Reduction of Neutrosophic Fuzzy Matrices Using Implication Operator

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Abstract: This research explores the reduction of Neutrosophic Fuzzy Matrices (NFMs) and highlights their significant properties, focusing particularly on nilpotent NFMs. The study examines the reduction of irreflexive and transitive NFMs and applies these principles to nilpotent NFMs, which are represented as acyclic graphs. These acyclic graphs play a critical role in defining consistent systems, especially when their union forms a cyclic graph that is isomorphic. Failure to meet these conditions results in non-isomorphic graphs. To demonstrate these ideas, numerical examples are provided, and the equivalent conditions for reduction are thoroughly established. Furthermore, the collection of s-transitive and w-transitive NFMs is shown to encompass the set of transitive NFMs for which reduction models have been verified. The properties of these reduction models are also proven to be applicable to s-transitive and w-transitive NFMs.

Keywords: Neutrosophic Fuzzy Matrix; Neutrosophic Fuzzy irreflexive matrix; Neutrosophic Fuzzy nilpotent matrix, s-transitive Neutrosophic Fuzzy matrix, w-transitive Neutrosophic Fuzzy matrix.

1. Introduction

Neutrosophic Fuzzy Matrices (NFMs) have garnered considerable interest in recent years for their ability to manage uncertainty, imprecision, and vagueness across diverse complex systems. These matrices extend the concepts of fuzzy sets and intuitionistic fuzzy sets (IFSs) by incorporating the degree of truth, indeterminacy, and falsity into a unified framework, as introduced by Smarandache [6]. Building upon the foundational work of Zadeh [1] on fuzzy sets and Atanassov [2] on IFSs, NFMs offer a versatile approach for modeling and analyzing systems characterized by incomplete and inconsistent information. Meenakshi [7] has extensively studied fuzzy matrix theory and its applications. The theoretical development of fuzzy matrices has been a topic of extensive research. For instance, studies by Shyamal and Pal [8] and Bhowmik and Pal [9,10] have explored operators and properties of fuzzy and intuitionistic fuzzy matrices. Further advancements include works by Meenakshi and Gandhimathi [11] and Sriram and Murugadas [12], which delve into the structural and algebraic properties of these matrices. These investigations laid the groundwork for extending the theory to neutrosophic fuzzy matrices, enabling the representation of more complex relationships.

Reduction models for fuzzy and intuitionistic fuzzy matrices have been extensively studied. Hashimoto [14,15] introduced the concepts of subinverses and the reduction of nilpotent fuzzy matrices, while Antonion et al. [21] explored methods for simplifying transitive fuzzy matrices. Similarly, Padder and Murugadas [22,24] extended these reduction techniques to intuitionistic fuzzy matrices, demonstrating their applications in various mathematical and computational contexts. These studies emphasize the importance of reduction techniques for simplifying the analysis and computation of large-scale fuzzy systems. A significant body of work has been devoted to exploring the properties of nilpotent matrices, which are integral to the characterization of acyclic graphs and consistent systems. Han et al. [31] and Lur et al. [32,34] investigated nilpotent fuzzy matrices and their algebraic properties, while Tan [35] examined their applications over distributive lattices. In the context of neutrosophic fuzzy matrices, these concepts have been extended to analyze systems with higher levels of uncertainty.

Recent studies have introduced new classes of NFMs, such as secondary k-column symmetric matrices and interval-valued secondary k-range symmetric matrices (Anandhkumar et al. [28,36]). These developments demonstrate the versatility of NFMs in modeling symmetrical relationships and interval-based uncertainties. Furthermore, the works on generalized symmetric Fermatean neutrosophic fuzzy matrices (Anandhkumar et al. [37]) highlight their applicability in advanced mathematical frameworks. The notion of transitivity plays a pivotal role in the study of NFMs. Transitive matrices are crucial for representing consistent systems and ensuring the reliability of conclusions drawn from such systems. The reduction of s-transitive and w-transitive neutrosophic fuzzy matrices, as explored by Padder and Murugadas [25,26], further enriches the theoretical understanding of these constructs. Additionally, the canonical forms of transitive intuitionistic fuzzy matrices, as discussed by Lee and Jeong [33], provide a basis for extending these ideas to neutrosophic systems. Mohamed et al. [38] have studied an efficient neutrosophic approach for evaluating possible Industry 5.0 enablers in consumer electronics: a case study. Salama et al. [39] have discussed a neutrosophic model for measuring and evaluating the role of digital transformation in improving sustainable performance using the balanced scorecard in Egyptian universities.

Anandhkumar et al [40] have studied Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. Radhika, et al [41] have presented on Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making on various Inverse of Neutrosophic Fuzzy Matrices. Anandhkumar et al [42] have studied Pseudo Similarity of Neutrosophic Fuzzy matrices. Radhika et al [43] have studied On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices. Prathab et al [44] have characterized Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses. Anandhkumar et al [45] have studied Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices. Punithavalli1 et al [46] have presented Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices. Anandhkumar et al [47] have analyzed Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices. Anandhkumar, et al [48] have studied Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices. Punithavalli, and Anandhkumar [49] have focused on Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices. Anandhkumar et al [50] have analyzed Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices. Anandhkumar et al [51] have studied Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices.

This article explores the reduction of NFMs and highlights their significant properties, focusing particularly on nilpotent NFMs. The study examines the reduction of irreflexive and transitive NFMs and applies these principles to nilpotent NFMs, which are represented as acyclic graphs. These acyclic graphs play a critical role in defining consistent systems, especially when their union forms a cyclic graph that is isomorphic. Failure to meet these conditions results in non-isomorphic graphs. To demonstrate these ideas, numerical examples are provided, and the equivalent conditions for reduction are thoroughly established. Furthermore, the collection of s-transitive and w-transitive NFMs is shown to encompass the set of transitive NFMs for which reduction models have been verified. The properties of these reduction models are also proven to be applicable to s-transitive and w-transitive NFMs.

Abbreviations

FM: Fuzzy Matrices

IFM: Intuitionistic Fuzzy Matrices

NFM **:** Neutrosophic Fuzzy Matrices

AI: Artificial intelligence

1.1 Literature Review

Evolution of Neutrosophic Fuzzy Matrices

Neutrosophic Fuzzy Matrices (NFMs) have emerged as an extension of fuzzy sets and intuitionistic fuzzy matrices (IFMs), addressing the limitations of traditional models in handling uncertainty, imprecision, and vagueness. Smarandache [6] introduced the concept of neutrosophic sets, which incorporate truth, indeterminacy, and falsity components into a unified framework. Building on the foundational theories of fuzzy sets by Zadeh [1] and IFSs by Atanassov [2], NFMs provide a robust mathematical tool for analysing complex systems characterized by incomplete or inconsistent information.

Development of Fuzzy and IFMs

Fuzzy matrices have been a significant area of research since their introduction, with extensive studies focusing on their structural properties and practical applications. Shyamal and Pal [8] explored fundamental operators and properties of fuzzy matrices, while Bhowmik and Pal [9,10] extended these studies to IFMs. Later, Meenakshi and Gandhimathi [11] and Sriram and Murugadas [12] advanced this work by delving into the algebraic and structural characteristics of these matrices. These foundational studies have paved the way for the application and theoretical extension of NFMs.

Reduction Techniques for Fuzzy and Intuitionistic Fuzzy Matrices

Reduction techniques have been central to simplifying and analysing large-scale fuzzy and intuitionistic fuzzy systems. Hashimoto [14,15] pioneered the concepts of sub inverses and the reduction of nilpotent fuzzy matrices, a concept further explored by Antonion et al. [21] in the context of transitive matrices. Padder and Murugadas [22,24] extended these reduction techniques to intuitionistic fuzzy matrices, demonstrating their practical applications in computational analysis.

1.2 Novelty

The novelty of this study lies in its extension of existing theoretical frameworks and its exploration of new techniques and applications of Neutrosophic Fuzzy Matrices (NFMs) in handling uncertainty, imprecision, and vagueness in complex systems. While previous research has focused on the foundational properties and reduction techniques for fuzzy and intuitionistic fuzzy matrices, this work introduces a novel approach by emphasizing the role of NFMs in representing systems with higher levels of uncertainty and more intricate relationships. Specifically, the study proposes the reduction of neutrosophic fuzzy matrices and their application to consistent systems, focusing on the properties of nilpotent NFMs in characterizing acyclic graphs. Additionally, the research explores the generalization of transitivity concepts through the examination of s-transitive and w-transitive neutrosophic fuzzy matrices, offering new perspectives for system modelling.

This paper's approach builds on and enhances prior research by incorporating reduction techniques, numerical examples, and advancing the theoretical understanding and practical utility of NFMs in various domains.

The referenced publications have significantly advanced the fields of fuzzy matrices, intuitionistic fuzzy matrices, and neutrosophic fuzzy matrices. Zadeh's foundational work on fuzzy sets [1] laid the groundwork for modelling uncertainty, while Atanassov extended this theory by introducing intuitionistic fuzzy sets [2], incorporating a degree of hesitation. Atanassov's later works further expanded on fuzzy logic with intuitionistic fuzzy implications [3] and fuzzy negations [4,5], introducing new logical tools for reasoning in complex systems. Smarandache [6] introduced neutrosophic sets, offering a comprehensive framework for handling truth, indeterminacy, and falsity, enabling better modelling of uncertainty.

In the realm of fuzzy matrices, Shyamal and Pal [8] introduced new operators, enhancing the mathematical tools available for modelling in fuzzy environments. Bhowmik and Pal's work on generalized intuitionistic fuzzy matrices [9,10] contributed novel results on the algebraic properties of these matrices, expanding their applicability in decision-making systems. Meenakshi and Gandhimathi [11] provided important methods for solving intuitionistic fuzzy relational equations, broadening the scope of IFSs in relational problem-solving. Sriram and Murugadas [12] investigated the algebraic properties of intuitionistic fuzzy matrices, particularly sub inverses, adding to the understanding of matrix operations in fuzzy systems.

Hashimoto's studies [13-15] on fuzzy matrix reductions, traces, and nilpotent matrices introduced essential techniques for simplifying complex fuzzy systems. Antonion et al. [21] focused on the reduction of transitive fuzzy matrices, providing key methods for analysing transitive properties in fuzzy relations. Murugadas and Padder [22,24] contributed significantly to the reduction techniques for intuitionistic fuzzy matrices, including nilpotent matrices, simplifying their computational analysis. The work of Padder and Murugadas [25,26] on s-transitive and w-transitive intuitionistic fuzzy matrices provided critical insights into the transitivity concepts in fuzzy systems, enriching the understanding of consistency and decision-making processes.

1.3 Main Contributions of the Study

This study makes the following key contributions:

Introduction of Advanced Reduction Techniques for NFMs

- Developed innovative methods for reducing Neutrosophic Fuzzy Matrices (NFMs), focusing on irreflexive, transitive, and nilpotent matrices.
- These reduction techniques simplify complex systems while preserving their structural and functional integrity.

Exploration of Nilpotent NFMs and Acyclic Graphs

- Highlighted the unique role of nilpotent NFMs in representing acyclic graphs, which are integral to defining consistent systems.
- Demonstrated the importance of the union of acyclic graphs forming cyclic graphs that are isomorphic, and analysed the consequences when these conditions are not met.

Equivalent Conditions for Reduction

- Established rigorous equivalent conditions for reducing NFMs, providing a mathematical framework to determine when reductions can be applied without loss of essential information.

Generalization of Transitivity Concepts

- Investigated and validated reduction models for s-transitive and w-transitive NFMs.
- Demonstrated that these models encompass general transitive NFMs, thus broadening the theoretical understanding of transitivity in neutrosophic systems.

Practical Demonstrations through Numerical Examples

- Presented detailed numerical examples to illustrate the application of reduction techniques.
- These examples offer a clear pathway for implementing the proposed methods in real-world scenarios.

Advancement of Theoretical Frameworks

- Extended the theoretical underpinnings of NFMs by integrating reduction techniques with structural properties of nilpotent, irreflexive, and transitive matrices.
- Enhanced the application potential of NFMs in decision-making, system modelling, and analysis of uncertain environments.

1.4 Expansion on Existing Problems

The study addresses several critical challenges and highlights existing gaps in the research on NFMs:

Computational Complexity in Reduction

Existing reduction methods for fuzzy and intuitionistic fuzzy matrices are not optimized for large-scale neutrosophic systems, resulting in high computational overhead. This study introduces more efficient reduction techniques, though further work is needed to optimize them for scalability in dynamic environments.

