Neutrosophic Rama Distribution with an Application

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Abstract: This paper introduces the Neutrosophic Rama Distribution (NRD) and explores its statistical properties, including neutrosophic moments, moment functions, and parameter estimation. The Rama distribution, a type of lifetime distribution, is useful for modeling lifetime data under varying hazard conditions, making it valuable in fields such as medical science, mechanical engineering, and industrial applications. In this study, we present the Neutrosophic Rama Distribution along with its key statistical characteristics. Parameter estimation is conducted using the maximum likelihood method, and the properties of the estimator are examined through a simulation study in a neutrosophic environment. Furthermore, the practical applicability of the NRD is demonstrated using a real-life dataset and compared with other neutrosophic models. The comparative analysis indicates that the Neutrosophic Rama Distribution outperforms the selected existing neutrosophic models.

Keywords: Neutrosophic Rama distribution; Maximum likelihood estimator; Neutrosophic moments; Statistical properties of distribution; Simulation study.

1. Introduction

Lifetime data analysis is essential in various applied sciences for multiple purposes. By analyzing and modeling lifetime data, we can predict trends and patterns, which can be valuable in fields such as biology, healthcare, engineering, and actuarial science, including insurance and risk management. For these situations, the two one-parameter lifetime distributions commonly used in statistics are the exponential and Lindley [1] distributions. However, Shanker et al. [2-7] conducted a comparative study of these distributions and reported that they often fail to adequately fit lifetime data because of their shape and other properties. In contrast, the Rama distribution, proposed by Rama Shanker [8], is more flexible and suitable for modeling lifetime data, offering improvements in terms of its hazard rate and shape.

The lifetime distributions are applied to a dataset considering the data as complete information about the characteristics measured. These measurement data are termed crisp data. The Rama distribution is traditionally applied only to crisp data. However, real-life datasets often contain indeterminate, vague, uncertain, or imprecise values alongside deterministic crisp values. The classical distributions fail to address situations in which indeterminacy, vagueness or imprecision are present in the data; for this purpose, we use neutrosophic logic. The extension of fuzzy logic and interval-based logic is neutrosophic logic, which can be used instead. Neutrosophic logic not only generalizes these approaches but also provides details about the degree of indeterminacy in the sample. Interval-based statistics capture data within intervals that include both deterministic and indeterministic parts. The principal of the neutrosophic set was properly explained by Smarandache [9, 10], with statistical analysis using neutrosophic sets.

Several studies have demonstrated that neutrosophic logic is more efficient than fuzzy-based logic. For example, Sumathi and Sweety [11] presented various methods to analyze trapezoidal neutrosophic data, whereas Abdel-Basset et al. [12, 13] explained how to use neutrosophic logic in a response system for renewable energy. Additionally, Zeema and Christopher [14] addressed how to optimize neutrosophic numbers alongside the way they perform in prediction tasks. When data contain no indeterministic or uncertain values, neutrosophic statistical analysis reverts to classical analysis. In such cases, the concepts of neutrosophic logic are far better than the methods of classical statistics.

In recent years, researchers have developed several neutrosophic probability distributions, such as the Alhasan and Smarandache distributions [15], which introduce various neutrosophic distributions, such as the neutrosophic Rayleigh distribution, neutrosophic Weibull distribution, neutrosophic three-parameter Weibull distribution, neutrosophic five-parameter Weibull distribution, neutrosophic inverse Weibull distribution and neutrosophic beta Weibull distribution. Neutrosophic uniform, neutrosophic Poisson and neutrosophic exponential distributions were proposed by Alhabib et al. [16]. The neutrosophic gamma distribution and neutrosophic lognormal distribution, which are applied to environmental data, were proposed by Zahid Khan et al. [17, 18]. Neutrosophic normal and binomial distributions are given by Patro and Smarandache [19]. The neutrosophic discrete Ramos–Louzada distribution was proposed by Ahsan-ul- Haq and Zafar [20]. A neutrosophic generalized Pareto model with applications in data modeling and quality control, in which neutrosophic data are used to construct an S-control chart, is also given by Zahid Khan et al. [21, 22]. For a better understanding of neutrosophic models and their applications in different areas, we studied these references [23-30]. Some recently developed neutrosophic distributions include the neutrosophic Laplace distribution, neutrosophic negative binomial distribution, neutrosophic logistic distribution—used in fuzzy data modeling—and the neutrosophic log-Gamma distribution, which has been applied to industrial growth data [31-34]. The Neutrosophic Poisson distribution series within the harmonic subclass of analytic functions, utilizing the Salagean derivative operator, was applied. Additionally, an investigative study was conducted on the Quick Switching System (QSS) using both fuzzy and Neutrosophic Poisson distributions [35, 36].

There is an extensive ensemble of works on different neutrosophic statistical distributions that can be used to model various types of data that contain some uncertainty or are vague or imprecise. In the present work, an attempt has been made to develop a neutrosophic Rama distribution (NRD) to explore its properties, such as neutrosophic moments, neutrosophic skewness, neutrosophic kurtosis, neutrosophic median, neutrosophic variance and neutrosophic moment-generating functions. The maximum likelihood estimation approach is applied for the estimation of parameters of the neutrosophic Rama distribution, and simulation-based estimates are obtained. A simulation study is conducted in neutrosophic environments. The proposed neutrosophic Rama distribution is also fitted the under the age of five mortality rates, covering the period 1995-2020 for Saudi Arabia [21], and it fits well in comparison with other neutrosophic probability distributions. The remaining framework of the present article is arranged as follows:

Section 2 contains a revisit of the neutrosophic Rama distribution. The statistical properties of the proposed distribution are discussed in Section 3, examples in section 4, and the estimation of the parameters of the proposed distribution is discussed in Section 5 with simulation study in section 6. The application of the proposed neutrosophic Rama distribution is presented in Section 7. Finally, the conclusions of the study are presented in section 8.

