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Quadripartitioned Neutrosophic Pythagorean fuzzy ideals in near-ring

RashmiKumar J [1], Hellen A[2], Babu A[3], Arulselvam A[4,*] and Bobin A[5]

[1]Associate Professor School of Arts Humanities and Management, Jeppiaar University; rashmilenny@gmail.com

[2]Assistant Professor, Acharya Institute of Technology, Banglore; hellenantonypushparaj@gmail.com

[3]Assistant Professor, Department of Mathematics, St.Joseph's Institute of Technology, Chennai; subashbabu18@gmail.com

[4]Assistant Professor, Bharath Institute of Higher Education and Research; arulselvam.a91@mail.com

[5]Assistant Professor, IFET Engineering College; bobinalbert@gmail.com

[*]Correspondence: arulselvam.a91@gmail.com; Tel.: (optional; include country code)

Abstract. This paper investigates the structural characteristics of quadri-partitioned neutrosophic Pythagorean fuzzy ideals (QNPFIs) and Bi-Ideals (QNPFBIs) within near-rings. It is established that the intersection of any collection of QNPFIs in a near-ring continues to exhibit the properties of a QNPFI. Similarly, we show that the intersection of QNPFBIs retains its bi-ideal structure. Moreover, an analysis of the power function of a QNPFI confirms that its fundamental properties remain intact under exponentiation. These findings contribute to the broader study of algebraic systems incorporating fuzzy and neutrosophic uncertainty principles.

Keywords: Quadri-partitioned; Quadri-Partitioned Neutrosophic; Pythagorean fuzzy; ideals; Near-ring

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1. **Introduction**

Mathematical frameworks incorporating fuzzy logic have been extensively explored due to their effectiveness in managing uncertainty, vagueness, and imprecision. The progression from classical fuzzy sets to more advanced generalizations, such as Intuitionistic Fuzzy Sets (IFS), Pythagorean Fuzzy Sets (PFS), and Neutrosophic Sets (NS), has led to significant developments in algebraic structures, particularly in near-rings. These advancements have broadened the scope of fuzzy algebra, facilitating applications in computational intelligence and decision-making. This study presents a comprehensive review of the key contributions in this domain, emphasizing the evolution of fuzzy ideals in near-rings and their extensions.

The foundation of fuzzy algebraic structures was established by Zadeh [30] through the introduction of fuzzy sets, which subsequently inspired research on fuzzy subgroups, subrings, and ideals. Pilz [18] provided a detailed study on near-rings, which later became a critical framework for incorporating fuzzy logic. Initial studies on fuzzy ideals in near-rings, including the works of S.D. Kim and H.S. Kim [12], focused on essential properties and conditions for prime fuzzy ideals. Basnet [6] introduced α, β-cut techniques in IFS, enhancing the analysis of fuzzy structures.

Atanassov [3] extended fuzzy set theory by introducing IFS, which accounts for both membership and non-membership values. Subsequent works [4, 5] explored algebraic operations in IFS, leading to the development of intuitionistic fuzzy subnear-rings [7, 16, 29] and intuitionistic Q-fuzzification approaches for algebraic structures. The study of intuitionistic fuzzy cosets and their homomorphic properties has further refined the algebraic understanding of these fuzzy systems [22]. However, the inherent limitations of IFS motivated the introduction of PFS [28], which offers a more flexible representation of uncertainty. Recent studies [2, 14] have analyzed Pythagorean fuzzy ideals and subrings, demonstrating their potential in extending classical fuzzy frameworks.

Homomorphism theorems play a fundamental role in algebraic structures, and their extensions to fuzzy environments have been widely investigated. Sharma [25] and other researchers [10, 11, 20] have examined homomorphic images and preimages of fuzzy ideals in near-rings, establishing conditions for prime fuzzy ideals. Further studies have introduced and characterized intuitionistic Q-fuzzification in subnear-rings [7, 16, 29] and applied α, β-cut techniques to analyze intuitionistic fuzzy subgroups [24, 31].

The concept of the neutrosophic set, which Smarandache [26, 27] introduced, is a generalisation of the intuitionistic fuzzy set.

The emergence of Neutrosophic and Quadripartitioned Fuzzy Sets (QFS) has provided an advanced approach to managing complex uncertainties by incorporating additional degrees of truth, falsity, indeterminacy, and contradiction. The Quadripartitioned Neutrosophic Pythagorean Fuzzy Set (QNPFS) proposed in [19] introduces a novel methodology for handling uncertainty in algebraic systems. These recent developments mark a significant step forward in the field, offering new avenues for further exploration.

