NSS

Neutrosophic Sets and Systems, Vol. 86, 2025

University of New Mexico

Optimal neutrosophic framework for population mean estimation under simple random sampling

Anoop Kumar[1], Priya[*,1] and Vrijesh Tripathi[2]

[1]Department of Statistics, Central University of Haryana, Mahendergarh, Haryana, India, 123031.

[2]Depatment of Mathematics and Statistics, The University of The West Indies St. Augustine, Trinidad and Tobago.

[*]Correspondence: priyav2698@gmail.com

Abstract. When conducting survey sampling, precise population mean estimation is essential, particularly when additional indeterminate data is available. The intrinsic ambiguity and uncertainty in the study and auxiliary variables are handled by using neutrosophic logic, which consists of truth, indeterminacy, and falsehood. In this paper, we extend the classical estimation techniques by incorporating bivariate auxiliary information within the neutrosophic framework, offering an optimal neutrosophic framework for population mean estimation under simple random sampling (SRS). Through simulation experiments and real-life datasets, the effectiveness of the proposed optimal neutrosophic framework is evaluated and compared with the adapted neutrosophic frameworks. The outcomes demonstrate that the suggested optimal neutrosophic framework demonstrates reduced mean square error (MSE) and enhanced efficiency in comparison to the adapted neutrosophic frameworks.

Keywords: Mean square error; Neutrosophic framework; Bivariate auxiliary information; Efficiency.

1. Introduction

In survey sampling, auxiliary data is essential since it greatly improves the accuracy and precision of the population mean estimation. By reducing the variance or MSE of the estimator, auxiliary variables associated with the study variable help to produce more accurate and efficient results. The estimation methods such as ratio, regression, product, logarithmic, exponential, and their modified versions use supplementary data to reduce the mean square error (MSE) and elevate the overall effectiveness of the estimation process. Many authors suggested different estimation procedures for estimating the population parameter based on several auxiliary information in survey sampling. [1] estimated the population mean using two auxiliary variables. Using two auxiliary variables in SRS, [2] created the best regression estimator for the population mean. [3] used two auxiliary variables to estimate the finite population mean in SRS and stratified random sampling. Using multiple auxiliary information, [4] investigated some new improved classes of estimators. A simulation analysis of the robust regression-ratio-type estimators of the mean using two auxiliary variables was presented by [5]. Using two auxiliary variables, [6] evaluated the effectiveness of the general class of ratio-exponential-log type estimators. [7] used multi-auxiliary information under ranked set sampling to construct a new class of efficient logarithmic estimators. [8] used two auxiliary variables under two-stage sampling to estimate the finite population mean by examining an improved generalized class of estimators. [9] introduced a few enhanced categories of estimators using bivariate auxiliary information under stratified sampling. With two auxiliary variables, [10] created an improved population mean estimation with probability proportionate to size sampling. Using multi-auxiliary data, [11] proposed a few optimal classes of estimators for the population mean.

These classical estimation methods presuppose precise and complete data. In contrast to this, indeterminate data is ambiguous, incomplete, or unsure; that frequently occurs in real-life surveys because of inadequate data gathering methods or inconsistent responses from respondents. Although being difficult, but indeterminate data can be managed by applying sophisticated techniques like fuzzy logic or neutrosophic theory, which include uncertainty into the analysis to generate more adaptable and realistic estimations. [12] developed the mathematical modeling and fuzzy availability analysis of stainless steel utensil manufacturing unit in steady state. [13] investigated shadowed type 2 fuzzy-based Markov model to predict shortest path with optimized waiting time. For handling indeterminate data, the survey sampling literature offers a few estimation techniques employing single auxiliary information. [14] presented the neutrosophic ratio-type estimators for population mean under SRS. [15] developed a generalized neutrosophic sampling strategy for enhanced population mean estimation. With the use of auxiliary data, [16] introduced the neutrosophic factor-type exponential estimators for improved population mean estimation. A neutrosophic robust ratio-type estimator was proposed by [17] for the estimation of the finite population mean. To estimate the neutrosophic finite median, [18] recommended employing robust parameters of the auxiliary variable. [19] determined the population mean using neutrosophic exponential-type estimator. [20] suggested the generalized robust-type neutrosophic ratio estimators of pharmaceutical daily stock prices. [21] proposed the ratio-type estimator for estimating the neutrosophic population mean in SRS under intuitionistic fuzzy cost function. [22] developed the neutrosophic regression cum ratio estimators for the population mean. [23] suggested the neutrosophic estimators in two-phase survey sampling. [24] constructed an almost unbiased estimator for population mean using neutrosophic theory. [25] computed the separate ratio and regression estimator under neutrosophic stratified sampling and provided an application of the methods using climate data. Under neutrosophic ranked set sampling (NRSS), [26] suggested the generalized estimator for computing the population mean. Later on, [27] suggested the generalized regressed exponential estimator for estimating the mean. Recently, [28] investigated the NRSS scheme for estimating the imprecise population mean. However, for estimating the imprecise population mean, some new modifications of ranked set sampling were suggested and demonstrated using demographic data by [29]. [30] suggested the new comprehensive imprecise mean estimation method using regression-cum-exponential type estimator. [31] estimated the population mean using the neutrosophic exponential estimators with real data application. [32] designed the neutrosophic mean estimators in the presence of extreme indeterminate observations.

In survey sampling, it is well-known that employing auxiliary information enhances the efficiency of the estimator. Often, data on multiple auxiliary variables are available, providing additional opportunities for improved estimation. Recently, [33] combined the two auxiliary variables for efficiently estimating the finite population mean under neutrosophic structure. This article introduces the following:

(1) Develops the methodology and notations consisting of bivariate auxiliary information under neutrosophic setup.
(2) Adapts some fundamental neutrosophic estimators under SRS based on bivariate auxiliary information. Additionally, considers the existing neutrosophic estimators based on bivariate auxiliary information under SRS.
(3) Proposes an optimal neutrosophic framework for the population mean estimation employing bivariate auxiliary information under SRS. This approach addresses challenges associated with vague, indeterminate, and uncertain data.
(4) Discusses the efficiency of the proposed optimal neutrosophic estimators using simulation study and a real-life application based on neutrosophic data.

1.1. *Notations under neutrosophic setup*

The statistical literature contains different kinds of neutrosophic data, including quantitative neutrosophic data which is based on a number existing in an unknown interval $[p,\ q]$. This unknown interval $[p,\ q]$ based on neutrosophic numbers can be expressed in different forms. In this paper, we have taken the neutrosophic interval values as $W_N = W_L + W_U I_N$ such that $I_N \in [I_L,\ I_U]$. This shows that the notations utilized for neutrosophic data are in an interval form $W_N \in [p,\ q]$, where p and q represent the lower and upper values of the neutrosophic data, respectively. For more deep study about neutrosophic notations, the reader may see [34].

Let a finite population (U=U_1, U_2,...,U_N) be based on N identifiable units from which a neutrosophic sample of size $n_N \in [n_L,\ n_U]$ is randomly selected. Let $y_N(i)$ be the i^{th} unit observation of the sample for the neutrosophic study variable y_N expressed as $y_N(i) \in [y_L,\ y_U]$, whereas corresponding to the neutrosophic study variable, data on neutrosophic auxiliary variables x_N and z_N are expressed as $x_N(i) \in [x_L,\ x_U]$ and $z_N(i) \in [z_L,\ z_U]$, respectively. Let $\bar{y}_N(i) \in [\bar{y}_L,\ \bar{y}_U]$, $\bar{x}_N(i) \in [\bar{x}_L,\ \bar{x}_U]$, and $\bar{z}_N(i) \in [\bar{z}_L,\ \bar{z}_U]$ be the neutrosophic sample means corresponding to the neutrosophic population means $\bar{Y}_N \in [\ \bar{Y}_L,\ \bar{Y}_U]$, $\bar{X}_N \in [\bar{X}_L,\ \bar{X}_U]$, and $\bar{Z}_N \in [\bar{Z}_L,\ \bar{Z}_U]$ for the neutrosophic study variable y_N and auxiliary variables x_N and z_N, respectively. The neutrosophic variation coefficients of variables y_N, x_N, and z_N are denoted as $C_{y_N} \in [C_{y_L},\ C_{y_U}]$, $C_{x_N} \in [C_{x_L},\ C_{x_U}]$, and $C_{z_N} \in [C_{z_L},\ C_{z_U}]$, respectively. The neutrosophic correlation coefficients between the neutrosophic variables $(x_N,\ y_N)$, $(y_N,\ z_N)$, and $(x_N,\ z_N)$ are denoted by $\rho_{xy_N} \in [\rho_{xy_L},\ \rho_{xy_U}]$, $\rho_{yz_N} \in [\rho_{yz_L},\ \rho_{yz_U}]$, and $\rho_{xz_N} \in [\rho_{xz_L},\ \rho_{xz_U}]$, respectively. The neutrosophic skewness and kurtosis coefficients of x_N are denoted by $\beta_1(x_N) \in [\beta_1(x_L),\ \beta_1(x_U)]$ and $\beta_2(x_N) \in [\beta_2(x_L),\ \beta_2(x_U)]$, respectively, while the neutrosophic skewness and kurtosis coefficients of z_N are denoted by $\beta_1(z_N) \in [\beta_1(z_L),\ \beta_1(z_U)]$ and $\beta_2(z_N) \in [\beta_2(z_L),\ \beta_2(z_U)]$, respectively.

