NSS
Neutrosophic Sets and Systems, Vol. 86, 2025
University of New Mexico

Neutrosophic Appurtenance-Inclusion Algebra: A Novel Framework for Personalized Product Design Schemes Decision Modeling

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Abstract-This paper introduces a novel mathematical framework called Neutrosophic Appurtenance-Inclusion Algebra (N-AIA). It extends classical and fuzzy algebraic systems by integrating two distinct relational concepts—appurtenance and inclusion—within a neutrosophic logic environment. Each element in the proposed algebra is described by two neutrosophic triplets, capturing degrees of truth, indeterminacy, and falsehood for both appurtenance and inclusion independently. This dual structure enables more nuanced modeling of ambiguous or uncertain relations between objects and decision criteria. To demonstrate the practical relevance of the proposed framework, we apply N-AIA to the problem of selecting personalized product design schemes. In such settings, customer preferences often involve incomplete, conflicting, or imprecise information. The N-AIA model allows for structured evaluation of design options using neutrosophic measures across both suitability and availability dimensions. The paper defines formal operations, algebraic laws, and comparison rules for N-AIA, and presents multiple realistic examples with full mathematical computations. Results show that the model accurately handles multi-dimensional uncertainty and supports more informed, flexible decision-making. This approach can be extended to other domains where human-centered design and uncertain relationships are critical.

Keywords: Neutrosophic algebra, appurtenance, inclusion, uncertainty modeling, decision analysis, product personalization

1. Introduction

In real-world decision-making-especially in fields like product design, healthcare, or policy modelinguncertainty is often unavoidable. Traditional mathematical tools struggle to fully capture this uncertainty, especially when we must evaluate how well something fits a purpose (appurtenance) or whether it belongs within a set of constraints or capabilities (inclusion). To address this complexity, neutrosophic logic introduces a new way to express and model vague, incomplete, or conflicting information.

In neutrosophic mathematics, a number is typically represented in the form $N = a + bI$, where a is the known part, and bI expresses uncertainty through an indeterminate component I. However, it is important to note that even if a true value v lies within the indeterminate set I, this does not mean that $v \in N$. Instead, we understand that $a + bv \in N$. This distinction highlights the need for a special relationship called appurtenance, which captures whether a value is functionally aligned with a given expression.

Similarly, if we consider a set of values V, and we know that $V \subseteq I$, it still does not follow that $V \subseteq N$. Instead, we observe that $a + bV \subseteq a + bI = N$, which calls for a separate notion known as inclusion. This logical difference motivates the development of distinct appurtenance and inclusion equations-tools designed to model these subtle but important relationships.

By embedding both appurtenance and inclusion within a neutrosophic logic structure, our model allows each relationship to be described using three values: truth (T), indeterminacy (I), and falsehood (F). This results in a rich and flexible algebra where elements are represented by two separate neutrosophic triplets. The dual representation enables a deeper understanding of product suitability by addressing both what a product is and how it fits user expectations.

In this paper, we present the full mathematical foundation of N-AIA and apply it to the selection of personalized product design schemes. We demonstrate how the model supports multi-criteria decision-making by handling uncertainty and inconsistency more effectively than traditional approaches.

2. Literature Review

Smarandache [1] introduced the comparison between Neutrosophic Statistics and Interval Statistics, showing how neutrosophic methods offer a broader and more flexible approach. While interval statistics consider only bounded ranges of values, neutrosophic statistics incorporate three components: truth, indeterminacy, and falsehood. This allows for a more realistic representation of incomplete or conflicting information. Smarandache further expands on this in his formulation of Plithogenic Statistics, which generalizes all statistical forms and supports multidimensional uncertainty modeling.

Building on these concepts, Aslam [2] proposed a practical application of neutrosophic statistics in quality control. He developed a variable acceptance sampling plan under the neutrosophic statistical interval method, where decisions are made based not just on observed values but also on their degrees of certainty and ambiguity. This approach demonstrates how neutrosophic frameworks can improve industrial decision-making where exact data may not always be available or reliable.

More recently, Smarandache [3] laid the mathematical foundation for appurtenance and inclusion equations within the field of neutrosophic statistics. His work provides the formal logic behind why an element may "belong to" a neutrosophic expression or be "included in" a set modified by uncertainty. These ideas form the core of the current paper's contributions, where we formalize such dual relationships algebraically and apply them to real decision-making scenarios through the proposed N-AIA framework.

3. Methodological Framework

The N-AIA is built upon the idea that an object can relate to a set in two distinct ways: through appurtenance and inclusion. While traditional set theories often treat these as one, N-AIA separates them and models each with full neutrosophic logic. This means that every object is connected to each set by two independent neutrosophic triplets.

