Even while fuzzy sets, IFS, and $IVIFS$ are highly effective in their respective fields, they are unable to capture the ambiguous and inconsistent information found in the actual world. In order to address ambiguity, imprecise, incomplete, and inconsistent data, Smarandache [4]- [20] introduced the idea of a neutrosophic set. Neurosophic set is a strong generic formal framework that expands on the ideas of the classic set, fuzzy set, $IFS, IVIFS$, and others. The factors defined by the neutrosophic set are highly appropriate for human cognition, as they account for the inherent imperfections in the knowledge individuals acquire (or observe) from the external world.

Clearly, the neutrosophic components are particularly well-suited for representing indeterminate and inconsistent information, making them highly applicable for addressing a wide range of decision-making problems. Decision-makers are increasingly confronted with complex decision-making challenges, characterized by a high degree of uncertainty. Consequently, a unified neutrosophic multi-criteria decision-making ($MCDM$) approach may prove effective in addressing numerous ambiguities. Thus the theory of neutrosophic sets has been widely applied in many real-life problems involving the identification of patterns, personnel selection, medical diagnosis, classification problems, etc. [9]- [14].

Neutrosophic set generalizes the above-mentioned sets from a philosophical point of view. It is necessary to specify the neutrosophic set and set-theoretic operators from a scientific or technical standpoint. If not, it will be challenging to use in actual applications. Therefore, Wang et al. (2010) proposed a single-valued neutrosophic set ($SVNS$) [15], which is an instance of the neutrosophic set, and provided the set-theoretic operators and various properties of $SVNS$. The theory of $SVNS$ can be used to scientific and technical domains since it is helpful for representing uncertain, imprecise, and inconsistent information. The ability of $SVNS$ to easily capture the ambiguous nature of subjective evaluations makes them suitable for capturing inconsistent, ambiguous, and imprecise information in multi-criteria decision analysis [16] - [20].

However, an $MCDM$ problem will become more challenging to solve with $SVNS$, though, if the criterion or evaluation level is raised. Therefore, this research introduces a new notion called 'single-valued neutrosophic multiple set' ($SVNMS$) which is an extended version of $SVNS$ and multiple set. Multiple set introduced by Shijina et al. [21] is a powerful tool to handle the uncertainty of an element together with its multiplicity. More precisely, a fuzzy membership function is used to represent each of an object's uncertain properties, and values are allocated to the function according to the object's multiplicity. The ability to condense all data into a single matrix is the main benefit of multiple sets. This means that each item in a multiple set is given a matrix, where each row corresponds to a unique fuzzy membership function that is determined by its attribute. As a result, it is broadly applicable to solve several practically existing $MCDM$ problems more easily with less time consumption. Prior research on multiple sets has demonstrated their use in personal selection, medical diagnosis, and pattern recognition [22]- [26]. Multiple sets have the flaw of not being able to characterize the degree of consistency. This restricts its application in numerous real-world problems. Thus, combining the theory of multiple sets and $SVNS$ results in the development of $SVNMS$, which can describe indeterminacy and imprecision more precisely. Additionally, its capacity to concurrently manage multiplicity and uncertainty aids in the resolution of a large number of $MCDM$ issues.

In this article we develop the theory of $SVNMS$ and define the concept of similarity measures between $SVNMS$. We have established an axiomatic definition for similarity measures between $SVNMS$ and proposed new similarity measures using max-min operators and fuzzy similarity measures. Additionally, it develops a multiple-attribute decision-making method based on the similarity measure of $SVNMS$ within a single-valued neutrosophic environment. The ranking order of each option may be established, and the best alternative can be quickly recognised using the similarity measures between each alternative and the ideal alternative. The structure of the article is as follows: Section 2 outlines the fundamental concepts essential for the study. Section 3 covers the basic concepts of $SVNMS$. Section 4 presents new similarity measures between $SVNMS$ using minimum and maximum operators and fuzzy similarity measures and explores their properties. Section 5 introduces a single-valued neutrosophic decision-making approach based on the similarity measure $SVNMS$ and provides a practical example to showcase the applications and effectiveness of the proposed decision-making approach. In Section 6, we conduct a comparative analysis of the proposed decision-making method against an existing method. Finally, Section 7 offers conclusions and suggestions for further research.

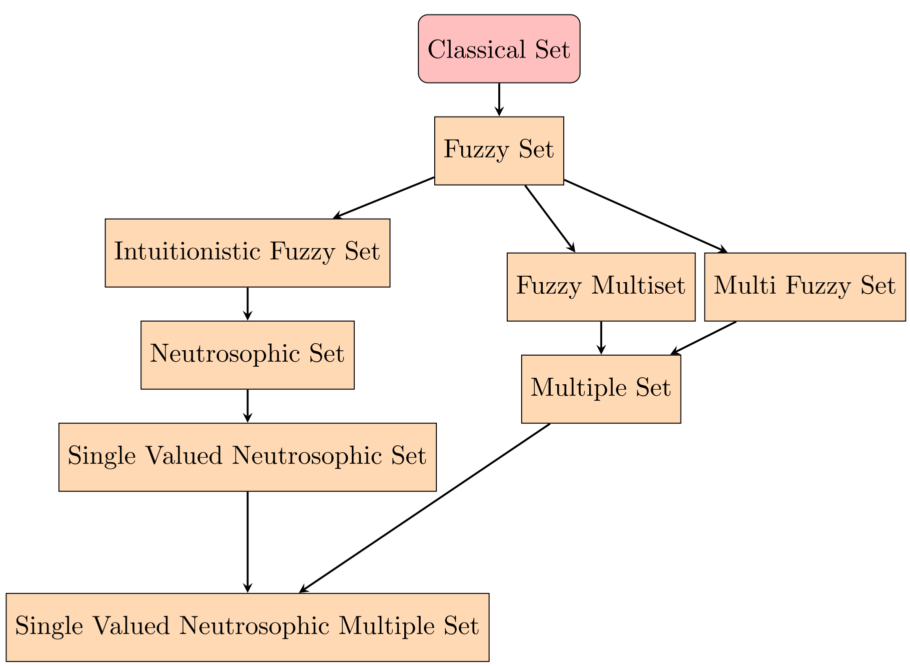

Figure 1 : *Conceptual development of neutrosophic multiple sets*

2. Preliminaries

This section describes the basic concepts of neutrosophic sets, single-valued neutrosophic sets, and multiple sets.

2.1. *Notation list*

The indices and notations used in this study are given as follows:

Indices

i	Index of u fuzzy membership functions
j	Index of multiplicities of a fixed fuzzy membership function
I	Index set

Notations

$\mathbf{MS}_{(u,v)}(X)$	The collection of all multiple sets of order (u, v) over X
$[0]_{(u,v)}$	Membership matrix with all entries 0
$[1]_{(u,v)}$	Membership matrix with all entries 1
N_v	$\{1, 2, 3, \ldots, v\}$

Definition 2.1. [5] Let X be a space of points; a neutrosophic set N in X is characterized by a truth-membership function $T_N(x')$, an indeterminacy-membership function $I_N(x')$, and a falsity-membership function $F_N(x')$. The functions $T_N(x')$, $I_N(x')$ and $F_N(x')$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $T_N(x') : X \longrightarrow]0^-, 1^+[, I_N(x') : X \longrightarrow]0^-, 1^+[$, and $F_N(x') : X \longrightarrow]0^-, 1^+[$.