To obtain neutrosophic $Bias_N \in [Bias_L,\ Bias_U]$ and neutrosophic $MSE_N \in [MSE_L,\ MSE_U]$ of the neutrosophic estimators, we take neutrosophic errors $\epsilon_{0N} = (\bar{y}_N - \bar{Y}_N)/\bar{Y}_N$, $\epsilon_{1N} = (\bar{x}_N - \bar{X}_N)/\bar{X}_N$, and $\epsilon_{2N} = (\bar{z}_N - \bar{Z}_N)/\bar{Z}_N$ such that $\epsilon_{0N} \in [\epsilon_{0L},\ \epsilon_{0U}]$, $\epsilon_{1N} \in [\epsilon_{1L},\ \epsilon_{1U}]$, and $\epsilon_{2N} \in [\epsilon_{2L},\ \epsilon_{2U}]$ together with their expectations given as:

$$\left.\begin{aligned} E(\epsilon_{0N}) = E(\epsilon_{1N}) = E(\epsilon_{2N}) &= 0,\\ E(\epsilon_{0N}^2) &= \varphi_N C_{y_N}^2,\\ E(\epsilon_{1N}^2) &= \varphi_N C_{x_N}^2,\\ E(\epsilon_{2N}^2) &= \varphi_N C_{z_N}^2,\\ E(\epsilon_{0N}\epsilon_{1N}) &= \varphi_N \rho_{xy_N} C_{x_N} C_{y_N},\\ E(\epsilon_{0N}\epsilon_{2N}) &= \varphi_N \rho_{yz_N} C_{y_N} C_{z_N},\\ \text{and } E(\epsilon_{1N}\epsilon_{2N}) &= \varphi_N \rho_{xz_N} C_{x_N} C_{z_N}, \end{aligned}\right\} \quad (1)$$

where $\varphi_N = 1/n_N$, $n_N \in [n_L,\ n_U]$, $C_{y_N} = \sigma_{y_N}/\bar{Y}_N$, $C_{x_N} = \sigma_{x_N}/\bar{X}_N$, $C_{z_N} = \sigma_{z_N}/\bar{Z}_N$, $\sigma_{x_N}^2 \in [\sigma_{x_L}^2, \sigma_{x_U}^2]$, $\sigma_{y_N}^2 \in [\sigma_{y_L}^2, \sigma_{y_U}^2]$, and $\sigma_{xy_N} \in [\sigma_{xy_L}, \sigma_{xy_U}]$.

The next section provides some adapted neutrosophic estimators based on bivarite auxiliary information with their characteristics. The proposed optimal neutrosophic estimators, their characteristics, and the conditions under which they dominate the adapted neutrosophic estimators are established in Section 3. A simulation study based on a hypothetically drawn normal population is presented in Section 4. In Section 5, the seasonal temperature data is used to illustrate the application of the proposed and adapted neutrosophic estimators. The article ends with the conclusions in Section 6.

2. Adapted neutrosophic estimators

This section adapts some well-known neutrosophic estimators for the population mean estimation under SRS employing bivariate auxiliary information.

If the auxiliary information is not available, then the neutrosophic mean per unit estimator is the obvious choice for the neutrosophic population mean $\bar{Y}_N$ given by

$$t_m = \bar{y}_N.$$

Theorem 2.1. *The variance of the estimator t_m is given by*

$$V(t_m) = \varphi_N \bar{Y}_N^2 C_{y_N}^2.$$

Proof. Consider the estimator t_m given as

$$t_m = \bar{y}_N.$$

Utilizing notations of (1), we rewrite t_m as follows:

$$t_m = \bar{Y}_N(1 + \epsilon_{0N}),$$
$$t_m - \bar{Y}_N = \bar{Y}_N \epsilon_{0N}. \tag{2}$$

Taking expectation both side to (2), we get

$$Bias(t_m) = 0.$$

This shows that the neutrosophic mean per unit estimator t_m is unbiased.

Again, squaring and taking expectation both sides of (2), we get

$$V(t_m) = \varphi_N \bar{Y}_N^2 C_{y_N}^2.$$

□

The neutrosophic generalized ratio estimator of population mean $\bar{Y}_N$ under SRS employing bivariate auxiliary information is given by

$$t_{gr} = \bar{y}_N \left(\frac{a_N \bar{X}_N + b_N}{a_N \bar{x}_N + b_N} \right) \left(\frac{c_N \bar{Z}_N + d_N}{c_N \bar{z}_N + d_N} \right),$$

where a_N, b_N, c_N, and d_N are either real values or known parameters of neutrosophic auxiliary variables x_N and z_N, namely, neutrosophic mean, neutrosophic standard deviation, neutrosophic correlation coefficient, neutrosophic variation coefficient, neutrosophic skewness coefficient, neutrosophic kurtosis coefficient, etc. A few members of the neutrosophic generalized ratio estimator t_{gr} based on bivariate auxiliary information are compiled in Table 1 for ready reference.

Table 1. Some sub-classes of the neutrosophic generalized ratio estimator t_{gr} based on bivariate auxiliary information

a_N	b_N	c_N	d_N	Some sub-classes of the estimator t_{gr}
1	0	1	0	$t^1_{gr}=\bar{y}_N\left(\frac{\bar{X}_N}{\bar{x}_N}\right)\left(\frac{\bar{Z}_N}{\bar{z}_N}\right)$
1	C_{x_N}	1	C_{z_N}	$t^2_{gr}=\bar{y}_N\left(\frac{\bar{X}_N+C_{x_N}}{\bar{x}_N+C_{x_N}}\right)\left(\frac{\bar{Z}_N+C_{z_N}}{\bar{z}_N+C_{z_N}}\right)$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$t^3_{gr}=\bar{y}_N\left(\frac{C_{x_N}\bar{X}_N+\beta_2(x_N)}{C_{x_N}\bar{x}_N+\beta_2(x_N)}\right)\left(\frac{C_{z_N}\bar{Z}_N+\beta_2(z_N)}{C_{z_N}\bar{z}_N+\beta_2(z_N)}\right)$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$t^4_{gr}=\bar{y}_N\left(\frac{\beta_2(x_N)\bar{X}_N+C_{x_N}}{\beta_2(x_N)\bar{x}_N+C_{x_N}}\right)\left(\frac{\beta_2(z_N)\bar{Z}_N+C_{z_N}}{\beta_2(z_N)\bar{z}_N+C_{z_N}}\right)$
1	ρ_{xy_N}	1	ρ_{yz_N}	$t^5_{gr}=\bar{y}_N\left(\frac{\bar{X}_N+\rho_{xy_N}}{\bar{x}_N+\rho_{xy_N}}\right)\left(\frac{\bar{Z}_N+\rho_{yz_N}}{\bar{z}_N+\rho_{yz_N}}\right)$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$t^6_{gr}=\bar{y}_N\left(\frac{\bar{X}_N+\beta_2(x_N)}{\bar{x}_N+\beta_2(x_N)}\right)\left(\frac{\bar{Z}_N+\beta_2(z_N)}{\bar{z}_N+\beta_2(x_N)}\right)$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$t^7_{gr}=\bar{y}_N\left(\frac{\bar{X}_N+\beta_1(x_N)}{\bar{x}_N+\beta_1(x_N)}\right)\left(\frac{\bar{Z}_N+\beta_1(z_N)}{\bar{z}_N+\beta_1(z_N)}\right)$

Theorem 2.2. *The bias and MSE of the neutrosophic generalized ratio estimator t_{gr} using bivariate auxiliary information are presented below as*

$$Bias(t_{gr}) = \bar{Y}_N\varphi_N\left(\begin{array}{l}\Psi_N^2C_{x_N}^2+\Pi_N^2C_{z_N}^2-\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-\Pi_N\rho_{yz_N}C_{y_N}C_{z_N}\\ +\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N}\end{array}\right),$$

$$MSE(t_{gr}) = \bar{Y}_N^2\varphi_N\left(\begin{array}{l}C_{y_N}^2+\Psi_N^2C_{x_N}^2+\Pi_N^2C_{z_N}^2-2\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-2\Pi_N\rho_{yz_N}C_{z_N}C_{y_N}\\ +2\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N}\end{array}\right),$$

where

$$\Psi_N=\left(\frac{a_N\bar{X}_N}{a_N\bar{X}_N+b_N}\right) \text{ and } \Pi_N=\left(\frac{c_N\bar{Z}_N}{c_N\bar{Z}_N+d_N}\right).$$

Proof. Consider the estimator t_{gr} as

$$t_{gr}=\bar{y}_N\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{x}_N+b_N}\right)\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{z}_N+d_N}\right).$$

Utilizing notations given in (1), we get

$$\begin{aligned} t_{gr} =&\bar{Y}_N(1+\epsilon_{0N})\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{X}_N(1+\epsilon_{1N})+b_N}\right)\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{Z}_N(1+\epsilon_{2N})+d_N}\right),\\ =&\bar{Y}_N(1+\epsilon_{0N})\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{X}_N+a_N\bar{X}_N\epsilon_{1N}+b_N}\right)\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{Z}_N+c_N\bar{Z}_N\epsilon_{2N}+d_N}\right),\\ =&\bar{Y}_N(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-1}(1+\Pi_N\epsilon_{2N})^{-1}.\end{aligned}$$

Using Taylor series expansion, multiplying right hand side terms and excluding error terms having power more than two, we get

$$t_{gr} = \bar{Y}_N\left(\begin{array}{l}1+\epsilon_{0N}-\Psi_N\epsilon_{1N}-\Pi_N\epsilon_{2N}+\Psi_N^2\epsilon_{1N}^2+\Pi_N^2\epsilon_{2N}^2-\Psi_N\epsilon_{0N}\epsilon_{1N}-\Pi_N\epsilon_{0N}\epsilon_{2N}\\ +\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N}\end{array}\right).$$

Subtracting $\bar{Y}_N$ both sides of the above expression, we get

$$t_{gr} - \bar{Y}_N = \bar{Y}_N \left(\begin{array}{l} \epsilon_{0N} - \Psi_N \epsilon_{1N} - \Pi_N \epsilon_{2N} + \Psi_N^2 \epsilon_{1N}^2 + \Pi_N^2 \epsilon_{2N}^2 - \Psi_N \epsilon_{0N} \epsilon_{1N} - \Pi_N \epsilon_{0N} \epsilon_{2N} \\ + \Psi_N \Pi_N \epsilon_{1N} \epsilon_{2N} \end{array} \right). \tag{3}$$