Inadequate Representation of Higher Uncertainty

Traditional fuzzy and intuitionistic fuzzy matrices struggle to handle high levels of uncertainty and indeterminacy. NFMs provide a more comprehensive framework, but the quantification and representation of complex uncertainties in real-time applications remain challenging.

Limited Understanding of Transitivity Variants

While transitivity is central to consistent system modelling, there is limited understanding of how s-transitivity and w-transitivity interact with general transitive properties. This work expands the theoretical framework but calls for further exploration of hybrid transitivity properties in practical applications.

Graph Isomorphism and Structural Analysis

The connection between acyclic graphs derived from nilpotent NFMs and cyclic graphs forming isomorphic structures is insufficiently explored. This study identifies these relationships, but a comprehensive classification of non-isomorphic graphs is required to strengthen consistency analysis.

Real-World Implementation Challenges

Despite their theoretical robustness, NFMs have limited implementation in practical domains like AI, engineering, and decision-making due to the lack of integrated software tools and frameworks. The proposed numerical examples bridge this gap partially, but more extensive case studies and tool development are needed to validate the methods in real-world settings.

Scalability of Reduction Models

Reduction techniques are often designed for static systems and lack adaptability for large, dynamic, or evolving datasets. Future work must focus on creating adaptive algorithms that can handle the evolving nature of data and relationships in neutrosophic systems.

Interdisciplinary Applications of NFMs

While NFMs are powerful, their application in interdisciplinary fields like healthcare, economics, and climate modelling is underexplored. This study lays a theoretical foundation that could inspire future research to apply NFMs in these and other domains, leveraging their ability to model uncertainty effectively.

1.5 Open Issues and Limitations

Scalability:

Existing reduction techniques face challenges in handling large-scale NFMs due to computational complexity. Efficient algorithms are needed to ensure scalability in real-world applications.

Generalization of Transitivity:

While this study extends transitivity concepts to s-transitive and w-transitive matrices, further exploration is required to handle systems with hybrid or dynamic transitivity properties.

Uncertainty Representation:

Although NFMs manage uncertainty effectively, representing and quantifying complex indeterminacies remain an open problem, particularly in real-time systems.

Practical Implementation:

Limited studies bridge the gap between theoretical reductions and their practical implementations in fields like AI, decision-making, and engineering.

Graph Isomorphism and System Consistency:

The relationship between cyclic graph formations and system consistency in NFMs is not fully resolved. Further investigation is needed to characterize non-isomorphic scenarios robustly.

1.6 Explanation of the Proposed Work

This research focuses on the reduction of Neutrosophic Fuzzy Matrices (NFMs) and the analysis of their properties, with a particular emphasis on nilpotent NFMs, irreflexive matrices, and transitive matrices. Below is a detailed description of the proposed work, its steps, limitations, and rationale.

Proposed Work

Key Objectives

- Develop and analyse reduction techniques for NFMs to simplify complex systems.
- Explore the role of nilpotent NFMs in defining acyclic graphs and their significance in consistent systems.
- Extend transitivity concepts by examining s-transitive and w-transitive NFMs.

Steps in the Proposed Work

Mathematical Framework Development

- Extend existing theories of fuzzy and intuitionistic fuzzy matrices to neutrosophic fuzzy systems.
- Define the properties and operators specific to NFMs, such as those for reduction, transitivity, and nilpotency.

Reduction Techniques

- **Irreflexive NFMs:** Formulate conditions for reduction by identifying and eliminating redundancies in the matrices.
- **Transitive NFMs:** Establish rules for maintaining transitivity during reduction processes.
- **Nilpotent NFMs:** Apply graph theory concepts to model nilpotent matrices as acyclic graphs. Develop conditions for reduction while preserving acyclic properties.

Graph Representation

- Represent nilpotent NFMs as acyclic graphs and examine the isomorphic properties of their unions when forming cyclic graphs.
- Analyse non-isomorphic cases and their implications for consistency in system representation.

Verification of s-Transitive and w-Transitive Properties

- Extend reduction models to encompass s-transitive and w-transitive matrices.
- Prove that these classes are subsets of general transitive NFMs.

Numerical Examples

- Provide detailed numerical examples to illustrate reduction methods, transitivity extensions, and graph isomorphism analyses.

Validation

- Verify the applicability of the reduction models and transitivity concepts through rigorous mathematical proofs and case studies.

1.7 Limitations of the Proposed Work

Scalability Challenges

- The proposed reduction methods are computationally intensive for large-scale NFMs.
- Handling high-dimensional matrices or dynamic systems with evolving uncertainty requires further optimization.

Complexity of Graph Isomorphism

- The analysis of cyclic and acyclic graphs, especially for large matrices, can be computationally challenging due to the NP-complete nature of the graph isomorphism problem.

Real-Time Applications

- The proposed methods are primarily theoretical and may require additional adaptation for real-time applications in domains like AI or decision-making systems.

Limited Interdisciplinary Case Studies

- While the methods are robust, their application to practical problems in diverse fields (e.g., healthcare or logistics) remains underexplored.

Assumptions in Reduction Models

- The reduction models assume ideal conditions, such as the availability of complete input data and precise matrix properties, which may not always hold in real-world scenarios.

1.8 Reasons for Choosing These Approaches

Addressing Gaps in Existing Research

- Traditional methods for fuzzy and intuitionistic fuzzy matrices are insufficient for handling higher degrees of uncertainty. NFMs, with their inclusion of truth, indeterminacy, and falsity components, provide a more comprehensive approach.

Enhancing Analytical Capabilities

- Reduction techniques simplify the computational analysis of NFMs, making them more accessible for modelling complex systems.

Graph Theory Integration

- The use of graph theory provides a visually intuitive and mathematically rigorous way to analyse properties like nilpotency and transitivity.

Generalization of Transitivity

- Expanding transitivity concepts (s-transitive and w-transitive) broadens the applicability of NFMs to various domains where consistency and relational hierarchies are critical.

Feasibility and Rigor

- The proposed methods build on well-established mathematical principles, ensuring theoretical robustness and a clear pathway for practical implementation.

1.9 Detailed Steps for the Work

Step 1: Define NFMs and Their Properties

- Define NFMs, their components (truth, indeterminacy, falsity), and their fundamental properties such as transitivity, irreflexivity, and nilpotency.

Step 2: Develop Reduction Methods

- Identify redundant elements in NFMs that do not contribute to system analysis.
- Formulate rules for reduction that preserve critical properties like transitivity and irreflexivity.

Step 3: Analyse Nilpotent NFMs

- Model nilpotent matrices as acyclic graphs.
- Explore the union of acyclic graphs and analyse conditions under which they form cyclic or isomorphic structures.

Step 4: Generalize Transitivity

- Define s-transitive and w-transitive matrices as extensions of transitive NFMs.
- Prove that the reduction models for these matrices are consistent and comprehensive.

Step 5: Provide Numerical Examples

- Use numerical matrices to demonstrate the reduction process.
- Show how equivalent conditions for reduction are applied to irreflexive, transitive, and nilpotent matrices.

Step 6: Validate Through Proofs and Applications

- Mathematically verify the properties of the reduced matrices and their applicability in complex systems.
- Apply the proposed methods to theoretical case studies and identify potential applications in decision-making, AI, and engineering systems.

1.10 Key Trends in Clarity of Results

Improved Simplification Through Reduction Techniques

- **Trend:** The proposed reduction techniques simplify large and complex neutrosophic fuzzy matrices (NFMs) while preserving key properties such as transitivity, irreflexivity, and nilpotency.
- **Key Results:** Reduction leads to faster computation and better interpretability in real-world applications, such as consistent system modelling.

Graphical Representation of Nilpotent NFMs

- **Trend:** Representing nilpotent NFMs as acyclic graphs offers a clear visualization of consistent systems and their behaviour when combined into cyclic graphs.
- **Key Results:** Demonstrating the structural differences between isomorphic and non-isomorphic graphs enhances understanding of system consistency.

Extension of Transitivity Concepts

- **Trend:** Generalizing transitivity to s-transitive and w-transitive matrices broadens the scope of NFMs for diverse applications.
- **Key Results:** Verification of reduction models for these classes of matrices ensures applicability to more complex systems.

Numerical Validation

- **Trend:** The numerical examples provided validate the practical utility of the reduction methods.
- **Key Results:** These examples highlight the scalability and effectiveness of the proposed techniques.

1.11 Comparison with Other Neutrosophic Methods (Tabular format)

This section provides a detailed comparison of the proposed methods for reducing Neutrosophic Fuzzy Matrices (NFMs) with existing approaches. The focus is on their computational efficiency, ability to handle uncertainty, and practical applicability. Key aspects such as transitivity, nilpotency, and structural preservation are discussed to highlight the advantages and limitations of each method.

Comparison Criteria

To ensure a comprehensive comparison, the following criteria are evaluated:

- **Reduction Efficiency:** The ability to simplify complex NFMs while retaining essential properties.
- **Support for Transitivity Extensions:** Whether the method accommodates s-transitive and w-transitive matrices.
- **Handling of Uncertainty:** The capacity to manage higher levels of indeterminacy and falsity.
- **Graph Representation:** Effectiveness in visualizing matrices as acyclic and cyclic graphs.
- **Computational Complexity:** The resources required for execution.

Existing Methods

Hashimoto's Reduction Techniques for Fuzzy Matrices

- **Key Features:** Pioneered reduction methods for nilpotent fuzzy matrices; primarily applied to simpler fuzzy systems.

- **Limitations:** Focused on fuzzy environments; lacks support for neutrosophic components (indeterminacy and falsity).
- **Comparison:** The proposed methods extend these concepts by incorporating neutrosophic components, making them applicable to more complex systems.

Padder and Murugadas' Reduction for Intuitionistic Fuzzy Matrices

- **Key Features:** Introduced reduction techniques for intuitionistic fuzzy matrices with extensions to s-transitivity and w-transitivity.
- **Limitations:** Limited adaptability to neutrosophic systems, which require simultaneous handling of three-valued logic (truth, indeterminacy, falsity).
- **Comparison:** Our approach generalizes these techniques to neutrosophic matrices, enabling broader applicability in systems with higher uncertainty.

Anandhkumar et al.'s Work on Symmetric Neutrosophic Fuzzy Matrices

- **Key Features:** Explored symmetry and interval-based uncertainties in neutrosophic matrices.
- **Limitations:** Focused on structural properties and symmetry rather than reduction or computational efficiency.
- **Comparison:** The proposed work complements these methods by emphasizing reduction techniques while preserving symmetry and structural integrity.

Lee and Jeong's Canonical Forms for Transitive Intuitionistic Fuzzy Matrices

- **Key Features:** Developed canonical forms for transitive matrices; improved consistency in system modelling.
- **Limitations:** Restricted to intuitionistic matrices; lacks the flexibility to address neutrosophic environments.
- **Comparison:** The proposed methods expand upon this work by verifying reduction models for s-transitive and w-transitive neutrosophic fuzzy matrices.

Proposed Method's Advantages

Criterion	**Proposed Method**	**Existing Methods**
Reduction Efficiency	Achieves significant matrix size reduction while preserving key properties.	Limited efficiency for large-scale matrices.

Support for Transitivity	Verifies reduction models for s-transitive and w-transitive matrices.	Focused on general transitivity; lacks extension.
Handling of Uncertainty	Incorporates neutrosophic components (truth, indeterminacy, falsity).	Primarily focused on fuzzy or intuitionistic systems.
Graph Representation	Visualizes nilpotent matrices as acyclic graphs with conditions for cyclic unions.	Graph-based interpretations not emphasized.
Computational Complexity	Optimized for large, complex systems with higher uncertainty levels.	Computational complexity increases with system size.

2. Preliminaries

Definition: 2.1 A NFSs R on the universe of discourse Y is well-defined as

$R=\left\{<y,r^{T}\left(y\right),r^{I}\left(y\right),r^{F}\left(y\right)>,y\in Y\right\}$, everywhere $r^{T},r^{I},r^{F}:Y\to]^{-}0,1^{+}[$ also $0\le r^{T}+r^{I}+r^{F}\le 3.$

Definition: 2.2 Let R and S be an n x n NFMs where $R=\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right)$ and $S=\left(s_{ij}{}^{T},s_{ij}{}^{I},s_{ij}{}^{F}\right)$ respectively.

