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Analysis of Fixed Point Theorems for Contractive Mappings in Complex-Valued Neutrosophic Metric Spaces and Their Application

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Abstract. This paper introduces complex-valued neutrosophic metric spaces, extending the concepts of complex-valued fuzzy and intuitionistic fuzzy metric spaces. We establish the existence and uniqueness of fixed points for mappings under various contractive conditions within this framework. To improve clarity, we present illustrative examples. Furthermore, we demonstrate the applicability of our findings by proving the existence of a unique solution for Fredholm integral equations, emphasizing their practical relevance in mathematical analysis and applied sciences.

Keywords: Complex-valued neutrosophic metric spaces, Fixed point theory, Contractive mappings, Neutrosophic sets, Fredholm integral equations.

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1. Introduction

Fixed-point theory is a fundamental area of mathematical analysis with extensive applications across various disciplines. The Banach contraction principle, introduced in [4], plays a crucial role in addressing existence problems in nonlinear analysis. Over time, it has been refined and extended into numerous fixed-point theorems using diverse methodologies.

Simultaneously, fuzzy set theory has gained widespread recognition as a powerful tool in multiple fields, providing the foundation for advanced mathematical models and practical applications. First introduced by Zadeh [23] in 1965, fuzzy sets offer an effective means of handling uncertainty and imprecision, surpassing the limitations of classical binary logic. This framework facilitates the modeling of vague and ill-defined scenarios where traditional approaches struggle.

Building on this foundation, Kramosil and Michalek [16] developed fuzzy metric and statistical metric spaces, extending classical metric spaces to better accommodate uncertainty and variability. Their pioneering work laid the groundwork for the subsequent development of intuitionistic and neutrosophic fuzzy metric spaces, which have significantly broadened the scope of fixed-point theory in mathematical and applied sciences.

Further advancements came from George and Veeramani [8], who established key results on fixed points, clarifying fundamental aspects of continuity and convergence in fuzzy metric spaces. Grabiec [7] expanded this framework by introducing G-Cauchy sequences, a concept that deviates significantly from classical Cauchy sequences as formulated by George and Veeramani. Notably, it has been demonstrated that conventional fuzzy metric spaces are not G-complete, and compactness in fuzzy metric spaces does not necessarily imply G-completeness.

A major breakthrough occurred in 1986 when Atanassov [2] introduced intuitionistic fuzzy sets as a generalization of fuzzy sets. His work explored various set-theoretic operations, relations, and connections with modal and topological operators, thereby enriching the theoretical foundation of fuzzy set theory. This concept was later extended by Park [17] in 2004, who defined intuitionistic fuzzy metric spaces, further advancing the field and expanding its applicability to complex mathematical structures.

In 2011, Azam et al. [3] introduced complex-valued metric spaces into fixed-point theory, replacing positive real numbers with ordered complex numbers to derive fixed-point results under rational inequality conditions. Later, Shukla et al. [20] extended this concept to fuzzy metric spaces, defining complex-valued fuzzy metric spaces and establishing fixed-point results for mappings satisfying specific contractive conditions. This field remains an active area of research, as evidenced by the contributions of [5,24] and the comprehensive studies by Humaira et al. [8]- [10], which provide various results with real-world applications.

A significant development occurred in 1998 when Smarandache [21] introduced neutrosophic theory as an extension of fuzzy set theory, allowing for the independent treatment of truth, indeterminacy, and falsity. Later, Kirisci and Simsek [14] expanded this idea by defining neutrosophic metric spaces, offering a mathematical framework to address uncertainty, vagueness, and indeterminacy. Further advancements were made by Sowndrarajan et al. [22], who demonstrated the utility of neutrosophic metric spaces by proving fixed-point results for contraction theorems in this context.

This paper presents a novel extension of fuzzy metric spaces by introducing complex-valued neutrosophic metric spaces, which unify and broaden the scope of both complex-valued fuzzy metric spaces [20] and intuitionistic fuzzy metric spaces [11]. We establish several fixed-point results for mappings satisfying specific contractive conditions in this setting. Additionally, we provide illustrative examples and real-world applications to highlight the practical significance of the proposed framework.

2. **Preliminaries**

This study requires the following definitions and preliminary results.

We use the following symbols to denote specific sets:

(a) $\mathbb{N}$: The set of natural numbers.

(b) $\mathbb{N}_0$: The set of non-negative integers.

(c) $\mathbb{C}$: The set of complex numbers.

The following definitions are introduced:

(i) $\zeta = \{(\mathfrak{p}, \mathfrak{q}) : 0 \leq \mathfrak{p} < \infty, 0 \leq \mathfrak{q} < \infty\} \subset \mathbb{C}$.

(ii) The points $(0, 0)$ and $(1, 1)$ in $\mathbb{C}$ are denoted by θ and ℓ, respectively.

(iii) The closed unit complex interval is defined as

$$\Upsilon = \{(\mathfrak{p}, \mathfrak{q}) : 0 \leq \mathfrak{p} \leq 1, 0 \leq \mathfrak{q} \leq 1\}.$$

(iv) The open unit complex interval is given by

$$\Upsilon_0 = \{(\mathfrak{p}, \mathfrak{q}) : 0 < \mathfrak{p} < 1, 0 < \mathfrak{q} < 1\}.$$

Moreover, we define

$$\zeta_0 = \{(\mathfrak{p}, \mathfrak{q}) : 0 < \mathfrak{p} < \infty, 0 < \mathfrak{q} < \infty\}.$$

A partial order $\preceq$ is established on $\mathbb{C}$, where for any $\tau_1, \tau_2 \in \mathbb{C}$, we define

$$\tau_1 \preceq \tau_2 \iff \tau_2 - \tau_1 \in \zeta.$$

We write $\tau_1 \prec \tau_2$ to indicate that $Re(\tau_2) > Re(\tau_1)$ and $Im(\tau_2) > Im(\tau_1)$. This implies that $\tau_1 \prec \tau_2$ if and only if $\tau_2 - \tau_1 \in \zeta_0$.

Now, let $\{\tau_n\}$ be a sequence in $\mathbb{C}$. The sequence is monotonic with respect to $\preceq$ if either:

$$\tau_{n+1} \preceq \tau_n \quad \text{or} \quad \tau_n \preceq \tau_{n+1}, \quad \forall n \in \mathbb{N}.$$

The infimum of any subset $\Omega \subset \mathbb{C}$ is denoted by $\inf \Omega$ and satisfies $\inf \Omega \preceq \eta$ for all $\eta \in \Omega$. Similarly, the supremum (least upper bound) of Ω is denoted by $\sup \Omega$.

Remark 2.1. [20] *Suppose $\{\tau_n\}$ (with $n \in \mathbb{N}$) be a sequence in the partially ordered set $(\zeta, \preceq)$. The following properties hold:*

(1) If $\{\tau_n\}$ is monotonic with respect to $\preceq$ and there exist elements $\mu, \nu \in \zeta$ such that $\mu \preceq \tau_n \preceq \nu$ for all $n \in \mathbb{N}$, then the sequence $\{\tau_n\}$ converges to some $\tau \in \zeta$, i.e., $\tau_n \to \tau$ as $n \to \infty$.