Taking expectation both sides of (3), we get

$$Bias(t_{gr}) = \bar{Y}_N \varphi_N \left(\begin{array}{l} \Psi_N^2 C_{x_N}^2 + \Pi_N^2 C_{z_N}^2 - \Psi_N \rho_{xy_N} C_{x_N} C_{y_N} - \Pi_N \rho_{yz_N} C_{y_N} C_{z_N} \\ + \Psi_N \Pi_N \rho_{xz_N} C_{x_N} C_{z_N} \end{array} \right).$$

Squaring and taking expectation both sides of (3), we get

$$MSE(t_{gr}) = \bar{Y}_N^2 \varphi_N \left(\begin{array}{l} C_{y_N}^2 + \Psi_N^2 C_{x_N}^2 + \Pi_N^2 C_{z_N}^2 - 2\Psi_N \rho_{xy_N} C_{x_N} C_{y_N} - 2\Pi_N \rho_{yz_N} C_{y_N} C_{z_N} \\ + 2\Psi_N \Pi_N \rho_{xz_N} C_{x_N} C_{z_N} \end{array} \right).$$

□

The neutrosophic generalized power ratio estimator t_{pr} for population mean under SRS employing bivariate auxiliary information is given by

$$t_{pr} = \bar{y}_N \left(\frac{a_N \bar{X}_N + b_N}{a_N \bar{x}_N + b_N} \right)^{\theta_{1N}} \left(\frac{c_N \bar{Z}_N + d_N}{c_N \bar{z}_N + d_N} \right)^{\theta_{2N}},$$

where θ_{1N} and θ_{2N} are suitably chosen scalars. Some sub-classes of the estimator t_{pr} are given in Table 2 for ready reference.

Table 2. Some sub-classes of the neutrosophic generalized power ratio estimator t_{pr} based on bivariate auxiliary information

a_N	b_N	c_N	d_N	Some sub-classes of the estimator t_{pr}
1	0	1	0	$t_{pr}^1 = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right)^{\theta_{1N}} \left(\frac{\bar{Z}_N}{\bar{z}_N} \right)^{\theta_{2N}}$
1	C_{x_N}	1	C_{z_N}	$t_{pr}^2 = \bar{y}_N \left(\frac{\bar{X}_N + C_{x_N}}{\bar{x}_N + C_{x_N}} \right)^{\theta_{1N}} \left(\frac{\bar{Z}_N + C_{z_N}}{\bar{z}_N + C_{z_N}} \right)^{\theta_{2N}}$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$t_{pr}^3 = \bar{y}_N \left(\frac{C_{x_N} \bar{X}_N + \beta_2(x_N)}{C_{x_N} \bar{x}_N + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{C_{z_N} \bar{Z}_N + \beta_2(z_N)}{C_{z_N} \bar{z}_N + \beta_2(z_N)} \right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$t_{pr}^4 = \bar{y}_N \left(\frac{\beta_2(x_N) \bar{X}_N + C_{x_N}}{\beta_2(x_N) \bar{x}_N + C_{x_N}} \right)^{\theta_{1N}} \left(\frac{\beta_2(z_N) \bar{Z}_N + C_{z_N}}{\beta_2(z_N) \bar{z}_N + C_{z_N}} \right)^{\theta_{2N}}$
1	ρ_{xy_N}	1	ρ_{yz_N}	$t_{pr}^5 = \bar{y}_N \left(\frac{\bar{X}_N + \rho_{xy_N}}{\bar{x}_N + \rho_{xy_N}} \right)^{\theta_{1N}} \left(\frac{\bar{Z}_N + \rho_{yz_N}}{\bar{z}_N + \rho_{yz_N}} \right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$t_{pr}^6 = \bar{y}_N \left(\frac{\bar{X}_N + \beta_2(x_N)}{\bar{x}_N + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{\bar{Z}_N + \beta_2(z_N)}{\bar{z}_N + \beta_2(x_N)} \right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$t_{pr}^7 = \bar{y}_N \left(\frac{\bar{X}_N + \beta_1(x_N)}{\bar{x}_N + \beta_1(x_N)} \right)^{\theta_{1N}} \left(\frac{\bar{Z}_N + \beta_1(z_N)}{\bar{z}_N + \beta_1(z_N)} \right)^{\theta_{2N}}$

Theorem 2.3. *The bias, MSE, and minimum MSE of the neutrosophic generalized power ratio estimator t_{pr} are given by*

$$Bias(t_{pr}) = \bar{Y}_N\varphi_N \left(\begin{array}{l} \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2 C_{x_N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2 C_{z_N}^2 - \theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N} \\ -\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} + \Psi_N\Pi_N\theta_{1N}\theta_{2N}\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right),$$

$$MSE(t_{pr}) = \bar{Y}_N^2\varphi_N \left(\begin{array}{l} C_{y_N}^2 + \theta_{1N}^2\Psi_N^2 C_{x_N}^2 + \theta_{2N}^2\Pi_N^2 C_{z_N}^2 - 2\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}Cy_N \\ -2\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} + 2\theta_{1N}\theta_{2N}\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right),$$

and $min.MSE(t_{pr}) = \bar{Y}_N^2\varphi_N C_{y_N}^2(1 - R_{y.xz_N}^2)$,

where $R_{y.xz_N}^2 = (\rho_{xy_N}^2 + \rho_{yz_N}^2 - 2\rho_{xy_N}\rho_{yz_N}\rho_{xz_N})/(1-\rho_{xz_N}^2)$ *is the multiple correlation coefficient.*

Proof. Consider the estimator t_{pr} as

$$t_{pr} = \bar{y}_N \left(\frac{a_N\bar{X}_N + b_N}{a_N\bar{x}_N + b_N}\right)^{\theta_{1N}} \left(\frac{c_N\bar{Z}_N + d_N}{c_N\bar{z}_N + d_N}\right)^{\theta_{2N}}.$$

Using notations given in (1), we can rewrite the estimator t_{pr} as

$$\begin{aligned} t_{pr} &= \bar{Y}_N(1+\epsilon_{0N})\left(\frac{a_N\bar{X}_N + b_N}{a_N\bar{X}_N(1+\epsilon_{1N}) + b_N}\right)^{\theta_{1N}} \left(\frac{c_N\bar{Z}_N + d_N}{c_N\bar{Z}_N(1+\epsilon_{2N}) + d_N}\right)^{\theta_{2N}}, \\ &= \bar{Y}_N(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-\theta_{1N}}(1+\Pi_N\epsilon_{2N})^{-\theta_{2N}}. \end{aligned}$$

Using Taylor series expansion, multiplying right hand side terms, and excluding error terms having power more than two, we get

$$t_{pr} = \bar{Y}_N \left(\begin{array}{l} 1 + \epsilon_{0N} - \theta_{1N}\Psi_N\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{1N} + \theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{array} \right). \tag{4}$$

Subtracting $\bar{Y}_N$ both sides of (4), we get

$$t_{pr} - \bar{Y}_N = \bar{Y}_N \left(\begin{array}{l} \epsilon_{0N} - \theta_{1N}\Psi_N\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{1N} + \theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{array} \right). \tag{5}$$

Taking expectation both sides of (5), we get

$$Bias(t_{pr}) = \bar{Y}_N\varphi_N \left(\begin{array}{l} \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2 C_{x_N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2 C_{z_N}^2 - \Psi_N\theta_{1N}\rho_{xy_N}C_{x_N}C_{y_N} \\ -\Pi_N\theta_{2N}\rho_{yz_N}C_{y_N}C_{z_N} + \theta_{1N}\theta_{2N}\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right).$$

Squaring and taking expectation both sides of (5), we have

$$MSE(t_{pr}) = \bar{Y}_N^2\varphi_N \left(\begin{array}{l} C_{y_N}^2 + \theta_{1N}^2\Psi_N^2 C_{x_N}^2 + \theta_{2N}^2\Pi_N^2 C_{z_N}^2 - 2\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}Cy_N \\ -2\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} + 2\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right). \tag{6}$$

The optimum values of θ_{1N} and θ_{2N} can be calculated by minimizing (6) as

$$\theta_{1N} = \frac{C_{y_N}}{C_{x_N}}\frac{(\rho_{xy_N} - \rho_{yz_N}\rho_{xz_N})}{(1-\rho_{xz_N}^2)} \text{ and } \theta_{2N} = \frac{C_{y_N}}{C_{z_N}}\frac{(\rho_{yz_N} - \rho_{xy_N}\rho_{xz_N})}{(1-\rho_{xz_N}^2)}.$$

Putting optimum values of θ_{1N} and θ_{2N} in (6), we get minimum MSE of t_{pr} as

$$min.MSE(t_{pr}) = \bar{Y}_N^2 \varphi_N C_{y_N}^2 (1 - R_{y.xz}^2).$$

□

Remark 2.4. Note that the minimum MSE of the estimator t_{pr} is independent of the values Ψ_N and Π_N which depend on different known parameters of auxiliary variables x_N and z_N. This shows that the MSE of the members $t_{pr}^i,\ i = 1, 2, ..., 7$ will be the same.

Combining the two auxiliary variables, [33] suggested a ratio-cum-product exponential type estimator of the finite population mean under neutrosophy as

$$t_{sg} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{t}_{1N}}{\bar{X}_N + \bar{t}_{1N}}\right) \exp\left(\frac{\bar{t}_{2N} - \bar{Z}_N}{\bar{t}_{2N} + \bar{Z}_N}\right),$$

In the estimator t_{sg}, $\bar{t}_{1N} = \bar{x}_N + \Delta_N(\bar{X}_N - \bar{x}_N)$ and $\bar{t}_{2N} = \bar{z}_N + \Lambda_N(\bar{Z}_N - \bar{z}_N)$, where Δ_N and Λ_N are suitably chosen constants.