(i) $R\vee S=\left(<r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}>\vee<s_{ij}{}^{T},s_{ij}{}^{I},s_{ij}{}^{F}>\right)=\left(<r_{ij}{}^{T}\vee s_{ij}{}^{T},r_{ij}{}^{I}\vee s_{ij}{}^{I},r_{ij}{}^{F}\wedge s_{ij}{}^{F}>\right)$

(ii) $R\wedge S=\left(<r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}>\wedge<s_{ij}{}^{T},s_{ij}{}^{I},s_{ij}{}^{F}>\right)=\left(<r_{ij}{}^{T}\wedge s_{ij}{}^{T},r_{ij}{}^{I}\wedge s_{ij}{}^{I},r_{ij}{}^{F}\vee s_{ij}{}^{F}>\right)$

(iii) $R\times S=\left(<r_{i1}{}^{T}\wedge s_{1j}{}^{T},r_{1i}{}^{I}\wedge s_{1j}{}^{I},r_{1i}{}^{F}\vee s_{1j}{}^{F}>\right)\vee\left(<r_{i2}{}^{T}\wedge s_{2j}{}^{T},r_{2i}{}^{I}\wedge s_{2j}{}^{I},r_{2i}{}^{F}\vee s_{2j}{}^{F}>\right)$

$\vee\ldots\vee\left(<r_{in}{}^{T}\wedge s_{nj}{}^{T},r_{ni}{}^{I}\wedge s_{nj}{}^{I},r_{ni}{}^{F}\vee s_{nj}{}^{F}>\right).$

(iv) $R^{k+1}=R^{k}\times R,\ \left(k=0,1,2,\ldots\right),$

Definition:2.3 A neutrosophic Fuzzy Matrices R is less than or equal to S (or R and S are comparable) i.e, $P\le Q$ if $\left(p_{ij}{}^{T},p_{ij}{}^{I},p_{ij}{}^{F}\right)\le\left(q_{ij}{}^{T},q_{ij}{}^{I},q_{ij}{}^{F}\right)$ means $p_{ij}{}^{T}\le q_{ij}{}^{T},p_{ij}{}^{I}\le q_{ij}{}^{I},p_{ij}{}^{F}\ge q_{ij}{}^{F}$.

Definition 2.4 Let $i, j, k \le n$ and let $R = \left[r_{ij} = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F} \right) \right]$ be an NFM. Then R is called

Transitive iff $R^2 = RR \le R$ $i.e., r_{ik}{}^{T} \wedge r_{kj}{}^{T} \le r_{ij}{}^{T}, r_{ik}{}^{I} \wedge r_{kj}{}^{I} \le r_{ij}{}^{I}$ and $r_{ik}{}^{F} \vee r_{kj}{}^{F} \ge r_{ij}{}^{F}$ for every $i, j, k \le n.$

Nilpotent iff $R^n = RR...R$ (n-times) = 0

consistent iff $r_{ij} \ge r_{ji}$ and $r_{jk} \ge r_{kj}$ implies $r_{ik} \ge r_{ki}$.

Weak transitive if and only if $u_{ij} > u_{ji}$ and $u_{jk} > u_{kj}$ implies $u_{ik} > u_{ki}$.

Definition 2.5 For any two comparable elements $\left(r^T, r^I, r^F \right), \left(s^T, s^I, s^F \right) \in \left(NFM \right)$ the operation $\left(r^T, r^I, r^F \right) \leftarrow \left(s^T, s^I, s^F \right)$ is defined as

$$\left(r^T, r^I, r^F \right) \leftarrow \left(s^T, s^I, s^F \right) = \begin{cases} \left(r^T, r^I, r^F \right) \; if \left(r^T, r^I, r^F \right) > \left(s^T, s^I, s^F \right), \\ \left(0, 0, 1 \right) \qquad if \left(r^T, r^I, r^F \right) \le \left(s^T, s^I, s^F \right). \end{cases}$$

Definition 2.6 (Isomorphism graph)

Two graphs are said to be isomorphic if they satisfy the following four conditions:

(i) **Equal Number of Vertices**: Both graphs must have the same number of vertices.

(ii) **Equal Number of Edges**: Both graphs must have the same number of edges.

(iii) **Equal Degree Sequence**: The degree sequence (the list of vertex degrees) must be identical for both graphs.

(iv) **Cycle Structure Consistency**: If one graph forms a cycle of length k using a set of vertices $\{v_1, v_2, v_3, \ldots, v_k\}$ the other graph must also form a cycle of the same length k with a corresponding set of vertices.

Or In graph theory, graph isomorphism is a way to determine if two graphs are equivalent or similar.

Definition 2.7 (Cyclic graphs and Acyclic graphs)

Cyclic graphs contain at least one cycle, which is a closed loop that allows you to return to the starting point by traversing a series of edges and nodes.

Acyclic graphs contain no cycles, which means there are no repeated edges or nodes in any path within the graph. Acyclic graphs are also known as Directed Acyclic Graphs (DAGs) when considering directed graphs.

Definition 2.8 Let $R=\left(r^T,r^I,r^F\right)$ and $S=\left(s^T,s^I,s^F\right)$ be two NFM then

(i) $R^{k+1}=R^k\times R,(k=1,2,3,...)$

(ii) $R\times S=\left[\bigcup_{k=1}^{n}\left(r_{ik}\wedge s_{kj}\right)\right].$

(iii) $R^T=\left[{r_{ji}}^T,{r_{ji}}^I,{r_{ji}}^F\right]$(the transpose of P)

(iv) R^2 = R (R is idempotent)

(v) R^k = O (R is nilpotent $k\in N$)

(vi) $R/S=R\overset{c}{\leftarrow}\left(R\times S\right),$

(vii) $\left(R\right)^+=R\vee R^2\vee...\vee R^n$

(viii) $R\le W$ iff $\left({r_{ij}}^T,{r_{ij}}^I,{r_{ij}}^F\right)\ge\left({w_{ij}}^T,{w_{ij}}^I,{w_{ij}}^F\right)$

(ix) $R\approx S$ iff $R\prec S$ and $S\prec R$

The following provides a clear explanation of the concepts described:

- It is straightforward to conclude that $\approx$ defines an equivalence relation on all m×n NFMs.
- Let $R\approx S$ imply that matrices R and S have the same number of zero-entries located at corresponding positions.
- A **zero matrix** $(0,0,1)$ is a matrix in which all entries are neutrosophic zero.
- An NFM R is said to be **irreflexive** if all its diagonal elements are zero.

i.e$\left({r_{ii}}^T,{r_{ii}}^I,{r_{ii}}^F\right)=(0,0,1)$ for all i and reflexive $\left({r_{ii}}^T,{r_{ii}}^I,{r_{ii}}^F\right)=(1,1,0)$ for all I, antisymmetric iff $\left({r_{ij}}^T,{r_{ij}}^I,{r_{ij}}^F\right)>\left(0,0,1\right)$ implies $\left({r_{ji}}^T,{r_{ji}}^I,{r_{ji}}^F\right)=\left(0,0,1\right)$ for all i ,j with $i\ne j$,max-min transitive iff $T^2\le T,$ w-transitive iff $\left({r_{ik}}^T\wedge {r_{kj}}^T,{r_{ik}}^I\wedge {r_{kj}}^I,{r_{ik}}^F\vee {r_{kj}}^F\right)>\left(0,0,1\right):$ implies $\left({r_{ij}}^T,{r_{ij}}^I,{r_{ij}}^F\right)>\left(0,0,1\right)$ for all i,j,k or equivalently iff $R^2\equiv R$, s-transitive iff $\left({r_{ik}}^T,{r_{ik}}^I,{r_{ik}}^F\right)>\left({r_{ki}}^T,{r_{ki}}^I,{r_{ki}}^F\right)$ and $\left({r_{kj}}^T,{r_{kj}}^I,{r_{kj}}^F\right)>\left({r_{jk}}^T,{r_{jk}}^I,{r_{jk}}^F\right)$ implies $\left({r_{ij}}^T,{r_{ij}}^I,{r_{ij}}^F\right)>\left({r_{ji}}^T,{r_{ji}}^I,{r_{ji}}^F\right)$ for any i,j,k such that $i\ne j,j\ne k,i\ne k$ or equivalently iff$\left(\Delta R\right)^2\prec\Delta R$. It is obvious that always positive matrix R, i.e., $\left({r_{ij}}^T,{r_{ij}}^I,{r_{ij}}^F\right)>\left(0,0,1\right)$ for every i,j is w-transitive.

3. Reduction of Neutrosophic Fuzzy Nilpotent Matrix

Theorem3.1. If R is an nxn irreflexive and transitive NFM, then $(R/R)^{+} = R.$

Proof: Since $W = \left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right) = R/R,\ W^{k} = \left(w_{ij}{}^{T^{(k)}}, w_{ij}{}^{I^{(k)}}, w_{ij}{}^{F^{(k)}}\right).$

Let R and W are nilpotent. Then, evidently, $(R/R)^{+} = (R/R)\vee(R/R)^{2}\vee\ldots\vee(R/R)^{n-1}\le R,$

To prove that $R \le (R/R)\vee(R/R)^{2}\vee\ldots\vee(R/R)^{n-1}$

That is, $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \le \left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right)$ for few $k(1\le k\le k-1).$

Assume that $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \ge \left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right)$ for every $k = 1,2,\ldots,n-1.$

(1) (a) Since $\left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right) \le \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, we get

$$\left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right) = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{il}{}^{T}\wedge r_{lj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}{}^{I}\wedge r_{lj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}{}^{F}\vee r_{lj}{}^{F}\right)\right)$$

That is, $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \le \left(\bigcup_{k=1}^{n}\left(r_{il}{}^{T}\wedge r_{lj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}{}^{I}\wedge r_{lj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}{}^{F}\vee r_{lj}{}^{F}\right)\right)$

Thus, we can find l_{11} such that $\left(r_{il_{11}}{}^{T}, r_{il_{11}}{}^{I}, r_{il_{11}}{}^{F}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ and

$\left(r_{il_{11}}{}^{T}, r_{il_{11}}{}^{I}, r_{il_{11}}{}^{F}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right).$

It follows that $\left(r_{ij}{}^{T^{(2)}}, r_{ij}{}^{I^{(2)}}, r_{ij}{}^{F^{(2)}}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) > (0,0,1).$

Therefore, $\left(r_{ij}{}^{T^{(2)}}, r_{ij}{}^{I^{(2)}}, r_{ij}{}^{F^{(2)}}\right) = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, since R is transitive.

We now prove that $\left(w_{il_{1_{p(1)}}}{}^{T}, w_{il_{1_{p(1)}}}{}^{I}, w_{il_{1_{p(1)}}}{}^{F}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ and

$\left(r_{l_{1_{p(1)}}}{}^{T}, r_{l_{1_{p(1)}}}{}^{I}, r_{l_{1_{p(1)}}}{}^{F}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ foe a few $l_{1_{p(1)}}$.

(b) If $\left(w_{il_{1_{p(1)}}}{}^{T}, w_{il_{1_{p(1)}}}{}^{I}, w_{il_{1_{p(1)}}}{}^{F}\right) \ge \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, then we put $p(1) = 1$.

If $\left(w_{il_{11}}{}^{T}, w_{il_{11}}{}^{I}, w_{il_{11}}{}^{F}\right) = (0,0,1)$ that is if

$$\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{il}{}^{T}\wedge r_{ll_{11}}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}{}^{I}\wedge r_{ll_{11}}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}{}^{F}\vee r_{ll_{11}}{}^{F}\right)\right) = (0,0,1),$$

Then $\left(r_{il_{11}}^{T}, r_{il_{11}}^{I}, r_{il_{11}}^{F}\right) \leq \left(\bigcup_{k=1}^{n}\left(r_{il}^{T} \wedge r_{ll_{11}}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}^{I} \wedge r_{ll_{11}}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}^{F} \vee r_{ll_{11}}^{F}\right)\right)$.

Since, $\left(r_{il_{11}}^{T}, r_{il_{11}}^{I}, r_{il_{11}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

we get $\left(r_{il_{12}}^{T}, r_{il_{12}}^{I}, r_{il_{12}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right), \left(r_{l_{12}l_{11}}^{T}, r_{l_{12}l_{11}}^{I}, r_{l_{12}l_{11}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$ and

$\left(r_{ij}^{T^{(3)}}, r_{ij}^{I^{(3)}}, r_{ij}^{F^{(3)}}\right) = \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$ for a few l_{12}.