2. Neutrosophic Rama Distribution:

Classical statistics do not account for uncertainty, imprecision, ambiguity, incompleteness, vagueness, or unknown values of the sample. Consequently, distributions or classical statistical theories are not applicable to the sample, which contains indeterminacy. To analyze such types of data, we use neutrosophic statistical theory, which has been developed and has become more popular in recent years. Neutrosophic theory is a generalization of classical statistical theory. Neutrosophic data combine classical data with an indeterminate part. Thus, let X_N be a neutrosophic number; then, X_N can be split into two parts such that

$$X_N = E + I$$

Where E is the exact or determinate portion of the data and where I is the exact or indeterminate portion of the data. X_N can be written in the form of an interval as $X_N \in [X_L, X_U]$. N is used as a subscript for neutrosophic random variables. For example, $X_N = 5 + I$, where $I \in [0, 0.5]$, is equivalent to $X_N \in [5, 5.5]$, so for sure $X_N \geq 5$ (meaning that the determinate part of X_N is 5), while the indeterminate part $I \in [0, 0.5]$ means the possibility for number X_N to be a little bigger than 5.

Let us assume that $X_{Nj} \in [X_{Lj}, X_{Uj}]$; j = 1, 2, 3, 4, … n. is a neutrosophic random variable that follows a neutrosophic Rama distribution (NRD) with a neutrosophic shape parameter $\theta_N \in [\theta_L, \theta_U]$.The neutrosophic cumulative distribution (NCDF) F (X_N, θ_N) and neutrosophic probability density function (NPDF) f (X_N, θ_N) of NRD are as follows:

$$F(X_N, \theta_N) = 1 - \left[1 + \frac{(\theta_N^3 x_N^3 + 3\theta_N^2 x_N^2 + 6\theta_N\ x_N)}{(\theta_N^3 + 6)}\right] e^{(-\theta_N\ x_N)};\quad \theta_N,\ x_N \geq 0 \tag{1}$$

And

$$f(X_N, \theta_N) = \frac{\theta_N^4}{(\theta_N^3 + 6)}(1 + x_N^3)e^{-\theta_N x_N};\quad \theta_N,\ x_N \geq 0 \tag{2}$$

Where θ_N is the shape parameter.

The survival function $S(X_N)$ and hazard function $h(X_N)$ of NRD can be expressed as, respectively given below

$$S(X_N) = 1 - F(X_N)$$

$$S(X_N, \theta_N) = \left[1 + \frac{(\theta_N^3 x_N^3 + 3\theta_N^2 x_N^2 + 6\theta_N\ x_N)}{(\theta_N^3 + 6)}\right] e^{(-\theta_N\ x_N)} \tag{3}$$

And $$h(X_N) = \frac{f(X_N, \theta_N)}{(S(X_N)}$$

$$h(X_N) = \frac{(\theta_N^4\ (1 + x_N^3))}{(\theta_N^3(1 + x_N^3) + 3\theta_N^2 x_N^2 + 6(1 + \theta_N x_N))} \tag{4}$$

Figure 1 illustrates the CDF of the NRD for various different values of shape parameters, whereas Figure 2 displays the PDF of the NRD, which is left-skewed for shape parameter [0.1, 0.35] and right-skewed for other different shape parameters such as [0.25, 0.75], [0.5, 1.25] and [1, 2] respectively. Figures 3 and 4 present the different behaviors of the hazard function and survival function respectively. We observe that, the hazard rate increases as the indeterminacy in parameter is being increased and approximately takes on a bathtub shape. Similarly, for the survival function of Rama distribution decreases as the indeterminacy in the parameter is being increased.

