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Some Remarks on Invertible Refined Neutrosophic Square Matrices

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Abstract. The main goal of this article is to define the adjoint of refined (resp., n-refined) neutrosophic square matrices, and to study the inverse of these matrices in terms of refined (resp., n-refined) neutrosophic determinants and adjoints. Also, it presents some modified results and examples on the inverse of refined (resp., n-refined) neutrosophic square matrices.

Keywords: Refined neutrosophic matrix; n-refined neutrosophic matrix; adjoint matrix; determinant; inverse.

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1. **Introduction**

Smarandache [15] introduced Neutrosophy as a new approach of generalized logic, aiming to address the uncertainties and indeterminacies present in real-world problems. He outlined the definition of the standard neutrosophic real number and established the conditions under which the division of two neutrosophic real numbers is possible.

In [10], Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures, based on the idea of indeterminacy (I). They extended several algebraic structures, such as groups, rings, fields and vector spaces for neutrosophic case.

By further subdividing the indeterminacy I into multiple levels such as $I_1, I_2, \cdots, I_n$, we obtain refined and n-refined neutrosophic structures, including groups [6], modules [13], and vector [9].

Recent works have significantly advanced the theoretical and applied aspects of fuzzy and neutrosophic structures, including the development of various neutrosophic ideals in ternary semigroups [16], the exploration of intuitionistic and fuzzy algebraic systems [17, 21, 24], novel weighted and trigonometric operators in complex fuzzy environments [18, 19], and the integration of neutrosophic models in graph theory and numerical analysis [22, 23]; these efforts are further extended by applications in intelligent systems, health technologies, and topological frameworks [20, 25–27, 29, 30].

Recently, the concept of vector spaces and linear transformations are generalized to neutrosophic matrices [7, 8, 11], refined neutrosophic matrices [2] and n-refined neutrosophic matrices [3]. Abobala et al. [4] examined the algebraic properties (inverse, determinant and diagonalization) of neutrosophic matrices and their representation through linear transformations [28].

By generalizing these concepts, Mohammad Abobala discussed existence of inverse for refined neutrosophic matrices in [2] and n-refined neutrosophic matrices in [3].

Neutrosophic matrix theory was studied by many researchers, where we find the inverse of neutrosophic square matrices using neutrosophic adjoint and neutrosophic determinant [12].

Inspired by these studies, the current article examines the challenge of determining the inverse of refined (resp., n-refined) neutrosophic square matrices, utilizing refined (resp., n-refined) neutrosophic adjoints and determinants.

Through examples, it is demonstrated that the converse of the statement "the refined (resp., n-refined) matrix M is invertible if and only if $det(M) \neq 0$" is false, a result previously established by Mohammad Abobala in references [2] and [3]. Additionally, various properties of the adjoint of refined (resp., n-refined) neutrosophic matrices are explored.

2. Preliminary

In this section, we revisits the concepts of neutrosophic matrices, refined neutrosophic matrices, n-refined neutrosophic matrices, along with the associated results.

Definition 2.1. [5] Let K be a field. Then the neutrosophic field corresponding to K associated with the inderminancy I is defined by $K(I) = \langle K \cup I \rangle$.

Definition 2.2. [14] T he neutrosophic matrix is defined by $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field.

Definition 2.3. [1] If X is a set, then $X(I_1, I_2) = \{(a, bI_1, cI_2) : a, b, c \in X\}$ is called the refined neutrosophic set generated by X, I_1 and I_2.

Definition 2.4. [5] Let $(R, +, \cdot)$ be a ring then $(R(I_1, I_2), +, \cdot)$ is called a refined neutrosophic ring generated by X, I_1 and I_2. Where $I_1^2 = I_1, I_2^2 = I_2, I_1I_2 = I_2I_1 = I_1$.

Definition 2.5. [2] Let $M = \{(a_{ij}) : a_{ij} \in K(I)\}$ where $K(I)$ is a refined neutrosophic field. We refer this to be the refined neutrosophic matrix.