Theorem 2.5. *The bias, MSE, and minimum MSE of the neutrosophic estimator t_{sg} are given by*

$$Bias(t_{sg}) = \bar{Y}_N \varphi_N \left[\begin{array}{l} \frac{3}{8}(1-\Delta_N)^2 C_{x_N}^2 - \frac{1}{8}(1-\Lambda_N)^2 C_{z_N}^2 - \frac{(1-\Delta_N)(1-\Lambda_N)}{4}\rho_{xz_N} C_{x_N} C_{z_N} \\ -\frac{(1-\Delta_N)}{2}\rho_{xy_N} C_{x_N} C_{y_N} + \frac{(1-\Lambda_N)}{2}\rho_{yz_N} C_{z_N} C_{y_N} \end{array}\right],$$

$$MSE(t_{sg}) = \bar{Y}_N^2 \varphi_N \left[\begin{array}{l} C_{y_N}^2 + \frac{(1-\Delta_N)^2}{4} C_{x_N}^2 + \frac{(1-\Lambda_N)^2}{4} C_{z_N}^2 - \frac{(1-\Delta_N)(1-\Lambda_N)}{2}\rho_{xz_N} C_{x_N} C_{z_2N} \\ +(1-\Lambda_N)\rho_{yz_N} C_{z_N} C_{y_N} - (1-\Delta_N)\rho_{xy_N} C_{x_N} C_{y_N} \end{array}\right],$$

where, $\Delta_N = 1 - \left[\frac{2C_{y_N}\rho_{xz_N}}{C_{x_N}(1-\rho_{xz_N}^2)}(\rho_{xy_N} - \rho_{yz_N}) + 2\rho_{xy_N}\frac{C_{y_N}}{C_{x_N}}\right]$
and $\Lambda_N = 1 - \left[\frac{2C_{y_N}\rho_{xz_N}}{C_{x_N}(1-\rho_{xz_N}^2)}(\rho_{xy_N} - \rho_{yz_N})\right]$.

Proof. Consider the estimator t_{sg} as

$$t_{sg} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{t}_{1N}}{\bar{X}_N + \bar{t}_{1N}}\right) \exp\left(\frac{\bar{t}_{2N} - \bar{Z}_N}{\bar{t}_{2N} + \bar{Z}_N}\right).$$

Using the notations given in (1), we can rewrite the estimator t_{sg} as

$$t_{sg} = \bar{Y}_N(1+\epsilon_{0N}) \exp\left[\frac{\bar{X}_N - \left(\bar{X}_N\left(1+\epsilon_{1N}(1-\Delta_N)\right)\right)}{\bar{X}_N + \left(\bar{X}_N\left(1+\epsilon_{1N}(1-\Delta_N)\right)\right)}\right] \exp\left[\frac{\bar{Z}_N\left(1+\epsilon_{2N}(1-\Lambda_N)\right) - \bar{Z}_N}{\bar{Z}_N\left(1+\epsilon_{2N}(1-\Lambda_N)\right) + \bar{Z}_N}\right],$$

$$= \bar{Y}_N(1+\epsilon_{0N}) \exp\left[-\frac{\epsilon_{1N}(1-\Delta_N)}{2\left(1+\frac{\epsilon_{1N}}{2}(1-\Delta_N)\right)}\right] \exp\left[\frac{\epsilon_{2N}(1-\Lambda_N)}{2\left(1+\frac{\epsilon_{2N}}{2}(1-\Lambda_N)\right)}\right].$$

Using Taylor series expansion, multiplying right hand side terms and excluding error terms having power more than two, we get

$$t_{sg} = \bar{Y}_N \left[\begin{array}{l} 1 - \frac{\epsilon_{1N}}{2}(1-\Delta_N) + \frac{3}{8}\epsilon_{1N}^2(1-\Delta_N)^2 + \frac{\epsilon_{2N}}{2}(1-\Lambda_N) \\ -\frac{\epsilon_{1N}\epsilon_{2N}}{4}(1-\Delta_N)(1-\Lambda_N) - \epsilon_{2N}^2\frac{(1-\Lambda_N)^2}{8} \\ +\epsilon_{0N} - \frac{\epsilon_{0N}\epsilon_{1N}}{2}(1-\Delta_N) + \frac{\epsilon_{0N}\epsilon_{2N}}{2}(1-\Lambda_N) \end{array}\right].$$

Subtracting $\bar{Y}_N$ both sides of the above expression, we get

$$t_{sg}-\bar{Y}_N=\bar{Y}_N\left[\begin{array}{l}1-\frac{\epsilon_{1N}}{2}(1-\Delta_N)+\frac{3}{8}\epsilon_{1N}^2(1-\Delta_N)^2+\frac{\epsilon_{2N}}{2}(1-\Lambda_N)\\-\frac{\epsilon_{1N}\epsilon_{2N}}{4}(1-\Delta_N)(1-\Lambda_N)-\epsilon_{2N}^2\frac{(1-\Lambda_N)^2}{8}\\+\epsilon_{0N}-\frac{\epsilon_{0N}\epsilon_{1N}}{2}(1-\Delta_N)+\frac{\epsilon_{0N}\epsilon_{2N}}{2}(1-\Lambda_N)\end{array}\right]. \quad (7)$$

Taking expectation both sides of (7), we get

$$Bias(t_{sg})=\bar{Y}_N\varphi_N\left[\begin{array}{l}1-\frac{3}{8}(1-\Delta_N)^2C_{x_N}^2-\frac{1}{8}(1-\Lambda_N)^2C_{z_N}^2-\frac{(1-\Delta_N)}{2}\rho_{xy_N}C_{x_N}C_{y_N}\\-\frac{1}{4}(1-\Delta_N)(1-\Lambda_N)\rho_{xz_N}C_{x_N}C_{z_N}+\frac{(1-\Lambda_N)}{2}\rho_{yz_N}C_{y_N}C_{Z_N}\end{array}\right].$$

Squaring and taking expectation both sides of (7), we have

$$MSE(t_{sg})=\bar{Y}_N^2\varphi_N\left[\begin{array}{l}C_{y_N}^2+\frac{(1-\Delta_N)^2}{4}C_{x_N}^2+\frac{(1-\Lambda_N)^2}{4}C_{z_N}^2-\frac{(1-\Delta_N)(1-\Lambda_N)}{2}\rho_{xz_N}C_{x_N}C_{z_N}\\+(1-\Lambda_N)\rho_{yz_N}C_{z_N}C_{y_N}-(1-\Delta_N)\rho_{xy_N}C_{x_N}C_{y_N}\end{array}\right]. \quad (8)$$

The optimum values of Δ_N and Λ_N can be obtained by minimizing (8) as

$$\Delta_{N(opt)}^*=1-\left[\frac{2C_{y_N}\rho_{xz_N}}{C_{x_N}(1-\rho_{xz_N}^2)}(\rho_{xy_N}-\rho_{yz_N})+2\rho_{xy_N}\frac{C_{y_N}}{C_{x_N}}\right] \quad (9)$$

$$\text{and } \Lambda_{N(opt)}^*=1-\left[\frac{2C_{y_N}\rho_{xz_N}}{C_{x_N}(1-\rho_{xz_N}^2)}(\rho_{xy_N}-\rho_{yz_N})\right]. \quad (10)$$

Putting these optimum values in (8), we get minimum MSE as

$$min.MSE(t_{sg})=\bar{Y}_N^2\varphi_N\left[\begin{array}{l}C_{y_N}^2+\frac{(1-\Delta_{N(opt)}^*)^2}{4}C_{x_N}^2+\frac{(1-\Lambda_{N(opt)}^*)^2}{4}C_{z_N}^2\\-\frac{(1-\Delta_{N(opt)}^*)(1-\Lambda_{N(opt)}^*)}{2}\rho_{xz_N}C_{x_N}C_{z_N}\\+(1-\Lambda_{N(opt)}^*)\rho_{yz_N}C_{z_N}C_{y_N}-(1-\Delta_{N(opt)}^*)\rho_{xy_N}C_{x_N}C_{y_N}\end{array}\right].$$

□

3. **Proposed optimal neutrosophic estimators**

The suggested optimal neutrosophic estimators are important because they improve the accuracy and reliability of statistical analysis under ambiguity, vagueness, and indeterminacy. These estimators are more successful than the conventional ones in dealing with imprecise, partial, or inconsistent data because they use neutrosophic sets. This strategy facilitates improved decision-making in real-world challenges including uncertainty. The optimal neutrosophic class of estimators t_k for population mean under SRS employing bivariate auxiliary information is given by

$$t_k=\alpha_N\bar{y}_N\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{x}_N+b_N}\right)^{\theta_{1N}}\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{z}_N+d_N}\right)^{\theta_{2N}},$$

where α_N, θ_{1N}, and θ_{2N} are suitably chosen scalars. Some sub-classes of the proposed estimator t_k are reported in Table 3 for ready reference.