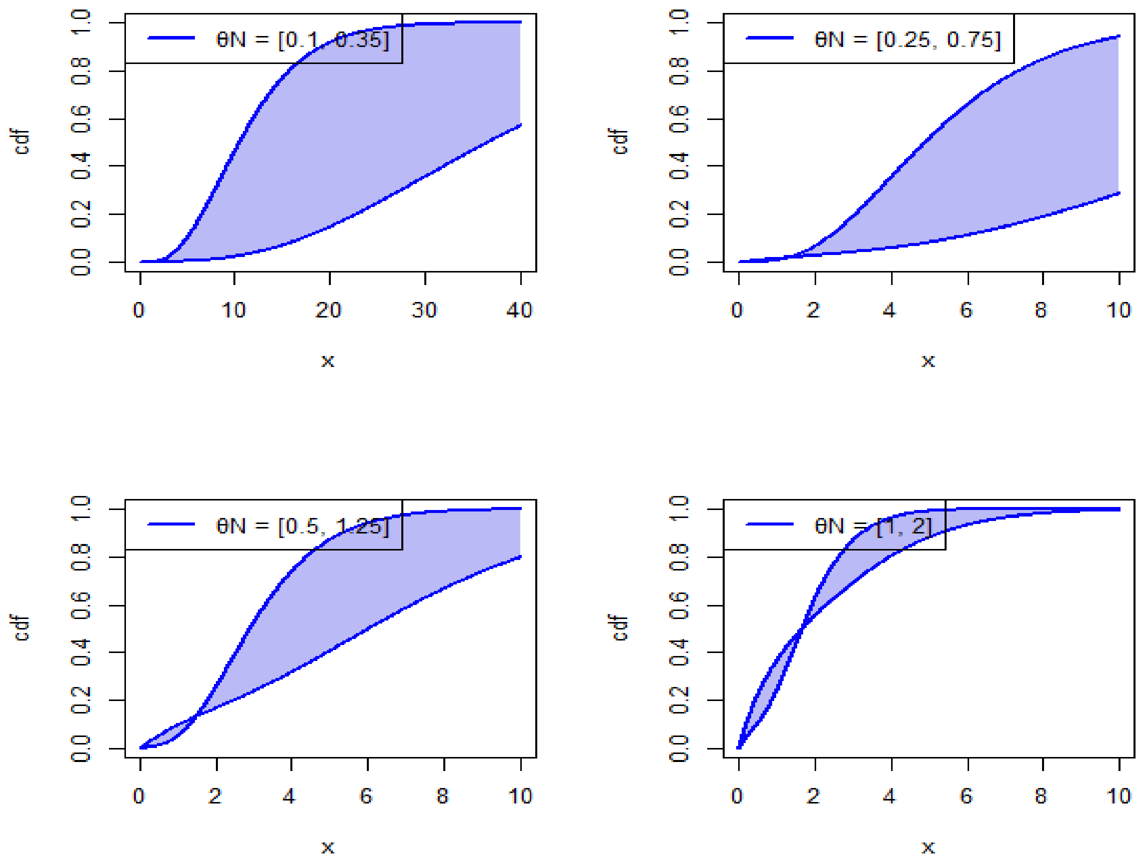


Figure 1, the various forms of the CDF curves of NRD are displayed for various shape parameters ($\theta_N = [\theta_L, \theta_U]$), such as [0.1, 0.35], [0.25, 0.75], [0.5, 1.25] and [1, 2].

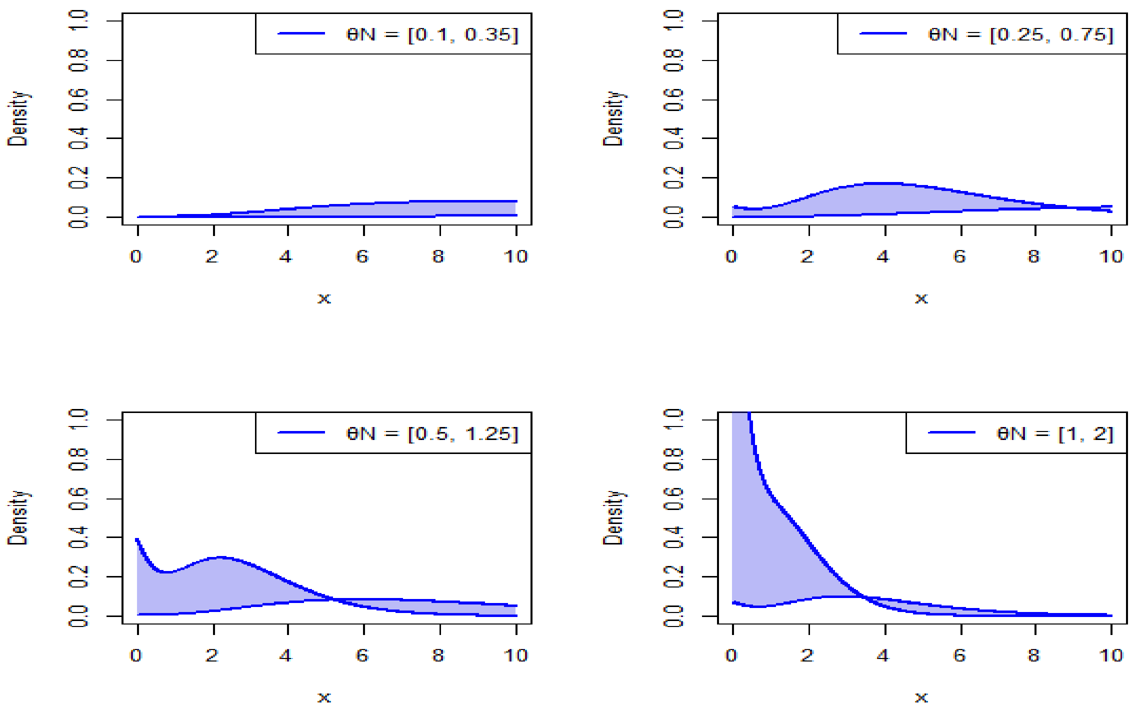


Figure 2 the various forms of the PDF curves of NRD are displayed for various shape parameters ($\theta_N = [\theta_L, \theta_U]$), such as [0.1, 0.35], [0.25, 0.75], [0.5, 1.25] and [1, 2].