1.1. *Motivation and Objectives*

The study of fuzzy and intuitionistic fuzzy algebraic structures has gained prominence due to their broad applications in decision-making, control systems, and artificial intelligence. While IFS and PFS have been extensively examined in group and ring theory, their application in near-ring structures remains an evolving research area. The introduction of QNPFS in near-rings provides an enriched framework for addressing uncertainty and imprecision in algebraic structures.

This research aims to: 1) Investigate the properties of Quadripartitioned Neutrosophic Pythagorean Fuzzy Ideals (QNPFIs) in near-rings. 2) Establish closure properties of QNPFIs and Quadripartitioned Neutrosophic Pythagorean Fuzzy Bi-Ideals (QNPFBIs) under intersection operations. 3) Analyze the impact of exponentiation on QNPFIs and determine conditions ensuring their structural integrity. 4) Provide mathematical characterizations of these fuzzy structures within near-rings, thereby expanding existing fuzzy algebra frameworks.

By addressing these objectives, this study contributes to the ongoing theoretical advancements in fuzzy algebra and its applicability in computational and decision-making contexts.

2. **Preliminaries**

Definition 2.1. [19] Let X be a universe of discourse, A **Quadripartitioned neutrosophic Pythagorean set** (QNPS) $A = \{z, \xi_A(z), \omega_A(z), \nu_A(z), \zeta_A(z)/z \in X\}$ where $\xi_A(z) + \zeta_A(z) \leq 1$, $\omega_A(z)+\nu_A(z) \leq 1$ and $0 \leq (\xi_A(z))^2+(\omega_A(z))^2+(\nu_A(z))^2+(\zeta_A(z))^2 \leq 2$ represent the degree of truth membership, contradiction membership, ignorance membership and false membership of the object $z \in X$.
$\xi_A(z)$ - Truth membership
$\omega_A(z)$ - Contradiction membership
$\nu_A(z)$ - Ignorance membership
$\zeta_A(z)$ - False membership

3. **Quadripartitioned Neutrosophic Pythagorean fuzzy ideals in near-rings**

Definition 3.1. A Quadripartitioned Neutrosophic Pythagorean fuzzy Set $A = (\xi_A, \omega_A, \nu_A, \zeta_A)$ in a near ring N is called a Quadripartitioned Neutrosophic Pythagorean fuzzy ideals of N if it satisfies the following axioms
(i)$\xi_A(n-q) \geq \min\{\xi_A(n), \xi_A(q)\}$; $\omega_A(n-q) \geq \min\{\omega_A(n), \eta_A(q)\}$
$\nu_A(n-q) \leq \max\{\nu_A(n), \nu_A(q)\}$; $\zeta_A(n-q) \leq \max\{\zeta_A(n), \zeta_A(q)\}$
(ii)$\xi_A(q+n-q) \geq \eta_A(n)$; $\omega_A(q+n-q) \geq \omega_A(n)$;
$\nu_A(q+n-q) \leq \nu_A(n)$; $\zeta_A(q+n-q) \leq \zeta_A(n)$
(iii) $\xi_A(nq) \geq \xi_A(q)$; $\omega_A(nq) \geq \omega_A(q)$

$\nu_A(nq) \leq \nu_A(q)$; $\zeta_A(nq) \leq \zeta_A(q)$

(iv) $\xi_A((n+z)q - nq) \geq \eta_A(z)$; $\omega_A((n+z)q - nq) \geq \omega_A(z)$;

$\nu_A((n+z)q - nq) \leq \nu_A(z)$; $\zeta_A((n+z)q - nq) \leq \zeta_A(z)$

Example 3.2. Let $A = \{a, b, c, d\}$ be a near ring with two binary operations '+' and '·' is defined as follows.

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	a	a	a
d	a	b	c	d

Define a Quadri-partitioned Neutrosophic Pythagorean fuzzy Set $A = (\xi_A, \omega_A, \nu_A, \zeta_A)$ where $\xi_A : X \to [0,1]$; $\omega_A : X \to [0,1]$; $\nu_A : X \to [0,1]$ and $\zeta_A : X \to [0,1]$ is given by

	$\xi_A(a), \omega_A(a), \nu_A(a), \zeta_A(a)$	$\xi_A(b), \omega_A(b), \nu_A(b), \zeta_A(b)$	$\xi_A(c), \omega_A(c), \nu_A(c), \zeta_A(c)$	$\xi_A(d), \omega_A(d), \nu_A(d), \zeta_A(d)$
1	(0.8,0.7,0.4,0.3)	(0.5,0.4,0.6,0.5)	(0.4,0.3,0.8,0.7)	(0.3,0.2,0.8,0.9)
2	(0.7,0.6,0.3,0.2)	(0.6,0.5,0.5,0.6)	(0.4,0.3,0.8,0.7)	(0.4,0.2,0.9,0.8)
3	(0.9,0.8,0.2,0.1)	(0.7,0.6,0.4,0.5)	(0.6,0.5,0.7,0.6)	(0.6,0.5,0.7,0.6)