Definition 2.2. [15] Let X be a universal set. A single-valued neutrosophic set N, in X is characterized by a truth-membership function $T_N(x')$, an indeterminacy-membership function $I_N(x')$, and a falsity-membership function $F_N(x')$. Then, an $SVNS, N$ can be denoted by

$$N = \{(x', T_N(x'), I_N(x'), F_N(x')) : x' \in X\},$$

where $T_N(x'), I_N(x'), F_N(x') \in [0, 1]$ for each point x' in X.

Definition 2.3. [15] The complement of an single-valued neutrosophic set N, denoted by N^c, is defined as $T_N^c(x') = F_N(x'), I_N^c(x') = 1 - I_N(x')$ and $F_N^c(x') = T_N(x')$. That is, $N^c = \{(x', F_N(x'), 1 - I_N(x'), T_N(x')) : x' \in X\}$.

Definition 2.4. [15] Let M and N be two $SVNS$. Then $M \subset N$ if $T_M(x') \leq T_N(x'), I_M(x') \geq I_N(x'), F_M(x') \geq F_N(x')$ for all x' in X.

Definition 2.5. [21] Let $N_1, N_2, \ldots, N_u$ be u distinct fuzzy sets on X and $N_i^1(x), N_i^2(x'), \ldots, N_i^v(x')$ denote v membership values of the fuzzy set N_i in the decreasing order for $i = 1, 2, \ldots, u$. Then, a multiple set $\mathbf{N}$ of order (u, v) over X is a collection $\{(x', \mathbf{N}(x)) : x' \in X\}$, where for each, $x' \in X$ its membership value is an $u \times v$ matrix given by

$$\mathbf{N(x')} = \begin{bmatrix} N_1^1(x') & N_1^2(x') & \cdots & N_1^v(x') \\ N_2^1(x') & N_2^2(x') & \cdots & N_2^v(x') \\ \cdots & \cdots & \cdots & \cdots \\ N_u^1(x') & N_u^2(x') & \cdots & N_u^v(x') \end{bmatrix}$$

Definition 2.6. [21] Let $\mathbf{R}, \mathbf{S} \in \mathbf{MS}_{(u,v)}(X)$ then,

(i) $\mathbf{R}$ subset of $\mathbf{S}$ denoted as, $\mathbf{R} \subseteq \mathbf{S}$ if and only if $\mathbf{R}(x') \leq \mathbf{S}(x')$ for every $x' \in X$.

(ii) The union of $\mathbf{R}$ and $\mathbf{S}$ is a multiple set in $\mathbf{MS}_{(u,v)}(X)$ denoted as, $\mathbf{R} \cup \mathbf{S}$ whose membership matrix is $(\mathbf{R} \cup \mathbf{S})(x') = \mathbf{R}(x') \vee \mathbf{S}(x')$ for every $x' \in X$, where $\vee$ denotes the maximum operator.

(iii) The intersection of $\mathbf{R}$ and $\mathbf{S}$ is a multiple set in $\mathbf{MS}_{(u,v)}(X)$ denoted as, $\mathbf{R} \cap \mathbf{S}$ whose membership matrix is $(\mathbf{R} \cap \mathbf{S})(x') = \mathbf{R}(x') \wedge \mathbf{S}(x')$ for every $x' \in X$ where $\wedge$ denotes the minimum operator.

(iv) The complement of $\mathbf{R} \in \mathbf{MS}_{(u,v)}(X)$, denoted by $\bar{\mathbf{R}}$, whose membership matrix for each $x' \in X$ is an $u \times v$ matrix $\bar{\mathbf{R}}(x') = [\bar{R}_i^j(x')]_{u \times v}$, where $\bar{R}_i^j(x') = 1 - R_i^{v-j+1}(x'), \forall i \in N_u$ and $j \in N_v$.

3. Single-valued neutrosophic multiple sets

This section introduces the concept of $SVNMS$ and examines its characteristics. The theory of $SVNMS$ develops as an extension of $SVNS$ and multiple sets.

Definition 3.1. Let X be a universal set. A single-valued neutrosophic multiple set $\mathbf{N}$ of order (u, v) is of the form

$$\mathbf{N} = \{(x, \mathbf{T_N}(x), \mathbf{I_N}(x), \mathbf{F_N}(x)) : x \in X\},$$

where $\mathbf{T_N}(x), \mathbf{I_N}(x)$ and $\mathbf{F_N}(x)$ denote the truth membership matrix, indeterminacy membership matrix, and falsity membership matrix of order (u, v) given by,

$$\mathbf{T_N}(x) = \begin{bmatrix} (T_N)_1^1(x) & (T_N)_1^2(x) & \cdots & (T_N)_1^v(x) \\ (T_N)_2^1(x) & (T_N)_2^2(x) & \cdots & (T_N)_2^v(x) \\ \cdots & \cdots & \cdots & \cdots \\ (T_N)_u^1(x) & (T_N)_u^2(x) & \cdots & (T_N)_u^v(x) \end{bmatrix}$$

$$\mathbf{I_N}(x) = \begin{bmatrix} (I_N)_1^1(x) & (I_N)_1^2(x) & \cdots & (I_N)_1^v(x) \\ (I_N)_2^1(x) & (I_N)_2^2(x) & \cdots & (I_N)_2^v(x) \\ \cdots & \cdots & \cdots & \cdots \\ (I_N)_u^1(x) & (I_N)_u^2(x) & \cdots & (I_N)_u^v(x) \end{bmatrix}$$

$$\mathbf{F_N}(x) = \begin{bmatrix} (F_N)_1^1(x) & (F_N)_1^2(x) & \cdots & (F_N)_1^v(x) \\ (F_N)_2^1(x) & (F_N)_2^2(x) & \cdots & (F_N)_2^v(x) \\ \cdots & \cdots & \cdots & \cdots \\ (F_N)_u^1(x) & (F_N)_u^2(x) & \cdots & (F_N)_u^v(x) \end{bmatrix}$$

Note that, for each $i \in N_u$ and $j \in N_v$, $(T_N)_i^j, (I_N)_i^j$ and $(F_N)_i^j$ denote a truth function, an indeterminacy function, and a falsity function, respectively. That is,

$$(T_N)_i^j : X \longrightarrow [0,1]$$
$$(I_N)_i^j : X \longrightarrow [0,1]$$
$$(F_N)_i^j : X \longrightarrow [0,1]$$

Now for any $x, i, j, 0 \leq (T_N)_i^j(x)+(I_N)_i^j(x)+(F_N)_i^j(x) \leq 3$ and for any fixed $i \in N_u, (T_N)_i^1(x) \subseteq (T_N)_i^2(x) \subseteq \ldots (T_N)_i^v(x)$, $(I_N)_i^1(x) \subseteq (I_N)_i^2(x) \subseteq \ldots (I_N)_i^v(x)$ and, $(F_N)_i^1(x) \subseteq (F_N)_i^2(x) \subseteq \ldots (F_N)_i^v(x)$.
The collection of all $SVNMS$ of order (u,v) over X is denoted as $\mathbf{nms}_*(\mathbf{X})_{(u,v)}$.

Definition 3.2. The complement of a single-valued neutrosophic multiple set $\mathbf{N}$ is $\overline{\mathbf{N}} = \{(x, \mathbf{T}_{\overline{\mathbf{N}}}(x), \mathbf{I}_{\overline{\mathbf{N}}}(x), \mathbf{F}_{\overline{\mathbf{N}}}(x)) : x \in X\}$, where for each $x \in X$ $(T_{\overline{N}})_i^j(x) = (F_N)_i^j(x)$, $(I_{\overline{N}})_i^j(x) = 1 - (I_N)_i^{v-j+1}(x)$ and $(F_{\overline{N}})_i^j(x) = (T_N)_i^j(x)$ for all x, i, j.