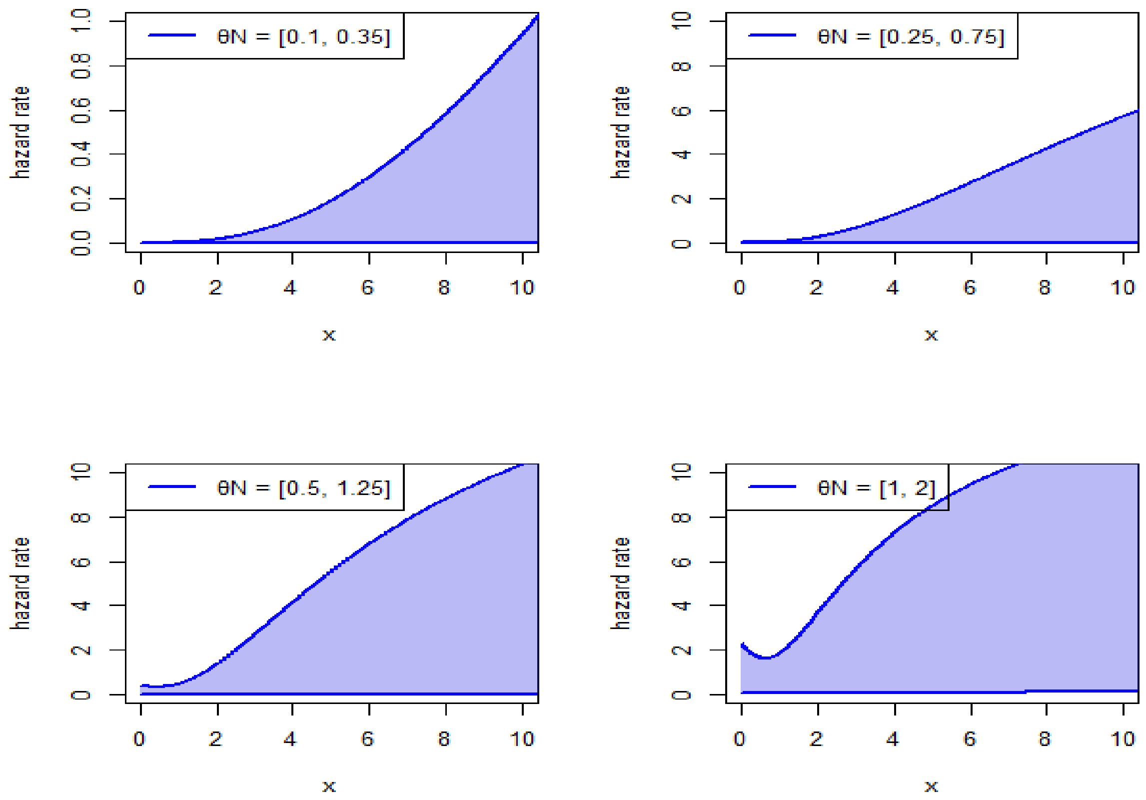


Figure 3 the various forms of hazard rates of NRD are displayed for various shape parameters ($\theta_N = [\theta_L, \theta_U]$), such as [0.1, 0.35], [0.25, 0.75], [0.5, 1.25] and [1, 2]

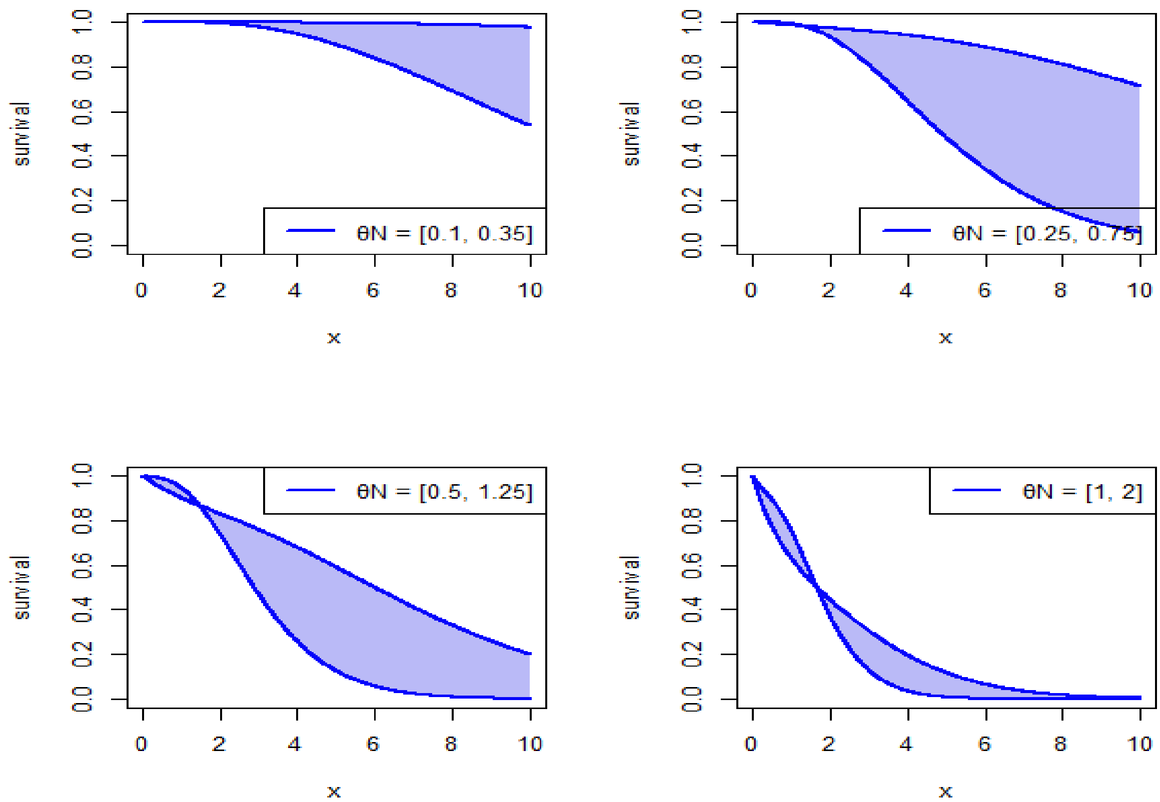


Figure 4 shows the various forms of the survival functions of NRD for various shape parameters ($\theta_N = [\theta_L, \theta_U]$), such as [0.1, 0.35], [0.25, 0.75], [0.5, 1.25] and [1, 2].