Theorem 3.3. *Let $A_i = \{(\xi_{A_i}, \omega_{A_i}, \nu_{A_i}, \zeta_{A_i}) | i \in I\}$ be a family of Quadripartitioned Neutrosophic Pythagorean fuzzy ideals in near ring N. Then $(\bigcap_{i\in I} A_i)$ is Quadripartitioned neutrosophic Pythagorean fuzzy ideal of N. Where $\{\bigcap_{i\in I} A_i\} = \{\bigcap_{i\in I} \xi_{A_i}, \bigcap_{i\in I} \eta_{A_i}, \bigcup_{i\in I} \mu_{A_i}, \bigcup_{i\in I} \lambda_{A_i}\}$.*

Proof. Let $A_i = \{(\xi_{A_i}, \eta_{A_i}, \mu_{A_i}, \lambda_{A_i}) | i \in I\}$ and for all $n, q, z \in N$.

$$\begin{aligned}\bigcap_{i\in I} \xi_{A_i}(n-q) &= \inf \xi_{A_i}(n-q) \\ &\geq \inf\{\min\{\xi_{A_i}(n), \xi_{A_i}(q)\}\} \\ &= \min\{\inf \xi_{A_i}(n), \inf \xi_{A_i}(q)\} \\ &= \min\{\xi_A(n), \xi_A(q)\}\end{aligned}$$

and

$$\begin{aligned}\bigcap_{i\in I} \omega_{A_i}(n-q) &= \inf \omega_{A_i}(n-q) \\ &\geq \inf\{\min\{\omega_{A_i}(n), \omega_{A_i}(q)\}\} \\ &= \min\{\inf \omega_{A_i}(n), \inf \omega_{A_i}(q)\} \\ &= \min\{\omega_A(n), \omega_A(q)\}\end{aligned}$$

and

$$\bigcup_{i\in I} \nu_{A_i}(n-q) = \sup \nu_{A_i}(n-q)$$

$$\leq \sup\{\max\{\nu_{A_i}(n), \nu_{A_i}(q)\}\}$$
$$= \max\{\sup \nu_{A_i}(n), \sup \nu_{A_i}(q)\}$$
$$= \max\{\nu_A(n), \nu_A(q)\}$$

and

$$\begin{aligned}\bigcup_{i\in I} \zeta_{A_i}(n-q) &= \sup \zeta_{A_i}(n-q)\\ &\leq \sup\{\max\{\zeta_{A_i}(n), \zeta_{A_i}(q)\}\}\\ &= \max\{\sup \zeta_{A_i}(n), \sup \zeta_{A_i}(q)\}\\ &= \max\{\zeta_A(n), \zeta_A(q)\}\end{aligned}$$

Next it is obtained

$$\begin{aligned}\bigcap_{i\in I} \xi_{A_i}(q+n-q) &= \inf \xi_{A_i}(q+n-q)\\ &\geq \inf\{\xi_{A_i}(n)\}\\ &= \bigcap_{i\in I} \xi_{A_i}(n).\end{aligned}$$

and

$$\begin{aligned}\bigcap_{i\in I} \omega_{A_i}(q+n-q) &= \inf \omega_{A_i}(q+n-q)\\ &\geq \inf\{\omega_{A_i}(n)\}\\ &= \bigcap_{i\in I} \omega_{A_i}(n).\end{aligned}$$

Thus

$$\begin{aligned}\bigcup_{i\in I} \nu_{A_i}(q+n-q) &= \sup \nu_{A_i}(q+n-q)\\ &\leq \sup\{\nu_{A_i}(n)\}\\ &= \bigcup_{i\in I} \nu_{A_i}(n).\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in I} \zeta_{A_i}(q+n-q) &= \sup \zeta_{A_i}(q+n-q)\\ &\leq \sup\{\zeta_{A_i}(n)\}\\ &= \bigcup_{i\in I} \zeta_{A_i}(n).\end{aligned}$$

Then

$$\begin{aligned}\bigcap_{i\in I} \xi_{A_i}(nq) &= \inf \xi_{A_i}(nq)\\ &\geq \inf \xi_{A_i}(q)\\ &= \bigcap_{i\in I} \xi_{A_i}(q).\end{aligned}$$