Further, since $\left(r_{l_{12}l_{11}}^{T} \wedge r_{l_{11}j}^{T}, r_{l_{12}l_{11}}^{I} \wedge r_{l_{11}j}^{I}, r_{l_{12}l_{11}}^{F} \vee r_{l_{11}j}^{F}\right) \leq \left(r_{l_{12}j}^{T}, r_{l_{12}j}^{I}, r_{l_{12}j}^{F}\right)$

and $\left(r_{l_{11}j}^{T}, r_{l_{11}j}^{I}, r_{l_{11}j}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$, We have $\left(r_{l_{12}j}^{T}, r_{l_{12}j}^{I}, r_{l_{12}j}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$.

(c) Moreover, if $\left(w_{il_{11}}^{T}, w_{il_{11}}^{I}, w_{il_{11}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$, then we put p(1) = 2.

If $\left(w_{il_{12}}^{T}, w_{il_{12}}^{I}, w_{il_{12}}^{F}\right) = (0,0,1)$

that is $\left(r_{il_{12}}^{T}, r_{il_{12}}^{I}, r_{il_{12}}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{il}^{T} \wedge r_{ll_{12}}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}^{I} \wedge r_{ll_{12}}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}^{F} \vee r_{ll_{12}}^{F}\right)\right) = (0,0,1)$

Then, $\left(r_{il_{12}}^{T}, r_{il_{12}}^{I}, r_{il_{12}}^{F}\right) \leq \left(\bigcup_{k=1}^{n}\left(r_{il}^{T} \wedge r_{ll_{12}}^{T}\right), \bigcup_{k=1}^{n}\left(r_{il}^{I} \wedge r_{ll_{12}}^{I}\right), \bigcap_{k=1}^{n}\left(r_{il}^{F} \vee r_{ll_{12}}^{F}\right)\right)$.

Since $\left(r_{il_{12}}^{T}, r_{il_{12}}^{I}, r_{il_{12}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

we get $\left(r_{il_{12}}^{T}, r_{il_{12}}^{I}, r_{il_{12}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right), \left(r_{l_{13}l_{12}}^{T}, r_{l_{13}l_{12}}^{I}, r_{l_{13}l_{12}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$ and

$\left(r_{ij}^{T^{(4)}}, r_{ij}^{I^{(4)}}, r_{ij}^{F^{(4)}}\right) = \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$ for some l_{13}.

Thus, since $\left(r_{l_{13}l_{12}}^{T} \wedge r_{l_{12}j}^{T}, r_{l_{13}l_{12}}^{I} \wedge r_{l_{12}j}^{I}, r_{l_{13}l_{12}}^{F} \vee r_{l_{12}j}^{F}\right) \leq \left(r_{l_{13}j}^{T}, r_{l_{13}j}^{I}, r_{l_{13}j}^{F}\right)$

and $\left(r_{l_{12}j}^{T}, r_{l_{12}j}^{I}, r_{l_{12}j}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

We have $\left(r_{l_{13}j}^{T}, r_{l_{13}j}^{I}, r_{l_{13}j}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$.

(d) By iterating the similar process, since R is nilpotent for some $l_{ip(1)}$ such that $p(1) < n-1$.

We obtain the following result $\left(w_{il_{1p(1)}}^{T}, w_{il_{1p(1)}}^{I}, w_{il_{1p(1)}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

$\left(r_{il_{1p(1)}}^{T}, r_{il_{1p(1)}}^{I}, r_{il_{1p(1)}}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$.

(2) Next, since $\left(w_{ij}^{T^{(2)}}, w_{ij}^{I^{(2)}}, w_{ij}^{F^{(2)}}\right) < \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

we get $\left(w^{T}{}_{l_{p(2)j}}, w^{I}{}_{l_{p(2)j}}, w^{F}{}_{l_{p(2)j}}\right) < \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$.

Then, by $\left(r^{T}{}_{l_{1p(2)j}}, r^{I}{}_{l_{1p(2)j}}, r^{F}{}_{l_{1p(2)j}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$ it follows that

$$\left(r_{il_{12}}{}^{T}, r_{il_{12}}{}^{I}, r_{il_{12}}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r^{T}{}_{l_{1p(1)l}} \wedge r_{ij}^{T}\right), \bigcup_{k=1}^{n}\left(r^{I}{}_{l_{1p(1)l}} \wedge r_{ij}^{I}\right), \bigcap_{k=1}^{n}\left(r^{F}{}_{l_{1p(1)l}} \vee r_{ij}^{F}\right)\right) = (0,0,1)$$

That is $\left(r^{T}{}_{l_{1p(1)j}}, r^{I}{}_{l_{1p(1)j}}, r^{F}{}_{l_{1p(1)j}}\right) \leq \left(\bigcup_{k=1}^{n}\left(r^{T}{}_{l_{1p(1)l}} \wedge r_{ij}^{T}\right), \bigcup_{k=1}^{n}\left(r^{I}{}_{l_{1p(1)l}} \wedge r_{ij}^{I}\right), \bigcap_{k=1}^{n}\left(r^{F}{}_{l_{1p(1)l}} \vee r_{ij}^{F}\right)\right)$

Since $\left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) \leq \left(r_{l_{1p(1)j}}{}^{T}, r_{l_{1p(1)j}}{}^{I}, r_{l_{1p(1)j}}{}^{F}\right)$,

we have $\left(r^{T}{}_{l_{1p(1)}l_{21}}, r^{I}{}_{l_{1p(1)}l_{21}}, r^{F}{}_{l_{1p(1)}l_{21}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right), \left(r_{l_{21j}}{}^{T}, r_{l_{21j}}{}^{I}, r_{l_{21j}}{}^{F}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

and $\left(r^{T^{(2)}}{}_{l_{1p(1)j}}, r^{I^{(2)}}{}_{l_{1p(1)j}}, r^{F^{(2)}}{}_{l_{1p(1)j}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$ for a few l_{21}.

Using the similar process as described in (1), the following result is obtained.

$\left(w^{T}{}_{l_{1p(1)}l_{2p(2)}}, w^{I}{}_{l_{1p(1)}l_{2p(2)}}, w^{F}{}_{l_{1p(1)}l_{2p(2)}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right), \left(r^{T}{}_{l_{2p(2)j}}, r^{I}{}_{l_{2p(2)j}}, r^{F}{}_{l_{2p(2)j}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$, and

$\left(w^{T^{(2)}}{}_{il_{2p(2)}}, w^{I^{(2)}}{}_{il_{2p(2)}}, w^{F^{(2)}}{}_{il_{2p(2)}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$. for some $l_{2p(2)}$.

(3) By continuing the process outlined above, we arrive at the following.

$\left(w^{T}{}_{l_{n-1p(n-1)}l_{np(n)}}, w^{I}{}_{l_{n-1p(n-1)}l_{np(n)}}, w^{F}{}_{l_{n-1p(n-1)}l_{np(n)}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right), \left(r^{T}{}_{l_{np(n)j}}, r^{I}{}_{l_{np(n)j}}, r^{F}{}_{l_{np(n)j}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$,

and $\left(w^{T^{(2)}}{}_{il_{np(n)}}, w^{I^{(2)}}{}_{il_{np(n)}}, w^{F^{(2)}}{}_{il_{np(n)}}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$.

Thus, a contradiction emerges under the assumption that S is nilpotent.

Thus, we obtained $\left(w^{T^{(k)}}{}_{ij}, w^{I^{(k)}}{}_{ij}, w^{F^{(k)}}{}_{ij}\right) \geq \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$ for a some $k\left(1 \leq k \leq n-1\right)$, so that

$W^{+} = R$

The subsequent illustration demonstrates that R/R is the reduced form of R, and it suffices to calculate the transitive closure of R/R rather than directly calculating the transitive closure of R.

Example3.1. Let R be the subsequent irreflexive and transitive NFM

$$R = \begin{bmatrix} (0,0,1) & (0.5,0.3,0.4) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) \\ (0.5,0.3,0.4) & (0.5,0.3,0.4) & (0,0,1) \end{bmatrix} \text{ (Irreflexive)}$$

$$R^2=\begin{bmatrix}(0,0,1) & (0,0,1) & (0,0,1)\\ (0,0,1) & (0,0,1) & (0,0,1)\\ (0,0,1) & (0.5,0.3,0.4) & (0,0,1)\end{bmatrix} \quad (\text{Transitive } R^2\le R)$$

Now, $R/R=R\overset{c}{\leftarrow}(R\times R)$,

$$R/R=\begin{bmatrix}(0,0,1) & (0.5,0.3,0.4) & (0,0,1)\\ (0,0,1) & (0,0,1) & (0,0,1)\\ (0.5,0.3,0.4) & (0,0,1) & (0,0,1)\end{bmatrix}$$

$(R/R)^+=(R/R)\vee(R/R)^2\vee(R/R)^3$,

$$(R/R)^+=\begin{bmatrix}(0,0,1) & (0.5,0.3,0.4) & (0,0,1)\\ (0,0,1) & (0,0,1) & (0,0,1)\\ (0.5,0.3,0.4) & (0.5,0.3,0.4) & (0,0,1)\end{bmatrix}$$

Therefore, $(R/R)^+=(R/R)\vee(R/R)^2\vee(R/R)^3$, (1)

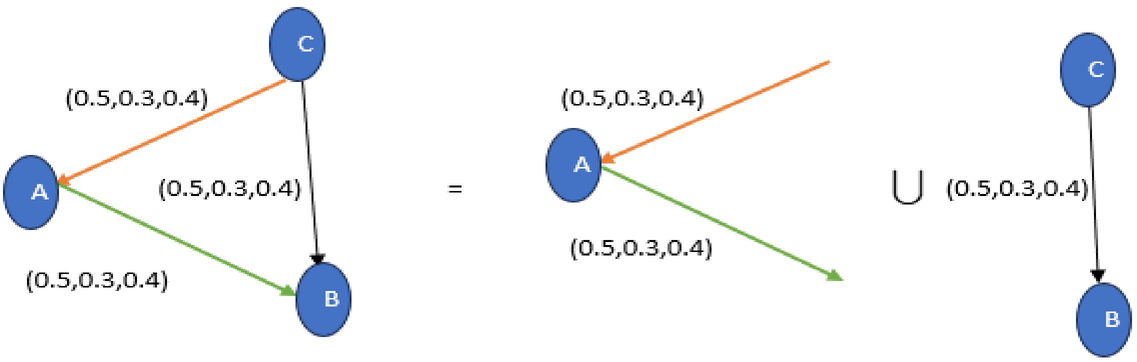

Figure 1

Figure 1: The graphical representation of equation (1) is isomorphism Graph

Remark 3.1 . In the above theorem, both irreflexivity and transitivity are crucial conditions. If either of these conditions fails, the result does not hold. This is demonstrated in the following example:

Example 3.2. Let R be the subsequent not irreflexive and transitive NFM

$$R=\begin{bmatrix}(0,0,1) & (0.3,0,4,0.5)\\ (0,0,1) & (0.2,0.4,0.8)\end{bmatrix}(\text{R is not irreflexive})$$

$$R^2=\begin{bmatrix}(0,0,1) & (0.3,0,4,0.5)\\ (0,0,1) & (0.2,0.4,0.8)\end{bmatrix}(\text{Transitive } R^2\le R)$$

Now, $R/R = R \overset{c}{\leftarrow} (R \times R)$,

$$R/R = \begin{bmatrix} (0,0,1) & (0.3,0,4,0.5) \\ (0,0,1) & (0,0,1) \end{bmatrix},\ (R/R)^2 = \begin{bmatrix} (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) \end{bmatrix}$$

$(R/R)^+ = (R/R) \vee (R/R)^2$,

Therefore, $R \neq (R/R)^+ = (R/R) \vee (R/R)^2$ (2)