3. Some Neutrosophic Statistical Properties of NRD:

In this section, we discuss several neutrosophic statistical properties of NRD, and the results are given below:

Theorem 3. 1. The rth moment about the origin of NRD is given as:

$$\mu_r' = \frac{\theta_N^3}{(\theta_N^3 \ +6)\theta_N^r}\left[\Gamma(r+1) + \frac{\Gamma(r+4)}{\theta_N^3}\right];\ r = 1, 2, 3, \ldots \tag{5}$$

Proof: The rth moment about the origin of NRD is defined as:

$$\mu_r' = E(x_N^r) = \int_0^\infty x_N^r * f(x_N, \theta_N)\, dx_N = \int_0^\infty X_N^r \frac{\theta_N^4}{\theta_N^4 + 6}(1 + X_N^3)e^{-(\theta_N X_N)} dX_N$$

$$\mu_r' = \frac{\theta_N^4}{(\theta_N^3 + 6)}\left[\int_0^\infty x_N^r\, e^{-x_N\theta_N}\, dx_N + \int_0^\infty x_N^{r+3}\, e^{-x_N\theta_N}\, dx_N\right] \tag{5.1}$$

Let $y_N = \theta_N x_N$ differentiating y_N with respect to x_N we obtain $dy_N = \theta_N dx_N$ putting all the values in equation (5.1), we obtained

$$\mu_r' = \frac{\theta_N^3}{\theta_N^3 + 6}\left[\int_0^\infty e^{-y_N}\left(\frac{y_N}{\theta_N}\right)^r dy_N + \int_0^\infty e^{-y_N}\left(\frac{y_N}{\theta_N}\right)^{r+3} dy_N\right]$$

Using the gamma integral function $\int_0^\infty x^{r-1}e^{-x}dx = \Gamma r$, we obtain

$$\mu_r' = \frac{\theta_N^3}{\theta_N^3 + 6}\left[\frac{\Gamma r + 1}{\theta_N^r} + \frac{\Gamma r + 4}{\theta_N^{r+3}}\right]$$

$$\mu_r' = \frac{\theta_N^3}{(\theta_N^3 + 6)\theta_N^r}\left[\Gamma r + 1 + \frac{\Gamma r + 4}{\theta_N^3}\right];\ r= 1, 2, 3\ldots$$

Putting r = 1 in equation (5), we obtain the first moment about the origin, i.e., the neutrosophic mean of NRD is given as

$$E(X_N) = \mu_1' = \frac{\theta_N^3 + 24}{(\theta_N^3 + 6)\theta_N}$$

Again, putting r=2 in equation (5), we obtain the second raw moment about the origin of NRD as

$$E(X_N^2) = \mu_2' = \frac{2(\theta_N^3 + 60)}{(\theta_N^3 + 6)\theta_N^2}$$

Therefore, the neutrosophic variance of NRD is given by

$$NVar(X_N) = \mu_2^{'} - (\mu_1^{'})^2$$

$$= \frac{\theta_N^6 + 84\theta_N^3 + 144}{(\theta_N^3 + 6)^2 \theta_N^2}$$

Theorem 3.2 Median function of the NRD.

Proof: Since we know that the median of random variable of x can be found through the CDF of that distribution function, it is given as

$$\text{Median} = F_X^{-1}(0.5)$$

Therefore, the neutrosophic median of NRD is given as

$$\text{Nmedian} = M_N = F_{X_N}^{-1}(0.5)$$

$$1 - \left[1 + \frac{(\theta_N^3 M_N^3 + 3\theta_N^2 M_N^2 + 6\theta_N M_N)}{(\theta_N^3 + 6)} \times e^{(-\theta_N M_N)}\right] = 0.5$$

$$\left[1 + \frac{(\theta_N^3 M_N^3 + 3\theta_N^2 M_N^2 + 6\theta_N M_N)}{(\theta_N^3 + 6)} \times e^{(-\theta_N M_N)}\right] = 0.5$$

$$[\theta_N^3 + 6 + (\theta_N^3 M_N^3 + 3\theta_N^2 M_N^2 + 6\theta_N\ M_N)\ ] - 0.5(\theta_N^3 + 6)e^{(\theta_N\ M_N)} = 0 \qquad (6)$$

After solving equation (6), we obtain the neutrosophic median M_N.

Theorem 3.1.1 Measure of skewness and kurtosis of NRD on the basis of moments

Proof: Since we know that the skewness is defined as

$$\text{Skewness} = \gamma_1 = \sqrt{\beta_1} \quad \text{Where} \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} \qquad (7)$$

Kurtosis is defined as

$$\text{Kurtosis} = \gamma_2 = \beta_2 - 3 \quad \text{Where} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} \qquad (8)$$

There is a relationship between the moment and the row moment such that

$$\mu_0 = \mu_0' = 1$$

$$\mu_1 = \mu_1' = \text{Mean}\ ; \ \mu_2 = \mu_2^{'} - (\mu_1^{'})^2 = \text{variance}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \qquad (9)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \qquad (10)$$

Since μ_1 and μ_2 are already calculated above, only μ_3 and μ_4 are derived. By setting r =3, 4 in equation (5), we obtain

$$\mu_3^{'} = \frac{6(\theta_N^3 + 120)}{(\theta_N^3 + 6)\theta_N^3} \text{ and } \mu_4^{'} = \frac{24(\theta_N^3 + 210)}{(\theta_N^3 + 6)\theta_N^4}$$

Putting the values of $\mu_4^{'}, \mu_3^{'}, \mu_2^{'}$ and $\mu_1^{'}$ into equations (9) and (10), we obtain

$$\mu_3 = \frac{2(\theta_N^9 + 198\theta_N^6 + 324\theta_N^3 + 864)}{(\theta_N^3 + 6)^3 \theta_N^3}$$

$$\mu_4 = \frac{9(\theta_N^{12} + 312\theta_N^9 + 2304\theta_N^6 + 10368\theta_N^3 + 10368)}{(\theta_N^3 + 6)^4 \theta_N^4}$$

By substituting the values of μ_2, μ_3 and μ_4 into equations (7) and (8), we obtain

$$\gamma_1 = \frac{2(\theta_N^9 + 198\theta_N^6 + 324\theta_N^3 + 864)}{(\theta_N^6 + 84\theta_N^3 + 144)^{\frac{3}{2}}}$$

$$\gamma_2 = \frac{9(\theta_N^{12} + 312\theta_N^9 + 2304\theta_N^6 + 10368\theta_N^3 + 10368)}{(\theta_N^6 + 84\theta_N^3 + 144)^2} - 3$$

Since $\gamma_1 > 0$ and $\gamma_2 > 0$ or$(\beta_2 > 3)$, this implies that NRD is a positively skewed and leptokurtic distribution.