Similarly

$$\begin{aligned}\bigcap_{i\in I} \omega_{A_i}(nq) &= \inf \omega_{A_i}(nq)\\ &\geq \inf \omega_{A_i}(q)\\ &= \bigcap_{i\in I} \omega_{A_i}(q).\end{aligned}$$

and $\bigcup_{i\in I} \nu_{A_i}(nq) = \sup \nu_{A_i}(nq)$

$$\leq \sup \nu_{A_i}(q)$$

$$= \bigcup_{i\in I} \nu_{A_i}(q).$$

Similarly

$$\begin{aligned}\bigcup_{i\in I} \zeta_{A_i}(nq) &= \sup \zeta_{A_i}(nq)\\ &\leq \sup \zeta_{A_i}(q)\\ &= \bigcup_{i\in I} \zeta_{A_i}(q).\end{aligned}$$

At last we write

$$\begin{aligned}\bigcap_{i\in I} \xi_{A_i}((n+z)q-nq) &= \inf \xi_{A_i}((n+z)q-nq)\\ &\geq \inf \xi_{A_i}(z)\\ &= \bigcap_{i\in I} \xi_{A_i}(z).\end{aligned}$$

Similarly

$$\begin{aligned}\bigcap_{i\in I} \omega_{A_i}((n+z)q-nq) &= \inf \omega_{A_i}((n+z)q-nq)\\ &\geq \inf \omega_{A_i}(z)\\ &= \bigcap_{i\in I} \omega_{A_i}(z).\end{aligned}$$

and $\bigcup_{i\in I} \nu_{A_i}((n+z)q-nq) = \sup \nu_{A_i}((n+z)q-nq)$

$$\begin{aligned}&\leq \sup \nu_{A_i}(z)\\ &= \bigcup_{i\in I} \nu_{A_i}(z).\end{aligned}$$

Similarly

$$\begin{aligned}\bigcup_{i\in I} \zeta_{A_i}((n+z)q-nq) &= \sup \zeta_{A_i}((n+z)q-nq)\\ &\leq \sup \zeta_{A_i}(z)\\ &= \bigcup_{i\in I} \zeta_{A_i}(z).\end{aligned}$$

Therefore $\bigcap_{i\in I} A_i$ is a family of Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring N. □

Theorem 3.4. *If A and B are Quadripartitioned neutrosophic Pythagorean fuzzy bi-ideal of near ring then the intersection of $A \cap B$ is also Quadripartitioned neutrosophic Pythagorean fuzzy bi-ideal of near ring.*

Proof. Let $A = (\xi_A, \omega_A, \nu_A, \zeta_A)$ be a Quadripartitioned neutrosophic Pythagorean fuzzy subset of N.

(i) For any $n, q, z \in N$ we have

$$\begin{aligned}\xi_{A\cap B}(n-q) &= \min\{\xi_A(n-q), \xi_B(n-q)\}\\ &\geq \min\{\min\{\xi_A(n), \xi_A(q)\}, \min\{\xi_B(n), \xi_B(q)\}\}\\ &= \min\{\min\{\xi_A(n), \xi_B(n)\}, \min\{\xi_A(q), \xi_B(q)\}\}\\ &= \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(q)\}\end{aligned}$$

Hence $\xi_{A\cap B}(n-q) \geq \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(q)\}$

and

$$\begin{aligned}\omega_{A\cap B}(n-q) &= \min\{\omega_A(n-q), \omega_B(n-q)\}\\ &\geq \min\{\min\{\omega_A(n), \omega_A(q)\}, \min\{\omega_B(n), \omega_B(q)\}\}\\ &= \min\{\min\{\omega_A(n), \omega_B(n)\}, \min\{\omega_A(q), \omega_B(q)\}\}\\ &= \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(q)\}\end{aligned}$$

Hence $\omega_{A\cap B}(n-q) \geq \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(q)\}$. Then

$$\begin{aligned}\nu_{A\cap B}(n-q) &= \max\{\nu_A(n-q), \nu_B(n-q)\}\\ &\leq \max\{\min\{\nu_A(n), \nu_A(q)\}, \max\{\nu_B(n), \xi_B(q)\}\}\\ &= \max\{\max\{\nu_A(n), \nu_B(n)\}, \max\{\nu_A(q), \xi_B(q)\}\}\\ &= \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(q)\}\end{aligned}$$

Hence $\nu_{A\cap B}(n-q) \leq \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(q)\}$

and

$$\begin{aligned}\zeta_{A\cap B}(n-q) &= \max\{\zeta_A(n-q), \zeta_B(n-q)\}\\ &\leq \max\{\max\{\zeta_A(n), \zeta_A(q)\}, \max\{\zeta_B(n), \zeta_B(q)\}\}\\ &= \max\{\max\{\zeta_A(n), \zeta_B(n)\}, \max\{\zeta_A(q), \zeta_B(q)\}\}\\ &= \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(q)\}\end{aligned}$$