Definition 2.6. [2] The determinant of a $n \times n$ refined neutrosophic matrix $M = M_0 + M_1I_1 + M_2I_2$ is defined as

$$detM = detM_0 + [det(M_0 + M_1 + M_2) - det(M_0 + M_2)]I_1 + [det(M_0 + M_2) - detM_0]I_2.$$

Theorem 2.7. *[2] The inverse of a $n\times n$ refined neutrosophic matrix $M = M_0+M_1I_1+M_2I_2$ exists if and only if M_0^{-1}, $(M_0 + M_2)^{-1}$ and $(M_0 + M_1 + M_2)^{-1}$ exists. The inverse of M is*

$$M^{-1} = M_0^{-1} + \left[(M_0 + M_1 + M_2)^{-1} - (M_0 + M_2)^{-1}\right] I_1 + \left[(M_0 + M_2)^{-1} - M_0^{-1}\right] I_2.$$

Theorem 2.8. *[2] Let $M = M_0 + M_1I_1 + M_2I_2$ and $N = N_0 + N_1I_1 + N_2I_2$ be two $n \times n$ refined neutrosophic matrices. Then the following results holds.*

(1) *M^{-1} exists if and only if $detM \neq 0$.*
(2) *$detMN = detMdetN$.*
(3) *$detM^{-1} = (detM)^{-1}$.*

Definition 2.9. [3] The n-refined neutrosophic matrix is defined by

$$A = \left\{(a_{ij})_{m\times n} : a_{ij} = a_0 + a_1I_1 + a_2I_2 + \cdots + a_nI_n \in R_n(I)\right\}$$

where $R_n(I)$ is a n-refined neutrosophic ring.

Definition 2.10. [3] The determinant of a $m \times m$ n-refined neutrosophic matrix $M = M_0 + M_1I_1 + \cdots + M_nI_n$, is defined by

$$detM = detA_0 + (detA_n - detA_0)I_n + \sum_{i=1}^{n-1}(detA_i - detA_{i+1})I_i$$

where, $A_0 = M_0$, $A_j = M_0 + M_j + M_{j+1} + \cdots + M_n$, $1 \le j \le n$.

Theorem 2.11. *[3] A $m \times m$ n-refined neutrosophic matrix $M = M_0 + M_1I_1 + M_2I_2 + \cdots + M_nI_n$ is invertible if and only if $A_j, 0 \le j \le n$ are invertible, where $A_0 = M_0$, $A_j = M_0 + M_j + M_{j+1} + \cdots + M_n$. The inverse of M is*

$$M^{-1} = (A_0)^{-1} + \left[(A_n)^{-1} - (A_0)^{-1}\right] I_n + \sum_{i=1}^{n-1}\left[(A_i)^{-1} - (A_{i+1})^{-1}\right] I_i$$

Theorem 2.12. *[3] Let $M = M_0 + M_1I_1 + M_2I_2 + \cdots + M_nI_n$ and $N = N_0 + N_1I_1 + N_2I_2 + \cdots + N_nI_n$ be $m \times m$ n-refined neutrosophic matrices. The the followig results holds.*

(a) M^{-1} *exists if and only if* $detM \neq 0$.

(b) $detMN = detMdetN$.

(c) $detM^{-1} = (detM)^{-1}$.

3. Invertible Refined Neutrosophic Square Matrices

Definition 3.1. The adjoint matrix of refined neutrosophic square matrix $M = M_0 + M_1I_1 + M_2I_2$ is defined as

$$adjM = adjM_0 + [adj(M_0 + M_1 + M_2) - adj(M_0 + M_2)]I_1 + [adj(M_0 + M_2) - adjM_0]I_2.$$

Example 3.2. Let

$$M = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$$

Here,

$$M_0 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, M_0 + M_2 = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}, M_0 + M_1 + M_2 = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix},$$

and,

$$adjM_0 = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}, adj(M_0 + M_2) = \begin{pmatrix} 1 & 0 \\ -7 & 5 \end{pmatrix}, adj(M_0 + M_1 + M_2) = \begin{pmatrix} 2 & 1 \\ -7 & 6 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} adjM &= adjM_0 + [adj(M_0 + M_1 + M_2) - adj(M_0 + M_2)]I_1 + [adj(M_0 + M_2) - adjM_0]I_2 \\ adjM &= \begin{pmatrix} 1 + I_1 & -1 + I_1 + I_2 \\ -3 - 4I_2 & 2 + I_1 + 3I_2 \end{pmatrix} \end{aligned}$$

Remark 3.3. The converse of the Theorem 2.8 is not true. We illustrate this with following examples.