Theorem 3.3 Moment generating function of NRD

Proof: The moment generating function of NRD is defined as

$$M_{X_N}(t) = E(e^{(X_N t)}) = \int_0^\infty e^{(X_N t)} \frac{\theta_N^4}{\theta_N^3 + 6}(1 + X_N^3)e^{(-\theta_N X_N)} dX_N$$

$$= \frac{\theta_N^4((\theta_N - t)^3 + 6)}{(\theta_N^3 + 6)(\theta_N - t)^4}; \text{ Where } t < \theta_N$$

4. Examples:

This section demonstrates the concept of NRD through examples drawn from survival analysis. **Example 1**. Let mortality rate under five age of rural residence is adequately followed by a neutrosophic Rama random variable with parameter θ_N = [2.1, 2.9]. What is the probability that the life span exceed 3 years?

Solution from CDF

$$P[T > 3] = 1 - F_N(3, \theta_N)$$

$$= \left[1 + \frac{\left(\theta_N^3 t^3 + 3\theta_N^2 t^2 + 6\theta_N t\right)}{(\theta_N^3 + 6)}\right] e^{(-\theta_N t)}$$

$$= \left[1 + \frac{(([2.1, 2.9] * 3)^3 + 3([2.1, 2.9] * 3)^2 + 6[2.1, 2.9] * 3)}{(([2.1, 2.9])^3 + 6)}\right] e^{(-[2.1,2.9]*3)}$$

$$= \left(1 + \frac{[406.917, 937.773]}{[15.261, 30.389]}\right) * [e^{-6.3}, e^{-8.7}]$$

$$= [14.3909, 62.4490] * [e^{-6.3}, e^{-8.7}]$$

$$= [0.0104, 0.0264]$$

Thus, the survival probability that lifetime exceeds 3 years with given neutrosophic parameters is approximately [1.04, 2.6] %.

Example 2. Let the mortality rate under five age group is modeled as neutrosophic Rama distribution with [3, 4] years being neutrosophic mean. Find the parameters θ_N of NRD.

Solution: Given that

$$\frac{\theta_N^3 + 24}{(\theta_N^3 + 6)\theta_N} = [3,4]$$

$$\left[\frac{\theta_l^3 + 24}{(\theta_u^3 + 6)\theta_u}, \frac{\theta_u^3 + 24}{(\theta_l^3 + 6)\theta_l}\right] = [3, 4]$$

$$3(\theta_u^3 + 6)\theta_u = (\theta_l^3 + 24) \tag{11}$$

$$4(\theta_l^3 + 6)\theta_l = (\theta_u^3 + 24) \tag{12}$$

After solving equation [11] and equation [12] we get the value of parameter θ_N = [0.932, 1.118]

5. Neutrosophic Parametric Estimation of NRD

Let $x_{N_1}, x_{N2}, x_{N3}, \ldots\ldots x_{Nn}$ be the sample of NRD; then, the likelihood function is given as

$$l(\theta_N) = n\log\left(\frac{\theta_N^4}{(\theta_N^3 + 6)}\right) + \sum_{i=1}^{n} \log(1 + x_{Ni}^3) - \sum_{i=1}^{n} \theta_N x_{Ni} \tag{13}$$

The NMLE (neutrosophic maximum likelihood estimate) of a parameter can be obtained by maximizing equation (13) with respect to θ_N .Therefore, differentiating equation (13) with respect to θ_N and equal to zero, we obtain

$$\theta_N^4 \overline{X}_N - \theta_N^3 + 6\theta_N \overline{X}_N - 24 = 0$$

Using the Newton–Raphson method, we can find the NMLE of θ_N .

6. Simulation studies

In Table 1, the neutrosophic mean, neutrosophic variance, neutrosophic median, neutrosophic skewness and neutrosophic kurtosis of the NRD are shown for various shape parameters. With increasing values of shape parameters with indeterminacy, all these statistical properties of NRD decrease except neutrosophic skewness and kurtosis. For example, when $[\theta_l, \theta_u] < 1$ and indeterminacy (I = 0.25), the neutrosophic mean, variance and median have higher indeterminacy, and neutrosophic skewness and kurtosis have lower indeterminacy than $[\theta_l, \theta_u] < 1$ with indeterminacy (I = 0.50). Similarly, when $[\theta_l, \theta_u] > 1$ with indeterminacy (I = 1), the neutrosophic mean, variance and median have minimum indeterminacy, but neutrosophic skewness and kurtosis have the highest indeterminacy.

Table 1 Neutrosophic mean, variance, median, skewness and kurtosis of the NRD for different values of the shape parameter.