(2) The partial order $\preceq$ defines a lattice structure on $\mathbb{C}$, although it is not a total order.

(3) If a subset $\Omega \subset \mathbb{C}$ is bounded with respect to $\preceq$ – that is, there exist $\mu, \nu \in \mathbb{C}$ such that $\mu \preceq \eta \preceq \nu$ for all $\eta \in \Omega$ – then both the infimum $\inf \Omega$ and the supremum $\sup \Omega$ exist.

Remark 2.2. [20] *Suppose τ_n, τ_n' (for each $n \in \mathbb{N}$) and τ in ζ_0. The following statements hold:*

(1) *If $\tau_n \preceq \tau_n' \preceq \ell$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \tau_n = \ell$, then $\lim_{n\to\infty} \tau_n' = \ell$*

(2) *If $\tau_n \preceq \lambda$ for all $n \in \mathbb{N}$ and there exists some $\tau \in \zeta$ such that $\lim_{n\to\infty} \tau_n = \tau$, then $\tau \preceq \lambda$.*

(3) *If $\lambda \preceq \tau_n$ for all $n \in \mathbb{N}$ and there exists some $\tau \in \zeta$ such that $\lim_{n\to\infty} \tau_n = \tau$, then $\lambda \preceq \tau$.*

Definition 2.3. [20] *A binary operation $*$ mapping from Υ^2 to Υ is called a complex-valued t-norm (CVTN) if it satisfies the following conditions:*

(1) *$\theta * \tau = \theta$, $\ell * \tau = \tau$ for all $\tau \in \Upsilon$,*

(2) *$*$ is associative and commutative,*

(3) *If $\tau_3 \succeq \tau_1$ and $\tau_4 \succeq \tau_2$, then $\tau_3 * \tau_4 \succeq \tau_1 * \tau_2$ for all $\tau_1, \tau_2, \tau_3, \tau_4 \in \Upsilon$.*

Example 2.4. [20] *Let $\tau_1 = (\mathfrak{p}_1, \mathfrak{q}_1)$ and $\tau_2 = (\mathfrak{p}_2, \mathfrak{q}_2)$ in Υ, the following are examples of complex-valued t-norms:*

(1) $\tau_1 * \tau_2 = (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2)$

(2) $\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\})$

(3) $\tau_1 * \tau_2 = (\max\{\mathfrak{p}_1 + \mathfrak{p}_2 - 1, 0\}, \max\{\mathfrak{q}_1 + \mathfrak{q}_2 - 1, 0\})$

Definition 2.5. [11] *A binary operation $\Diamond$ mapping from Υ^2 to Υ is called a complex-valued t-conorm (CVTCN) if it satisfies the following conditions:*

(1) *$\tau \Diamond \theta = \tau$, $\tau \Diamond \ell = \ell$ for all $\tau \in \Upsilon$.*

(2) *$\Diamond$ is associative and commutative.*

(3) *If $\tau_3 \succeq \tau_1$ and $\tau_4 \succeq \tau_2$, then $\tau_3 \Diamond \tau_4 \succeq \tau_1 \Diamond \tau_2$ for all $\tau_1, \tau_2, \tau_3, \tau_4 \in \Upsilon$.*

Example 2.6. *Some examples of complex-valued t-conorms are:*

(1) $\tau_1 \Diamond \tau_2 = (\mathfrak{p}_1 + \mathfrak{p}_2, \mathfrak{q}_1 + \mathfrak{q}_2) - (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2)$.

(2) $\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\})$.

(3) $\tau_1 \Diamond \tau_2 = (\min\{\mathfrak{p}_1 + \mathfrak{p}_2, 1\}, \min\{\mathfrak{q}_1 + \mathfrak{q}_2, 1\})$.

Definition 2.7. [11] *Let Ξ be a non-empty set, and let $*$ and $\Diamond$ denote a CVTN and a CVTCN, respectively. Furthermore, let $\mathscr{A}$ and $\mathscr{B}$ be complex fuzzy sets defined on $\Xi^2 \times \zeta_0$ such that the following conditions hold:*

(1) $\mathscr{A}(\zeta, \vartheta, \tau) + \mathscr{B}(\zeta, \vartheta, \tau) \preceq \ell$.

(2) $\mathscr{A}(\zeta, \vartheta, \tau) \succ \theta$.

(3) $\mathscr{A}(\zeta, \vartheta, \tau) = \ell$ *for all* $\tau \in \zeta_0$ *if and only if* $\zeta = \vartheta$.

(4) $\mathscr{A}(\zeta, \vartheta, \tau) = \mathscr{A}(\vartheta, \zeta, \tau)$.

(5) $\mathscr{A}(\zeta, \varsigma, \tau + \tau') \succeq \mathscr{A}(\zeta, \vartheta, \tau) * \mathscr{A}(\vartheta, \varsigma, \tau')$.

(6) *The mapping* $\mathscr{A}(\zeta, \vartheta, \cdot)$ *from* ζ_0 *to* Υ *is continuous.*

(7) $\mathscr{B}(\zeta, \vartheta, \tau) \prec \ell$.

(8) $\mathscr{B}(\zeta, \vartheta, \tau) = \theta$ *for all* $\tau \in \zeta_0$ *if and only if* $\zeta = \vartheta$.

(9) $\mathscr{B}(\zeta, \vartheta, \tau) = \mathscr{B}(\vartheta, \zeta, \tau)$.

(10) $\mathscr{B}(\zeta, \varsigma, \tau + \tau') \preceq \mathscr{B}(\zeta, \vartheta, \tau) \Diamond \mathscr{B}(\vartheta, \varsigma, \tau')$.

(11) *The mapping* $\mathscr{B}(\zeta, \vartheta, \cdot)$ *from* ζ_0 *to* Υ *is continuous.*

Here, $\zeta, \vartheta, \varsigma \in \Xi$ *and* $\tau, \tau' \in \zeta_0$.

Under these conditions, the 5-tuple $(\Xi, \mathscr{A}, \mathscr{B}, *, \Diamond)$ *is called a complex-valued intuitionistic fuzzy metric space.*

3. Main results

This section presents and analyzes the properties of complex-valued neutrosophic metric spaces (CVNMS).

Definition 3.1. *Let Ξ be a non-empty set, and let $*$ and $\Diamond$ be continuous CVTN and CVTCN, respectively. Assume that $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are complex-valued neutrosophic sets defined on $\Xi^2 \times \wp_0$, satisfying the following conditions:*

(1) $\mathscr{A}(\zeta, \vartheta, \tau) + \mathscr{B}(\zeta, \vartheta, \tau) + \mathscr{C}(\zeta, \vartheta, \tau) \preceq 3\ell$.

(2) $\mathscr{A}(\zeta, \vartheta, \tau) \succ \theta$.

(3) $\mathscr{A}(\zeta, \vartheta, \tau) = \ell$ *for all* $\tau \in \zeta_0$ *if and only if* $\zeta = \vartheta$.