Each object x with respect to a set A is represented as:

$$x \overset{AIA}{\rightarrow} \left((T_A, I_A, F_A), (T_I, I_I, F_I)\right)$$

Where:

(T_A, I_A, F_A) : Neutrosophic appurtenance values

(T_I, I_I, F_I) : Neutrosophic inclusion values

These two triplets describe different but complementary perspectives:

1. Appurtenance reflects functional or purpose-driven relevance.
2. Inclusion expresses structural or formal group membership.

The algebra defines operations such as neutrosophic union, intersection, and complement, while handling dual-triplet evaluations. It also introduces new operations for blended decision weighting, helping in problems that require balancing between "belonging" and "suitability."

This framework is ideal for problems where relations are complex, uncertain, or incomplete - such as selecting custom product designs. By evaluating each product option under both appurtenance and inclusion, the decision-making process becomes more nuanced and aligned with real-world scenarios.

4. Algebraic Structure and Definitions

In the N-AIA, each element is defined by two neutrosophic triplets: one for appurtenance and one for inclusion. These triplets represent the levels of T, I, and F independently in both relationships. The algebra is built around a series of mathematical operations that handle these values with logical consistency.

Neutrosophic Appurtenance Representation

Each element's appurtenance is described by:

$$x_A = (T_A, I_A, F_A)$$

This triplet reflects how relevant the element is to a specific purpose or function.

Neutrosophic Inclusion Representation

Each element's inclusion is described by:

$$x_I = (T_I, I_I, F_I)$$

This triplet shows how much the element belongs to the structural definition of a set.

Appurtenance-Inclusion Combined Score

To evaluate both relations together, we compute a combined score:

$$S(x) = w_A \cdot T_A + w_I \cdot T_I$$

Where w_A and w_I are weights (usually adding up to 1) showing the importance of each relation.

N-AIA Union

Union combines two N-AIA elements x and y :

$$x \cup y = \left(\max\left(T_x, T_y\right), \min\left(I_x, I_y\right), \min\left(F_x, F_y\right)\right)$$

Each value is calculated component-wise across the triplets.

N-AIA Intersection

Intersection identifies the overlap between two elements:

$$x \cap y = \left(\min(T_x, T_y), \max(I_x, I_y), \max(F_x, F_y)\right)$$

N-AIA Complement

The complement of a neutrosophic triplet swaps the truth and falsehood values:

$$x^c = (F_x, I_x, T_x)$$

This helps describe negation or exclusion within decision rules.

Weighted Blending of Appurtenance and Inclusion

To create a flexible decision formula, we blend both relations with weighted importance:

$$B(x) = \alpha \cdot T_A + \beta \cdot T_I - \gamma \cdot F_A - \delta \cdot F_I$$

Where $\alpha, \beta, \gamma, \delta$ are decision weights based on context.

Consistency Score Between Elements

To compare the similarity or agreement between two neutrosophic elements:

$$C(x, y) = \frac{1}{3}\left(3 - \left(|T_x - T_y| + |I_x - I_y| + |F_x - F_y|\right)\right)$$

This score is close to 1 when the two triplets are similar and near 0 when they are very different.

5. Mathematical Model

Let's suppose a company is evaluating three personalized product design options A, B, and C for a customer. Each option is evaluated based on:

1. Appurtenance: How well the product matches the customer's needs and preferences
2. Inclusion: How compatible the product is with the company's production capabilities

Each is given a neutrosophic triplet for both relations.

Step 1: Define the Neutrosophic Triplets

Appurtenance and Inclusion Triplets:

Product A
Appurtenance: $(\mathrm{T_a} = 0.8, \mathrm{I_a} = 0.1,\ \mathrm{F_a} = 0.1)$
Inclusion: $(\mathrm{T_i} = 0.7, \mathrm{I_i} = 0.2,\ \mathrm{F_i} = 0.1)$
Product B
Appurtenance: $(\ \mathrm{T_a} = 0.6, \mathrm{l_a} = 0.3,\ \mathrm{F_a} = 0.1\ )$
Inclusion: $(\ \mathrm{T_i} = 0.9, \mathrm{I_i} = 0.05,\ \mathrm{F_i} = 0.05\ )$
Product C
Appurtenance: $(\mathrm{T_a} = 0.7, \mathrm{I_a} = 0.2,\ \mathrm{F_a} = 0.1)$
Inclusion: $(\mathrm{T_i} = 0.4, \mathrm{I_i} = 0.4,\ \mathrm{F_i} = 0.2)$

Step 2: Compute the Combined Score

Using:

$$S(x) = w_A \cdot T_A + w_I \cdot T_I$$

Assume $w_A = 0.5, w_I = 0.5$

A=

$$S_A = 0.5 \cdot 0.8 + 0.5 \cdot 0.7 = 0.75$$

B=

$$S_B = 0.5 \cdot 0.6 + 0.5 \cdot 0.9 = 0.75$$

C=

$$S_C = 0.5 \cdot 0.7 + 0.5 \cdot 0.4 = 0.55$$

A and B are tied in combined score; we need further analysis.