θ_N	Mean	Variance	Median	Skewness	Kurtosis
[0.1, 0.35]	[11.35, 40.06]	[32.21, 409.87]	[10.45, 36.72]	[0.96, 1.02]	[3.3344, 3.639]
[0.25,0.75]	[4.99, 16.24]	[6.26, 79.41]	[4.69, 14.66]	[0.72, 1.18]	[2.2832, 4.9403]
[0.5, 1.25]	[2.43, 8.47]	[1.56, 33.25]	[2.40, 7.25]	[0.33, 2.35]	[0.8372, 11.663]
[1, 2]	[0.89, 4.57]	[0.29, 17.96]	[0.83, 3.31]	[0.11, 9.60]	[0.2037, 54.672]

Next, we present a simulation study under a neutrosophic environment to estimate the NMLE of the parameters via R software (R version 4.3.3, in 2024). For the simulation, we generate N=10000 random samples of size n= 30, 50, 100, 200 and 500 from the neutrosophic Rama distribution for various shape parameters, i.e., [01, 0.35], [0.25, 0.75], [0.5, 1.25] and [1, 2], respectively. The simulation results are obtained and presented in Table 2.

Table 2 Simulation study of NRD for various actual shape parameters.

n	AEs	ABs	MSEs
$\theta_N = [0.1, 0.35]$			
30	[0.1421, 0.3539]	[0.421, 0.0039]	[0.0019, 0.0010]
50	[0.1417, 0.3529]	[0.0417, 0.0029]	[0.0018, 0.0006]
100	[0.1412, 0.3518]	[0.0412, 0.0018]	[0.0017, 0.0030]
200	[0.1410, 0.3516]	[0.041, 0.0016]	[0.0017, 0.0002]
500	[0.1407, 0.3515]	[0.0407, 0.0015]	[0.0016, .00006]
$\theta_N = [0.25, 0.75]$			
30	[0.2583, 0.7533]	[0.0083, 0.0033]	[0.0006, 0.0045]
50	[0.2578, 0.7507]	[0.0078, 0.0066]	[0.0034, 0.0026]
100	[0.2578, 0.7497]	[0.0078,-0.0003]	[0.0002, 0.0013]
200	[0.2575, 0.7498]	[0.0075,-0.0002]	[0.0001, 0.0007]
500	[0.2575, 0.7495]	[0.0075,-0.0005]	[0.000086,0.00027]
$\theta_N = [0.5, 1.25]$			
30	[0.5032, 1.2554]	[0.0032, 0.0054]	[0.0019, 0.0115]
50	[0.5015, 1.2541]	[0.0015, 0.0041]	[0.0012, 0.0067]
100	[0.5009, 1.2531]	[0.0009, 0.0031]	[0.0006, 0.0034]
200	[0.5007, 1.2519]	[0.0007, 0.0019]	[0.0003, 0.0016]
500	[0.5004, 1.2516]	[0.0004, 0.0016]	[0.0001, 0.0007]
$\theta_N = [1, 2]$			
30	[1.0043, 2.0197]	[0.0043, 0.0197]	[0.0072, 0.0335]
50	[1.0014, 2.0128]	[0.0014, 0.0128]	[0.0043, 0.0194]
100	[1.0002, 2.0071]	[0.0002, 0.0071]	[0.0023, 0.0088]

200	[1.00002,2.0031]	[0.000028,0.0031]	[0.0012, 0.0047]
500	[1.0005, 2.0016]	[0.0005, 0.0016]	[0.00045, 0.0018]

Table 2 presents the results of the simulation study, including the average estimates (AEs), average bias (ABs), and mean square errors (MSEs) for various chosen shape parameters. As the sample size increases, the ABs and MSEs decrease, and the AEs approach the true values of the shape parameters. This indicates that the NMLE behaves as a consistent estimator. With a sufficiently large sample size, it provides estimates of the parameters that are approximately accurate.

7. Real data applications:

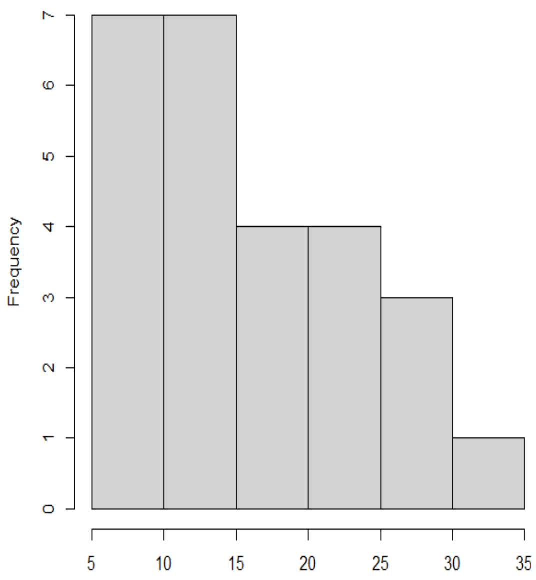
In this section, the neutrosophic Rama distribution is applied to a real dataset; for this purpose, we use the under the age of five mortality rates, covering the period 1995-2020 for Saudi Arabia, as represented in Table 3 [21]. The parameter estimations under different neutrosophic models that use the same dataset are presented in Table 4 for model comparison. The AIC, BIC, and K-S test criteria are used to determine the compatibility of the proposed model with several existing neutrosophic models, such as the neutrosophic exponential distribution (NED), neutrosophic Weibull distribution (NWD) and neutrosophic Rayleigh distribution (NRAD). In Figure 5, a visual examination of the data suggests that the Rama distribution fits the mortality data well, as the observations closely follow the straight line. However, since interval mortality rates for the under-five age group are used in this study, the conventional Rama distribution analysis is suitable. The proposed model allows for data summarization while accounting for inherent uncertainties. Table 5 provides a descriptive summary of the mortality statistics using the proposed neutrosophic model.

Table 3. Under age five mortality rates (in neutrosophic form) of Saudi Arabia 1995- 2020.