Definition 3.3. Let $\mathbf{M}$ and $\mathbf{N}$ be two $SNNMS$ of order (u,v), then $\mathbf{M} \subseteq \mathbf{N}$ if $(T_M)_i^j(x) \leq (T_N)_i^j(x), (I_M)_i^j(x) \geq (I_N)_i^j(x), (F_M)_i^j(x) \geq (F_N)_i^j(x) \;\; \forall \;\; x, i, j$.

Theorem 3.4. *Let $\mathbf{M}$ and $\mathbf{N}$ be two $SVNMS$ of same of the same then $\mathbf{M} \subseteq \mathbf{N}$ if and only if $\overline{\mathbf{N}} \subseteq \overline{\mathbf{M}}$.*

Proof. $\mathbf{M} \subseteq \mathbf{N}$ if and only if $(T_M)_i^j(x) \leq (T_N)_i^j(x), (I_M)_i^j(x) \geq (I_N)_i^j(x)$, $(F_M)_i^j(x) \geq (F_N)_i^j(x) \;\; \forall \;\; x, i, j$. Now, $(T_{\overline{N}})_i^j(x) = (F_N)_i^j(x) \leq (F_M)_i^j(x) = (T_{\overline{M}})_i^j(x)$, $(I_{\overline{N}})_i^j(x) = 1 - (I_N)_i^{v-j+1}(x) \geq 1 - (I_M)_i^{v-j+1}(x) = (I_{\overline{M}})_i^j(x)$, $(F_{\overline{N}})_i^j(x) = (I_N)_i^j(x) \geq (I_M)_i^j(x) = (F_{\overline{M}})_i^j(x) \;\; \forall \;\; x, i, j$ if and only if $\overline{\mathbf{N}} \subseteq \overline{\mathbf{M}}$. □

Definition 3.5. The union of two single-valued neutrosophic multiple sets $\mathbf{M}$ and $\mathbf{N}$ is $\mathbf{M} \cup \mathbf{N} = \{(x, \mathbf{T_{M\cup N}}, \mathbf{I_{M\cup N}}, \mathbf{F_{M\cup N}}) : x \in X\}$ where,

$$(T_{M\cup N})_i^j(x) = max\{(T_M)_i^j(x), (T_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(I_{M\cup N})_i^j(x) = max\{(I_M)_i^j(x), (I_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(F_{M\cup N})_i^j(x) = max\{(F_M)_i^j(x), (F_N)_i^j(x)\} \ \forall \ x, i, j.$$

Theorem 3.6. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two single-valued neutrosophic multiple sets of same order then,* $\mathbf{M} \cup \mathbf{N}$ *is the smallest* $SVNMS$ *containing both* $\mathbf{M}$ *and* $\mathbf{N}$.

Proof. The proof follows in a similar way as in $SVNS$. □

Definition 3.7. The intersection of two single-valued neutrosophic multiple sets $\mathbf{M}$ and $\mathbf{N}$ is $\mathbf{M} \cap \mathbf{N} = \{(x, \mathbf{T_{M\cap N}}, \mathbf{I_{M\cap N}}, \mathbf{F_{M\cap N}}) : x \in X\}$, where

$$(T_{M\cap N})_i^j(x) = min\{(T_M)_i^j(x), (T_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(I_{M\cap N})_i^j(x) = min\{(I_M)_i^j(x), (I_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(F_{M\cap N})_i^j(x) = min\{(F_M)_i^j(x), (F_N)_i^j(x)\} \ \forall \ x, i, j.$$

Theorem 3.8. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two single-valued neutrosophic multiple sets of the same order, then* $\mathbf{M} \cap \mathbf{N}$ *is the largest* $SVNMS$ *contained in both* $\mathbf{M}$ *and* $\mathbf{N}$.

Proof. The proof follows in a similar way as in $SVNS$. □

Definition 3.9. The difference of two single-valued neutrosophic multiple sets $\mathbf{M}$ and $\mathbf{N}$ is $\mathbf{M}/\mathbf{N} = \{(x, \mathbf{T_{M/N}}, \mathbf{I_{M/N}}, \mathbf{F_{M/N}}) : x \in X\}$ where

$$(T_{M/N})_i^j(x) = min\{(T_M)_i^j(x), (F_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(I_{M/N})_i^j(x) = min\{(I_M)_i^j(x), 1 - (I_N)_i^j(x)\} \ \forall \ x, i, j,$$
$$(F_{M/N})_i^j(x) = min\{(F_M)_i^j(x), (T_N)_i^j(x)\} \ \forall \ x, i, j.$$

Definition 3.10. The truth favorite of an $SVNMS$ $\mathbf{N}$ is $\triangle \mathbf{N} = \{(x, \triangle \mathbf{T}, \triangle \mathbf{I}, \triangle \mathbf{F}) : x \in X\}$ where

$$(\triangle T)_i^j(x) = min\{(T_N)_i^j(x) + (I_N)_i^j(x), 1\} \ \forall \ x, i, j,$$
$$(\triangle I)_i^j(x) = 0 \ \forall \ x, i, j,$$
$$(\triangle F)_i^j(x) = (F_N)_i^j(x) \ \forall \ x, i, j.$$

Definition 3.11. The falsity favorite of an $SVNMS$ $\mathbf{N}$ is $\nabla\mathbf{N} = \{(x, \nabla\mathbf{T}, \nabla\mathbf{I}, \nabla\mathbf{F}) : x \in X\}$ where,

$$(\nabla T)_i^j(x) = (T_N)_i^j(x)\ \forall\ x, i, j,$$
$$(\nabla I)_i^j(x) = 0\ \forall\ x, i, j,$$
$$(\nabla F)_i^j(x) = min\{(F_N)_i^j(x) + (I_N)_i^j(x), 1\}\ \forall\ x, i, j.$$

4. Similarity measures of $SVNMS$

In this section, the similarity measure between two $SVNMS$ is defined axiomatically, and we propose similarity measures using max-min operators and fuzzy similarity measures.

Definition 4.1. $S : \mathbf{nms}_*(X)_{(u,v)} \longrightarrow [0, \infty)$ is a similarity measure of $SVNMS$ if,

(i) $S(\mathbf{M}, \mathbf{N}) = S(\mathbf{N}, \mathbf{M})\ \forall\ \mathbf{M}, \mathbf{N} \in \mathbf{nms}_*(X)_{(u,v)}$;

(ii) $S(\mathbf{M}, \mathbf{M}) = \underset{\mathbf{P},\mathbf{Q}\in\mathbf{nms}_*(X)_{(u,v)}}{max} S(\mathbf{P}, \mathbf{Q})$;

(iii) $S(\mathbf{M}, \overline{\mathbf{M}}) = 0$ for every $\mathbf{M} \in \tau_{(u,v)}(X)$ where $\tau_{(u,v)}(X)$ denotes the collection of all $SVNMSs$ with $\mathbf{T_M}(x) = [1]_{(u,v)}, \mathbf{I_M}(x) = [1]_{(u,v)}/[0]_{(u,v)}$, $\mathbf{F_M}(x) = [0]_{(u,v)}$ or $\mathbf{T_M}(x) = [0]_{(u,v)}, \mathbf{I_M}(x) = [1]_{(u,v)}/[0]_{(u,v)}, \mathbf{F_M}(x) = [0]_{(u,v)}$;

(iv) If $\mathbf{M_1} \subseteq \mathbf{M_2} \subseteq \mathbf{M_3}$, then $S(\mathbf{M_1}, \mathbf{M_2}) \geq S(\mathbf{M_1}, \mathbf{M_3})$ and $S(\mathbf{M_1}, \mathbf{M_3}) \leq S(\mathbf{M_2}, \mathbf{M_3})$.