Hence $\zeta_{A\cap B}(n-q) \leq \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(q)\}$. Then

$$\begin{aligned}\xi_{A\cap B}(nqz) &= \min\{\xi_A(klm), \xi_B(nqz)\}\\ &\geq \min\{\min\{\xi_A(n), \xi_A(z)\}, \min\{\xi_B(n), \xi_B(z)\}\}\\ &= \min\{\min\{\xi_A(n), \xi_B(n)\}, \min\{\xi_A(z), \xi_B(z)\}\}\\ &= \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(z)\}\end{aligned}$$

Hence $\xi_{A\cap B}(nqz) \geq \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(z)\}$

and

$$\begin{aligned}\omega_{A\cap B}(nqz) &= \min\{\omega_A(nqz), \omega_B(nqz)\}\\ &\geq \min\{\min\{\omega_A(n), \omega_A(z)\}, \min\{\omega_B(n), \omega_B(z)\}\}\\ &= \min\{\min\{\omega_A(n), \omega_B(n)\}, \min\{\omega_A(z), \omega_B(z)\}\}\\ &= \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(z)\}\end{aligned}$$

Hence $\omega_{A\cap B}(nqz) \geq \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(z)\}$. Then

$$\begin{aligned}\nu_{A\cap B}(nqz) &= \max\{\nu_A(nqz), \nu_B(nqz)\}\\ &\leq \max\{\min\{\nu_A(n), \nu_A(z)\}, \max\{\nu_B(n), \xi_B(z)\}\}\\ &= \max\{\max\{\nu_A(n), \nu_B(n)\}, \max\{\nu_A(z), \xi_B(z)\}\}\\ &= \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(z)\}\end{aligned}$$

Hence $\nu_{A\cap B}(nqz) \leq \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(z)\}$

and

$$\begin{aligned}\zeta_{A\cap B}(nqz) &= \max\{\zeta_A(nqz), \zeta_B(nqz)\}\\ &\leq \max\{\max\{\zeta_A(n), \zeta_A(z)\}, \max\{\zeta_B(n), \zeta_B(z)\}\}\end{aligned}$$

$$= \max\{\max\{\zeta_A(n), \zeta_B(n)\}, \max\{\zeta_A(z), \zeta_B(z)\}\}$$
$$= \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(z)\}$$

Hence $\zeta_{A\cap B}(nqz) \leq \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(z)\}$.

Hence $A \cap B$ is a Quadripartitioned neutrosophic Pythagorean fuzzy bi-ideal in near ring N.

□

Theorem 3.5. *If A and B are Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring then the intersection of $A \cap B$ is also Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring.*

Proof. Let $A = (\xi_A, \omega_A, \nu_A, \zeta_A)$ be a Quadripartitioned neutrosophic Pythagorean fuzzy subset of N.

(i) For any $n, q, z \in N$ we have

$$\xi_{A\cap B}(n-q) = \min\{\xi_A(n-q), \xi_B(n-q)\}$$
$$\geq \min\{\min\{\xi_A(n), \xi_A(q)\}, \min\{\xi_B(n), \xi_B(q)\}\}$$
$$= \min\{\min\{\xi_A(n), \xi_B(n)\}, \min\{\xi_A(q), \xi_B(q)\}\}$$
$$= \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(q)\}$$

Hence $\xi_{A\cap B}(n-q) \geq \min\{\xi_{A\cap B}(n), \xi_{A\cap B}(q)\}$

and

$$\omega_{A\cap B}(n-q) = \min\{\omega_A(n-q), \omega_B(n-q)\}$$
$$\geq \min\{\min\{\omega_A(n), \omega_A(q)\}, \min\{\omega_B(n), \omega_B(q)\}\}$$
$$= \min\{\min\{\omega_A(n), \omega_B(n)\}, \min\{\omega_A(q), \omega_B(q)\}\}$$
$$= \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(q)\}$$

Hence $\omega_{A\cap B}(n-q) \geq \min\{\omega_{A\cap B}(n), \omega_{A\cap B}(q)\}$. Then

$$\nu_{A\cap B}(n-q) = \max\{\nu_A(n-q), \nu_B(n-q)\}$$
$$\leq \max\{\min\{\nu_A(n), \nu_A(q)\}, \max\{\nu_B(n), \xi_B(q)\}\}$$
$$= \max\{\max\{\nu_A(n), \nu_B(n)\}, \max\{\nu_A(q), \xi_B(q)\}\}$$
$$= \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(q)\}$$