Example 3.4. Let

$$M = \begin{pmatrix} 2I_1 + I_2 & 4 + 3I_1 \\ -I_1 + I_2 & 2 + 4I_2 \end{pmatrix}.$$

Then $detM = 16I_1 + 2I_2 \neq 0$. Here, $M_0 = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$ and $detM_0 = 0$. Therefore, M is not invertible and hence M is not invertible by Theorem 2.7.

Example 3.5. Let

$$M = \begin{pmatrix} 1 + I_1 & -2 + I_2 \\ -I_2 & 1 + 2I_1 \end{pmatrix}.$$

Then $detM = 1 + 5I_1 - I_2 \neq 0$.
Here, $M_0 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$M_0 + M_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $det(M_0 + M_2) = 0$, this implies that the inverse of $M_0 + M_2$ does not exits. Hence by Theorem 2.7, M is not invertible

Example 3.6. Let

$$M = \begin{pmatrix} 1 + I_1 & -2 + I_2 \\ 0 & 1 + 2I_1 - 3I_2 \end{pmatrix}.$$

$detM = 1 + 2I_1 - 3I_2 \neq 0$.
Here, $M_0 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}$.
Therefore, $M_0 + M_1 + M_2 = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$, $det(M_0 + M_1 + M_2) = 0$, this implies that the inverse of $M_0 + M_1 + M_2$ does not exits. Hence by Theorem 2.7, M is not invertible

Remark 3.7. Let $a + bI_1 + cI_2$ and $p + qI_1 + rI_2$ be two refined neutrosophic numbers. As in the Definition 3.1, we can prove that

$$\frac{a + bI_1 + cI_2}{p + qI_1 + rI_2}$$

is exists if and only if $p \neq 0$, $p + r \neq 0$ and $p + q + r \neq 0$.

We can revise the Theorem 2.7 as follows:

Theorem 3.8. *A square refined neutrosophic matrix $M = M_0 + M_1I_1 + M_2I_2$ is invertible if and only if $detM_0 \neq 0, det(M_0 + M_2) \neq 0$, $det(M_0 + M_1 + M_2) \neq 0$ and*

$$M^{-1} = \frac{1}{detM}(adjM).$$

Proof. By Remark 3.7,

$$\frac{1}{detM} = \frac{1}{detM_0 + [det(M_0 + M_1 + M_2) - det(M_0 + M_2)]I_1 + [det(M_0 + M_2) - det(M_0)]I_2}$$

exists only if

$$detM_0 \neq 0, \quad det(M_0 + M_2) \neq 0 \quad \text{and} \quad det(M_0 + M_1 + M_2) \neq 0.$$

Also,

$$\begin{aligned}
\frac{1}{detM}(adjM) &= \left(\frac{1}{detM + 0 + [det(M_0+M_1+M_2) - det(M_0+M_2)]I_1 + [det(M_0+M_2) - det(M_0)]I_2}\right) \\
&\quad (adjM_0 + [adj(M_0+M_1+M_2) - adj(M_0+M_2)]I_1 + [adj(M_0+M_2) - adjM_0]I_2) \\
&= \frac{adjM_0}{detM_0} + \left[\frac{adj(M_0+M_1+M_2)}{det(M_0+M_1+M_2)} - \frac{adj(M_0+M_2)}{det(M_0+M_2)}\right] I_1 + \left[\frac{adj(M_0+M_2)}{det(M_0+M_2)} - \frac{adjM_0}{detM_0}\right] I_2 \\
&= {M_0}^{-1} + \left[(M_0+M_1+M_2)^{-1} - (M_0+M_2)^{-1}\right] I_1 + \left[(M_0+M_2)^{-1} - {M_0}^{-1}\right] I_2 \\
&= M^{-1}
\end{aligned}$$

Therefore, the result follows from Theorem 2.7. □

Example 3.9. Consider the refined neutrosophic matrix

$$M = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix}$$

where, $M_0 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $M_0 + M_2 = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$ and $M_0 + M_1 + M_2 = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$.

$detM_0 = -1$, $det(M_0+M_2) = 5$ and $det(M_0+M_1+M_2) = 19$.
Since $detM_0$, $det(M_0+M_2)$ and $det(M_0+M_1+M_2)$ are non-zero, M is invertible.