Let $\mathbf{M}$ and $\mathbf{N}$ be two $SVNMS$ of order (u, v) defined over the same domain $X = \{x_1, x_2, \ldots, x_n\}$. A similarity measure S between $\mathbf{M}$ and $\mathbf{N}$ is defined as

$$S^*(\mathbf{M}, \mathbf{N}) = \frac{1}{u}\sum_{i=1}^{u} \underset{j\in N_v}{max}\, S(M_i^j, N_i^j), \tag{1}$$

where $S(M_i^j, N_i^j)$ denotes the similarity measure between the neutrosophic sets M_i^j and N_i^j calculated as

$$\begin{aligned} S(M_i^j, N_i^j) = \frac{1}{3n}\sum_{s=1}^{n}\Bigg(&\frac{min[T_{M_i^j}(x_s), T_{N_i^j}(x_s)]}{max[T_{M_i^j}(x_s), T_{N_i^j}(x_s)]} + \frac{min[I_{M_i^j}(x_s), I_{N_i^j}(x_s)]}{max[I_{M_i^j}(x_s), I_{N_i^j}(x_s)]} \\ &+ \frac{min[F_{M_i^j}(x_s), F_{N_i^j}(x_s)]}{max[F_{M_i^j}(x_s), F_{N_i^j}(x_s)]}\Bigg) \end{aligned} \tag{2}$$

Proposition 4.2. *The similarity measure S defined above satisfies all the conditions of $SVNMS$.*

Proof. It is easy to remark that S^* satisfies the axioms (i) and (ii) of Definition 4.1.
Let $\mathbf{M} \in \tau_{(u,v)}(X)$ with $\mathbf{T_M}(x_s) = [1]_{(u,v)}, \mathbf{I_M}(x_s) = [1]_{(u,v)}, \mathbf{F_M}(x_s) = [0]_{(u,v)}$, then $\mathbf{T_{\overline{M}}}(x_s) = [0]_{(u,v)}, \mathbf{I_{\overline{M}}}(x_s) = [0]_{(u,v)}, \mathbf{F_{\overline{M}}}(x_s) = [1]_{(u,v)}$. Now, $S(M_i^j, N_i^j) = 0\ \forall\ i, j$. Thus $S^*(\mathbf{M}, \overline{\mathbf{M}}) = 0$.
Let, $\mathbf{M_1} \subseteq \mathbf{M_2} \subseteq \mathbf{M_3}$ then $(T_{M_1})_i^j(x_s) \leq (T_{M_2})_i^j(x_s) \leq (T_{M_3})_i^j(x_s), (I_{M_1})_i^j(x_s) \geq (I_{M_2})_i^j(x_s) \geq$

$(I_{M_3})_i^j(x_s)$ and $(F_{M_1})_i^j(x_s) \geq (F_{M_2})_i^j(x_s) \geq (F_{M_3})_i^j(x_s)\ \forall\ x_s, i, j$.
Then,

$$
\begin{aligned}
S^*(\mathbf{M_1}, \mathbf{M_2}) &= \frac{1}{u}\sum_{i=1}^{u} \max_{j \in N_v} S((M_1)_i^j, (M_2)_i^j), \text{ where} \\
S((M_1)_i^j, (M_2)_i^j) &= \frac{1}{3n}\sum_{s=1}^{n}\left(\frac{min[(T_{M_1})_i^j(x_s), (T_{M_2})_i^j(x_s)]}{max[(T_{M_1})_i^j(x_s), (T_{M_2})_i^j(x_s)]} + \frac{min[(I_{M_1})_i^j(x_s), (I_{M_2})_i^j(x_s)]}{max[(I_{M_1})_i^j(x_s), (I_{M_2})_i^j(x_s)]}\right. \\
&\quad \left. + \frac{min[(F_{M_1})_i^j(x_s), (F_{M_2})_i^j(x_s)]}{max[(F_{M_1})_i^j(x_s), (F_{M_2})_i^j(x_s)]}\right) \\
&= \frac{1}{3n}\sum_{s=1}^{n}\left(\frac{(T_{M_1})_i^j(x_s)}{(T_{M_2})_i^j(x_s)} + \frac{(I_{M_2})_i^j(x_s)}{(I_{M_1})_i^j(x_s)} + \frac{(F_{M_2})_i^j(x_s)}{(F_{M_1})_i^j(x_s)}\right) \\
&\geq \frac{1}{3n}\sum_{s=1}^{n}\left(\frac{(T_{M_2})_i^j(x_s)}{(T_{M_3})_i^j(x_s)} + \frac{(I_{M_3})_i^j(x_s)}{(I_{M_2})_i^j(x_s)} + \frac{(F_{M_3})_i^j(x_s)}{(F_{M_2})_i^j(x_s)}\right) \\
&= S((M_2)_i^j, (M_3)_i^j)
\end{aligned}
$$

This is true for all i, j. Hence,
$S^*(\mathbf{M_1}, \mathbf{M_2}) = \frac{1}{u}\sum_{i=1}^{u} \max_{j \in N_v} S((M_1)_i^j, (M_2)_i^j) \geq \frac{1}{u}\sum_{i=1}^{u} \max_{j \in N_v} S((M_2)_i^j, (M_3)_i^j) = S(\mathbf{M_1}, \mathbf{M_2})$.
That is, $S^*(\mathbf{M_1}, \mathbf{M_2}) \geq S^*(\mathbf{M_2}, \mathbf{M_3})$. Similarly we can show, $S^*(\mathbf{M_1}, \mathbf{M_3}) \leq S^*(\mathbf{M_2}, \mathbf{M_3})$. □

Example 4.3. Let $X = \{x_1\}$ and $\mathbf{M}$ and $\mathbf{N}$ be two $SVNMS$ of order $(2, 2)$ defined over X with
$\mathbf{T_M}(x_1) = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$, $\mathbf{I_M}(x_1) = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$, $\mathbf{F_M}(x_1) = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$
$\mathbf{T_N}(x_1) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.4 \end{bmatrix}$, $\mathbf{I_N}(x_1) = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$, $\mathbf{F_N}(x_1) = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}$

$$
\begin{aligned}
\text{Now}, S^*(\mathbf{M}, \mathbf{N}) &= \frac{1}{2}\sum_{i=1}^{u} \max_{j \in N_v} S(M_i^j, N_i^j) \\
&= \frac{1}{2}\Big(\frac{1}{3}(max\Big[s(M_1^1, N_1^1), s(M_1^2, N_1^2)\Big] + max\Big[s(M_2^1, N_2^1), s(M_2^2, N_2^2)\Big])\Big) \\
&= \frac{1}{6}(max[0.82, 0.83] + max[0.72, 0.65]) \\
&= \frac{1}{6}(0.83 + 0.72) \\
&= 0.258.
\end{aligned}
$$

Let s be any fuzzy similarity measure. For any $SVNMS$ $\mathbf{M}$ and $\mathbf{N}$ of order (u,v) over X, the similarity measure between $\mathbf{M}$ and $\mathbf{N}$ induced by s is defined as

$$S(\mathbf{M},\mathbf{N}) = \frac{1}{3}\Big(\sum_{i=1}^{u} \max_{j\in N_v} s((T_M)_i^j,(T_N)_i^j) + \max_{j\in N_v} s((I_M)_i^j,(I_N)_i^j) + \max_{j\in N_v} s((F_M)_i^j,(F_N)_i^j)\Big). \tag{3}$$

Proposition 4.4. *For any two $SVNMS$, $\mathbf{M}$ and $\mathbf{N}$, $S(\mathbf{M},\mathbf{N})$ given in equation (3) is a similarity measure between $\mathbf{M}$ and $\mathbf{N}$.*

Proof. It is evident that S satisfies the axiom (i) given in Definition 4.1.