Hence $\nu_{A\cap B}(n-q) \leq \max\{\nu_{A\cap B}(n), \nu_{A\cap B}(q)\}$

and

$$\zeta_{A\cap B}(n-q) = \max\{\zeta_A(n-q), \zeta_B(n-q)\}$$
$$\leq \max\{\max\{\zeta_A(n), \zeta_A(q)\}, \max\{\zeta_B(n), \zeta_B(q)\}\}$$
$$= \max\{\max\{\zeta_A(n), \zeta_B(n)\}, \max\{\zeta_A(q), \zeta_B(q)\}\}$$
$$= \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(q)\}$$

Hence $\zeta_{A\cap B}(n-q) \leq \max\{\zeta_{A\cap B}(n), \zeta_{A\cap B}(q)\}$. Next we obtained.

$$\xi_{A\cap B}(q+n-q) = \min\{\xi_A(q+n-q), \xi_B(q+n-q)\}$$
$$\geq \min\{\xi_A(n), \xi_B(n)\}$$

$= \xi_{A\cap B}(n).$

Similarly we prove

$$\begin{aligned}
\omega_{A\cap B}(q+n-q) &= \min\{\omega_A(q+n-q), \omega_B(q+n-q)\}\\
&\geq \min\{\omega_A(n), \omega_B(n)\}\\
&= \omega_{A\cap B}(n)\\
\nu_{A\cap B}(q+n-q) &= \max\{\nu_A(q+n-q), \nu_B(q+n-q)\}\\
&\leq \max\{\nu_A(n), \nu_B(n)\}\\
&= \nu_{A\cap B}(n).
\end{aligned}$$

and

$$\begin{aligned}
\zeta_{A\cap B}(q+n-q) &= \max\{\zeta_A(q+n-q), \zeta_B(q+n-q)\}\\
&\leq \max\{\zeta_A(n), \zeta_B(n)\}\\
&= \zeta_{A\cap B}(n).
\end{aligned}$$

Further more we deduce that

$$\begin{aligned}
\xi_{A\cap B}(nq) &= \min\{\xi_A(nq), \xi_B(nq)\}\\
&\geq \min\{\xi_A(q), \xi_B(q)\}\\
&= \xi_{A\cap B}(q)
\end{aligned}$$

Similarly we prove that

$\omega_{A\cap B}(nq)$, $\nu_{A\cap B}(nq)$ and $\zeta_{A\cap B}(nq)$.

Finally we deduce that

$$\begin{aligned}
\xi_{A\cap B}((n+z)q-nq) &= \min\{\xi_A((n+z)q-nq), \xi_B((n+z)q-nq)\}\\
&\geq \min\{\xi_A(z), \xi_B(z)\}\\
&= \xi_{A\cap B}(z).
\end{aligned}$$

Similarly we prove

$\omega_{A\cap B}((n+z)q-nq)$, $\nu_{A\cap B}((n+z)q-nq)$ and $\zeta_{A\cap B}((n+z)q-nq)$.

Therefore $A\cap B$ is a Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring N.

□

Theorem 3.6. *Let $A = (\xi_A, \omega_A, \nu_A, \zeta_A)$ be a Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring N. Then $A^m = \{z, \xi_{A^m}, \omega_{A^m}, \nu_{A^m}, \zeta_{A^m} : z \in N\}$ is a QNPFI of N. Where m is a positive integer and $\xi_{A^m}(z) = (\xi_A(z))^m$, $\omega_{A^m}(z) = (\omega_A(z))^m$, $\nu_{A^m}(z) = (\nu_A(z))^m$ and $\zeta_{A^m}(z) = (\zeta_A(z))^m$.*

Proof. Let A be a QNPFI of N. Let $n, q, z \in N$. Then the following are observed.

For truth membership grade, we can write

$$\begin{aligned}
\xi_{A^m}(n-q) &= (\xi_A(n-q))^m\\
&\geq \min\{\xi_A(n), \xi_A(q)\}^m\\
&= \min\{(\xi_A(n))^m, (\xi_A(q))^m\}
\end{aligned}$$

$$= \min\{\xi_{A^m}(n), \xi_{A^m}(q)\}.$$

For contradiction membership grade,

$$\begin{aligned}\omega_{A^m}(n-q) &= (\omega_A(n-q))^m \\ &\geq \min\{\omega_A(n), \omega_A(q)\}^m \\ &= \min\{(\omega_A(n))^m, (\omega_A(q))^m\} \\ &= \min\{\omega_{A^m}(n), \omega_{A^m}(q)\}.\end{aligned}$$