Also, $detM = -1 + 14I_1 + 6I_2$ and $adjM = \begin{pmatrix} 1+I_1 & -1+I_1+I_2 \\ -3-4I_2 & 2+I_1+3I_2 \end{pmatrix}$.
Then,

$$\begin{aligned}
M^{-1} &= \frac{1}{detM}(adjM) \\
&= \frac{1}{-1+14I_1+6I_2}\begin{pmatrix} 1+I_1 & -1+I_1+I_2 \\ -3-4I_2 & 2+I_1+3I_2 \end{pmatrix} \\
&= \left(-1 - \frac{14}{95}I_1 + \frac{6}{5}I_2\right)\begin{pmatrix} 1+I_1 & -1+I_1+I_2 \\ -3-4I_2 & 2+I_1+3I_2 \end{pmatrix} \\
&= \begin{pmatrix} -1 - \frac{9}{95}I_1 + \frac{6}{5}I_2 & 1 + \frac{1}{19}I_1 - I_2 \\ 3 + \frac{98}{95}I_1 - \frac{22}{5}I_2 & -2 - \frac{13}{19}I_1 + 3I_2 \end{pmatrix}
\end{aligned}$$

Theorem 3.10. *If $M = M_0 + M_1I_1 + M_2I_2$ and $N = N_0 + N_1I_1 + N_2I_2$ are two invertible refined neutrosophic square matrices, then MN is also invertible and $(MN)^{-1} = N^{-1}M^{-1}$.*

Proof. By Theorem 3.8, if M is invertible then

$$det(M_0) \neq 0,\ det(M_0+M_2) \neq 0 \text{ and } det(M_0+M_1+M_2) \neq 0.$$

Similarly, if N is invertible then

$$detN_0 \neq 0 \text{ , } det(N_0+N_2) \neq 0 \text{ and } det(N_0+N_1+N_2) \neq 0.$$

This implies that,

$$det(M_0N_0) = detM_0 \; detN_0 \neq 0, \quad det[(M_0+M_2)(N_0+N_2)] = det(M_0+M_2) \; det(N_0+N_2) \neq 0$$
$$\text{and} \quad det[(M_0+M_1+M_2)(N_0+N_1+N_2)] = det(M_0+M_1+M_2) \; det(N_0+N_1+N_2) \neq 0.$$

$$det(MN) =$$
$$det(M_0N_0) + [det((M_0+M_1+M_2)(N_0+N_1+N_2))]I_1 + [det((M_0+M_2)(N_0+N_2))]I_2 \neq 0$$

Therefore, MN is invertible. By associate law of matrix multiplication, we have

$$(MN)(N^{-1}M^{-1}) = M(NN^{-1})M^{-1} = U_{n\times n}$$
$$(N^{-1}M^{-1})(MN) = N^{-1}(M^{-1}M)N = U_{n\times n}.$$

Thus, $(MN)^{-1} = N^{-1}M^{-1}$. □

4. **Invertible n-Refined Neutrosophic Square Matrices**

Definition 4.1. The adjoint matrix of a $m \times m$ n-refined neutrosophic matrix $M = M_0 + M_1I_1 + \cdots + M_nI_n$ is defined by

$$adjM = adjA_0 + (adjA_n - adjA_0)I_n + \sum_{i=1}^{n-1}(adjA_i - adjA_{i+1})I_i$$

where, $A_0 = M_0$, $A_j = M_0 + M_j + M_{j+1} + \cdots + M_n$ for $1 \leq j \leq n$.

Example 4.2. Let

$$M = \begin{pmatrix} 2+I_1+3I_2+I_3 & 1-I_1-I_2-I_3 \\ 3+4I_2 & 1+I_1+2I_3 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} I_3.$$

Here, $M_0 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$ and $M_3 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Therefore,

$adjM = adjM_0 + [adj(M_0+M_1+M_2+M_3) - adj(M_0+M_2+M_3)]I_1 + [adj(M_0+M_2+M_3) - adj(M_0+M_3)]I_2 + [adj(M_0+M_3) - adj(M_0)]I_3 = \begin{pmatrix} 1+I_1+2I_3 & -1+I_1+I_2+I_3 \\ -3-4I_2 & 2+I_1+3I_2+I_3 \end{pmatrix}$

Remark 4.3. The result in the Theorem 2.12, "an n-refined neutrosophic square matrix X is invertible if and only if $detX \neq 0$" does not hold in reverse. The following examples illustrate this.

Example 4.4. Let

$$M = \begin{pmatrix} 2I_1 + I_2 + I_3 & 3 + I_2 \\ 0 & I_1 \end{pmatrix}.$$

Then $detM = 2I_1 + I_1 + I_1 = 4I_1 \neq 0$.