TABLE 3. Some sub-classes of the optimal neutrosophic ratio type estimator t_k based on bivariate auxiliary information

a_N	b_N	c_N	d_N	Sub-classes of the proposed estimator t_k
1	0	1	0	$t_k^1=\alpha_N\bar{y}_N\left(\frac{\bar{X}_N}{\bar{x}_N}\right)^{\theta_{1N}}\left(\frac{\bar{Z}_N}{\bar{z}_N}\right)^{\theta_{2N}}$
1	C_{x_N}	1	C_{z_N}	$t_k^2=\alpha_N\bar{y}_N\left(\frac{\bar{X}_N+C_{x_N}}{\bar{x}_N+C_{x_N}}\right)^{\theta_{1N}}\left(\frac{\bar{Z}_N+C_{z_N}}{\bar{z}_N+C_{z_N}}\right)^{\theta_{2N}}$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$t_k^3=\alpha_N\bar{y}_N\left(\frac{C_{x_N}\bar{X}_N+\beta_2(x_N)}{C_{x_N}\bar{x}_N+\beta_2(x_N)}\right)^{\theta_{1N}}\left(\frac{C_{z_N}\bar{Z}_N+\beta_2(z_N)}{C_{z_N}\bar{z}_N+\beta_2(z_N)}\right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$t_k^4=\alpha_N\bar{y}_N\left(\frac{\beta_2(x_N)\bar{X}_N+C_{x_N}}{\beta_2(x_N)\bar{x}_N+C_{x_N}}\right)^{\theta_{1N}}\left(\frac{\beta_2(z_N)\bar{Z}_N+C_{z_N}}{\beta_2(z_N)\bar{z}_N+C_{z_N}}\right)^{\theta_{2N}}$
1	ρ_{xy_N}	1	ρ_{yz_N}	$t_k^5=\alpha_N\bar{y}_N\left(\frac{\bar{X}_N+\rho_{xy_N}}{\bar{x}_N+\rho_{xy_N}}\right)^{\theta_{1N}}\left(\frac{\bar{Z}_N+\rho_{yz_N}}{\bar{z}_N+\rho_{yz_N}}\right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$t_k^6=\alpha_N\bar{y}_N\left(\frac{\bar{X}_N+\beta_2(x_N)}{\bar{x}_N+\beta_2(x_N)}\right)^{\theta_{1N}}\left(\frac{\bar{Z}_N+\beta_2(z_N)}{\bar{z}_N+\beta_2(x_N)}\right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$t_k^7=\alpha_N\bar{y}_N\left(\frac{\bar{X}_N+\beta_1(x_N)}{\bar{x}_N+\beta_1(x_N)}\right)^{\theta_{1N}}\left(\frac{\bar{Z}_N+\beta_1(z_N)}{\bar{z}_N+\beta_1(z_N)}\right)^{\theta_{2N}}$

Theorem 3.1. *The bias, MSE, and minimum MSE of the proposed optimal neutrosophic estimator t_k are given by*

$$
\begin{aligned}
Bias(t_k) =&\bar{Y}_N\left[\alpha_N\left\{\begin{array}{l}1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\varphi_N C_{x_N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\varphi_N C_{z_N}^2\\ -\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N}\\ +\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N}\end{array}\right\}-1\right],\\
MSE(t_k) =&\bar{Y}_N^2\left[\begin{array}{l}1+\alpha_N^2\left(\begin{array}{l}1+\varphi_N C_{y_N}^2+2\theta_{1N}^2\Psi_N^2\varphi_N C_{x_N}^2+2\theta_{2N}^2\Pi_N^2\varphi_N C_{z_N}^2\\ +\theta_{1N}\Psi_N^2\varphi_N C_{x_N}^2+\theta_{2N}\Pi_N^2\varphi_N C_{z_N}^2-4\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}\\ -4\theta_{2N}\Pi_N\varphi_N\rho_{yz_N}C_{y_N}C_{z_N}+4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\varphi_N\rho_{xz_N}C_{x_N}C_{z_N}\end{array}\right)\\ -2\alpha_N\left\{\begin{array}{l}1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\varphi_N C_{x_N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\varphi_N C_{z_N}^2\\ -\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N}\\ +\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N}\end{array}\right\}\end{array}\right],
\end{aligned}
$$

$$min.MSE(t_k)=\bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right).$$

Proof. Consider the estimator t_k as

$$t_k=\alpha_N\bar{y}_N\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{x}_N+b_N}\right)^{\theta_{1N}}\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{z}_N+d_N}\right)^{\theta_{2N}}.$$

Using notations given in (1), we get

$$
\begin{aligned}
t_k =&\alpha_N\bar{Y}_N(1+\epsilon_{0N})\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{X}_N(1+\epsilon_{1N})+b_N}\right)^{\theta_{1N}}\left(\frac{c_N\bar{Z}_N+d_N}{c_N\bar{Z}_N(1+\epsilon_{2N})+d_N}\right)^{\theta_{2N}}\\
=&\alpha_N\bar{Y}_N(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-\theta_{1N}}(1+\Pi_N\epsilon_{2N})^{-\theta_{2N}}.
\end{aligned}
$$

Using Taylor series expansion, multiplying right hand terms and excluding error terms having power more than two, we get

$$t_k = \alpha_N \bar{Y}_N \begin{pmatrix} 1+\epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\Psi_N\theta_{1N}\epsilon_{0N}\epsilon_{1N}-\Pi_N\theta_{2N}\epsilon_{0N}\epsilon_{1N}+\Psi_N\Pi_N\theta_{1N}\theta_{2N}\epsilon_{1N}\epsilon_{2N} \end{pmatrix}.$$

Subtracting $\bar{Y}_N$ both sides of the above expression, we get

$$t_k-\bar{Y}_N = \bar{Y}_N \left[\alpha_N \begin{pmatrix} 1+\epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 \\ +\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2-\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{1N} \\ +\theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{pmatrix} -1 \right]. \quad (11)$$

Taking expectation both sides of (11), we get

$$Bias(t_k) = \bar{Y}_N \left[\alpha_N \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\varphi_N C_{x_N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\varphi_N C_{z_N}^2 \\ -\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} \\ +\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right\} -1 \right].$$

Squaring and taking expectation both sides of (11), we get

$$\begin{aligned} MSE(t_k) &= \bar{Y}_N^2 \left[\begin{array}{l} 1+\alpha_N^2 \begin{pmatrix} 1+\varphi_N C_{y_N}^2+2\theta_{1N}^2\Psi_N^2\varphi_N C_{x_N}^2+2\theta_{2N}^2\Pi_N^2\varphi_N C_{z_N}^2 \\ +\theta_{1N}\Psi_N^2\varphi_N C_{x_N}^2+\theta_{2N}\Pi_N^2\varphi_N C_{z_N}^2-4\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N} \\ -4\theta_{2N}\Pi_N\varphi_N\rho_{yz_N}C_{y_N}C_{z_N}+4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\varphi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{pmatrix} \\ -2\alpha_N \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\varphi_N C_{x_N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\varphi_N C_{z_N}^2 \\ -\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} \\ +\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right\} \end{array} \right], \\ &= \bar{Y}_N^2(1+\alpha_N^2 A_N-2\alpha_N B_N). \end{aligned} \quad (12)$$

where,

$$A_N = \begin{pmatrix} 1+\varphi_N C_{y_N}^2+2\theta_{1N}^2\Psi_N^2\varphi_N C_{x_N}^2+2\theta_{2N}^2\Pi_N^2\varphi_N C_{z_N}^2+\theta_{1N}\Psi_N^2\varphi_N C_{x_N}^2+\theta_{2N}\Pi_N^2\varphi_N C_{z_N}^2 \\ -4\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N}-4\theta_{2N}\Pi_N\varphi_N\rho_{yz_N}C_{y_N}C_{z_N}+4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\varphi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{pmatrix},$$

$$B_N = \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\varphi_N C_{x_N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\varphi_N C_{z_N}^2-\theta_{1N}\Psi_N\rho_{xy_N}C_{x_N}C_{y_N} \\ -\theta_{2N}\Pi_N\rho_{yz_N}C_{y_N}C_{z_N}+\theta_{1N}\theta_{2N}\Pi_N\Psi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{array} \right\}.$$

Minimizing (12) with respect to α_N, we get

$$\alpha_{N(opt)} = \frac{B_N^2}{A_N}. \quad (13)$$

Putting $\alpha_{N(opt)}$ in (12), we get

$$min.MSE(t_k) = \bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right).$$

It is remarkable that the simultaneous optimization of α_N, θ_{1N}, and θ_{2N} is not possible. The optimum values of θ_{1N}, and θ_{2N} can be obtained by putting $\alpha_N = 1$ in the estimator t_k and optimizing the MSE. The optimum values of θ_{1N}, and θ_{2N} are given as follows:

$$\theta_{1N} = \frac{C_{y_N}}{C_{x_N}} \frac{(\rho_{xy_N} - \rho_{yz_N}\rho_{xz_N})}{(1-\rho^2_{xz_N})} \text{ and } \theta_{2N} = \frac{C_{y_N}}{C_{z_N}} \frac{(\rho_{yz_N} - \rho_{xy_N}\rho_{xz_N})}{(1-\rho^2_{xz_N})}.$$

□

The efficiency comparison of proposed and adapted estimators in survey sampling is critical for determining which estimator produces more exact and trustworthy estimates with the least variance or MSE. It assures that the new estimator outperforms existing estimators by decreasing MSE and enhancing accuracy. Such comparisons offer a foundation for using more effective estimators, which improves the overall quality of data gathering and decision-making processes in research and practice.