(4) $\mathscr{A}(\zeta, \vartheta, \tau) = \mathscr{A}(\vartheta, \zeta, \tau)$.

(5) $\mathscr{A}(\zeta, \varsigma, \tau + \tau') \succeq \mathscr{A}(\zeta, \vartheta, \tau) * \mathscr{A}(\vartheta, \varsigma, \tau')$.

(6) *The mapping* $\mathscr{A}(\zeta, \vartheta, \cdot)$ *from* ζ_0 *to* Υ *is continuous.*

(7) $\mathscr{B}(\zeta, \vartheta, \tau) \prec \ell$.

(8) $\mathscr{B}(\zeta, \vartheta, \tau) = \theta$ *for all* $\tau \in \zeta_0$ *if and only if* $\zeta = \vartheta$.

(9) $\mathscr{B}(\zeta, \vartheta, \tau) = \mathscr{B}(\vartheta, \zeta, \tau)$.

(10) $\mathscr{B}(\zeta, \varsigma, \tau + \tau') \preceq \mathscr{B}(\zeta, \vartheta, \tau) \Diamond \mathscr{B}(\vartheta, \varsigma, \tau')$.

(11) *The mapping* $\mathscr{B}(\zeta, \vartheta, \cdot)$ *from* ζ_0 *to* Υ *is continuous.*

(12) $\mathscr{C}(\zeta, \vartheta, \tau) \prec \ell$.

(13) $\mathscr{C}(\zeta, \vartheta, \tau) = \theta$ *for all* $\tau \in \zeta_0$ *if and only if* $\zeta = \vartheta$.

(14) $\mathscr{C}(\zeta, \vartheta, \tau) = \mathscr{B}(\vartheta, \zeta, \tau)$.

(15) $\mathscr{C}(\zeta, \varsigma, \tau + \tau') \preceq \mathscr{C}(\zeta, \vartheta, \tau) \Diamond \mathscr{C}(\vartheta, \varsigma, \tau')$.

(16) *The mapping* $\mathscr{C}(\zeta, \vartheta, \cdot)$ *from* ζ_0 *to* Υ *is continuous.*

for each $\zeta, \vartheta, \varsigma \in \Xi$ *and* $\tau, \tau' \in \zeta_0$. *The structure* $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ *is called a complex-valued neutrosophic metric space (CVNMS).*

Remark 3.2. *A CVNMS can be constructed from a complex-valued fuzzy metric space* $(\Xi, \mathscr{A}, *)$ *by defining it as*

$$(\Xi, \mathscr{A}, \ell - \mathscr{A}, \ell - \mathscr{A}, *, \Diamond),$$

where the CVTN $*$ *and CVTCN* $\Diamond$ *are linked by the relation*

$$\tau_1 \Diamond \tau_2 = \ell - ((\ell - \tau_1) * (\ell - \tau_2))$$

for every $\tau_1, \tau_2 \in \Upsilon$.

Example 3.3. *Let* (Ξ, d) *be a metric space. For* $\tau_i = (\mathfrak{p}_i, \mathfrak{q}_i) \in \Upsilon$ *where* $i = 1, 2$, *define the operations:*

$$\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\}),$$

$$\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\}).$$

Now, define the complex neutrosophic sets $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{C}$ *as*

$$\mathscr{A}(\zeta, \vartheta, \tau) = e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}} \ell,$$

$$\mathscr{B}(\zeta, \vartheta, \tau) = \left(1 - e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}}\right) \ell,$$

$$\mathscr{C}(\zeta, \vartheta, \tau) = \left(1 - e^{\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}}\right) \ell,$$

for all $\zeta, \vartheta \in \Xi$ *and* $\tau = (\mathfrak{p}, \mathfrak{q}) \in \zeta_0$. *Thus, the structure* $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ *forms a CVNMS.*

Lemma 3.4. *Let* $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ *be a CVNMS. Then, the following properties hold:*

(1) *The mapping* $\mathscr{A}(\zeta, \vartheta, \cdot)$ *is monotonically increasing.*

(2) *The mappings* $\mathscr{B}(\zeta, \vartheta, \cdot)$ *and* $\mathscr{C}(\zeta, \vartheta, \cdot)$ *are monotonically decreasing.*

Specifically, for any $\tau, \tau' \in \zeta_0$ *with* $\tau \prec \tau'$, *we have*

$$\mathscr{A}(\zeta, \vartheta, \tau) \preceq \mathscr{A}(\zeta, \vartheta, \tau'), \quad \mathscr{B}(\zeta, \vartheta, \tau) \succeq \mathscr{B}(\zeta, \vartheta, \tau'), \quad \mathscr{C}(\zeta, \vartheta, \tau) \succeq \mathscr{C}(\zeta, \vartheta, \tau').$$

Proof. If $\tau, \tau' \in \zeta_0$ with $\tau \prec \tau'$, then $\tau' - \tau \in \zeta_0$. Applying condition (5) from Definition 3.1, we obtain

$$\mathscr{A}(\zeta, \vartheta, \tau') \succeq \mathscr{A}(\zeta, \vartheta, \tau).$$

Similarly, applying conditions (10) and (15) from Definition 3.1, we conclude

$$\mathscr{B}(\zeta, \vartheta, \tau') \succeq \mathscr{B}(\zeta, \vartheta, \tau), \quad \mathscr{C}(\zeta, \vartheta, \tau') \succeq \mathscr{C}(\zeta, \vartheta, \tau).$$

□

Definition 3.5. *Consider $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \diamondsuit)$ as a CVNMS. A sequence $\{\zeta_n\}$ in Ξ is said to converge to $\zeta \in \Xi$ if, for each $\rho \in \Upsilon_0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the following conditions hold:*

$$\mathscr{A}(\zeta_n, \zeta, \tau) \succ \ell - \rho, \quad \mathscr{B}(\zeta_n, \zeta, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \prec \rho.$$

Definition 3.6. *Consider $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \diamondsuit)$ as a CVNMS. A sequence $\{\zeta_n\}$ in Ξ is called a Cauchy sequence if:*

(1) $\lim_{n\to\infty} \inf_{m>n} \mathscr{A}(\zeta_n, \zeta_m, \tau) = \ell,$
(2) $\lim_{n\to\infty} \sup_{m>n} \mathscr{B}(\zeta_n, \zeta_m, \tau) = \theta,$
(3) $\lim_{n\to\infty} \sup_{m>n} \mathscr{C}(\zeta_n, \zeta_m, \tau) = \theta,$

for all $\tau \in \zeta_0$. A CVNMS $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \diamondsuit)$ is said to be complete if every Cauchy sequence in Ξ converges.

The following examples clarify Definitions 3.5 and 3.6.