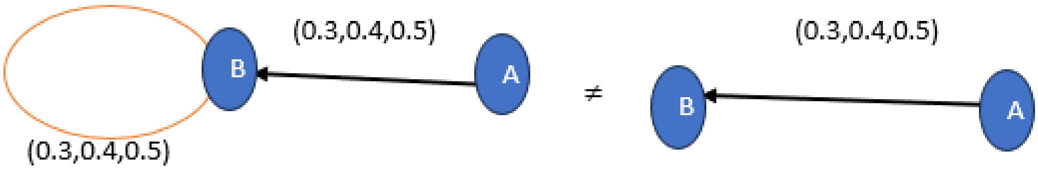

Figure 2

Figure 2: The graphical representation of equation (2) non isomorphism graph

Example 3.3 Let R be the following irreflexive and not transitive NFM

$$R = \begin{bmatrix} (0,0,1) & (0.5,0.3,0.4) & (0.3,0.4,0.5) \\ (0,0,1) & (0,0,1) & (0,0,1) \\ (0.5,0.3,0.4) & (0.5,0.3,0.4) & (0,0,1) \end{bmatrix} \quad \text{(Irreflexive)}$$

$$R^2 = \begin{bmatrix} (0.3,0.3,0.5) & (0.3,0.3,0.5) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) \\ (0.5,0.3,0.4) & (0.5,0.3,0.4) & (0.3,0.3,0.5) \end{bmatrix} \text{(Not transitive)}$$

Now, $R/R = R \overset{c}{\leftarrow} (R \times R),\ (R/R)^+ = (R/R) \vee (R/R)^2 \vee (R/R)^3$,

$$R/R = \begin{bmatrix} (0,0,1) & (0.5,0.3,0.4) & (0.3,0.4,0.5) \\ (0,0,1) & (0,0,1) & (0,0,1) \\ (0.5,0.3,0.4) & (0,0,1) & (0,0,1) \end{bmatrix}$$

$$(R/R)^+ = \begin{bmatrix} (0.3,0.4,0.5) & (0.5,0.3,0.4) & (0.3,0.4,0.5) \\ (0,0,1) & (0,0,1) & (0,0,1) \\ (0.5,0.3,0.4) & (0.5,0.3,0.4) & (0.3,0.4,0.5) \end{bmatrix}$$

Therefore, $R \neq (R/R)^+ = (R/R) \vee (R/R)^2 \vee (R/R)^3$ (3)

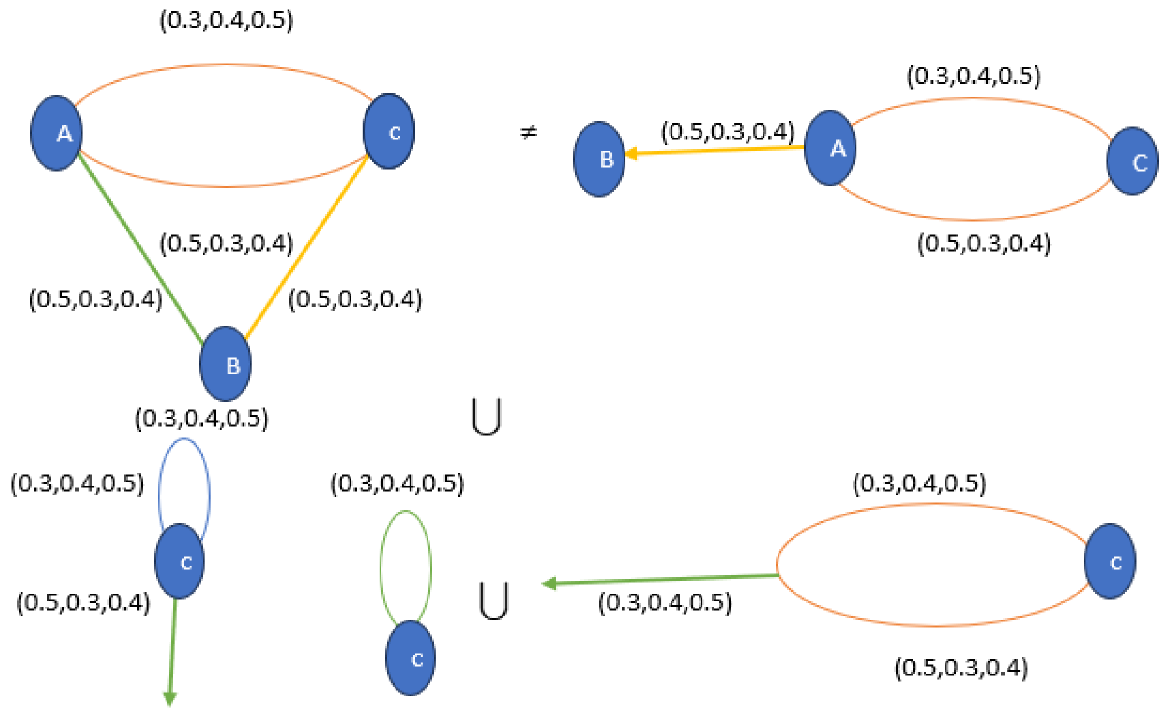

Figure 3

Figure 3: The graphical representation of equation (3) non isomorphism graph

Theorem3.2. Let R be a nxn irreflexive and transitive matrix. Then, the subsequent conditions are equal

(i) $R/R \le S \le R$

(ii) $S^{+} = R$ for any nxn NFMS

Proof: Let $S^{k} = \left(s^{T^{(k)}}{}_{ij}, s^{I^{(k)}}{}_{ij}, s^{F^{(k)}}{}_{ij} \right)$ and $T = \left(t^{T}{}_{ij}, t^{I}{}_{ij}, t^{F}{}_{ij} \right) = R/R$. That is

$$\left(t^{T}{}_{ij}, t^{I}{}_{ij}, t^{F}{}_{ij} \right) = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F} \right) \leftarrow \left(\bigcup_{k=1}^{n} \left(r_{ik}{}^{T} \wedge r_{kj}{}^{T} \right), \bigcup_{k=1}^{n} \left(r_{ik}{}^{I} \wedge r_{kj}{}^{I} \right), \bigcap_{k=1}^{n} \left(r_{ik}{}^{F} \vee r_{kj}{}^{F} \right) \right).$$

(1) Assume that $R/R \le S \le R$ evidently from Theorem 3.1, $S^{+} = R$

(2) Assume that $S^{+} = R$ then we get $S \le R$

(a) Let n=1. The only irreflexive matrix is $(0,0,1)$. Thus,

(b) Let n = 2. Since S is nilpotent $S^{2} = (0,0,1)$,we get $R/R \le R = S^{+} = S \vee S^{2} = S$.

(c) Let $n \le 3$.Suppose that $\left(s^{T}{}_{ij}, s^{I}{}_{ij}, s^{F}{}_{ij} \right) < \left(t^{T}{}_{ij}, t^{I}{}_{ij}, t^{F}{}_{ij} \right)$, then

$\left(t^T{}_{ij}, t^I{}_{ij}, t^F{}_{ij}\right) = \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right) > (0,0,1)$ and

$$\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^T \wedge r_{kj}{}^T\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^I \wedge r_{kj}{}^I\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^F \vee r_{kj}{}^F\right)\right) \le \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right).$$

Now we obtained,

$\left(s^T{}_{il_1}, s^I{}_{il_1}, s^F{}_{il_1}\right) \wedge \left(s^T{}_{l_1l_2}, s^I{}_{l_1l_2}, s^F{}_{l_1l_2}\right) \wedge \ldots \wedge \left(s^T{}_{l_hj}, s^I{}_{l_hj}, s^F{}_{l_hj}\right) = \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$ for right indices $l_1, l_2, \ldots, l_h \left(1 \le h \le n-2\right)$,

So that, $\left(r^T{}_{il_1}, r^I{}_{il_1}, r^F{}_{il_1}\right) \wedge \left(r^T{}_{l_1l_2}, r^I{}_{l_1l_2}, r^F{}_{l_1l_2}\right) \wedge \ldots \wedge \left(r^T{}_{l_hj}, r^I{}_{l_hj}, r^F{}_{l_hj}\right) \le \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$.

Thus $\left(r^T{}_{il_1}, r^I{}_{il_1}, r^F{}_{il_1}\right) \ge \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$ and $\left(r^T{}_{l_1j}, r^I{}_{l_1j}, r^F{}_{l_1j}\right) \ge \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$ for l_1.

Then, $\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^T \wedge r_{kj}{}^T\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^I \wedge r_{kj}{}^I\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^F \vee r_{kj}{}^F\right)\right) \ge \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$.

$$\left(t^T{}_{ij}, t^I{}_{ij}, t^F{}_{ij}\right) = \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^T \wedge r_{kj}{}^T\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^I \wedge r_{kj}{}^I\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^F \vee r_{kj}{}^F\right)\right) = (0,0,1).$$

This contradicts the fact that $\left(t^T{}_{ij}, t^I{}_{ij}, t^F{}_{ij}\right) > (0,0,1)$. Thus $T \le S$, so that $R/R \le S \le R$ .By the properties of Theorem3.3 above, R/R is minimal in the set of NFMs such that $S^+ = R$.

Theorem3.3. Let R be an nxn irreflexive and transitive NFM. Then, the subsequent conditions are equal:

(i) $R/R \le S \le R$

(ii) $R/R = S/R$

Proof: Let $F = \left(f^T{}_{ij}, f^I{}_{ij}, f^F{}_{ij}\right) = R/R$ and $G = \left(g^T{}_{ij}, g^I{}_{ij}, g^F{}_{ij}\right) = S/R$ .Then,

$$\left(f^T{}_{ij}, f^I{}_{ij}, f^F{}_{ij}\right) = \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^T \wedge r_{kj}{}^T\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^I \wedge r_{kj}{}^I\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^F \vee r_{kj}{}^F\right)\right),$$

$$\left(g^T{}_{ij}, g^I{}_{ij}, g^F{}_{ij}\right) = \left(s_{ij}{}^T, s_{ij}{}^I, s_{ij}{}^F\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(s_{ik}{}^T \wedge s_{kj}{}^T\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^I \wedge r_{kj}{}^I\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^F \vee s_{kj}{}^F\right)\right).$$

(i) implies (ii): Assume that $R/R \le S \le R$, so that

$\left(f^T{}_{ij}, f^I{}_{ij}, f^F{}_{ij}\right) \le \left(s_{ij}{}^T, s_{ij}{}^I, s_{ij}{}^F\right) \le \left(r_{ij}{}^T, r_{ij}{}^I, r_{ij}{}^F\right)$.

(a) To prove that $\left(f^T{}_{ij}, f^I{}_{ij}, f^F{}_{ij}\right) \le \left(g^T{}_{ij}, g^I{}_{ij}, g^F{}_{ij}\right)$. Let $\left(f^T{}_{ij}, f^I{}_{ij}, f^F{}_{ij}\right) > (0,0,1)$.

Then, $\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)=\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)>(0,0,1),$, so that $\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)=\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ and

$$\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(r_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(r_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)<\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right).$$

Since $\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right)\geq\left(s_{ik}{}^{T}, s_{ik}{}^{I}, s_{ik}{}^{F}\right)$,

we have $\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(s_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(s_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)<\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right).$

Consequently,

$$\left(g^{T}{}_{ij}, g^{I}{}_{ij}, g^{F}{}_{ij}\right)=\left(s_{ik}{}^{T}, s_{ik}{}^{I}, s_{ik}{}^{F}\right)\leftarrow\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(s_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(s_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)$$

$$=\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right)\leftarrow\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(s_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(s_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)$$

so that $\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)\leq\left(g^{T}{}_{ij}, g^{I}{}_{ij}, g^{F}{}_{ij}\right).$

(b) To prove that $\left(g^{T}{}_{ij}, g^{I}{}_{ij}, g^{F}{}_{ij}\right)\leq\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)$

Let $\left(g^{T}{}_{ij}, g^{I}{}_{ij}, g^{F}{}_{ij}\right)>(0,0,1)$, then $\left(g^{T}{}_{ij}, g^{I}{}_{ij}, g^{F}{}_{ij}\right)=\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)>(0,0,1)$, and hence,

$\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(s_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(s_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)<\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right).$ Recall that

$$\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)=\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)\leftarrow\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(r_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(r_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)$$

$$\leq\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)\leq\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right).$$

We have since $\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)>(0,0,1)$, $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)>(0,0,1)$.

We shall Prove, if $\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)\leq\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, then there is a contradiction.

Assume that $\left(f^{T}{}_{ij}, f^{I}{}_{ij}, f^{F}{}_{ij}\right)<\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Then, $\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(r_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(r_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right)\geq(0,0,1),$

So that $\left(r_{ik(1)}{}^{T}, r_{ik(1)}{}^{I}, r_{ik(1)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right), \left(r_{k(1)j}{}^{T}, r_{k(1)j}{}^{I}, r_{k(1)j}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ and

$\left(r_{ik(1)}{}^{T^{(2)}}, r_{ik(1)}{}^{I^{(2)}}, r_{ik(1)}{}^{F^{(2)}}\right) = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) > \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ for few k (1).