Let $\mathbf{M} \in \mathbf{nms}_*(X)_{(u,v)}$. Clearly, $S(\mathbf{M},\mathbf{M}) \leq \max_{\mathbf{P},\mathbf{Q}\in\mathbf{nms}_*(X)_{(u,v)}} S(\mathbf{P},\mathbf{Q})...(a)$. For fuzzy similarity measure s, $s((T_M)_i^j,(T_M)_i^j) \geq s((T_P)_i^j,(T_Q)_i^j)$,
$s((I_M)_i^j,(I_M)_i^j) \geq s((I_P)_i^j,(I_Q)_i^j), s((F_M)_i^j,(F_M)_i^j) \geq s((F_P)_i^j,(F_Q)_i^j)$ for all i,j. This implies $S(\mathbf{M},\mathbf{M}) \geq \max_{\mathbf{P},\mathbf{Q}\in\mathbf{nms}_*(X)_{(u,v)}} S(\mathbf{P},\mathbf{Q})...(b)$. From (a) and (b) we get $S(\mathbf{M},\mathbf{M}) = \max_{\mathbf{P},\mathbf{Q}\in\mathbf{nms}_*(X)_{(u,v)}} S(\mathbf{P},\mathbf{Q})$.

Let $\mathbf{M} \in \tau_{(u,v)}(X)$ where $\mathbf{T_M}(x) = [1]_{(u,v)}, \mathbf{I_M}(x) = [0]_{(u,v)}, \mathbf{F_M}(x) = [0]_{(u,v)}$. Then $\mathbf{T_{\overline{M}}}(x) = [0]_{(u,v)}, \mathbf{I_{\overline{M}}}(x) = [1]_{(u,v)}, \mathbf{F_{\overline{M}}}(x) = [1]_{(u,v)}$.
For all $i,j;$ $(T_M)_i^j,(T_{\overline{M}})_i^j,(I_M)_i^j,(I_{\overline{M}})_i^j,(F_M)_i^j,(F_{\overline{M}})_i^j$ denotes fuzzy membership functions with $(T_M)_i^j(x) = 1, (T_{\overline{M}})_i^j(x) = 0, (I_M)_i^j(x) = 0, (I_{\overline{M}})_i^j(x) = 1, (F_M)_i^j(x) = 0, (F_{\overline{M}})_i^j(x) = 1\ \forall\ x \in X$. Hence for each $i,j, s((T_M)_i^j,(T_{\overline{M}})_i^j) = s((I_M)_i^j,(I_{\overline{M}})_i^j) = s((F_M)_i^j,(F_{\overline{M}})_i^j) = 0$. Hence $S(\mathbf{M},\overline{\mathbf{M}}) = 0$ for every $\mathbf{M} \in \tau_{(u,v)}(X)$.

Let $\mathbf{M_1} \subseteq \mathbf{M_2} \subseteq \mathbf{M_3}$. That is $(T_{M_1})_i^j(x) \leq (T_{M_2})_i^j(x) \leq (T_{M_3})_i^j(x)$,
$(I_{M_1})_i^j(x) \geq (I_{M_2})_i^j(x) \geq (I_{M_3})_i^j(x), (F_{M_1})_i^j(x) \geq (F_{M_2})_i^j(x) \geq (F_{M_3})_i^j(x)$
$\forall\ x,i,j$. This implies $(T_{M_1})_i^j \subset (T_{M_2})_i^j \subset (T_{M_2})_i^j$ for all i,j.
Thus $s((T_{M_1})_i^j,(T_{M_2})_i^j) \geq s((T_{M_1})_i^j,T_{M_3})_i^j)...(c)$. Similarly
$s((I_{M_1})_i^j,(I_{M_2})_i^j) \geq s((I_{M_1})_i^j,I_{M_3})_i^j)...(d)$,
$s((F_{M_1})_i^j,(F_{M_2})_i^j) \geq s((F_{M_1})_i^j,F_{M_3})_i^j)...(e)$. This is true for all i,j. Hence from $(c),(d)$ and (e) we get $S(\mathbf{M_1},\mathbf{M_2}) \geq S(\mathbf{M_1},\mathbf{M_3})$. Similarly we can show $S(\mathbf{M_1},\mathbf{M_3}) \leq S(\mathbf{M_2},\mathbf{M_3})$. □

Example 4.5. Consider the $SVNMSs$ $\mathbf{M}$ and $\mathbf{N}$ given in Example 4.3. Let s_1 be a fuzzy similarity measure defined as $s_1(M,N) = 1 - \max_{x\in X}|M(x) - N(x)|$. Let S_1 be the similarity measure induced by s_1.

$$S_1(\mathbf{M},\mathbf{N}) = \frac{1}{3}\bigg(\sum_{i=1}^{2} \underset{j\in N_2}{max}\, s((T_M)_i^j,(T_N)_i^j) + \underset{j\in N_2}{max}\, s((I_M)_i^j,(I_N)_i^j)$$
$$+ \underset{j\in N_2}{max}\, s((F_M)_i^j,(F_N)_i^j)\bigg)$$

$$= \frac{1}{3}\bigg(max\{s((T_M)_1^1,(T_N)_1^1), s((T_M)_1^2,(T_N)_1^2)\}$$
$$+ (max\{s((T_M)_2^1,(T_N)_2^1), s((T_M)_2^2,(T_N)_2^2)\}$$
$$+ max\{s((I_M)_1^1,(I_N)_1^1), s((I_M)_1^2,(I_N)_1^2)\}$$
$$+ (max\{s((I_M)_2^1,(I_N)_2^1), s((I_M)_2^2,(I_N)_2^2)\}$$
$$+ max\{s((F_M)_1^1,(F_N)_1^1), s((F_M)_1^2,(F_N)_1^2)\}$$
$$+ (max\{s((F_M)_2^1,(F_N)_2^1), s((F_M)_2^2,(F_N)_2^2)\}\bigg)$$

$$= \frac{1}{3}\bigg(max\{0.8, 0.9\} + max\{1, 0.9\} + max\{0.9, 0.9\}$$
$$+ max\{0.9, 0.9\} + max\{0.9, 0.9\} + max\{1, 1\} + max\{0.9, 0.9\}\bigg)$$
$$= \frac{1}{3}(0.9 + 1 + 0.9 + 0.9 + 1 + 0.9)$$
$$= 1.86.$$

5. Application of similarity measures of $SVNMSs$ in decision-making process

There are many situations in real life where choosing one of the available options turns into a time-consuming task. Because it must address all of the characteristics and aspects, the selection process is difficult. This usually happens during the “decision-making process.” The decision-making process is a methodical procedure that people or organizations use to weigh their options and make decisions that will result in the outcomes they want. During this process, information is analyzed, options are considered, and the best course of action is chosen. This process can be more easily described mathematically by utilizing $SVNMS$. Truth, indeterminacy, and falsity membership matrices for each option can be developed according to the problem's criteria.