Here, $A_0 = M_0 = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ and $detA_0 = 0$. Therefore the inverse of A_0 does not exists. By Theorem 2.11, M is not invertible .

Example 4.5. Let

$$M = \begin{pmatrix} I_1 + I_3 & 0 \\ I_2 + 2I_3 & I_2 \end{pmatrix}.$$

Then $detM = I_1 + I_2 \neq 0$.

Here, $A_3 = M_0 + M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, $det(A_3) = 0$, A_3 is not invertible hence M is not invertible by Theorem 2.11.

Remark 4.6. Let $a_0 + a_1I_1 + a_2I_2 + \cdots + a_nI_n$ and $b_0 + b_1I_1 + b_2I_2 + \cdots + b_nI_n$ be two n-refined neutrosophic numbers. We can prove that

$$\frac{a_0 + a_1I_1 + a_2I_2 + \cdots + a_nI_n}{b_0 + b_1I_1 + b_2I_2 + \cdots + b_nI_n}$$

is exists if and only if $m_i \neq 0,\ 0 \leq i \leq n$, where $m_0 = b_0, m_j = b_0 + b_j + b_{j+1} + \cdots + b_n,\ \ 1 \leq j \leq n$.

Now, we can revise the Theorem 2.11 in the following manner.

Theorem 4.7. *A $m \times m$ n-refined neutrosophic matrix $M = M_0 + M_1I_+ \cdots + M_nI_n$ is invertible if and only if $detA_j \neq 0, 0 \leq j \leq n$, where, $A_0 = M_0$, $A_j = M_0 + M_j + M_{j+1} + \cdots + M_n$ and*

$$M^{-1} = \frac{1}{detM}(adjM).$$

Proof. By Remark 4.6,

$$\frac{1}{detM} = \frac{1}{detA_0 + (detA_n - detA_0)I_n + \sum_{i=1}^{n-1}(detA_i - detA_{i+1})I_i}$$

exists only if $detA_0 \neq 0$, and $detA_j \neq 0, 1 \leq j \leq n$.

Also,

$$
\begin{aligned}
\frac{1}{detM}(adjM) &= \left(\frac{1}{detA_0 + (detA_n - detA_0)I_n + \sum_{i=1}^{n-1}(detA_i - detA_{i+1})I_i}\right) \\
&\quad \left(adjA_0 + (adjA_n - adjA_0)I_n + \sum_{i=1}^{n-1}(adjA_i - adjA_{i+1})I_i\right) \\
&= \frac{adjA_0}{detA_0} + \left[\frac{adjA_n}{detA_n} - \frac{adjA_0}{detA_0}\right] I_n + \sum_{i=1}^{n-1}\left[\frac{adjA_i}{detA_i} - \frac{adjA_{i+1}}{detA_{i+1}}\right] I_i \\
&= (A_0)^{-1} + \left[(A_n)^{-1} - (A_0)^{-1}\right] I_n + \sum_{i=1}^{n-1}\left[(A_i)^{-1} - (A_{i+1})^{-1}\right] I_i \\
&= M^{-1}
\end{aligned}
$$

□

Theorem 4.8. *If M and N are $m \times m$ n-refined neutrosophic invertible matrices then MN is also invertible and $(MN)^{-1} = N^{-1}M^{-1}$.*

Proof. The proof is similar to that of Theorem 3.10. □

Theorem 4.9. *Consider two $m \times m$ refined (resp., n-refined) neutrosophic invertible neutrosophic matrices M and N. Then we have:*

(1) $det(adjM) = (detM)^{m-1}$.
(2) $adj(MN) = adjM\ adjN$.
(3) $adj(M^k) = (adjM)^k$ *for any positive integer* k.
(4) $adj(M^T) = (adjM)^T$.
(5) $adj(adjM) = (detM)^{m-2}M$

Proof. We can easily prove the above results. □

5. Conclusions

In this article, we explored the concept of the adjoint of refined (or n-refined) neutrosophic square matrices and analyzed their properties. Also, we established the existence conditions for invertibility. Furthermore, we present multiple illustrative examples to validate our results.

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References

[1] Abdel-Basset, M., Gamal, A., Son, L. H., and Smarandache, F., *A bipolar neutrosophic multi criteria decision making framework for professional selection*, Applied Sciences, Vol. 10, pp. 1-22, 2020.