We have $\left(s_{ik(1)}{}^{T}, s_{ik(1)}{}^{I}, s_{ik(1)}{}^{F}\right) \leq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, since $\left(r_{k(1)j}{}^{T}, r_{k(1)j}{}^{I}, r_{k(1)j}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

and $\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right) \leq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Therefore, $F \leq S$ and $\left(f_{ik(1)}{}^{T}, f_{ik(1)}{}^{I}, f_{ik(1)}{}^{F}\right) < \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Furthermore, $\left(f_{ik(1)}{}^{T}, f_{ik(1)}{}^{I}, f_{ik(1)}{}^{F}\right) < \left(r_{ik(1)}{}^{T}, r_{ik(1)}{}^{I}, r_{ik(1)}{}^{F}\right)$,

since $\left(r_{ik(1)}{}^{T}, r_{ik(1)}{}^{I}, r_{ik(1)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Thus, $\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Therefore,

$\left(r_{ik(2)}{}^{T}, r_{ik(2)}{}^{I}, r_{ik(2)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right), \left(r^{T}{}_{k(2)k(2)}, r^{I}{}_{k(2)k(2)}, r^{F}{}_{k(2)k(2)}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

and $\left(r_{ij}{}^{T^{(3)}}, r_{ij}{}^{I^{(3)}}, r_{ij}{}^{F^{(3)}}\right) > \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) > (0,0,1)$ for some k(2).

Since $\left(r_{k(1)j}{}^{T}, r_{k(1)j}{}^{I}, r_{k(1)j}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ and $\left(r^{T}{}_{k(2)k(1)}, r^{I}{}_{k(2)k(1)}, r^{F}{}_{k(2)k(1)}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

We have $\left(r_{k(2)j}{}^{T}, r_{k(2)j}{}^{I}, r_{k(2)j}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

so that $\left(s_{ik(2)}{}^{T}, s_{ik(2)}{}^{I}, s_{ik(2)}{}^{F}\right) < \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

since $\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right) < \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$. Then,

Therefore, $F \leq S$ and $\left(f_{ik(2)}{}^{T}, f_{ik(2)}{}^{I}, f_{ik(2)}{}^{F}\right) < \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Moreover,

$\left(f_{ik(2)}{}^{T}, f_{ik(2)}{}^{I}, f_{ik(2)}{}^{F}\right) < \left(r_{ik(2)}{}^{T}, r_{ik(2)}{}^{I}, r_{ik(2)}{}^{F}\right)$, since $\left(r_{ik(2)}{}^{T}, r_{ik(2)}{}^{I}, r_{ik(2)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Thus,

$\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r_{kk(2)}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r_{kk(2)}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r_{kk(2)}{}^{F}\right)\right) \geq \left(r_{ik(2)}{}^{T}, s_{ik(2)}{}^{I}, s_{ik(2)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Therefore,

$\left(r_{ik(3)}{}^{T}, s_{ik(3)}{}^{I}, s_{ik(3)}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, $\left(r^{T}{}_{k(3)k(2)}, r^{I}{}_{k(3)k(2)}, s^{F}{}_{k(3)k(2)}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$, and

$\left(r_{ij}{}^{T^{(4)}}, r_{ij}{}^{I^{(4)}}, r_{ij}{}^{F^{(4)}}\right) = \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) > (0,0,1)$, for a some k(3).

we obtain $\left(r_{ij}{}^{T^{(n)}}, r_{ij}{}^{I^{(n)}}, r_{ij}{}^{F^{(n)}}\right) > (0,0,1)$ a result that leads to a contradiction, since R is nilpotent,

Thus, $\left(f_{ij}{}^{T}, f_{ij}{}^{I}, f_{ij}{}^{F}\right) \geq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

so that $\left(g_{ij}{}^{T}, g_{ij}{}^{I}, g_{ij}{}^{F}\right) = \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leq \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \leq \left(f_{ij}{}^{T}, f_{ij}{}^{I}, f_{ij}{}^{F}\right)$.

(2) $\Rightarrow$ (1) Assume that $R/R = S/R$.

(a) It is obvious that $R/R = S/R \leq S$.

(b) We prove that $S \leq Q$. Assume that $\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$.

Since $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right)$

We take $\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right) = (0,0,1)$,

so that $\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kj}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r_{kj}{}^{I}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r_{kj}{}^{F}\right)\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > (0,0,1)$.

Since, $\left(s_{ik(1)}{}^{T}, s_{ik(1)}{}^{I}, s_{ik(1)}{}^{F}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$, $\left(r_{k(1)j}{}^{T}, r_{k(1)j}{}^{I}, r_{k(1)j}{}^{F}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$,

and $\left(r^{T^{(1)}}{}_{k(1)j}, r^{I^{(1)}}{}_{k(1)j}, r^{F^{(1)}}{}_{k(1)j}\right) = \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > (0,0,1)$ for few k(1).

Since $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) < \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$

and $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \geq \left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) \wedge \left(r^{T}{}_{k(1)j}, r^{I}{}_{k(1)j}, r^{F}{}_{k(1)j}\right)$,

We have $\left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) < \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$,

so that $\left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) < \left(s^{T}{}_{ik(1)}, s^{I}{}_{ik(1)}, s^{F}{}_{ik(1)}\right)$.

Since $\left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r^{T}{}_{kk(1)}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r^{I}{}_{kk(1)}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r^{F}{}_{kk(1)}\right)\right)$

$$\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kk(1)}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r^{I}{}_{kk(1)}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r^{F}{}_{kk(1)}\right)\right).$$

$$\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kk(1)}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r^{I}{}_{kk(1)}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r^{F}{}_{kk(1)}\right)\right) \geq \left(s^{T}{}_{ik(1)}, s^{I}{}_{ik(1)}, s^{F}{}_{ik(1)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right).$$

Then, $\left(s_{ik(2)}{}^{T}, s_{ik(2)}{}^{I}, s_{ik(2)}{}^{F}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$,

$\left(r_{k(2)k(1)}{}^{T}, r_{k(2)k(1)}{}^{I}, r_{k(2)k(1)}{}^{F}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$,

and $\left(r^{T^{(2)}}{}_{k(2)j}, r^{I^{(2)}}{}_{k(2)j}, r^{F^{(2)}}{}_{k(2)j}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > (0,0,1)$ for a few k(2).

Since $\left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) < \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$

and $\left(r^{T}{}_{ik(1)}, r^{I}{}_{ik(1)}, r^{F}{}_{ik(1)}\right) \geq \left(r^{T}{}_{ik(2)}, r^{I}{}_{ik(2)}, r^{F}{}_{ik(2)}\right) \wedge \left(r^{T}{}_{k(2)k(1)}, r^{I}{}_{k(2)k(1)}, r^{F}{}_{k(2)k(1)}\right)$,

We get $\left(r^{T}{}_{ik(2)}, r^{I}{}_{ik(2)}, r^{F}{}_{ik(2)}\right) < \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right), \left(r^{T}{}_{ik(2)}, r^{I}{}_{ik(2)}, r^{F}{}_{ik(2)}\right) < \left(s^{T}{}_{ik(2)}, s^{I}{}_{ik(2)}, s^{F}{}_{ik(2)}\right)$

Since $\left(r^{T}{}_{ik(2)}, r^{I}{}_{ik(2)}, r^{F}{}_{ik(2)}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T} \wedge r^{T}{}_{kk(2)}\right), \bigcup_{k=1}^{n}\left(r_{ik}{}^{I} \wedge r^{I}{}_{kk(2)}\right), \bigcap_{k=1}^{n}\left(r_{ik}{}^{F} \vee r^{F}{}_{kk(2)}\right)\right)$

$$\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) \leftarrow \left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kk(1)}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r^{I}{}_{kk(1)}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r^{F}{}_{kk(1)}\right)\right)$$

We get

$$\left(\bigcup_{k=1}^{n}\left(s_{ik}{}^{T} \wedge r_{kk(2)}{}^{T}\right), \bigcup_{k=1}^{n}\left(s_{ik}{}^{I} \wedge r^{I}{}_{kk(2)}\right), \bigcap_{k=1}^{n}\left(s_{ik}{}^{F} \vee r^{F}{}_{kk(2)}\right)\right) \geq \left(s^{T}{}_{ik(2)}, s^{I}{}_{ik(2)}, s^{F}{}_{ik(2)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right).$$

Then, $\left(s^{T}{}_{ik(3)}, s^{I}{}_{ik(3)}, s^{F}{}_{ik(3)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right), \left(s^{T}{}_{k(3)k(2)}, s^{I}{}_{k(3)k(2)}, s^{F}{}_{k(3)k(2)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$ and

$\left(r^{T^{(3)}}{}_{k(2)j}, r^{I^{(3)}}{}_{k(2)j}, r^{F^{(3)}}{}_{k(2)j}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > (0,0,1)$ for some k(3).

By following the same reasoning, we encounter a
contradiction.

$\left(s^{T}{}_{ik(n)}, s^{I}{}_{ik(n)}, s^{F}{}_{ik(n)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right), \left(s^{T}{}_{k(3)k(n-1)}, s^{I}{}_{k(3)k(n-1)}, s^{F}{}_{k(3)k(n-1)}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right)$ and

$\left(r^{T^{(n)}}{}_{k(2)j}, r^{I^{(n)}}{}_{k(2)j}, r^{F^{(n)}}{}_{k(2)j}\right) \geq \left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) > (0,0,1)$,

This contradicts the fact R is nilpotent. Hence $\left(s_{ij}{}^{T}, s_{ij}{}^{I}, s_{ij}{}^{F}\right) < \left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ for all i,j.

4.Reduction of an w-transitive and s-transitive NFM

In this section we examine the general reduction system of NFM concerning a product of three NFM. If A is an mxn, R is an nxn and S is an mXm NFM respectively. Also, we prove some properties of reduction of nilpotent NFMs of [30] remain valid or w-transitive and s-transitive NFMS.

Lemma 4.1. : Let R be antisymmetric NFM then R is w-transitive NFM iff R is s-transitive NFM.
Proof. Let R be w-transitive

Then $\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) \wedge \left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > (0,0,1) \Rightarrow \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$

Since R is antisymmetric $\left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1) \Rightarrow \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right) = (0,0,1)$

Now let, $\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) > \left(r_{ki}^{T}, r_{ki}^{I}, r_{ki}^{F}\right)$ and $\left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right)$.

To prove that $\left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right)$

$\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) > \left(r_{ki}^{T}, r_{ki}^{I}, r_{ki}^{F}\right)$ and $\left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right)$.

$\Rightarrow \left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) \wedge \left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > (0,0,1) \Rightarrow \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$

$\Rightarrow \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1) \Rightarrow \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right) > (0,0,1)$

Therefore, $\left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right)$

Conversely let R be s-transitive, then $\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) > \left(r_{ki}^{T}, r_{ki}^{I}, r_{ki}^{F}\right)$ and

$\left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right)$.

$\Rightarrow \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right)$

To prove: $\left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > (0,0,1)$

Let $\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) \wedge \left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > (0,0,1)$

$\Rightarrow \left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) > (0,0,1), \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right) > (0,0,1)$

$\Rightarrow \left(r_{ki}^{T}, r_{ki}^{I}, r_{ki}^{F}\right) = (0,0,1), \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right) = (0,0,1)$ (because R is antisymmetric)

$\left(r_{ik}^{T}, r_{ik}^{I}, r_{ik}^{F}\right) > \left(r_{ki}^{T}, r_{ki}^{I}, r_{ki}^{F}\right)$ and

$\left(r_{kj}^{T}, r_{kj}^{I}, r_{kj}^{F}\right) > \left(r_{jk}^{T}, r_{jk}^{I}, r_{jk}^{F}\right) \Rightarrow \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right) > \left(r_{ji}^{T}, r_{ji}^{I}, r_{ji}^{F}\right)$

$\Rightarrow\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)>(0,0,1)$ (By antisymmetric property)

Lemma4.2. If R is max-min transitive NFM, then R is w-transitive NFM.

Proof. Let $R^2 \leq R$

$\Rightarrow\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right) \wedge\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right) \leq\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

$\Rightarrow\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right) \wedge\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)>(0,0,1)$

$\Rightarrow\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)>(0,0,1)$. Therefore, R is w-transitive.

Lemma 4.3. If $R=\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ is max-min transitive NFM then $R=\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ is s-transitive NFM.