In this section, we present a handling method for the multi-criteria decision-making problem under a single-valued neutrosophic environment (or called a single-valued neutrosophic multi-criteria decision-making method) by means of the similarity measure between $SVNMS$. Selecting this approach will make it much simpler to compile and summarize the components.

Algorithm.

Let $A = \{A_1, A_2, \ldots, A_r\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria. Assume that the weight allotted to each alternative be w_s such that $\sum\limits_s w_s = 1$.

1. Define r, $SVNMS$, $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_r}$ of order (u, v) over C corresponding to the r alternatives $\{A_1, A_2, \ldots, A_r\}$. Here u denotes the number of experts involved in the evaluation purpose and v denotes the number of evaluations.

2. Define an ideal alternative A' and a $SVNMS, \mathbf{A'}$ over C, corresponding to the alternative where, $\mathbf{T}_{\mathbf{A'}}(C_s) = [1]_{(u,v)}, \mathbf{I}_{\mathbf{A'}}(C_s) = [0]_{(u,v)}$ and $\mathbf{F}_{\mathbf{A'}}(C_s) = [0]_{(u,v)}$ for all $C_s \in C$.

3. Define weighted similarity measure S^* and evaluate the similarity S^* between $\mathbf{A_k}; k = 1, 2, \ldots, r$ and $\mathbf{A'}$ as,

$$
\begin{aligned}
S^*(\mathbf{A_k}, \mathbf{A'}) &= \frac{1}{u}\sum_{i=1}^{u} \max_{j \in N_v} S((\mathbf{A_k})_i^j, (\mathbf{A'})_i^j) \text{ where,} \\
S((\mathbf{A_k})_i^j, (\mathbf{A'})_i^j) &= \frac{1}{3n}\sum_{s=1}^{n} w_s \left(\frac{min[(T_{A_k})_i^j(C_s), (T_{A'})_i^j(C_s)]}{max[(T_{A_k})_i^j(C_s), (T_{A'})_i^j(C_s)]} + \frac{min[(I_{A_k})_i^j(C_s), (I_{A'})_i^j(C_s)]}{max[(I_{A_k})_i^j(C_s), (I_{A'})_i^j(C_s)]} \right. \\
&\qquad \left. + \frac{min[(F_{A_k})_i^j(C_s), (F_{A'})_i^j(C_s)]}{max[(F_{A_k})_i^j(C_s), (F_{A'})_i^j(C_s)]} \right)
\end{aligned}
$$

4. Rank $S^*(\mathbf{A_k}, \mathbf{A'})$ for $k = 1, 2, \ldots, r$ and select the alternative showing the greatest value.

5.1. *Illustrative example*

This section illustrates the use of the suggested decision-making approach and its efficacy using an example of a "personnel selection problem." In [24] we have solved a personnel selection problem using ordered weighted aggregation operators of multiple sets. However, the proposed method is insufficient for handling indeterminate and inconsistent information. Here we modify the same problem by adding inconsistent data and apply the above mentioned algorithm involving the weighted similarity measure of $SVNS$ to solve it.

Problem. "A multinational corporation is conducting interviews for the position of HR manager. Two specialists make up the interview panel, and the interviews are divided into two tiers based on the following three criteria: C_1 - critical thinking, C_2 - general awareness, and C_3 - communication skill. Three applicants A_1, A_2 and A_3 were chosen for the last round of interview. The weight vector of the criteria is given by $\mathbf{w} = (0.35, 0.30, 0.35)$. Three candidates are evaluated with respect to the above three criteria by the two experts and the observation is consolidated in to two single-valued neutrosophic decision table.

Candidates	Critical thinking (C_1)	General awareness (C_2)	Communication skill (C_3)
A_1	$< 0.4, 0.2, 0.3 >$	$< 0.5, 0.2, 0.2 >$	$< 0.2, 0.3, 0.5 >$
A_2	$< 0.6, 0.2, 0.1 >$	$< 0.6, 0.1, 0.1 >$	$< 0.5, 0.1, 0.2 >$
A_3	$< 0.3, 0.3, 0.2 >$	$< 0.5, 0.2, 0.3 >$	$< 0.5, 0.2, 0.1 >$

Table 1: *Evaluation of the decision maker 1.*

Candidates	Critical thinking(C_1)	General awareness (C_2)	Communication skill (C_3)
A_1	$< 0.5, 0.2, 0.2 >$	$< 0.5, 0.2, 0.3 >$	$< 0.1, 0.2, 0.4 >$
A_2	$< 0.6, 0.1, 0.1 >$	$< 0.6, 0.2, 0.2 >$	$< 0.5, 0.2, 0.1 >$
A_3	$< 0.3, 0.2, 0.3 >$	$< 0.5, 0.2, 0.3 >$	$< 0.5, 0.1, 0.1 >$

Table 2: *Evaluation of the decision maker 2.*

Let $\mathbf{A_1}, \mathbf{A_2}$ and $\mathbf{A_3}$ be 3, $SVNMS$ of order $(2, 1)$ corresponding to the three candidates in the given problem defined over the criteria set $C = \{C_1, C_2, C_3\}$. Considering the truth, indeterminacy, and falsity values given in Tables 1 and 2. Define the truth, indeterminacy, and falsity membership matrices for each $SVNMS$ as",

$$\mathbf{T_{A_1}}(C_1) = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, \mathbf{I_{A_1}}(C_1) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_1}}(C_1) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}$$

$$\mathbf{T_{A_1}}(C_2) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \mathbf{I_{A_1}}(C_2) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_1}}(C_2) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T_{A_1}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{I_{A_1}}(C_3) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_1}}(C_3) = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$$

$$\mathbf{T_{A_2}}(C_1) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \mathbf{I_{A_2}}(C_1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{F_{A_2}}(C_1) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$$\mathbf{T_{A_2}}(C_2) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \mathbf{I_{A_2}}(C_2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_2}}(C_2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$\mathbf{T_{A_2}}(C_3) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \mathbf{I_{A_2}}(C_3) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_2}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$\mathbf{T_{A_3}}(C_1) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}, \mathbf{I_{A_3}}(C_1) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_3}}(C_1) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T_{A_3}}(C_2) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \mathbf{I_{A_3}}(C_2) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \mathbf{F_{A_3}}(C_2) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T_{A_3}}(C_3) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \mathbf{I_{A_3}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{F_{A_3}}(C_3) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

Now define the ideal $SVNMS$ $\mathbf{A'}$ with membership matrices,
$\mathbf{T}_{\mathbf{A'}}(C_s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{I}_{\mathbf{A'}}(C_s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{F}_{\mathbf{A'}}(C_s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $s = 1, 2, 3$ and calculate $S^*(\mathbf{A_k}, \mathbf{A'})$ for $k = 1, 2, 3$.