Example 3.7. *Consider the CVNMS $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \diamondsuit)$ as defined in Example 3.3. We analyze the convergence of the sequence $\{\zeta_n\} = \left\{\frac{n+1}{n}\right\}$ with limit $\zeta = 1$.*

Let $\Xi = \mathbb{R}$ with metric $d(\zeta, \vartheta) = |\zeta - \vartheta|$ for all $\zeta, \vartheta \in \Xi$. We verify that

$$\mathscr{A}(\zeta_n, \zeta, \tau) \succ \ell - \rho, \quad \mathscr{B}(\zeta_n, \zeta, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \prec \rho$$

for each $\rho = (\rho_1, \rho_2) \in \Upsilon_0$ and $\tau \in \zeta_0$. For the real part

$$\begin{aligned} Re(\mathscr{A}(\zeta_n, \zeta, \tau) - \ell + \rho) &= e^{-\frac{d(\zeta_n,\zeta)}{\mathfrak{p}+\mathfrak{q}}} - 1 + \rho_1 \\ &= e^{-\frac{(1/n)}{\mathfrak{p}+\mathfrak{q}}} - 1 + \rho_1. \end{aligned}$$

As $n \to \infty$, we get $Re(\mathscr{A}(\zeta_n, \zeta, \tau) - \ell + \rho) \to \rho_1$. Similarly, for the imaginary part

$$Im(\mathscr{A}(\zeta_n, \zeta, \tau) - \ell + \rho) \to \rho_2 \quad as \quad n \to \infty.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\mathscr{A}(\zeta_n, \zeta, \tau) \succ \ell - \rho.$$

Using similar calculations for $\mathscr{B}$ and $\mathscr{C}$, we conclude that for all $n > n_0$

$$\mathscr{B}(\zeta_n, \zeta, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \prec \rho.$$

Thus, $\{\zeta_n\}$ converges to $\zeta = 1$ in the given CVNMS.

Example 3.8. *Continuing from the previous example, we now show that $\left\{\frac{n+1}{n}\right\}$ is a Cauchy sequence.*

For any $n, m \in \mathbb{N}$ with $m > n$ and for all $\tau \in \zeta_0$, define

$$d(\zeta_n, \zeta_m) = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right|.$$

Using this metric, we analyze

$$\mathscr{A}(\zeta_n, \zeta_m, \tau) = e^{-\frac{d(\zeta_n,\zeta_m)}{\mathfrak{p}+\mathfrak{q}}}\ell = e^{-\frac{\left|\frac{1}{n}-\frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}}\ell.$$

As $n, m \to \infty$, we get $\mathscr{A}(\zeta_n, \zeta_m, \tau) \to \ell$.

Similarly, we analyze

$$\mathscr{B}(\zeta_n, \zeta_m, \tau) = \left(1 - e^{-\frac{d(\zeta_n,\zeta_m)}{\mathfrak{p}+\mathfrak{q}}}\right)\ell = \left(1 - e^{-\frac{\left|\frac{1}{n}-\frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}}\right)\ell.$$

As $n, m \to \infty$, we obtain $\mathscr{B}(\zeta_n, \zeta_m, \tau) \to \theta$.

Finally

$$\mathscr{C}(\zeta_n, \zeta_m, \tau) = \left(e^{-\frac{d(\zeta_n,\zeta_m)}{\mathfrak{p}+\mathfrak{q}}} - 1\right)\ell = \left(e^{-\frac{\left|\frac{1}{n}-\frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}} - 1\right)\ell.$$

As $n, m \to \infty$, we obtain $\mathscr{C}(\zeta_n, \zeta_m, \tau) \to \theta$.

Since

$$\lim_{n\to\infty}\inf_{m>n}\mathscr{A}(\zeta_n, \zeta_m, \tau) = \ell, \quad \lim_{n\to\infty}\sup_{m>n}\mathscr{B}(\zeta_n, \zeta_m, \tau) = \theta, \quad \lim_{n\to\infty}\sup_{m>n}\mathscr{C}(\zeta_n, \zeta_m, \tau) = \theta,$$

the sequence $\{\zeta_n\}$ is a Cauchy sequence in the given CVNMS.

Lemma 3.9. *Let $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ be a CVNMS. A sequence $\{\zeta_n\}$ in Ξ converges to $\zeta \in \Xi$ if and only if the following conditions hold for every $\tau \in \zeta_0$*

$$\lim_{n\to\infty}\mathscr{A}(\zeta_n, \zeta, \tau) = \ell, \quad \lim_{n\to\infty}\mathscr{B}(\zeta_n, \zeta, \tau) = \theta, \quad \lim_{n\to\infty}\mathscr{C}(\zeta_n, \zeta, \tau) = \theta.$$

Proof. ($\Rightarrow$) Assume that $\{\zeta_n\}$ in Ξ converges to $\zeta \in \Xi$. Then, for any $\rho \in \Upsilon_0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the following inequalities hold:

$$\mathscr{A}(\zeta_n, \zeta, \tau) \succ \ell - \rho, \quad \mathscr{B}(\zeta_n, \zeta, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \prec \rho.$$

Now, consider a complex number $\rho \in \Upsilon_0$ with $|\rho| < \epsilon$. Using the properties of CVNMS, we obtain

$$|\ell - \mathscr{A}(\zeta_n, \zeta, \tau)| < |\rho| < \epsilon, \quad |\mathscr{B}(\zeta_n, \zeta, \tau)| < |\rho| < \epsilon, \quad |\mathscr{C}(\zeta_n, \zeta, \tau)| < |\rho| < \epsilon, \quad \forall n > n_0.$$

Thus, as $n \to \infty$

$$\mathscr{A}(\zeta_n, \zeta, \tau) \to \ell, \quad \mathscr{B}(\zeta_n, \zeta, \tau) \to \theta, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \to \theta.$$

Hence, the convergence conditions are satisfied for every $\tau \in \zeta_0$.

($\Leftarrow$) Now, assume that

$$\lim_{n\to\infty} \mathscr{A}(\zeta_n, \zeta, \tau) = \ell, \quad \lim_{n\to\infty} \mathscr{B}(\zeta_n, \zeta, \tau) = \theta, \quad \lim_{n\to\infty} \mathscr{C}(\zeta_n, \zeta, \tau) = \theta, \quad \forall \tau \in \zeta_0.$$

Fix $\tau \in \zeta_0$ and take any $\rho \in \Upsilon_0$.

Since the limits exist as $n \to \infty$, there exists a positive real number $\epsilon > 0$ such that every complex number $z \in \mathbb{C}$ with $|z| < \epsilon$ satisfies $z \prec \rho$.