- Comparison of the proposed optimal neutrosophic ratio type estimator t_k with the neutrosophic sample mean $\bar{y}_N$.

$$\begin{aligned}
min.MSE(t_k) &< V(\bar{y}_N) \\
\bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right) &< \bar{Y}_N^2\varphi_N C^2_{y_N} \\
1-\frac{B_N^2}{A_N} &< \varphi_N C^2_{y_N} \\
\frac{B_N^2}{A_N} &> 1-\varphi_N C^2_{y_N}
\end{aligned}$$

- Comparison of the proposed optimal neutrosophic ratio type estimator t_k with the neutrosophic generalized ratio estimator t_{gr}.

$$\begin{aligned}
&min.MSE(t_k) < MSE(t_{gr}) \\
&\bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right) < \bar{Y}_N^2\varphi_N \begin{pmatrix} C^2_{y_N} + \Psi_N^2 C^2_{x_N} + \Pi_N^2 C^2_{z_N} - 2\Psi_N\rho_{xy_N}C_{x_N}C_{y_N} \\ -2\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} + 2\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{pmatrix} \\
&\frac{B_N^2}{A_N} > 1-\varphi_N \begin{pmatrix} C^2_{y_N} + \Psi_N^2 C^2_{x_N} + \Pi_N^2 C^2_{z_N} - 2\Psi_N\rho_{xy_N}C_{x_N}C_{y_N} \\ -2\Pi_N\rho_{yz_N}C_{y_N}C_{z_N} + 2\Psi_N\Pi_N\rho_{xz_N}C_{x_N}C_{z_N} \end{pmatrix}
\end{aligned}$$

- Comparison of the proposed optimal neutrosophic ratio type estimator t_k with the neutrosophic generalized power ratio estimator t_{pr}.

$$\begin{aligned}
min.MSE(t_k) &< min.MSE(t_{pr}) \\
\bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right) &< \bar{Y}_N^2\varphi_N C^2_{y_N}(1-R^2_{y.xz}) \\
\frac{B_N^2}{A_N} &> 1-\varphi_N C^2_{y_N}(1-R^2_{y.xz})
\end{aligned}$$

- Comparison of the proposed optimal neutrosophic ratio type estimator t_k with the neutrosophic ratio-cum-product exponential type estimator t_{sg}.

$$min.MSE(t_k) < min.MSE(t_{sg})$$

$$\bar{Y}_N^2\left(1-\frac{B_N^2}{A_N}\right) < \bar{Y}_N^2\varphi_N \left[\begin{array}{l} C_{y_N}^2 + \frac{(1-\Delta^*_{N(opt)})^2}{4}C_{x_N}^2 + \frac{(1-\Lambda^*_{N(opt)})^2}{4}C_{z_N}^2 \\ -\frac{(1-\Delta^*_{N(opt)})(1-\Lambda^*_{N(opt)})}{2}\rho_{xz_N}C_{x_N}C_{z_N} \\ +(1-\Lambda^*_{N(opt)})\rho_{yz_N}C_{z_N}C_{y_N} - (1-\Delta^*_{N(opt)})\rho_{xy_N}C_{x_N}C_{y_N} \end{array}\right]$$

$$\frac{B_N^2}{A_N} > 1-\varphi_N \left[\begin{array}{l} C_{y_N}^2 + \frac{(1-\Delta^*_{N(opt)})^2}{4}C_{x_N}^2 + \frac{(1-\Lambda^*_{N(opt)})^2}{4}C_{z_N}^2 \\ -\frac{(1-\Delta^*_{N(opt)})(1-\Lambda^*_{N(opt)})}{2}\rho_{xz_N}C_{x_N}C_{z_N} \\ +(1-\Lambda^*_{N(opt)})\rho_{yz_N}C_{z_N}C_{y_N} - (1-\Delta^*_{N(opt)})\rho_{xy_N}C_{x_N}C_{y_N} \end{array}\right]$$

4. **Simulation study**

A simulation study is essential for evaluating the effectiveness of the suggested estimators in controlled conditions, as it allows researchers to test numerous situations and measure flexibility. It reveals insights into estimators behaviour when real-world data is unavailable or unsuitable. Therefore, we conduct a simulation study to examine the performance of the adapted and suggested neutrosophic estimators using an artificially created normal population. The trivariate normal population of size $N = 500$ is generated from R software using the parameters $\bar{Y}_N \in [10,\ 20]$, $\bar{X}_N \in [15,\ 25]$, $\bar{Z}_N \in [20,\ 30]$, $\sigma_{y_N} \in [24,\ 26]$, $\sigma_{x_N} \in [25,\ 28]$, $\sigma_{z_N} \in [26,\ 30]$, and different values of correlation coefficients ρ_{xy_N}, ρ_{yz_N}, and ρ_{xz_N}. Based on 15,000 iterations, the bias, MSE, and PRE are computed using the following formulas:

$$Bias(t^*) = \frac{1}{15,000}\sum_{i=1}^{15,000}(t^* - \bar{Y}_N),\ \text{where } t^* = t_m,\ t_{gr},\ t_{pr},\ \text{and } t_k, \tag{14}$$

$$MSE(t^*) = \frac{1}{15,000}\sum_{i=1}^{15,000}(t^* - \bar{Y}_N)^2, \tag{15}$$

$$\text{and } PRE(t_m,\ t^*) = \frac{MSE(t_m)}{MSE(t^*)} \times 100. \tag{16}$$

Utilizing the necessary parameters of the neutrosophic normal population and for different values of correlation coefficients, the bias, MSE, and PRE results are obtained and compiled in Table 4, Table 5, and Table 6, respectively.

From the bias results of Table 4, it can be observed that the members $t_k^i,\ i = 1, 2, ..., 7$ of the proposed optimal neutrosophic estimator t_k are negatively biased for each combination of correlation coefficients. From Table 5, it can be seen that the MSE of the members $t_k^i,\ i = 1, 2, ..., 7$ of the proposed optimal neutrosophic estimator t_k is less than the mean per unit estimator t_m, the members t_{gr}^i of the generalized ratio estimator t_{gr}, and the members t_{pr}^i of the neutrosophic generalized power ratio estimator t_{pr} for every combination of correlation coefficients. The PRE results of Table 6 indicate that the members $t_k^i,\ i = 1, 2, ..., 7$ of the proposed optimal neutrosophic estimator t_k outperform the mean per unit estimator t_m, the members t_{gr}^i of the generalized ratio estimator t_{gr}, and the members t_{pr}^i of the neutrosophic generalized power ratio estimator t_{pr} for every combination of correlation coefficients. Moreover, it is also observed that the MSE and PRE of each member of the proposed estimator decrease and increase as the values of correlation coefficients increase.

TABLE 4. $Bias_N \in [Bias_L,\ Bias_U]$ of neutrosophic estimators using neutrosophic normal population

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
$\bar{y}_N$	(0, 0)	(0, 0)	(0, 0)	(0, 0)
t_{gr}^1	(0.3014, 0.3867)	(0.1737, 0.3106)	(0.0622, 0.2337)	(-0.0071, 0.15482)
t_{gr}^2	(0.2131, 0.3446)	(0.0970, 0.2697)	(0.0239, 0.1940)	(-0.0429, 0.1164)
t_{gr}^3	(0.2225, 0.2605)	(0.1184, 0.1659)	(0.0167, 0.0710)	(-0.0640, -0.0257)
t_{gr}^4	(0.2951, 0.3806)	(0.1717, 0.3051)	(0.0625, 0.2288)	(-0.0058, 0.1507)
t_{gr}^5	(0.2988, 0.3832)	(0.1716, 0.3047)	(0.0605, 0.2261)	(-0.0075, 0.1461)
t_{gr}^6	(0.1766, 0.3832)	(0.0653, 0.2214)	(-0.0039, 0.1477)	(-0.0696, 0.0728)
t_{gr}^7	(0.2985, 0.3832)	(0.1716, 0.3095)	(0.0628, 0.2327)	(-0.0064, 0.1539)
t_{sg}	(0.0117, 0.0003)	(-0.0713, -0.0030)	(-0.0371, -0.0140)	(-0.0866, -0.0373)
$t_{pr}^i,\ i = 1, 2, ..., 7$	(0.0274, 0.0359)	(0.0002, 0.0344)	(-0.0324, 0.0148)	(-0.0755, -0.0179)
t_k^1	(-0.0830, -0.0441)	(-0.1868, -0.0976)	(-0.2618, -0.1644)	(-0.3481, -0.2411)
t_k^2	(-0.0830, -0.0441)	(-0.1871, -0.0976)	(-0.2620, -0.1645)	(-0.3485, -0.2412)
t_k^3	(-0.0830, -0.0441)	(-0.1871, -0.09763)	(-0.2620, -0.1645)	(-0.3485, -0.2412)
t_k^4	(-0.0830, -0.0441)	(-0.1869, -0.0976)	(-0.2619, -0.1644)	(-0.3483, -0.2411)
t_k^5	(-0.0830, -0.0441)	(-0.1869, -0.0976)	(-0.2619, -0.1644)	(-0.3483, -0.2411)
t_k^6	(-0.0831, -0.0441)	(-0.1872, -0.0976)	(-0.2621, -0.1645)	(-0.3487, -0.2412)
t_k^7	(-0.0830, -0.0441)	(-0.1868, -0.0976)	(-0.2618, -0.1644)	(-0.3481, -0.2411)

5. **Real data applications**

In March 2012, the Indian Government has published the national data sharing and accessibility policy (NDSAP). The national policy is expected to increase the accessibility and easier sharing of non-sensitive data amongst the registered users and their availability for scientific, economic and social developmental purposes. Over the years, various data holding organizations of Central and State Governments have published their

Table 5. $MSE_N \in [MSE_L,\ MSE_U]$ of neutrosophic estimators using neutrosophic normal population