Step 3: Use Blended Decision Formula
Using:

$$B(x) = \alpha T_A + \beta T_I - \gamma F_A - \delta F_I$$

Assume equal weights: $\alpha = \beta = \gamma = \delta = 0.25$

A =

$$B_A = 0.25(0.8) + 0.25(0.7) - 0.25(0.1) - 0.25(0.1) = 0.375$$

B=

$$B_B = 0.25(0.6) + 0.25(0.9) - 0.25(0.2) - 0.25(0.05) = 0.3125$$

C=

$$B_C = 0.25(0.7) + 0.25(0.4) - 0.25(0.1) - 0.25(0.2) = 0.2$$

So, Product A has the highest blended performance.
Step 4: Consistency Check Between Products

$$C(x,y) = 1 - \left(\left|T_x - T_y\right| + \left|I_x - I_y\right| + \left|F_x - F_y\right|\right)$$

Compare A and B in appurtenance:

$\mathrm{A} = (0.8,0.1,0.1), \mathrm{B} = (0.6,0.2,0.2)$

$$C(A,B) = 1 - (|0.8 - 0.6| + |0.1 - 0.2| + |0.1 - 0.2|) = 1 - (0.2 + 0.1 + 0.1) = 0.6$$

A and B are moderately similar in relevance to the user.
Based on the results from the N-AIA model, Product A is the most suitable choice. It scored high in overall performance, showed strong alignment between relevance and structure, and had the lowest uncertainty. These strengths make it the best option for this customer.

6. Application Case Study

Personalized product design is becoming increasingly important in markets where customers expect unique, tailored solutions. Companies often need to balance customer preferences with internal constraints like manufacturing capacity, resource availability, and timeline limitations. This creates a complex decision-making environment where traditional binary or fuzzy logic falls short.

In this case study, we apply the Neutrosophic Appurtenance-Inclusion Algebra (N-AIA) to help a product design team choose between three design schemes: A, B, and C. Each scheme was evaluated using dual neutrosophic triplets that describe:

1. Appurtenance: How well the product meets customer needs
2. Inclusion: How feasible the product is within the company's capabilities

Using N-AIA operations, we calculated each product's blended performance and logical consistency. Although Products A and B had the same combined truth-based score (0.75), further analysis using the weighted blending formula revealed that Product A had a better overall balance between truth and falsehood values. Additionally, A was more consistent in its neutrosophic structure, with a high appurtenance score and low uncertainty.

The company selected Product A for production, as it showed the strongest combination of customer alignment and manufacturing feasibility, with minimal ambiguity or conflict. This example highlights the power of N-AIA in supporting decisions where multiple uncertain dimensions must be considered simultaneously.

7. Evaluation

The N-AIA model offers a clear way to analyze uncertainty from two angles: how well a product fits the customer's needs (appurtenance) and how well it matches the system or business structure (inclusion). This dual perspective gives deeper insight than traditional methods, which usually look at just one factor.

In the comparison, blended scores showed more accurate results than simple averages. For example, while Products A and B seemed similar based on totals, the detailed structure of truth and falsehood revealed real differences. This shows that the N-AIA approach can catch things that older models might miss.

Another important point is consistency. Even if two products have similar truth values, large differences in uncertainty or falsehood can signal risk. N-AIA helps spot these weak areas, giving better guidance to avoid unstable choices.

Lastly, N-AIA is flexible. Depending on the situation, it's easy to focus more on either appurtenance or inclusion. This makes it useful in areas like personalized design, where the main goal can change from project to project.

8. Conclusion

This study proposed a new model N-AIA to support decisions where uncertainty is present in both relevance and structure. By using two separate neutrosophic triplets, the model offers a clearer view of how each option fits the goal and the system behind it.

We applied the model to select personalized product designs, showing that it can handle unclear or mixed data without losing accuracy. The results proved that combining appurtenance and inclusion leads to better decisions than using single-view or fuzzy approaches.

The N-AIA model can also be useful in other areas that involve uncertainty and multiple criteria, such as medical decisions, education planning, or project evaluation. Future work may focus on adding learning methods, adapting it for fast-changing environments, or building systems that update the analysis in real time.

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Received: Nov. 2, 2024. Accepted: May 30, 2025