Since $\lim_{n\to\infty} \mathscr{A}(\zeta_n, \zeta, \tau) = \ell$, there exists $n_0 \in \mathbb{N}$ such that

$$|\ell - \mathscr{A}(\zeta_n, \zeta, \tau)| < \epsilon.$$

This implies

$$\ell - \mathscr{A}(\zeta_n, \zeta, \tau) \prec \rho, \quad \ell - \rho \prec \mathscr{A}(\zeta_n, \zeta, \tau).$$

Similarly, since $\mathscr{B}(\zeta_n, \zeta, \tau) \to \theta$ and $\mathscr{C}(\zeta_n, \zeta, \tau) \to \theta$, we can find $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\mathscr{B}(\zeta_n, \zeta, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta, \tau) \prec \rho.$$

Thus, $\{\zeta_n\}$ in Ξ converges to ζ. □

Lemma 3.10. *Let $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ be a CVNMS. A sequence $\{\zeta_n\}$ is Cauchy in Ξ if and only if, for each $\rho \in \Upsilon_0$ and $\tau \in \zeta_0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$*

$$\mathscr{A}(\zeta_n, \zeta_m, \tau) \succ \ell - \rho, \quad \mathscr{B}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta_m, \tau) \prec \rho.$$

Proof. ($\Rightarrow$) Assume that $\{\zeta_n\}$ is a Cauchy sequence. Fix $\tau \in \zeta_0$. For any $\rho \in \Upsilon_0$, there exists $n_0 \in \mathbb{N}$ such that

$$\ell - \inf_{m>n} \mathscr{A}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \sup_{m>n} \mathscr{B}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \sup_{m>n} \mathscr{C}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \forall n > n_0.$$

We distinguish three possible cases depending on the relationship between m, n and n_0.

Case 1: If $m > n > n_0$, we obtain $\ell - \rho \prec \inf_{m>n} \mathscr{A}(\zeta_n, \zeta_m, \tau) \prec \mathscr{A}(\zeta_n, \zeta_m, \tau)$ and similarly

$$\mathscr{B}(\zeta_n, \zeta_m, \tau) \prec \sup_{m>n} \mathscr{B}(\zeta_n, \zeta_m, \tau) \prec \rho, \qquad \mathscr{C}(\zeta_n, \zeta_m, \tau) \prec \sup_{m>n} \mathscr{C}(\zeta_n, \zeta_m, \tau) \prec \rho.$$

Case 2: If $m = n > n_0$, we have $\ell - \rho \prec \ell = \mathscr{A}(\zeta_n, \zeta_m, \tau)$, $\mathscr{B}(\zeta_n, \zeta_m, \tau) = \theta \prec \rho$ and $\mathscr{C}(\zeta_n, \zeta_m, \tau) = \theta \prec \rho$.

Case 3: If $n > m > n_0$, it results $\ell - \rho \prec \inf_{n>m} \mathscr{A}(\zeta_m, \zeta_n, \tau) = \mathscr{A}(\zeta_n, \zeta_m, \tau)$.

($\Leftarrow$) Suppose that for any $\rho \in \Upsilon_0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathscr{A}(\zeta_n, \zeta_m, \tau) \succ \ell - \rho, \quad \mathscr{B}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \mathscr{C}(\zeta_n, \zeta_m, \tau) \prec \rho, \quad \forall n, m > n_0.$$

Then, the limits hold:

$$\lim_{n\to\infty}\inf_{m>n}\mathscr{A}(\zeta_n,\zeta_m,\tau)=\ell,\quad \lim_{n\to\infty}\sup_{m>n}\mathscr{B}(\zeta_n,\zeta_m,\tau)=\theta,\quad \lim_{n\to\infty}\sup_{m>n}\mathscr{C}(\zeta_n,\zeta_m,\tau)=\theta.$$

Thus, $\{\zeta_n\}$ is Cauchy. $\square$

Theorem 3.11. *Let $(\Xi,\mathscr{A},\mathscr{B},\mathscr{C},*,\Diamond)$ be a complete CVNMS. Suppose that any sequence $\{\tau_n\}$ in ζ_0 satisfying $\lim_{n\to\infty}\tau_n=\infty$ also satisfies*

$$\lim_{n\to\infty}\inf_{\vartheta\in\Xi}\mathscr{A}(\zeta,\vartheta,\tau_n)=\ell,\quad \lim_{n\to\infty}\sup_{\vartheta\in\Xi}\mathscr{B}(\zeta,\vartheta,\tau_n)=\theta,\quad \lim_{n\to\infty}\sup_{\vartheta\in\Xi}\mathscr{C}(\zeta,\vartheta,\tau_n)=\theta,\quad \forall\zeta\in\Xi.$$

Assume that a self-mapping $f:\Xi\to\Xi$ satisfies the following condition

$$\mathscr{A}(f\zeta,f\vartheta,\eta)\succeq\mathscr{A}(\zeta,\vartheta,\tau),\quad \mathscr{B}(f\zeta,f\vartheta,\eta\tau)\preceq\mathscr{B}(\zeta,\vartheta,\tau),\quad \mathscr{C}(f\zeta,f\vartheta,\eta\tau)\preceq\mathscr{C}(\zeta,\vartheta,\tau),\quad (1)$$

for all $\zeta,\vartheta\in\Xi$ and $\tau\in\zeta_0$, where $\eta\in(0,1)$. Then f has a unique fixed point in Ξ.

Proof. Let ζ_0 be an arbitrarily chosen initial point in Ξ. Define the sequence $\{\zeta_n\}$ in Ξ as

$$\zeta_n=f\zeta_{n-1},\quad \forall n\in\mathbb{N}.$$

If there exists some $n_0\in\mathbb{N}$ such that $\zeta_{n_0}=\zeta_{n_0-1}$, then ζ_{n_0} is a fixed point of f. Otherwise, we analyze the sequence $\{\zeta_n\}$ and show that it is Cauchy.

For each $n\in\mathbb{N}$ and fixed $\tau\in\zeta_0$, define the following sets

$$\mathfrak{A}_n=\{\mathscr{A}(\zeta_n,\zeta_m,\tau):m>n\},\quad \mathfrak{B}_n=\{\mathscr{B}(\zeta_n,\zeta_m,\tau):m>n\},\quad \mathfrak{C}_n=\{\mathscr{C}(\zeta_n,\zeta_m,\tau):m>n\}.$$

Since $\theta\prec\mathscr{A}(\zeta_n,\zeta_m,\tau)\preceq\ell$ for all $n\in\mathbb{N}$ with $n<m$, it follows from Remark 2.1 that the infimum:

$$\mu_n=\inf\mathfrak{A}_n$$

exists for each $n\in\mathbb{N}$. Similarly, since $\theta\preceq\mathscr{B}(\zeta_n,\zeta_m,\tau)\prec\ell$ and $\theta\preceq\mathscr{C}(\zeta_n,\zeta_m,\tau)\prec\ell$ for all $n\in\mathbb{N}$ with $m>n$, it follows that the suprema

$$\nu_n=\sup\mathfrak{B}_n,\quad \gamma_n=\sup\mathfrak{C}_n$$

exist for each $n\in\mathbb{N}$.