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
$\bar{y}_N$	(5.3664, 6.3157)	(5.3439, 6.2731)	(6.0146, 6.2923)	(5.8616, 6.3965)
t_{gr}^1	(6.9095, 11.8385)	(4.9639, 9.0733)	(3.3643, 6.3644)	(1.4265, 3.6615)
t_{gr}^2	(6.2520, 11.1024)	(4.4883, 8.3975)	(3.0891, 5.7489)	(1.2091, 3.1069)
t_{gr}^3	(6.3071, 10.4377)	(4.5119, 7.7881)	(2.9753, 5.1977)	(1.1023, 2.6169)
t_{gr}^4	(6.6440, 11.5662)	(4.7695, 8.8249)	(3.2481, 6.1379)	(1.3338, 3.4549)
t_{gr}^5	(6.7967, 11.6679)	(4.8149, 8.7922)	(3.2097, 5.9970)	(1.2645, 3.2319)
t_{gr}^6	(5.9734, 10.2322)	(4.2882, 7.6031)	(2.8972, 5.0373)	(1.0580, 2.4856)
t_{gr}^7	(6.8853, 11.8154)	(4.948, 9.0530)	(3.3694, 6.3482)	(1.4326, 3.6487)
t_{sg}	(4.9010, 5.7798)	(4.0440, 4.7580)	(3.035, 3.2601)	(1.1136, 1.2367)
$t_{pr}^i,\ i=1,2,...,7$	(4.6688, 5.5090)	(3.6444, 4.2911)	(2.4857, 2.7518)	(0.8542, 0.9699)
t_k^1	(4.2008, 5.3831)	(3.3626, 4.2100)	(2.4431, 2.7206)	(0.8531, 0.9682)
t_k^2	(4.2111, 5.3850)	(3.3757, 4.2124)	(2.4469, 2.7226)	(0.8528, 0.9687)
t_k^3	(4.2098, 5.3868)	(3.3739, 4.2145)	(2.4482, 2.7243)	(0.8528, 0.9691)
t_k^4	(4.2047, 5.3838)	(3.3676, 4.2108)	(2.4446, 2.7213)	(0.8530, 0.9684)
t_k^5	(4.2027, 5.3836)	(3.3664, 4.2110)	(2.4451, 2.7218)	(0.8530, 0.9686)
t_k^6	(4.2158, 5.3874)	(3.3815, 4.2153)	(2.4498, 2.7249)	(0.8525, 0.9683)
t_k^7	(4.2009, 5.3831)	(3.3628, 4.2100)	(2.4431, 2.7207)	(0.8531, 0.9795)

non-sensitive data, keeping in view, the broad guidelines delineated in the right to information (RTI) Act 2005. Under NDSAP, in January, 2023, the India Meteorological Department (IMD) also published the data which is based on Seasonal and Annual Minimum/Maximum Temperature series for the period 1901-2021. This data can be assessed from the publicly available website `https://www.data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperature-series-period-1901-2021`. The temperature of months October-December during 1901-2021 is taken as neutrosophic study variable y_N, while the temperature of months June-September and March-May during 1901-2021 is taken as neutrosophic auxiliary variables x_N and z_N, respectively. The important characteristics related to this data are compiled in Table 7.

This real dataset is used to examine the performance of the adapted and proposed neutrosophic estimators in terms of bias, MSE, and PRE. From the findings of real data compiled in Table 8, it is observed that the members $t_k^i,\ i=1,2,...,7$ of the proposed optimal neutrosophic estimator t_k are negatively biassed. The findings of Table 8 also show that the members t_k^i of the proposed neutrosophic estimator t_k obtain the least MSE and highest PRE than the

TABLE 6. $PRE_N \in [PRE_L,\ PRE_U]$ of neutrosophic estimators using neutrosophic normal population

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
$\bar{y}_N$	(100.0000, 100.0000)	(100.0000, 100.0009)	(100.0000, 100.0000)	(100.0000, 100.0000)
t_{gr}^1	(77.6675, 53.3488)	(107.6560, 69.1381)	(178.7782, 98.8676)	(410.8916, 174.6957)
t_{gr}^2	(85.8342, 56.8861)	(119.0636, 74.7021)	(194.7050, 109.4534)	(484.7652, 205.8810)
t_{gr}^3	(85.0846, 60.5086)	(118.4403, 80.5473)	(202.1504, 121.0606)	(531.7292, 244.4284)
t_{gr}^4	(80.7701, 54.6049)	(112.0444, 71.0843)	(185.1730, 102.5166)	(439.4532, 185.1448)
t_{gr}^5	(78.9554, 54.1292)	(110.9866, 71.3482)	(187.3867, 104.9256)	(463.5458, 197.9191)
t_{gr}^6	(89.8383, 61.7242)	(124.6201, 82.5068)	(207.5966, 124.9151)	(553.9966, 257.3452)
t_{gr}^7	(77.9403, 53.4531)	(107.9928, 69.2932)	(178.5046, 99.1204)	(409.1544, 175.3112)
t_{sg}	(109.4961, 109.2726)	(132.1436, 131.8424)	(198.1643, 193.0072)	(526.3398, 517.2179)
$t_{pr}^i,\ i = 1, 2, ..., 7$	(114.9416, 114.6424)	(146.6347, 146.1879)	(241.9613, 228.6578)	(686.2144, 659.4919)
t_k^1	(127.7474, 117.3239)	(158.9198, 149.0046)	(246.1812, 231.2789)	(687.0395, 660.6028)
t_k^2	(127.4350, 117.2824)	(158.3024, 161.5663)	(245.8035, 231.1135)	(687.3027, 660.2914)
t_k^3	(127.4736, 117.2443)	(158.3891, 148.8429)	(245.6697, 230.9710)	(687.2713, 660.0523)
t_k^4	(127.6288, 117.3089)	(158.685, 148.9740)	(246.0303, 231.2194)	(687.1212, 660.4837)
t_k^5	(127.6887, 117.3128)	(158.7405, 148.9688)	(245.9860, 231.1831)	(687.1460, 660.3635)
t_k^6	(127.2924, 117.2308)	(158.0313, 148.8145)	(245.5129, 230.9163)	(687.5282, 659.9868)
t_k^7	(127.7447, 117.3239)	(158.9115, 149.0036)	(246.1845, 231.2762)	(687.0472, 660.5961)

members t_{gr}^i of the generalized ratio estimator t_{gr}, and the members t_{pr}^i of the neutrosophic generalized power ratio estimator t_{pr}.

Table 9 displays bias, MSE, and PRE for the classical data. The members of the suggested estimator t_k are the most efficient among all estimators, with the lowest MSE and highest PRE. When comparing neutrosophic findings in Table 8 with classical findings in Table 9, we can conclude that in situations where data are not clear and crisp, instead of relying on a single value in the case of classical estimators, we have an interval to rely on for better results, as we can accept the output if it falls in between these values, because we are handling uncertain data.

TABLE 7. Descriptive values of neutrosophic and classical parameters for real dataset

Neutrosophic parameter	Neutrosophic values	Classical parameter	Classical values
N_N	121	N	121
n_N	(30, 30)	n	30
$\bar{Y}_N$	(17.431, 28.478)	$\bar{Y}$	22.954
$\bar{X}_N$	(24.092, 31.684)	$\bar{X}$	27.888
$\bar{Z}_N$	(21.747, 33.209)	$\bar{Z}$	27.478
C_{y_N}	(0.028, 0.019)	C_y	0.020
C_{x_N}	(0.009, 0.013)	C_x	0.010
C_{z_N}	(0.019, 0.012)	C_z	0.016
ρ_{xy_N}	(0.452, 0.694)	ρ_{xy}	0.649
ρ_{xz_N}	(0.427, 0.357)	ρ_{xz}	0.314
ρ_{yz_N}	(0.247, 0.405)	ρ_{yz}	0.317
$\beta_1(x_N)$	(3.121, 2.997)	$\beta_1(x)$	0.239
$\beta_2(x_N)$	(-0.122, 0.074)	$\beta_2(x)$	2.975
$\beta_1(z_N)$	(2.828, 3.136)	$\beta_1(z)$	0.030
$\beta_2(z_N)$	(0.183, -0.163)	$\beta_2(z)$	3.132

TABLE 8. $Bias_N \in [Bias_L,\ Bias_U]$, $MSE_N \in [MSE_L,\ MSE_U]$, and $PRE_N \in [PRE_L,\ PRE_U]$ of the neutrosophic estimators for real data set

Estimators	$(Bias_L, Bias_U)$	(MSE_L, MSE_U)	(PRE_L, PRE_U)
$\bar{y}_N$	(0, 0)	(0.0079, 0.0103)	(100.0000, 100.0000)
t_{gr}^1	(0.00016, 0.00008)	(0.0091, 0.0074)	(87.0250, 139.0112)
t_{gr}^2	(0.00016, 0.00008)	(0.0091, 0.0074)	(87.0844, 139.0788)
t_{gr}^3	(0.00019, 0.00002)	(0.0093, 0.0126)	(84.7288, 82.0142)
t_{gr}^4	(0.00016, 0.00008)	(0.0090, 0.0074)	(87.3052, 139.0359)
t_{gr}^5	(0.00016, 0.00007)	(0.0090, 0.0073)	(87.8766, 141.5893)
t_{gr}^6	(0.00016, 0.00008)	(0.0090, 0.0074)	(87.5085, 138.5515)
t_{gr}^7	(0.00016, 0.00006)	(0.0089, 0.0072)	(88.3912, 142.5806)
t_{sg}	(-0.00001, -0.00001)	(0.0066, 0.0050)	(119.5000, 153.2665)
t_{pr}^i	(0.00001, -0.00009)	(0.0064, 0.0053)	(122.7051, 192.8389)
t_k^1	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2444, 204.3487)
t_k^2	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2444, 204.3487)
t_k^3	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2449, 204.3487)
t_k^4	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2444, 204.3487)
t_k^5	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2444, 204.3487)
t_k^6	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2444, 204.3487)
t_k^7	(-0.00009, -0.00012)	(0.0062, 0.0038)	(126.2443, 204.3486)