$$\begin{aligned}
S^*(\mathbf{A_1}, \mathbf{A'}) &= \frac{1}{2}\sum_{i=1}^{2} \max_{j\in N_1} S((\mathbf{A_1})_i^j, (\mathbf{A'})_i^j) \\
&= \frac{1}{2}\Big(S((\mathbf{A_1})_1^1, (\mathbf{A'})_1^1) + S((\mathbf{A_1})_2^1, (\mathbf{A'})_2^1) \Big) \\
&= \frac{1}{18}\sum_{s=1}^{3} w_s \Bigg(\frac{min[(T_{A_k})_i^j(C_s), (T_{A'})_i^j(C_s)]}{max[(T_{A_k})_i^j(C_s), (T_{A'})_i^j(C_s)]} + \frac{min[(I_{A_k})_i^j(C_s), (I_{A'})_i^j(C_s)]}{max[(I_{A_k})_i^j(C_s), (I_{A'})_i^j(C_s)]} \\
&+ \frac{min[(F_{A_k})_i^j(C_s), (F_{A'})_i^j(C_s)]}{max[(F_{A_k})_i^j(C_s), (F_{A'})_i^j(C_s)]} \Bigg) \\
&= \frac{1}{18}(0.72) \\
&= 0.04.
\end{aligned}$$

Similarly we compute, $S^*(\mathbf{A_2}, \mathbf{A'}) = 0.062$ and $S^*(\mathbf{A_3}, \mathbf{A'}) = 0.047$. Hence, $S(\mathbf{A_1}, \mathbf{A'}) \leq S(\mathbf{A_3}, \mathbf{A'}) \leq S(\mathbf{A_2}, \mathbf{A'})$. We can infer from this ranking that $\mathbf{A_2}$ is more similar to $\mathbf{A'}$. Hence the candidate A_2 is identified as the most suitable choice for the job.

6. **Comparative Study**

In this section, we present a comprehensive evaluation of our proposed method for addressing the decision-making problem, comparing it with the established methods outlined in [16]. In 2013, Jun Ye [16] introduced the correlation and correlation coefficient of single-valued neutrosophic sets. This was done by extending the correlation of intuitionistic fuzzy sets, and it was shown that the cosine similarity measure is a particular instance of the correlation coefficient in $SVNS$. Then, using the weighted correlation coefficient or the weighted cosine similarity measure of $SVNS$, a decision-making process is suggested. They employed the suggested approach to resolve a decision-making problem that was modified from [17].

The method proposed by Jun Ye becomes insufficient for addressing the same problem when the level of evaluation or the number of evaluators increases. In this study, we modify the problem presented in [17] by expanding the number of decision-makers. We then apply the proposed algorithm, which incorporates a weighted similarity measure of $SVNS$, to simplify the decision-making process.

Problem. "There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1)A_1

is a car company; (2)A_2 is a food company; (3)A_3 is a computer company; and (4)A_4 is an arms company. The investment company must take a decision according to the following three criteria: (1)C_1 is the risk analysis; (2)C_2 is the growth analysis; and (3)C_3 is the environmental impact analysis. The weight vector of the criteria is given by $W = (0.35, 0.25, 0.40)$. The four possible alternatives, evaluated based on the three criteria mentioned above, were assessed by two experts, resulting in the following single-valued neutrosophic decision matrices, D_1 and D_2.

$$D_1 = \begin{bmatrix} < 0.4, 0.2, 0.3 > & < 0.4, 0.2, 0.3 > & < 0.2, 0.2, 0.5 > \\ < 0.6, 0.1, 0.2 > & < 0.6, 0.1, 0.2 > & < 0.5, 0.2, 0.2 > \\ < 0.3, 0.2, 0.3 > & < 0.5, 0.2, 0.3 > & < 0.5, 0.3, 0.2 > \\ < 0.7, 0.0, 0.1 > & < 0.6, 0.1, 0.2 > & < 0.4, 0.3, 0.2 > \end{bmatrix}$$

$$D_2 = \begin{bmatrix} < 0.5, 0.1, 0.3 > & < 0.3, 0.2, 0.3 > & < 0.1, 0.1, 0.5 > \\ < 0.6, 0.1, 0.1 > & < 0.5, 0.1, 0.1 > & < 0.6, 0.2, 0.1 > \\ < 0.2, 0.3, 0.3 > & < 0.6, 0.1, 0.2 > & < 0.5, 0.2, 0.2 > \\ < 0.7, 0.1, 0.2 > & < 0.5, 0.1, 0.1 > & < 0.4, 0.1, 0.2 > \end{bmatrix}$$

In this context, each entry of the decision matrix represents the degree of truthfulness, indeterminacy, and falsity assigned by the decision maker to each alternative A_i with respect to each criterion C_i".

Solution. Construct two tables based on the single-valued neutrosophic decision matrices D_1 and D_2.

Alternatives	Risk analysis(C_1)	Growth analysis (C_2)	Environmental impacts (C_3)
Car company (A_1)	$(0.4, 0.2, 0.3)$	$(0.4, 0.2, 0.3)$	$(0.2, 0.2, 0.5)$
Food company (A_2)	$(0.6, 0.1, 0.2)$	$(0.6, 0.1, 0.2)$	$(0.5, 0.2, 0.2)$
Computer company (A_3)	$(0.3, 0.2, 0.3)$	$(0.5, 0.2, 0.3)$	$(0.5, 0.3, 0.2)$
Arms company (A_4)	$(0.7, 0.0, 0.1)$	$(0.6, 0.1, 0.2)$	$(0.4, 0.3, 0.2)$

Table 1: *Evaluation of the decision maker 1.*

Alternatives	Risk analysis(C_1)	Growth analysis (C_2)	Environmental impacts (C_3)
Car company (A_1)	$(0.5, 0.1, 0.3)$	$(0.3, 0.2, 0.3)$	$(0.1, 0.1, 0.5)$
Food company (A_2)	$(0.6, 0.1, 0.1)$	$(0.5, 0.1, 0.1)$	$(0.6, 0.2, 0.1)$
Computer company (A_3)	$(0.2, 0.3, 0.3)$	$(0.6, 0.1, 0.2)$	$(0.5, 0.2, 0.2)$
Arms company (A_4)	$(0.7, 0.1, 0.2)$	$(0.5, 0.1, 0.1)$	$(0.4, 0.1, 0.2)$

Table 1: *Evaluation of the decision maker 2.*

Let $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ and $\mathbf{A_4}$ be 4, $SVNMS$ of order $(2, 1)$ corresponding to the four alternatives in the given problem defined over the criteria set $C = \{C_1, C_2, C_3\}$. Considering the truth, indeterminacy and falsity values given in table 1 and 2 we define the truth, indeterminacy and falsity matrices for each $SVNMS$ as

$$\mathbf{T}_{\mathbf{A_1}}(C_1) = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, \mathbf{I}_{\mathbf{A_1}}(C_1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_1}}(C_1) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_1}}(C_2) = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \mathbf{I}_{\mathbf{A_1}}(C_2) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \mathbf{F}_{\mathbf{A_1}}(C_2) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_1}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{I}_{\mathbf{A_1}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_1}}(C_3) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_2}}(C_1) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \mathbf{I}_{\mathbf{A_2}}(C_1) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_2}}(C_1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_2}}(C_2) = \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix}, \mathbf{I}_{\mathbf{A_2}}(C_2) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_2}}(C_2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_2}}(C_3) = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}, \mathbf{I}_{\mathbf{A_2}}(C_3) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \mathbf{F}_{\mathbf{A_2}}(C_3) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_3}}(C_1) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \mathbf{I}_{\mathbf{A_3}}(C_1) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \mathbf{F}_{\mathbf{A_3}}(C_1) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_3}}(C_2) = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}, \mathbf{I}_{\mathbf{A_3}}(C_2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_3}}(C_2) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_3}}(C_3) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \mathbf{I}_{\mathbf{A_3}}(C_3) = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \mathbf{F}_{\mathbf{A_3}}(C_3) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

$$\mathbf{T}_{\mathbf{A_4}}(C_1) = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}, \mathbf{I}_{\mathbf{A_4}}(C_1) = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, \mathbf{F}_{\mathbf{A_4}}(C_1) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$\mathbf{T_{A_4}}(C_2) = \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix}$, $\mathbf{I_{A_4}}(C_2) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $\mathbf{F_{A_4}}(C_2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$
$\mathbf{T_{A_4}}(C_3) = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}$, $\mathbf{I_{A_4}}(C_3) = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$,$\mathbf{F_{A_4}}(C_3) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$