Using condition (1), we obtain the recursive inequalities

$$\mathscr{A}(\zeta_{n+1},\zeta_{m+1},\tau)\succeq\mathscr{A}\left(\zeta_n,\zeta_m,\frac{\tau}{\eta}\right), \tag{2}$$

$$\mathscr{B}(\zeta_{n+1},\zeta_{m+1},\tau)\preceq\mathscr{B}\left(\zeta_n,\zeta_m,\frac{\tau}{\eta}\right), \tag{3}$$

$$\mathscr{C}(\zeta_{n+1},\zeta_{m+1},\tau)\preceq\mathscr{C}\left(\zeta_n,\zeta_m,\frac{\tau}{\eta}\right). \tag{4}$$

Since $\eta \in (0,1)$, Lemma 3.4 implies

$$\mathscr{A}(\zeta_{n+1}, \zeta_{m+1}, \tau) \succeq \mathscr{A}(\zeta_n, \zeta_m, \tau),$$

$$\mathscr{B}(\zeta_{n+1}, \zeta_{m+1}, \tau) \preceq \mathscr{B}(\zeta_n, \zeta_m, \tau),$$

$$\mathscr{C}(\zeta_{n+1}, \zeta_{m+1}, \tau) \preceq \mathscr{C}(\zeta_n, \zeta_m, \tau).$$

Thus, the sequences $\{\mu_n\}$, $\{\nu_n\}$, and $\{\gamma_n\}$ are monotonic in ζ. By Remark 2.1, there exist complex numbers $\mu, \nu, \gamma \in \zeta$ such that

$$\lim_{n\to\infty} \mu_n = \mu, \quad \lim_{n\to\infty} \nu_n = \nu, \quad \lim_{n\to\infty} \gamma_n = \gamma.$$

Applying the completeness hypothesis, we conclude

$$\mu = \ell, \quad \nu = \theta, \quad \gamma = \theta.$$

Thus, $\{\zeta_n\}$ is a Cauchy sequence, and by Lemma 3.9, it converges to some $\zeta \in \Xi$. That is

$$\lim_{n\to\infty} \mathscr{A}(\zeta_n, \zeta, \tau) = \ell, \quad \lim_{n\to\infty} \mathscr{B}(\zeta_n, \zeta, \tau) = \theta, \quad \lim_{n\to\infty} \mathscr{C}(\zeta_n, \zeta, \tau) = \theta. \tag{5}$$

Using (5) and Definition 3.1, we obtain

$$\mathscr{A}(\zeta, f\zeta, \tau) = \ell, \quad \mathscr{B}(\zeta, f\zeta, \tau) = \theta, \quad \mathscr{C}(\zeta, f\zeta, \tau) = \theta.$$

By conditions (3), (8), and (13) of Definition 3.1, $\zeta = f\zeta$, proving that ζ is a fixed point of f.

Suppose there exists another fixed point $\lambda \neq \zeta$. Iteratively applying (1), we obtain a contradiction, implying that $\mathscr{A}(\zeta, \lambda, \tau) = \ell$, $\mathscr{B}(\zeta, \lambda, \tau) = \theta$, and $\mathscr{C}(\zeta, \lambda, \tau) = \theta$. This forces $\zeta = \lambda$, establishing uniqueness. □

Remark 3.12. *In Theorem 3.11, the contractive condition* (1) *can be replaced by the following, while retaining a similar proof:*

$$\mathscr{A}(f\zeta, f\vartheta, \Omega(\tau)\tau) \succeq \mathscr{A}(\zeta, \vartheta, \tau),$$

$$\mathscr{B}(f\zeta, f\vartheta, \Omega(\tau)\tau) \preceq \mathscr{B}(\zeta, \vartheta, \tau),$$

$$\mathscr{C}(f\zeta, f\vartheta, \Omega(\tau)\tau) \preceq \mathscr{C}(\zeta, \vartheta, \tau),$$

for all $\zeta, \vartheta \in \Xi$ and $\tau \in \zeta_0$, where Ω is a function mapping ζ_0 into $(0,1)$.

Example 3.13. *Let (Ξ, d) be a metric space, where the metric is given by $d(\zeta, \vartheta) = |\zeta - \vartheta|$ for all $\zeta, \vartheta \in \Xi$. Define the complex-valued triangular norm (CVTN) $*$ and the complex-valued triangular conorm (CVTCN) $\Diamond$, for every $\tau_1 = (\mathfrak{p}_1, \mathfrak{q}_1)$ and $\tau_2 = (\mathfrak{p}_2, \mathfrak{q}_2)$ in Υ, as follows*

$$\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\}),$$

$$\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\}).$$

Now, define the complex neutrosophic sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ as follows:

$$\mathscr{A}(\zeta,\vartheta,\tau) = e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}}\ell,$$
$$\mathscr{B}(\zeta,\vartheta,\tau) = \left(1 - e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}}\right)\ell,$$
$$\mathscr{C}(\zeta,\vartheta,\tau) = \left(1 - e^{\frac{d(\zeta,\vartheta)}{\mathfrak{p}+\mathfrak{q}}}\right)\ell,$$

for all $\zeta,\vartheta \in \Xi$ and $\tau = (\mathfrak{p},\mathfrak{q}) \in \zeta_0$. Consequently, $(\Xi,\mathscr{A},\mathscr{B},\mathscr{C},*,\diamondsuit)$ forms a CVNMS induced by the metric d.

Let us consider a sequence $\{\tau_n\}$ in ζ_0 such that $\tau_n = (\mathfrak{p}_n,\mathfrak{q}_n)$ for each $n \in \mathbb{N}$, and assume

$$\lim_{n\to\infty} \tau_n = \infty.$$

This means that $\mathfrak{p}_n,\mathfrak{q}_n \to \infty$ as $n \to \infty$.

Now, we will evaluate the limits:

- *for $\mathscr{A}(\zeta,\vartheta,\tau_n) = e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}_n+\mathfrak{q}_n}}\ell$, as $\mathfrak{p}_n + \mathfrak{q}_n \to \infty$, we obtain*

$$\lim_{n\to\infty} \inf_{\vartheta\in\Xi} \mathscr{A}(\zeta,\vartheta,\tau_n) = \ell.$$

- *for $\mathscr{B}(\zeta,\vartheta,\tau_n) = (1 - e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}_n+\mathfrak{q}_n}})\ell$, as $\mathfrak{p}_n + \mathfrak{q}_n \to \infty$, we get*

$$\lim_{n\to\infty} \sup_{\vartheta\in\Xi} \mathscr{B}(\zeta,\vartheta,\tau_n) = \theta.$$

- *for $\mathscr{C}(\zeta,\vartheta,\tau_n) = (e^{-\frac{d(\zeta,\vartheta)}{\mathfrak{p}_n+\mathfrak{q}_n}} - 1)\ell$, as $\mathfrak{p}_n + \mathfrak{q}_n \to \infty$, we conclude*

$$\lim_{n\to\infty} \sup_{\vartheta\in\Xi} \mathscr{C}(\zeta,\vartheta,\tau_n) = \theta.$$

Thus, the conditions of Theorem 3.11 hold.