For ignorance membership grade,

$$\begin{aligned}\nu_{A^m}(n-q) &= (\nu_A(n-q))^m \\ &\leq \max\{\nu_A(n), \nu_A(q)\}^m \\ &= \max\{(\nu_A(n))^m, (\nu_A(q))^m\} \\ &= \max\{\nu_{A^m}(n), \nu_{A^m}(q)\}.\end{aligned}$$

For false membership grade,

$$\begin{aligned}\zeta_{A^m}(n-q) &= (\zeta_A(n-q))^m \\ &\leq \max\{\zeta_A(n), \zeta_A(q)\}^m \\ &= \max\{(\zeta_A(n))^m, (\zeta_A(q))^m\} \\ &= \max\{\zeta_{A^m}(n), \zeta_{A^m}(q)\}.\end{aligned}$$

Next it is obtained that

$$\begin{aligned}\xi_{A^m}(q+n-q) &= (\xi_A(q+n-q))^m \\ &\geq (\xi_A(n))^m \\ &= \xi_{A^m}(n).\end{aligned}$$

$$\begin{aligned}\omega_{A^m}(q+n-q) &= (\omega_A(q+n-q))^m \\ &\geq (\omega_A(n))^m \\ &= \omega_{A^m}(n).\end{aligned}$$

$$\begin{aligned}\nu_{A^m}(q+n-q) &= (\nu_A(q+n-q))^m \\ &\leq (\nu_A(n))^m \\ &= \nu_{A^m}(n).\end{aligned}$$

and

$$\begin{aligned}\zeta_{A^m}(q+n-q) &= (\zeta_A(q+n-q))^m \\ &\leq (\zeta_A(n))^m \\ &= \zeta_{A^m}(n).\end{aligned}$$

Also

$$\begin{aligned}\xi_{A^m}(nq) &= (\xi_A(nq))^m \\ &\geq (\xi_A(q))^m \\ &= \xi_{A^m}(q)\end{aligned}$$

$$\begin{aligned}\omega_{A^m}(nq) &= (\omega_A(nq))^m \\ &\geq (\omega_A(q))^m\end{aligned}$$

$$
\begin{aligned}
&= \omega_{A^m}(q) \\
\nu_{A^m}(nq) &= (\nu_A(nq))^m \\
&\leq (\nu_A(q))^m \\
&= \nu_{A^m}(q).
\end{aligned}
$$

Finally

$$
\begin{aligned}
\zeta_{A^m}(nq) &= (\zeta_A(nq))^m \\
&\leq (\zeta_A(q))^m \\
&= \zeta_{A^m}(q).
\end{aligned}
$$

At last, we write

$$
\begin{aligned}
\xi_{A^m}((n+z)q - nq) &= (\xi_A((n+z)q - nq))^m \\
&\geq (\xi_A(z))^m \\
&= \xi_{A^m}(z)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_{A^m}((n+z)q - nq) &= (\omega_A((n+z)q - nq))^m \\
&\geq (\omega_A(z))^m \\
&= \omega_{A^m}(z)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{A^m}((n+z)q - nq) &= (\nu_A((n+z)q - nq))^m \\
&\leq (\nu_A(z))^m \\
&= \nu_{A^m}(z)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\zeta_{A^m}((n+z)q - nq) &= (\zeta_A((n+z)q - nq))^m \\
&\leq (\zeta_A(z))^m \\
&= \zeta_{A^m}(z).
\end{aligned}
$$

Therefore A^m is a Quadripartitioned neutrosophic Pythagorean fuzzy ideal of near ring N^m.

□

Theorem 3.7. *Let $A_i = \{(\xi_{A_i}, \omega_{A_i}, \nu_{A_i}, \zeta_{A_i}) | i \in I\}$ be a family of Quadripartitioned Neutrosophic Pythagorean fuzzy bi-ideals in near ring N. Then $(\bigcap_{i\in I} A_i)$ is Quadripartitioned neutrosophic Pythagorean fuzzy bi-ideal of N. Where $\{\bigcap_{i\in I} A_i\} = \{\bigcap_{i\in I} \xi_{A_i}, \bigcap_{i\in I} \eta_{A_i}, \bigcup_{i\in I} \mu_{A_i}, \bigcup_{i\in I} \lambda_{A_i}\}$.*

Proof. Let $A_i = \{(\xi_{A_i}, \eta_{A_i}, \mu_{A_i}, \lambda_{A_i}) | i \in I\}$ and for all $n, q, z \in N$.