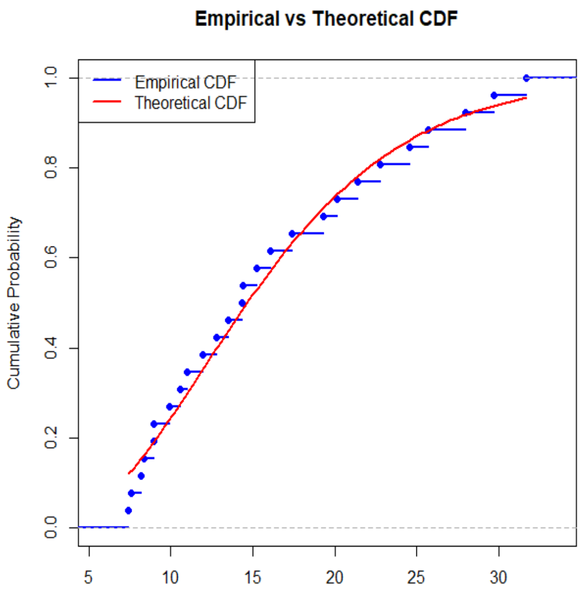
Under age five mortality rates
[31.53, 31.81], [20.66, 22.09], [14.51, 15.92], [10.12, 11.03], [7.23, 7.98], [29.33, 30.08], [19.74, 20.59], [13.92, 14.71], [9.12, 10.69], [6.81, 8.06], [27.23, 28.67], [18.57, 20.03], [12.73, 14.32], [8.47, 9.42], [25.09, 26.34], [18.04, 18.77], [12.20, 13.35], [8.59, 9.28], [24.20, 24.88], [16.89, 17.89], [11.18, 12.68], [7.65, 9.03], [22.00, 23.50], [15.92, 16.21], [10.21, 11.75], [7.77, 8.59]

Table 4. Estimates and goodness-of-fit statistics for the underage five mortality rates dataset.

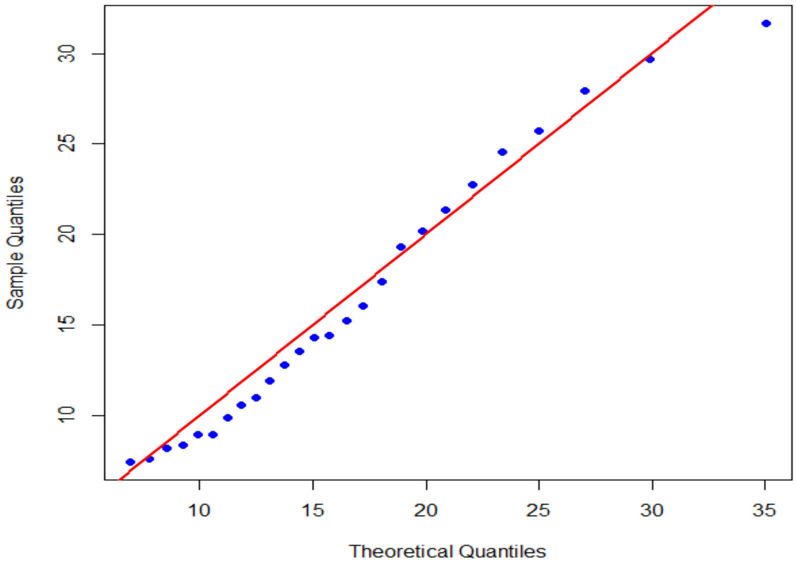
Model (Parameter)	**Estimates**	**Log likelihood**	**AIC**	**BIC**	**K-S test**
NED (scale)	[0.059,0.065]	[-97.271,-99.506]	[196.543,201.012]	[197.801,202.27]	[0.333,0.403]
NWD (shape and scale)	[2.289,2.55] [17.538,19.46]	[-86.831,-86.910]	[177.663,177.819]	[180.335,180.179]	[0.551,0.661]
NRAD (scale)	[12.081,12.93]	[-87.212,-88.062]	[176.423,178.124]	[177.681,179.382]	[0.555,0.566]
NRD (shape)	[0.235, 0.258]	[-98.509,-99.827]	[175.018,176.619]	[176.276,177.877]	[0.801, 0.872]



(a)



(b)



(c)

Figure 5. (a) Histogram of mortality rate under age five data; **(b)** Empirical and theoretical CDF plot and **(c)** The quantile-quantile plot of the mortality rate under age five data.

Table 5. Descriptive statistics of mortality rate under age five group data set using suggested distribution

Descriptive statistics	Estimated values
Mean	[15.46799 16.99669]
Standard deviation	[7.759022 8.534710]
Estimated parameter	[0.235, 0.258]

Table 5 presents the estimated uncertainty bounds for key statistics based on the proposed distribution. Since the dataset under study contains inherent imprecision, all estimated values are expressed as intervals. This makes the proposed model more flexible and better suited for effectively analyzing imprecise datasets**.**

8. Conclusions

In this paper, the neutrosophic Rama distribution is proposed. The statistical properties of the neutrosophic Rama distribution are discussed in the form of the neutrosophic mean, neutrosophic variance, neutrosophic median, neutrosophic moment generating function and neutrosophic rth raw moments of NRD along with neutrosophic skewness and kurtosis. A simulation study is applied for parameter estimation, and the average bias and MSEs of the neutrosophic Rama distribution are given. On the basis of the simulation study, the NMLE is observed as a consistent estimator and converges to the true value of the parameter as the sample size increases. Finally, the proposed neutrosophic distribution is fitted to the neutrosophic mortality rates under- five age group of Saudi Arabia and compared with selected existing neutrosophic models, such as the neutrosophic exponential distribution, neutrosophic Weibull distribution and neutrosophic Rayleigh distribution. On the basis of the AIC, BIC and K-S test results, the proposed model was better than the other selected models.

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