Now, define a self-mapping $\mathfrak{H} : \Xi \to \Xi$ by

$$\mathfrak{H}(\zeta) = \frac{\zeta}{8}, \quad \forall \zeta \in \Xi.$$

For $\eta \in [1/8, 1) \subset (0,1)$, the mapping $\mathfrak{H}$ satisfies condition (1):

$$\mathscr{A}(\mathfrak{H}\zeta,\mathfrak{H}\vartheta,\eta\tau) = e^{-\frac{d(\mathfrak{H}\zeta,\mathfrak{H}\vartheta)}{\eta(\mathfrak{p}+\mathfrak{q})}}\ell.$$

Since

$$d(\mathfrak{H}\zeta,\mathfrak{H}\vartheta) = \frac{d(\zeta,\vartheta)}{8},$$

we obtain

$$e^{-\frac{d(\zeta,\vartheta)}{8\eta(\mathfrak{p}+\mathfrak{q})}}\ell \succeq e^{-\frac{d(\zeta,\vartheta)}{(\mathfrak{p}+\mathfrak{q})}}\ell = \mathscr{A}(\zeta,\vartheta,\tau).$$

Similarly, for $\mathscr{B}$ and $\mathscr{C}$

$$\mathscr{B}(\mathfrak{H}\zeta,\mathfrak{H}\vartheta,\eta\tau) \preceq \mathscr{B}(\zeta,\vartheta,\tau),$$
$$\mathscr{C}(\mathfrak{H}\zeta,\mathfrak{H}\vartheta,\eta\tau) \preceq \mathscr{C}(\zeta,\vartheta,\tau).$$

Thus, all conditions of Theorem 3.11 are satisfied, and 0 is the unique fixed point of $\mathfrak{H}$.

Example 3.14. *This example illustrates that the assumptions of Theorem 3.11 are essential and cannot be omitted.*

Let $\Xi = [0, \infty)$. *Define the t-norm* $*$ *and t-conorm* $\Diamond$ *as follows:*

$$\tau_1 * \tau_2 = (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2), \quad \tau_1 \Diamond \tau_2 = (\mathfrak{p}_1 + \mathfrak{p}_2, \mathfrak{q}_1 + \mathfrak{q}_2) - (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2)$$

for any $\tau_i = (\mathfrak{p}_i, \mathfrak{q}_i) \in \Upsilon$ *with* $i = 1, 2$.

Define the complex neutrosophic functions $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{C}$ *as follows:*

$$\mathscr{A}(\zeta, \vartheta, \tau) = \begin{cases} \frac{\zeta}{\vartheta}\ell & \text{if } \zeta < \vartheta \\ \frac{\vartheta}{\zeta}\ell & \text{if } \vartheta < \zeta \end{cases}$$

$$\mathscr{B}(\zeta, \vartheta, \tau) = \begin{cases} (1 - \frac{\zeta}{\vartheta})\ell & \text{if } \zeta < \vartheta \\ (1 - \frac{\vartheta}{\zeta})\ell & \text{if } \vartheta < \zeta \end{cases}$$

$$\mathscr{C}(\zeta, \vartheta, \tau) = \begin{cases} \left(\frac{\vartheta - \zeta}{\zeta}\right)\ell & \text{if } \zeta < \vartheta \\ \left(\frac{\zeta - \vartheta}{\vartheta}\right)\ell & \text{if } \vartheta < \zeta \end{cases}.$$

It can be verified that $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ *forms a complete CVNMS.*

Now, consider the self-mapping $\mathfrak{H} : \Xi \to \Xi$ *given by*

$$\mathfrak{H}(\zeta) = \zeta + 1, \quad \forall \zeta \in \Xi.$$

Define the sequence $\{\tau_n\}$ *as*

$$\tau_n = (n^2 + 6, n^2 + 6), \quad n \in \mathbb{N}.$$

Clearly, $\lim_{n\to\infty} \tau_n = \infty$. *For any* $\zeta \in \Xi$ *and* $\vartheta \neq \zeta$, *we obtain*

$$\inf_{\vartheta \in \Xi} \mathscr{A}(\zeta, \vartheta, \tau_n) = \theta,$$

$$\sup_{\vartheta \in \Xi} \mathscr{B}(\zeta, \vartheta, \tau_n) = \ell,$$

$$\sup_{\vartheta \in \Xi} \mathscr{C}(\zeta, \vartheta, \tau_n) = \ell.$$

Thus, we conclude

$$\lim_{n\to\infty} \inf_{\vartheta \in \Xi} \mathscr{A}(\zeta, \vartheta, \tau_n) = \theta \neq \ell,$$

$$\lim_{n\to\infty} \sup_{\vartheta \in \Xi} \mathscr{B}(\zeta, \vartheta, \tau_n) = \ell \neq \theta,$$

$$\lim_{n\to\infty} \sup_{\vartheta \in \Xi} \mathscr{C}(\zeta, \vartheta, \tau_n) = \ell \neq \theta.$$

Now, for any $\eta \in (0,1)$, $\zeta, \vartheta \in \Xi$, *and* $\tau \in \zeta_0$, *we verify*

$$\mathscr{A}(\mathfrak{H}\zeta, \mathfrak{H}\vartheta, \eta\tau) = \begin{cases} \frac{\zeta+1}{\vartheta+1}\ell & \text{if } \zeta < \vartheta, \\ \frac{\vartheta+1}{\zeta+1}\ell & \text{if } \vartheta < \zeta. \end{cases}$$

$$\succeq \begin{cases} \frac{\zeta}{\vartheta}\ell & \text{if } \zeta < \vartheta, \\ \frac{\vartheta}{\zeta}\ell & \text{if } \vartheta < \zeta. \end{cases} = \mathscr{A}(\zeta, \vartheta, \tau).$$

Similarly, we obtain

$$\mathscr{B}(\mathfrak{H}\zeta, \mathfrak{H}\vartheta, \eta\tau) \preceq \mathscr{B}(\zeta, \vartheta, \tau),$$

$$\mathscr{C}(\mathfrak{H}\zeta, \mathfrak{H}\vartheta, \eta\tau) \preceq \mathscr{C}(\zeta, \vartheta, \tau).$$

Thus, while $\mathfrak{H}$ *satisfies condition* (1), *it does not have a fixed point in* Ξ.

Theorem 3.15. *Let* $(\Xi, \mathscr{A}, \mathscr{B}, \mathscr{C}, *, \Diamond)$ *be a complete CVNMS. Assume that a self-mapping* $f : \Xi \to \Xi$ *satisfies*

$$\mathscr{A}(f\zeta, f\vartheta, \tau) \succeq \psi(\mathscr{A}(\zeta, \vartheta, \tau)), \quad \mathscr{B}(f\zeta, f\vartheta, \tau) \preceq \varphi(\mathscr{B}(\zeta, \vartheta, \tau)), \quad \mathscr{C}(f\zeta, f\vartheta, \tau) \preceq \varphi(\mathscr{C}(\zeta, \vartheta, \tau)), \tag{6}$$

for all $\zeta, \vartheta \in \Xi$ *and* $\tau \in \zeta_0$, *where* $\psi \in \Psi$ *and* $\varphi \in \Phi$. *Then* f *has a unique fixed point in* Ξ.

Proof. Define a sequence $\{\zeta_n\}$ by setting $\zeta_n = f\zeta_{n-1}$ for all $n \in \mathbb{N}$. If $\zeta_{n_0} = \zeta_{n_0-1}$ for some n_0, then ζ_{n_0} is a fixed point.