$$
\begin{aligned}
\bigcap_{i\in I} \xi_{A_i}(n-q) &= \inf \xi_{A_i}(n-q) \\
&\geq \inf\{\min\{\xi_{A_i}(n), \xi_{A_i}(q)\}\} \\
&= \min\{\inf \xi_{A_i}(n), \inf \xi_{A_i}(q)\} \\
&= \min\{\bigcap_{i\in I} \xi_A(n), \bigcap_{i\in I} \xi_A(q)\}
\end{aligned}
$$

and

$$\begin{aligned}\bigcap_{i\in I}\omega_{A_i}(n-q) &= \inf\omega_{A_i}(n-q)\\ &\geq \inf\{\min\{\omega_{A_i}(n),\omega_{A_i}(q)\}\}\\ &= \min\{\inf\omega_{A_i}(n),\inf\omega_{A_i}(q)\}\\ &= \min\{\bigcap_{i\in I}\omega_A(n),\bigcap_{i\in I}\omega_A(q)\}\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in I}\nu_{A_i}(n-q) &= \sup\nu_{A_i}(n-q)\\ &\leq \sup\{\max\{\nu_{A_i}(n),\nu_{A_i}(q)\}\}\\ &= \max\{\sup\nu_{A_i}(n),\sup\nu_{A_i}(q)\}\\ &= \max\{\bigcap_{i\in I}\nu_A(n),\bigcap_{i\in I}\nu_A(q)\}\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in I}\zeta_{A_i}(n-q) &= \sup\zeta_{A_i}(n-q)\\ &\leq \sup\{\max\{\zeta_{A_i}(n),\zeta_{A_i}(q)\}\}\\ &= \max\{\sup\zeta_{A_i}(n),\sup\zeta_{A_i}(q)\}\\ &= \max\{\bigcap_{i\in I}\zeta_A(n),\bigcap_{i\in I}\zeta_A(q)\}\end{aligned}$$

At last we obtained,

$$\begin{aligned}\bigcap_{i\in I}\xi_{A_i}(nqz) &= \inf\xi_{A_i}(nqz)\\ &\geq \inf\{\min\{\xi_{A_i}(n),\xi_{A_i}(z)\}\}\\ &= \min\{\inf\xi_{A_i}(n),\inf\xi_{A_i}(z)\}\\ &= \min\{\bigcap_{i\in I}\xi_A(n),\bigcap_{i\in I}\xi_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\bigcap_{i\in I}\omega_{A_i}(nqz) &= \inf\omega_{A_i}(nqz)\\ &\geq \inf\{\min\{\omega_{A_i}(n),\omega_{A_i}(z)\}\}\\ &= \min\{\inf\omega_{A_i}(n),\inf\omega_{A_i}(z)\}\\ &= \min\{\bigcap_{i\in I}\omega_A(n),\bigcap_{i\in I}\omega_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in I}\nu_{A_i}(nqz) &= \sup\nu_{A_i}(nqz)\\ &\leq \sup\{\max\{\nu_{A_i}(n),\nu_{A_i}(z)\}\}\\ &= \max\{\sup\nu_{A_i}(n),\sup\nu_{A_i}(z)\}\\ &= \max\{\bigcap_{i\in I}\nu_A(n),\bigcap_{i\in I}\nu_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in I}\zeta_{A_i}(nqz) &= \sup\zeta_{A_i}(nqz)\\ &\leq \sup\{\max\{\zeta_{A_i}(n),\zeta_{A_i}(z)\}\}\\ &= \max\{\sup\zeta_{A_i}(n),\sup\zeta_{A_i}(z)\}\end{aligned}$$

$$= \max\{\bigcap_{i\in I}\zeta_A(n), \bigcap_{i\in I}\zeta_A(z)\}.$$

Therefore $\bigcap_{i\in I} A_i$ is a family of Quadripartitioned neutrosophic Pythagorean fuzzy bi-ideals of near ring N. □

4. **Conclusion**

This research explores the core characteristics of quadri-partitioned neutrosophic Pythagorean fuzzy ideals (QNPFIs) and bi-ideals (QNPFBIs) within the framework of near-rings. We have demonstrated that the intersection of these structures retains their fundamental fuzzy ideal properties. Furthermore, our findings establish that exponentiation of a QNPFI preserves its intrinsic features.

Future investigations could focus on broadening these concepts to encompass more generalized algebraic frameworks, including semirings and near rings with additional structural constraints. Another significant avenue for exploration is the practical implementation of these fuzzy structures in computational intelligence and optimization-based decision-making processes. Additionally, examining homomorphic properties and their influence on algebraic structures may contribute to a deeper theoretical understanding of QNPFIs in advanced mathematical settings.

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