Proof. We must show that if R is max-min transitive NFM, $\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right)>\left(r_{ki}{}^{T}, r_{ki}{}^{I}, r_{ki}{}^{F}\right)$ and $\left(r_{ji}{}^{T}, r_{ji}{}^{I}, r_{ji}{}^{F}\right)>\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$ then $\left(r_{jk}{}^{T}, r_{jk}{}^{I}, r_{jk}{}^{F}\right)>\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)$.

Suppose if $\left(r_{jk}{}^{T}, r_{jk}{}^{I}, r_{jk}{}^{F}\right) \leq\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)$

Now

$\left(r_{jk}{}^{T}, r_{jk}{}^{I}, r_{jk}{}^{F}\right) \geq\left(r_{ji}{}^{T}, r_{ji}{}^{I}, r_{ji}{}^{F}\right) \wedge\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right)>\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \wedge\left(r_{ki}{}^{T}, r_{ki}{}^{I}, r_{ki}{}^{F}\right)$

$\Rightarrow\left(r_{jk}{}^{T}, r_{jk}{}^{I}, r_{jk}{}^{F}\right) \geq\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right) \wedge\left(r_{ji}{}^{T}, r_{ji}{}^{I}, r_{ji}{}^{F}\right) \wedge\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

$=\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right) \wedge\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) \Rightarrow\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)>\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

On the other hand, $\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right)>\left(r_{ki}{}^{T}, r_{ki}{}^{I}, r_{ki}{}^{F}\right) \geq\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right) \wedge\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

Since $\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)>\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$

$\Rightarrow\left(r_{ik}{}^{T}, r_{ik}{}^{I}, r_{ik}{}^{F}\right) \wedge\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)>\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right)$,

Which contradicts the max-min transitive of R.

Hence $\left(r_{jk}{}^{T}, r_{jk}{}^{I}, r_{jk}{}^{F}\right)>\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right)$.

Theorem 4.1. If R antisymmetric and s-transitive NFM, implies ΔT w-transitive and nilpotent NFM.

Proof. Let R be antisymmetric $\Rightarrow \Delta R=R.$

Since R is s-transitive R is w-transitive by Lemma 4.1

$i.e., R^2 \approx R \Rightarrow R^2 \prec R$

$R \prec R^2$

$\Rightarrow (\Delta R)^2 \approx \Delta R \Rightarrow (\Delta R)^2 \prec \Delta R$

$\Delta R \prec (\Delta R)^2$ (By antisymmetric property).

Hence ΔR is w-transitive.

Let $(\Delta R)^n = \left(r_{ij}^{T^{\Delta,n}}, r_{ij}^{I^{\Delta,n}}, r_{ij}^{F^{\Delta,n}}\right)$.Let us consider that there exist indices $i, j \in \{1,2,...,n\}$

So that $\left(r_{ij}^{T^{\Delta,n}}, r_{ij}^{I^{\Delta,n}}, r_{ij}^{F^{\Delta,n}}\right) > (0,0,1)$.

Then

$$\left(r_{ij}^{T^{\Delta,n}}, r_{ij}^{I^{\Delta,n}}, r_{ij}^{F^{\Delta,n}}\right) = \left(r^{T^{\Delta}}{}_{h_0h_1}, r^{I^{\Delta}}{}_{h_0h_1}, r^{F^{\Delta}}{}_{h_0h_1}\right) \wedge \left(r^{T^{\Delta}}{}_{h_1h_2}, r^{I^{\Delta}}{}_{h_1h_2}, r^{F^{\Delta}}{}_{h_1h_2}\right)$$

$$\wedge \ldots \wedge \left(r^{T^{\Delta}}{}_{h_{n-1}h_n}, r^{I^{\Delta}}{}_{h_{n-1}h_n}, r^{F^{\Delta}}{}_{h_{n-1}h_n}\right) > (0,0,1).$$

For a few integers $h_0, h_1, h_2, \ldots h_n \in \{1,2,...,n\}$ so that $h_0 = i$ and $h_n = j$.

Then $h_a = h_b$ for a and b $(a < b)$

and $\left(r^{T^{\Delta}}{}_{h_ah_{a+1}}, r^{I^{\Delta}}{}_{h_ah_{a+1}}, r^{F^{\Delta}}{}_{h_ah_{a+1}}\right) > (0,0,1) = \left(r^{T^{\Delta}}{}_{h_{a+1}h_a}, r^{I^{\Delta}}{}_{h_{a+1}h_a}, r^{F^{\Delta}}{}_{h_{a+1}h_a}\right)$,

$\left(r^{T^{\Delta}}{}_{h_{a+1}h_{a+1}}, r^{I^{\Delta}}{}_{h_{a+1}h_{a+1}}, r^{F^{\Delta}}{}_{h_{a+1}h_{a+1}}\right) > (0,0,1) = \left(r^{T^{\Delta}}{}_{h_{a+2}h_{a+1}}, r^{I^{\Delta}}{}_{h_{a+2}h_{a+1}}, r^{F^{\Delta}}{}_{h_{a+2}h_{a+1}}\right)$,

$\ldots\left(r^{T^{\Delta}}{}_{h_{b-1}h_b}, r^{I^{\Delta}}{}_{h_{b-1}h_b}, r^{F^{\Delta}}{}_{h_{b-1}h_b}\right) > (0,0,1) = \left(r^{T^{\Delta}}{}_{h_bh_{b-1}}, r^{I^{\Delta}}{}_{h_bh_{b-1}}, r^{F^{\Delta}}{}_{h_bh_{b-1}}\right)$

By applying the s-transitivity of NFM ΔR we obtain the following result.

$$\left(r^{T^{\Delta,n}}{}_{h_ah_a}, r^{I^{\Delta,n}}{}_{h_ah_a}, r^{F^{\Delta,n}}{}_{h_ah_a}\right) = \left(r^{T^{\Delta,n}}{}_{h_ah_b}, r^{I^{\Delta,n}}{}_{h_ah_b}, r^{F^{\Delta,n}}{}_{h_ah_b}\right) > \left(r^{T^{\Delta,n}}{}_{h_bh_a}, r^{I^{\Delta,n}}{}_{h_bh_a}, r^{F^{\Delta,n}}{}_{h_bh_a}\right)$$

Which is not possible.

Theorem 4.2. Let R be any w-transitive and irrefexive NFM then $R^n = (0,0,1)$

Proof. Assume that $R^n > (0,0,1)$. Then there exists $l_1, l_2, \ldots, l_{n-1}$ such that

$$\left(r_{il_1}{}^T, r_{il_1}{}^I, r_{il_1}{}^F\right) \wedge \left(r_{l_1l_2}{}^T, r_{l_1l_2}{}^I, r_{l_1l_2}{}^F\right) \wedge \left(r^T{}_{l_{n-1}j}, r^I{}_{l_{n-1}j}, r^F{}_{l_{n-1}j}\right) > (0,0,1).$$

Put $l_0 = i$ and $l_n = j$ for some a and b such that $(a < b)$.

$$\Rightarrow \left(r^T{}_{l_al_{a+1}}, r^I{}_{l_al_{a+1}}, r^F{}_{l_al_{a+1}}\right) \wedge \ldots \wedge \left(r^T{}_{l_al_{a+1}}, r^I{}_{l_al_{a+1}}, r^F{}_{l_al_{a+1}}\right) > (0,0,1).$$

$\Rightarrow \left(r^T{}_{l_al_a}, r^I{}_{l_al_a}, r^F{}_{l_al_a}\right)$ contradicts with fact that R is irreflexive.

Therefore $R^n = (0,0,1)$

Corollary 4.1 Let R and P be w-transitive square NFMs of order n. If P is irreflexive NFM and $P \le R$, then $(A/P)R = AR$ for any mxn matrix A.

Example:4.1 Let $A = \begin{bmatrix} <0.6,0.4,0.2> & <0.3,0.4,0.2> \\ <0.7,0.4,0.1> & <0.4,0.4,0.3> \\ <0.5,0.4,0.2> & <0.6,0.4,0.1> \end{bmatrix}$

$$R = \begin{bmatrix} <1,1,0> & <0.6,0.4,0.1> \\ <0.6,0.4,0.1> & <1,1,0> \end{bmatrix}$$

We assume R be a similarity matrix where $R = \left(r_{ij}^{T}, r_{ij}^{I}, r_{ij}^{F}\right)$ denotes the degree.

Now, let

$P = \begin{bmatrix} <0,0,1> & <0,0,1> \\ <0.6,0.4,0.1> & <0,0,1> \end{bmatrix} \le R$ be nilpotent NFM by means of which we reduce A.

$AP = \begin{bmatrix} <0.3,0.4,0.2> & <0,0,1> \\ <0.4,0.4,0.3> & <0,0,1> \\ <0.6,0.4,0.1> & <0,0,1> \end{bmatrix}$. Hence $A/P = A \overset{c}{\longleftarrow} (AP)$

$$A/P = \begin{bmatrix} <0.6,0.4,0.2> & <0.3,0.4,0.2> \\ <0.7,0.4,0.1> & <0.4,0.4,0.3> \\ <0,0,1> & <0.6,0.4,0.1> \end{bmatrix}$$

$$(A/P)R = \begin{bmatrix} <0.6,0.4,0.2> & <0.6,0.4,0.2> \\ <0.7,0.4,0.1> & <0.6,0.4,0.1> \\ <0.6,0.4,0.1> & <0.6,0.4,0.1> \end{bmatrix}$$

$$AR = \begin{bmatrix} <0.6,0.4,0.2> & <0.6,0.4,0.2> \\ <0.7,0.4,0.1> & <0.6,0.4,0.1> \\ <0.6,0.4,0.1> & <0.6,0.4,0.1> \end{bmatrix}$$

Therefore, $(A/P)R = AR$

Theorem 4.3 Let R is irrefexive and w-transitive NFM, then $(R/R)^{+} \approx R$.

Proof: Let $N=\left[\left(n^{T}{}_{ij},n^{I}{}_{ij},n^{F}{}_{ij}\right)\right]=R/R,$

$$\left(n^{T}{}_{ij},n^{I}{}_{ij},n^{F}{}_{ij}\right)=\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right)\leftarrow\left(\bigcup_{k=1}^{n}\left(r_{ik}{}^{T}\wedge r_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(r_{ik}{}^{I}\wedge r_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(r_{ik}{}^{F}\vee r_{kj}{}^{F}\right)\right).$$

To prove $N^{+}\prec R,$ suppose $\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right)=(0,0,1)$ and $\left(n_{ij}{}^{T^{k}},n_{ij}{}^{I^{k}},n_{ij}{}^{F^{k}}\right)>(0,0,1)$ for some k.

$\left(n^{T}{}_{ih_1},n^{I}{}_{ih_1},n^{F}{}_{ih_1}\right)>(0,0,1),\left(n^{T}{}_{h_1h_2},n^{I}{}_{h_1h_2},n^{F}{}_{h_1h_2}\right)>(0,0,1),\ldots,\left(n^{T}{}_{h_{k-1}j},n^{I}{}_{h_{k-1}j},n^{F}{}_{h_{k-1}j}\right)>(0,0,1)$ for a few indices $h_0=i,h_1,h_2,\ldots,h_{k-1},h_k=j$ which implies

$\left(r_{ih_1}{}^{T},r_{ih_1}{}^{I},r_{ih_1}{}^{F}\right)>(0,0,1),\left(r_{h_1h_2}{}^{T},r_{h_1h_2}{}^{I},r_{h_1h_2}{}^{F}\right)>(0,0,1),\ldots,\left(r_{h_{k-1}j}{}^{T},r_{h_{k-1}j}{}^{I},r_{h_{k-1}j}{}^{F}\right)>(0,0,1).$

By w-transitivity of R, we get $\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right)>(0,0,1)$,a contradiction, now we have to prove that, $R\prec N^{+}.$ Therefore, R and N is nilpotent, since $N\le R.$ Assume that $\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right)>(0,0,1)$ and $\left(n_{ij}{}^{T^{k}},n_{ij}{}^{I^{k}},n_{ij}{}^{F^{k}}\right)=(0,0,1)\Rightarrow\left(n_{ij}{}^{T^{k}},n_{ij}{}^{I^{k}},n_{ij}{}^{F^{k}}\right)>(0,0,1)$ for every k=1,2,…,n-1. Since $\left(n^{T}{}_{ij},n^{I}{}_{ij},n^{F}{}_{ij}\right)>(0,0,1).$ We obtain $\left(r_{ih_1}{}^{T},r_{ih_1}{}^{I},r_{ih_1}{}^{F}\right)\ge\left(r_{ij}{}^{T},r_{ij}{}^{I},r_{ij}{}^{F}\right),\left(r_{h_1j}{}^{T},r_{h_1j}{}^{I},r_{h_1j}{}^{F}\right)>(0,0,1)$ for some h_1 consequently $\left(r_{ij}{}^{T^{(2)}},r_{ij}{}^{I^{(2)}},r_{ij}{}^{F^{(2)}}\right)>(0,0,1)$.