TABLE 9. Bias, MSE, and PRE of estimators under classical data

Estimators	$Bias$	MSE	PRE
$\bar{y}$	0.0000	0.0070	100.0000
t_{gr}^{1}	0.0001	0.0069	101.5228
t_{gr}^{2}	0.0001	0.0069	101.5880
t_{gr}^{3}	-0.000009	0.0062	111.5913
t_{gr}^{4}	0.0001	0.0069	101.5437
t_{gr}^{5}	0.0001	0.0068	102.9575
t_{gr}^{6}	0.00009	0.0061	113.3949
t_{gr}^{7}	0.00009	0.0062	112.8295
t_{sg}	-0.000006	0.0051	137.4965
$t_{pr}^{i},\ i = 1, 2, ..., 7$	-0.00001	0.0040	172.4487
t_{k}^{1}	-0.0001	0.0039	177.1242
t_{k}^{2}	-0.0001	0.0039	177.1242
t_{k}^{3}	-0.0001	0.0039	177.1235
t_{k}^{4}	-0.0001	0.0039	177.1242
t_{k}^{5}	-0.0001	0.0039	177.1242
t_{k}^{6}	-0.0001	0.0039	177.1241
t_{k}^{7}	-0.0001	0.0039	177.1242

6. **Conclusions**

In this paper, we presented an optimal neutrosophic framework for estimating the population mean under SRS by employing bivariate auxiliary information. The use of neutrosophic sets enabled us to account for the uncertainty, indeterminacy, and imprecision inherent in real-world survey data. By incorporating neutrosophic logic into classical sampling theory, we created an optimal neutrosophic estimation method that outperforms the adapted neutrosophic estimation methods in cases with high uncertainty. A simulation study is undertaken on an artificially generated normal population. The simulation findings reveal that the suggested neutrosophic estimator outperforms the adapted neutrosophic estimators, with lower MSE and higher PRE. In addition, the suggested neutrosophic framework outperformed the adapted ones in terms of accuracy as well as reliability when applied to real-life temperature data. This demonstrates the practical application of the suggested neutrosophic estimators in statistical estimating procedures, which provide considerable benefits in uncertain contexts.

In future studies, we may look at expanding this study to more complex sampling designs, including stratified and double sampling. Furthermore, real-life case studies would help to evaluate the efficiency of the neutrosophic framework and broaden its applicability in a variety of survey research and data analysis fields.

Funding: No funding was received for conducting this study.

Acknowledgments: The authors are highly thankful to the editor and reviewers for their valuable suggestions to improve the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Data availability statement: Data included in article/supplementary material/referenced in article.

References

[1] Gupta, S.; Shabbir, J. On the use of transformed auxiliary variables in estimating population mean by using two auxiliary variables. J. Stat. Plan. Inference. **2007**, 137(5), 1606-1611.

[2] Awan, W.H.; Shabbir, J. On optimum regression estimator for population mean using two auxiliary variables in simple random sampling. Commun. Stat. Simul. Comput. **2014**, 43(6), 1508-1522.

[3] Muneer, S.; Shabbir, J.; Khalil, A. Estimation of finite population mean in simple random sampling and stratified random sampling using two auxiliary variables. Commun. Stat. Theory Methods **2017**, 46(5), 2181-2192.

[4] Bhushan, S.; Gupta, R.; Singh, S.; Kumar, A. Some new improved classes of estimators using multiple auxiliary information. Glob. J. Pure Appl. Math. **2020**, 16(3), 515-528.

[5] Zaman, T.; Dunder, E.; Audu, A.; Alilah, D.A.; Shahzad, U.; Hanif, M. Robust regressionratiotype estimators of the mean utilizing two auxiliary variables: a simulation study. Math. Probl. Eng. **2021**, 2021(1), 6383927.

[6] Shabbir, J.; Ahmed, S.; Sanaullah, A.; Onyango, R. Measuring performance of ratioexponentiallog type general class of estimators using two auxiliary variables. Math. Probl. Eng. **2021**, 2021(1), 5245621.

[7] Bhushan, S.; Kumar, A. New efficient logarithmic estimators using multiauxiliary information under ranked set sampling. Concurr. Comput. **2022**, 34(27), e7337.

[8] Ahmad, S.; Hussain, S.; Shabbir, J.; Aamir, M.; El-Morshedy, M.; Ahmad, Z.; Alrajhi, S. Improved generalized class of estimators in estimating the finite population mean using two auxiliary variables under two-stage sampling. AIMS Mathematics. **2022**, 7(6), 10609-10624.

[9] Bhushan, S.; Kumar, A.; Onyango, R.; Singh, S. Some improved classes of estimators in stratified sampling using bivariate auxiliary information. J. Probab. Stat. **2022**, 2022(1), 2660114.

[10] Ahmad, S.; Zahid, E.; Shabbir, J.; Aamir, M.; Onyango, R. Enhanced estimation of the population mean using two auxiliary variables under probability proportional to size sampling. Math. Probl. Eng. **2023**, 2023(1), 5564360.

[11] Bhushan, S.; Kumar, A.; Alsadat, N.; Mustafa, M.S.; Alsolmi, M.M. Some optimal classes of estimators based on multi-auxiliary information. Axioms. **2023**, 12(6), 515.

[12] Chhoker, P.K.; Nagar, A. Mathematical modeling and fuzzy availability analysis of stainless steel utensil manufacturing unit in steady state: a case study. Int. J. Syst. Assur. Eng. Manag. **2015**, 6, 304-318.

[13] Kumar, P.; Dudeja, C. Shadowed type 2 fuzzy-based Markov model to predict shortest path with optimized waiting time. Soft Comput. **2021**, 25, 995-1005.

[14] Tahir, Z.; Khan, H.; Aslam, M.; Shabbir, J.; Mahmood; Smarandache, F. Neutrosophic ratio-type estimators for estimating the population mean. Complex Intell. Syst. **2021**, 7(6), 2991-3001.

[15] Yadav, S.K.; Smarandache, F. Generalized neutrosophic sampling strategy for elevated estimation of population mean. Neutrosophic Sets Syst. **2023**, 53, 1-20.

[16] Yadav, V.K.; Prasad, S. Neutrosophic estimators for estimating the population mean in survey sampling. Measurement: Interdiscip Res. **2024**. DOI:10.1080/15366367.2023.2267835.

[17] Alqudah, M.A.; Zayed, M.; Subzar, M.; Wani, S.A. Neutrosophic robust ratio type estimator for estimating finite population mean. Heliyon. **2024**, 10(8), 1-12.

[18] Masood, S.; Ibrar, B.; Shabbir, J.; Shokri, A.; Movaheedi, Z. Estimating neutrosophic finite median employing robust measures of the auxiliary variable. Sci. Rep. **2024**, 14(1), 10255.

[19] Kumar, S.; Kour, S.P.; Choudhary, M.; Sharma, V. Determination of population mean using neutrosophic exponential-type estimator. Lobachevskii J. Math. **2022**, 43(11), 3359-3367.

[20] Singh, R.; Smarandache, F.; Mishra, R. Generalized robust-type neutrosophic ratio estimators of pharmaceutical daily stock prices. In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics. **2023**, 417-429. Academic Press.

[21] Ullah, A.; Shabbir, J.; Alomair, A.M.; Alomair, M.A. Ratio-type estimator for estimating the neutrosophic population mean in simple random sampling under intuitionistic fuzzy cost function. Axioms. **2023**, 12(9), 890.

[22] Singh, A.; Aslam, M.; Vishwakarma, G.K.; Dhital, A.; Patrascu, I. Neutrosophic regression cum ratio estimators for the population mean: an application in medical science. In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics. **2023**, 313-333. Academic Press.

[23] Yadav, V.K.; Prasad, S.P. Neutrosophic estimators in two-phase survey sampling. Neutrosophic Sets Syst. **2023**, 61(1), 29.

[24] Singh, R.; Kumari, A.; Smarandache, F.; Tiwari, S.N. Construction of almost unbiased estimator for population mean using neutrosophic information. Neutrosophic Sets Syst. **2025**, 76(1), 449-463.

[25] Singh, A.; Kulkarni, H.; Smarandache, F.; Vishwakarma, G.K. Computation of separate ratio and regression estimator under Neutrosophic stratified sampling: an application to climate data. J. Fuzzy Ext. Appl. **2024**, 5(4), 605-621.

[26] Vishwakarma, G.K.; Singh, A. Generalized estimator for computation of population mean under neutrosophic ranked set technique: An application to solar energy data. Comput. Appl. Math. **2022**, 41(4), 144.

[27] Khanna, B. Generalized regressed exponential estimator for estimation of mean under neutrosophic ranked set sampling. Natl. Acad. Sci. Lett. **2024**, 1-7.

[28] Singh, R.; Kumari, A. Neutrosophic ranked set sampling scheme for estimating population mean: An application to demographic data. Neutrosophic Sets Syst. **2024**, 68, 246-270.

[29] Kumari, A.; Singh, R.; Smarandache, F. New modification of ranked set sampling for estimating population mean: Neutrosophic median ranked set sampling with an application to demographic data. Int. J. Comput. Intell. Syst. **2024**, 17(1), 210.

[30] Alomair, A.M.; Ahmad, S. New comprehensive mean estimation using regression-cum-exponential type estimator: Application with neutrosophic data. Kuwait J. Sci. **2025**, 52(1), 100346.

[31] Singh, A.; Singh, P.; Sharma, P.; Aloraini, B. Estimation of population mean using neutrosophic exponential estimators with application to real data. Int. J. Neutrosophic Sci. **2025**, 25(3), 322-338.

[32] Yadav, V.K.; Majhi, D.; Alkhathami, A.A.; Prasad, S. Neutrosophic mean estimators using extreme indeterminate observations in sample surveys. Neutrosophic Sets Syst. **2025**, 80, 86-117.

[33] Singh, P.; Gupta, S. Combining two auxiliary variables for elevated estimation of finite population mean under neutrosophic framework. Neutrosophic Sets Syst. **2025**, 76, 275-287.

[34] Smarandache, F. Introduction to neutrosophic statistics. Sitech & Education Publishing, **2014**.

Received: Nov. 14, 2024. Accepted: May 25, 2025