Now define the ideal $SVNMS$ $\mathbf{A'}$ with membership matrices,
$\mathbf{T_{A'}}(C_s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{I_{A'}}(C_s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{F_{A'}}(C_s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $s = 1, 2, 3$ and calculate $S^*(\mathbf{A_k}, \mathbf{A'})$ for $k = 1, 2, 3, 4$.

$$\begin{aligned} S^*(\mathbf{A_1}, \mathbf{A'}) &= \frac{1}{2}\sum_{i=1}^{2} \underset{j\in N_1}{max}\, S((\mathbf{A_1})_i^j, (\mathbf{A'})_i^j) \\ &= S((\mathbf{A_1})_1^1, (\mathbf{A'})_1^1) + S((\mathbf{A_1})_2^1, (\mathbf{A'})_2^1) \\ &= \frac{1}{2}\Big(0.035 + 0.032\Big) \\ &= \frac{1}{2}(0.067) \\ &= 0.0335 \end{aligned}$$

Similarly we compute, $S^*(\mathbf{A_2}, \mathbf{A'}) = 0.064$, $S^*(\mathbf{A_3}, \mathbf{A'}) = 0.0465$ and
$S^*(\mathbf{A_4}, \mathbf{A'}) = 0.06$. Hence, $S^*(\mathbf{A_2}, \mathbf{A'}) \leq S^*(\mathbf{A_4}, \mathbf{A'}) \leq S^*(\mathbf{A_3}, \mathbf{A'}) \leq S^*(\mathbf{A_1}, \mathbf{A'})$. We can infer from this ranking that $\mathbf{A_2}$ is more similar to $\mathbf{A'}$ hence it is the best alternative. It is therefore more advantageous to invest in a food company.

The decision-making problem discussed in [16] can be effectively solved using the proposed algorithm. In this approach, each alternative A_i is treated as a multiple set of order $(1, 1)$, and the ordered weighted similarity $S^*(\mathbf{A_i}, \mathbf{A'})$ is computed, where $\mathbf{A'}$ represents the multiple set corresponding to the ideal alternative. We compute, $S^*(\mathbf{A_1}, \mathbf{A'}) = 0.036$, $S^*(\mathbf{A_2}, \mathbf{A'}) = 0.0622$, $S^*(\mathbf{A_3}, \mathbf{A'}) = 0.047$ and $S^*(\mathbf{A_4}, \mathbf{A'}) = 0.0621$. Hence, $S(\mathbf{A_2}, \mathbf{A'}) \leq S(\mathbf{A_4}, \mathbf{A'}) \leq S(\mathbf{A_3}, \mathbf{A'}) \leq S(\mathbf{A_1}, \mathbf{A'})$. We can infer from this ranking that $\mathbf{A_2}$ is more similar to $\mathbf{A'}$. The optimal alternative and the ranking order obtained are consistent with those in [16], as illustrated in Figures 2 and 3. Thus, the method applied in this research serves as an ideal approach for solving complex real-life decision-making problems.

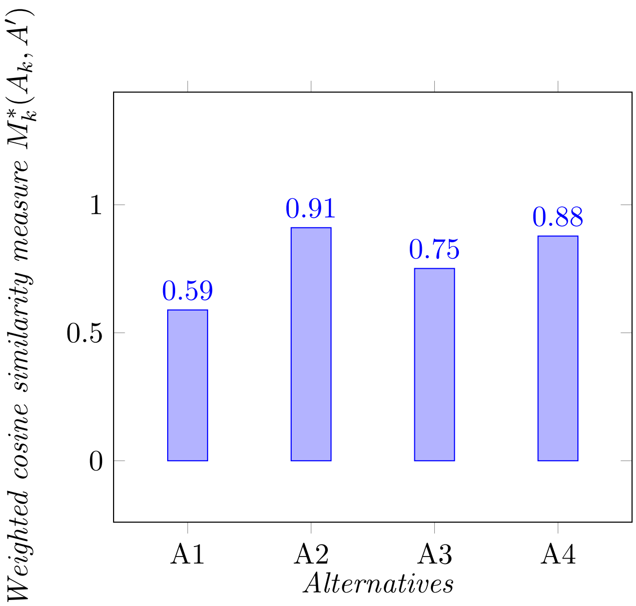

Figure 2 : *Comparison of weighted cosine similarity measure of different alternative*

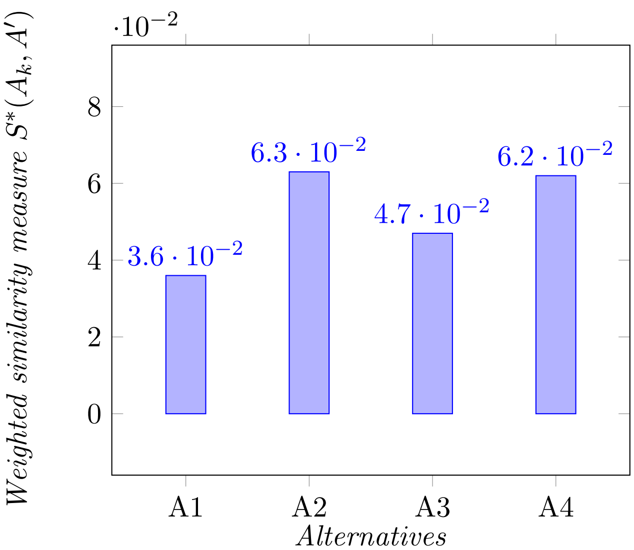

Figure 3 : *Comparison of weighted similarity measure of different alternative*

7. Conclusion

In this paper we introduce the theory of $SVNMS$ as an extension of $SVNS$ and multiple sets. $SVNMS$ is a new mathematical structure capable of handling indeterminacy and imprecision in a more simpler way. It is a much better tool to represent a practical problem in a more systematic manner. Secondly, we have provided an axiomatic definition for similarity measure between $SVNMS$ and proposed similarity measures using max-min operators and fuzzy similarity measures. A multi-criteria decision-making method in a neutrosophic context has been subsequently developed through the application of similarity measures. This method compares each alternative to the ideal alternative. It is easy to identify the best option and calculate the ranking order of all the options using the similarity measures. The application of the developed approach has been illustrated through an example. Finally, a comparative analysis was conducted between the proposed method and the approach described by Jun Ye [16], concluding that the proposed method is significantly more user-friendly.

7.1. *Discussion and future work.*

The superiority of the suggested multi-criteria decision-making method lies in its ability to handle uncertainty and multiplicity of an item simultaneously. This makes it more flexible and practical compared to existing decision-making methods in real-world decision-making scenarios. It can also be utilized to solve decision-making problems involving more criteria and decision-makers. Future work will focus on solving complex decision-making problems, including group decision-making problems with uncertain weights of criteria, as well as problems from other domains like medical informatics, bioinformatics, expert systems, etc.

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