Now we must prove that $\left(n^{T}{}_{ip},n^{I}{}_{ip},n^{F}{}_{ip}\right)>(0,0,1)$ and $\left(r^{T}{}_{pj},r^{I}{}_{pj},r^{F}{}_{pj}\right)>(0,0,1)$ for a few p.

If $\left(n^{T}{}_{ih_1},n^{I}{}_{ih_1},n^{F}{}_{ih_1}\right)=(0,0,1)$ then

$\left(r_{ih_2}{}^{T},r_{ih_2}{}^{I},r_{ih_2}{}^{F}\right)\ge\left(r_{ih_1}{}^{T},r_{ih_1}{}^{I},r_{ih_1}{}^{F}\right)>(0,0,1),\ \left(r_{h_2h_1}{}^{T},r_{h_2h_1}{}^{I},r_{h_2h_1}{}^{F}\right)\ge\left(r_{ih_1}{}^{T},r_{ih_1}{}^{I},r_{ih_1}{}^{F}\right)>(0,0,1)$ for a few h2 and consequently $\left(r_{ij}{}^{T^{(3)}},r_{ij}{}^{I^{(3)}},r_{ij}{}^{F^{(3)}}\right)>(0,0,1)$ Of course, $\left(r_{h_2j}{}^{T},r_{h_2j}{}^{I},r_{h_2j}{}^{F}\right)>(0,0,1).$ If

$\left(n^{T}{}_{ih_2},n^{I}{}_{ih_2},n^{F}{}_{ih_2}\right)=(0,0,1)$ then

$\left(n^{T}{}_{ih_3},n^{I}{}_{ih_3},n^{F}{}_{ih_3}\right)\ge\left(n^{T}{}_{ih_2},n^{I}{}_{ih_2},n^{F}{}_{ih_2}\right)>(0,0,1),\left(n^{T}{}_{h_3h_2},n^{I}{}_{h_3h_2},n^{F}{}_{h_3h_2}\right)\ge\left(n^{T}{}_{ih_2},n^{I}{}_{ih_2},n^{F}{}_{ih_2}\right)>(0,0,1)$ for a few h3 and consequently $\left(r_{ij}{}^{T^{(4)}},r_{ij}{}^{I^{(4)}},r_{ij}{}^{F^{(4)}}\right)>(0,0,1)$ Of course, $\left(r_{h_3j}{}^{T},r_{h_3j}{}^{I},r_{h_3j}{}^{F}\right)>(0,0,1).$ By repeating the same process, we have $\left(r_{ij}{}^{T^{(n)}},r_{ij}{}^{I^{(n)}},r_{ij}{}^{F^{(n)}}\right)>(0,0,1)$

which is impossible because R is nilpotent. So we get $\left(n^{T}{}_{ih_p}, n^{I}{}_{ih_p}, n^{F}{}_{ih_p}\right) > (0,0,1)$ and $\left(r_{ij}{}^{T}, r_{ij}{}^{I}, r_{ij}{}^{F}\right) > (0,0,1)$ for a few p.

Since $\left(n_{ij}{}^{T^{(2)}}, n_{ij}{}^{I^{(2)}}, n_{ij}{}^{F^{(2)}}\right) = (0,0,1)$, we obtain $\left(n^{T^{(2)}}{}_{h_p j}, n^{I^{(2)}}{}_{h_p j}, n^{F^{(2)}}{}_{h_p j}\right) = (0,0,1)$ and consequently $\left(r_{h_p k}{}^{T}, r_{h_p k}{}^{I}, r_{h_p k}{}^{F}\right) \geq \left(r_{h_p j}{}^{T}, r_{h_p j}{}^{I}, r_{h_p j}{}^{F}\right) > (0,0,1)$, $\left(r_{kj}{}^{T}, r_{kj}{}^{I}, r_{kj}{}^{F}\right) \geq \left(r_{h_p j}{}^{T}, r_{h_p j}{}^{I}, r_{h_p j}{}^{F}\right) > (0,0,1)$. By repeating the above process, we get $\left(n^{T}{}_{h_p l_2}, n^{I^{(2)}}{}_{h_p l_2}, n^{F^{(2)}}{}_{h_p l_2}\right) > (0,0,1)$ for a few l_2 and consequently $\left(n^{T^{(2)}}{}_{il_2}, n^{I^{(2)}}{}_{il_2}, n^{F^{(2)}}{}_{il_2}\right) > (0,0,1)\left(l_1 = h_p\right)$. By continuing this procedure, we would have $\left(n^{T^{(n)}}{}_{il_n}, n^{I^{(n)}}{}_{il_n}, n^{F^{(n)}}{}_{il_n}\right) > (0,0,1)$ this leads to a contradiction, as it conflicts with the fact that R is nilpotent.

Example.4.2 Consider irreflexive and w-transitive NFM, whose graph is depicted easily

$$R = \begin{bmatrix} (0,0,1) & (0.3,0.4,0.6) & (0.5,0.4,0.4) & (0.3,0.4,0.6) \\ (0,0,1) & (0,0,1) & (0.4,0.4,0.5) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0.4,0.4,0.5) & (0.8,0.4,0.1) & (0,0,1) \end{bmatrix}$$

. Then reduction of R and its transitive

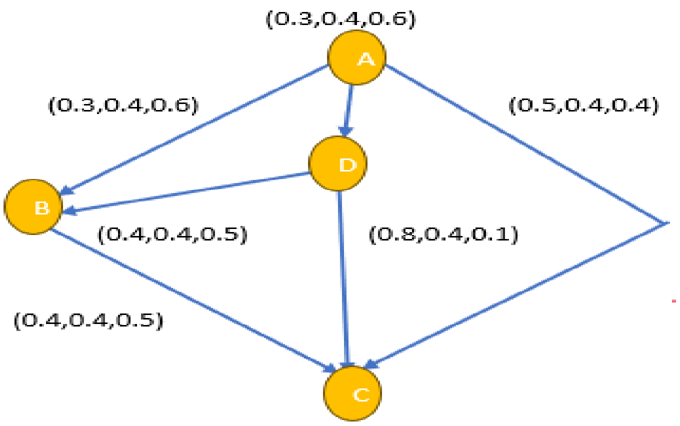

Figure 4: Matrix (R/R)

Now, $R/R = R \overset{c}{\leftarrow} \left(R \times R\right)$,

$$RR=\begin{bmatrix}(0,0,1) & (0.4,0.4,0.5) & (0.6,0.4,0.3) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0.4,0.4,0.5) & (0,0,1)\end{bmatrix}$$

Then, $R/R=\begin{bmatrix}(0,0,1) & (0,0,1) & (0,0,1) & (0.6,0.4,0.3)\\(0,0,1) & (0,0,1) & (0.4,0.4,0.5) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0.4,0.4,0.5) & (0.8,0.4,0.1) & (0,0,1)\end{bmatrix}$

$$\left(R/R\right)^{+}=\left(R/R\right)\vee\left(R/R\right)^{2}\vee\left(R/R\right)^{3}\vee\left(R/R\right)^{4},$$

$$\left(R/R\right)^{+}=\begin{bmatrix}(0,0,1) & (0,0,1) & (0,0,1) & (0.6,0.4,0.3)\\(0,0,1) & (0,0,1) & (0.4,0.4,0.5) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0.4,0.4,0.5) & (0.8,0.4,0.1) & (0,0,1)\end{bmatrix}$$

$$\vee\begin{bmatrix}(0,0,1) & (0.4,0.4,0.5) & (0.6,0.4,0.3) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0.4,0.4,0.5) & (0.4,0.4,0.5) & (0,0,1)\end{bmatrix}$$

$$\vee\begin{bmatrix}(0,0,1) & (0.4,0.4,0.5) & (0.6,0.4,0.3) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0.4,0.4,0.5) & (0.4,0.4,0.5) & (0,0,1)\end{bmatrix}\vee\begin{bmatrix}(0,0,1) & (0.4,0.4,0.5) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\end{bmatrix}$$

$$\vee\begin{bmatrix}(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\end{bmatrix}$$

$$\left(R/R\right)^{+}=\begin{bmatrix}(0,0,1) & (0.4,0.4,0.5) & (0.6,0.4,0.3) & (0.6,0.4,0.3)\\(0,0,1) & (0,0,1) & (0.4,0.4,0.5) & (0,0,1)\\(0,0,1) & (0,0,1) & (0,0,1) & (0,0,1)\\(0,0,1) & (0.4,0.4,0.5) & (0.8,0.4,0.1) & (0,0,1)\end{bmatrix}$$

$$\left(R/R\right)^{+}\approx R$$

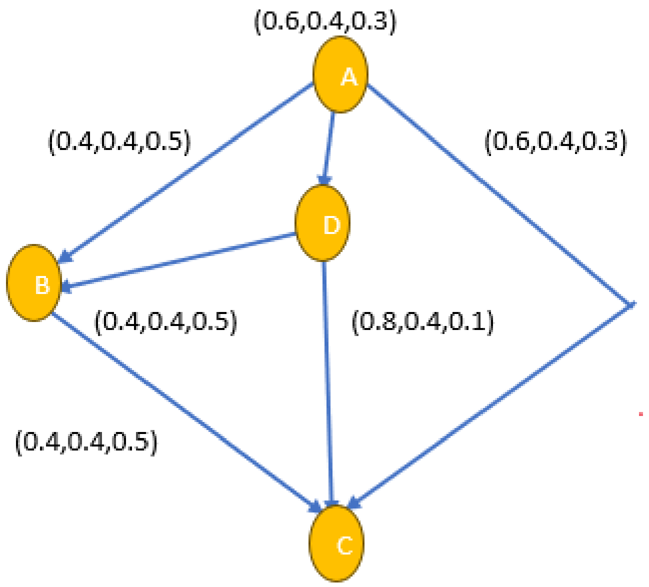

Figure 4: Matrix (R/R)⁺

5. Conclusion and Future Work

This research offers an in-depth analysis of the reduction of Neutrosophic Fuzzy Matrices (NFMs), with a specific emphasis on nilpotent NFMs and their representation as acyclic graphs. By examining the reduction of irreflexive, transitive, and nilpotent NFMs, we have established key insights into the structural behavior of these matrices in complex systems. The work highlights the importance of the union of acyclic graphs forming isomorphic cyclic graphs for defining consistent systems and emphasizes the role of transitivity in simplifying and analyzing NFMs. We have also presented equivalent conditions for reduction, verified the applicability of reduction models for s-transitive and w-transitive NFMs, and demonstrated these concepts through numerical examples. The study contributes to the theoretical understanding and practical applications of NFMs, providing valuable tools for handling uncertainty and complexity in a variety of systems. The verified reduction models offer a robust framework for analyzing large-scale neutrosophic fuzzy systems, making them more accessible and easier to compute.

The current research provides a significant foundation for the study and application of Neutrosophic Fuzzy Matrices (NFMs), yet several opportunities for future exploration remain. One promising direction is the development of advanced reduction techniques tailored for large-scale and dynamically evolving NFMs. This includes integrating adaptive learning models and optimization methods to enhance computational efficiency, particularly for real-time systems. Additionally, there is considerable potential in combining NFMs with machine learning frameworks to address challenges such as feature extraction and classification in uncertain environments, with applications in healthcare, agriculture, and smart systems. Another avenue involves extending the proposed methodologies to interval-valued and multi-attribute NFMs, enabling more effective multi-criteria decision-making in complex scenarios like supply chain optimization and urban planning. Despite its contributions, the current work has limitations, such as scalability concerns and assumptions of ideal conditions, which may limit its applicability to real-world datasets. Addressing these issues through noise tolerance mechanisms and distributed computing will be critical. Furthermore, the exploration of graph-theoretic properties of NFMs presents opportunities for breakthroughs in network analysis and optimization. The findings of this research not only enhance theoretical understanding but also hold the potential to revolutionize decision-making processes across diverse domains by providing robust tools for managing uncertainty and complexity.

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