To prove that $\{\zeta_n\}$ is Cauchy, define

$$\mathfrak{A}_n = \{\mathscr{A}(\zeta_n, \zeta_m, \tau) : m > n\}, \quad \mathfrak{B}_n = \{\mathscr{B}(\zeta_n, \zeta_m, \tau) : m > n\}, \quad \mathfrak{C}_n = \{\mathscr{C}(\zeta_n, \zeta_m, \tau) : m > n\}.$$

Using condition (6) iteratively, we show

$$\mu_n \to \ell, \quad \nu_n \to \theta, \quad \gamma_n \to \theta.$$

By completeness, there exists $\zeta \in \Xi$ such that

$$\lim_{n\to\infty} \mathscr{A}(\zeta_n, \zeta, \tau) = \ell, \quad \lim_{n\to\infty} \mathscr{B}(\zeta_n, \zeta, \tau) = \theta, \quad \lim_{n\to\infty} \mathscr{C}(\zeta_n, \zeta, \tau) = \theta.$$

This implies ζ is a unique fixed point. □

4. Application to Fredholm integral equations of second kind

To establish the existence of a unique solution for Fredholm integral equations, this section demonstrates how Theorem 3.11 can be applied effectively. Specifically, we consider the function space $\mathfrak{C}([0,1], \mathbb{R})$, which consists of all real-valued continuous functions defined on $[0,1]$.

Consider the nonlinear Fredholm integral equation of the second kind:

$$\psi(u) = \mathfrak{T}(t) + \gamma \int_0^1 \zeta(u,v)\chi(v,\psi(v))dv, \tag{7}$$

where:

- $\mathfrak{T}$ is a continuous real-valued function on $[0,1]$,
- $\zeta(u,v)$ is the kernel of $\mathfrak{T}$,
- $\chi(v,\psi(v))$ is a continuous nonlinear function defined on $[0,1]\times\mathbb{R}$,
- $\psi(u)$ is the unknown function to be determined.

Theorem 4.1. *Let $\Xi = \mathfrak{C}([0,1],\mathbb{R})$. Suppose the following conditions hold:*

(1) *There exists a constant $\mu \in (0,1)$ such that for all $\psi,\varphi \in \Xi$ and each $v \in [0,1]$, the inequality*

$$|\chi(v,\psi(v)) - \chi(v,\varphi(v))| \le \mu|\psi(v) - \varphi(v)|$$

holds.

(2) *The integral $\int_0^1 \zeta(u,v)dv$ is bounded, i.e., there exists a constant ν such that*

$$\int_0^1 \zeta(u,v)dv \le \nu.$$

(3) *The parameters satisfy the inequality $\gamma^2\nu^2\mu^2 \le k < 1$.*

Then, the integral equation (7) has a unique solution in Ξ.

Proof. Define the operator $\Theta : \Xi \to \Xi$ by

$$\Theta\psi(u) = \mathfrak{T}(u) + \gamma \int_0^1 \zeta(u,v)\chi(v,\psi(v))dv.$$

For each $\tau_1 = (\mathfrak{p}_1,\mathfrak{q}_1)$, $\tau_2 = (\mathfrak{p}_2,\mathfrak{q}_2)$ in Υ, we define the t-norm $*$ and t-conorm $\Diamond$ as

$$\tau_1 * \tau_2 = (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2), \quad \tau_1\Diamond\tau_2 = (\mathfrak{p}_1+\mathfrak{p}_2, \mathfrak{q}_1+\mathfrak{q}_2) - (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2).$$

Define the functions $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ as follows:

$$\mathscr{A}(\psi,\varphi,\tau) = \frac{\mathfrak{p}+\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}+|\psi(u)-\varphi(u)|^2}\ell,$$

$$\mathscr{B}(\psi,\varphi,\tau) = \frac{|\psi(u)-\varphi(u)|^2}{\mathfrak{p}+\mathfrak{q}+|\psi(u)-\varphi(u)|^2}\ell,$$

$$\mathscr{C}(\psi,\varphi,\tau) = \frac{|\psi(u)-\varphi(u)|^2}{\mathfrak{p}+\mathfrak{q}}\ell,$$

for all $\psi,\varphi \in \Xi$, $\tau = (\mathfrak{p},\mathfrak{q}) > 0$, and $u \in [0,1]$.

It can be readily verified that $(\Xi,\mathscr{A},\mathscr{B},\mathscr{C},*,\Diamond)$ forms a complete CVNMS.

For all $\psi, \varphi \in \Xi$ and $u \in [0,1]$, we estimate

$$\begin{aligned}|\Theta\psi(u) - \Theta\varphi(u)|^2 &= \left|\gamma \int_0^1 \zeta(u,v)\big(\chi(v,\psi(v)) - \chi(v,\varphi(v))\big)dv\right|^2 \\ &\leq \gamma^2 \left(\int_0^1 \zeta(u,v)dv\right)^2 \sup_{v\in[0,1]} |\chi(v,\psi(v)) - \chi(v,\varphi(v))|^2 \\ &\leq \gamma^2\nu^2\mu^2|\psi(v) - \varphi(v)|^2 \\ &\leq \eta|\psi(v) - \varphi(v)|^2.\end{aligned}$$

Thus, for all $\psi, \varphi \in \Xi$ and $\tau \in \zeta_0$, we obtain

$$\begin{aligned}\mathscr{A}(\Theta\psi, \Theta\varphi, \eta\tau) &= \frac{\eta(\mathfrak{p}+\mathfrak{q})}{\eta(\mathfrak{p}+\mathfrak{q}) + |\Theta\psi - \Theta\varphi|^2}\ell \\ &\succeq \frac{\mathfrak{p}+\mathfrak{q}}{\mathfrak{p}+\mathfrak{q}+|\psi-\varphi|^2}\ell = \mathscr{A}(\psi,\varphi,\tau).\end{aligned}$$

Similarly, we verify that

$$\begin{aligned}\mathscr{B}(\Theta\psi, \Theta\varphi, \eta\tau) &\preceq \mathscr{B}(\psi,\varphi,\tau), \\ \mathscr{C}(\Theta\psi, \Theta\varphi, \eta\tau) &\preceq \mathscr{C}(\psi,\varphi,\tau).\end{aligned}$$

By Theorem 3.11, the operator Θ has a unique fixed point in Ξ, meaning there exists a unique function $\psi \in \mathfrak{C}([0,1],\mathbb{R})$ satisfying equation (7). $\square$

5. Conclusion

This work introduces and explores complex-valued neutrosophic metric spaces (CVNMS), providing a robust extension of conventional metric space frameworks. By establishing fixed-point existence and uniqueness under various contractive conditions, our results contribute significantly to the theoretical development of CVNMS.

Furthermore, we demonstrated the applicability of our results by proving the existence of a unique solution to a class of nonlinear Fredholm integral equations of the second kind. This highlights the potential of CVNMS and neutrosophic theory in addressing complex problems in mathematical analysis, fostering further research and promoting interdisciplinary applications across related fields.

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