[2] Abobala, M., *On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations*, Journal Of Mathematics, Article ID 5531093, 2021.

[3] Abobala, M., *On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations*, Mathematical Problems in Engineering, Article ID 5573072, 2021.

[4] Abobala, M., Hatip, A., Olgun, N., Broumi, S., Salama, A. A., and Khaled, E. H., *The Algebraic Creativity In The Neutrosophic Square Matrices*, Neutrosophic Sets and Systems, Vol. 40, pp. 1-11, 2021.

[5] Adeleke, E. O., Agboola, A. A. A., and Smarandache, F., *Refined neutrosophic rings II*, International Journal of Neutrosophic Science, Vol. 2, pp. 89-94, 2020.

[6] Agboola, A.A.A., *On Refined Neutrosophic Algebraic Structures*, Neutrosophic Sets and Systems, Vol. 10, pp. 99-101, 2015.

[7] Das, R., Smarandache, F., and Tripathy, B., *Neutrosophic Fuzzy Matrices and Some Algebraic Operations*, Neutrosophic Sets and Systems, Vol. 32, pp. 401-409, 2020.

[8] Dhar M., Broumi S., and Smarandache F., *A Note on Square Neutrosophic Fuzzy Matrices*, Neutrosophic Sets and Systems, Vol. 3, pp. 37-41, 2014.

[9] Ibrahim, M.A., Agboola, A.A.A, Badmus, B.S., and Akinleye, S.A., *On refined Neutrosophic Vector Spaces I*, International Journal of Neutrosophic Science, Vol. 7, pp. 97-109, 2020.

[10] Kandasamy, W. B. V., and Smarandache, F., *Some Neutrosophic Algebraic Structures and Neutrosophic n-Algebraic Structures*, (Arizona: Hexis Phoenix), 2006.

[11] Khaled, H., and Younus, A., and Mohammad, A., *The Rectangle Neutrosophic Fuzzy Matrices*, Faculty of Education Journal, Vol. 15, pp. 3015-3024, 2019. (Arabic version)

[12] Prabakaran, P., Gustavo Alvarez Gomez, Rita Azucena Diaz Vasquez, and Andres Leon Yacelga, *A Note on Invertible Neutrosophic Square Matrices*, International Journal of Neutrosophic Science, Vol. 22, pp. 56-62, 2023.

[13] Sankari, H., and Abobala, M., *n-Refined Neutrosophic Modules*, Neutrosophic Sets and Systems, Vol. 36, pp. 1-11, 2020.

[14] Sankari, H., and Abobala, M., *Neutrosophic Linear Diophantine Equations With Two Variables*, Neutrosophic Sets and Systems, Vol. 38, pp. 399-408, 2020.

[15] Smarandache, F., *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, American Research Press. Rehoboth, 2003.

[16] Rajalakshmi, A., Hatamleh, R., Al-Husban, A., Muthu Kumaran, K. L., & Malchijah Raj, M. S. (2025). Various (ζ_1, ζ_2) neutrosophic ideals of an ordered ternary semigroups. *Communications on Applied Nonlinear Analysis*, **32**(3), 400–417.

[17] Hatamleh, R., Al-Husban, A., Sundarakannan, N., & Malchijah Raj, M. S. (2025). Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism. *Communications on Applied Nonlinear Analysis*, **32**(3), 418–435.

[18] Hatamleh, R., Al-Husban, A., Palanikumar, M., & Sundareswari, K. (2025). Different Weighted Operators such as Generalized Averaging and Generalized Geometric based on Trigonometric $\wp$-rung Interval-Valued Approach. *Communications on Applied Nonlinear Analysis*, **32**(5), 91–101.

[19] Hatamleh, R., Al-Husban, A., Sundareswari, K., Balaj, G., & Palanikumar, M. (2025). Complex Tangent Trigonometric Approach Applied to (γ, τ)-Rung Fuzzy Set using Weighted Averaging, Geometric Operators and its Extension. *Communications on Applied Nonlinear Analysis*, **32**(5), 133–144.

[20] Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Al-Qadri, M. O., & Al-Husban, A. (2024). On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2 \leq 3$. *Neutrosophic Sets and Systems*, **68**, 8–25.

[21] Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Hijazeen, R. B. Y., Al-Qadri, M. O., & Al-Husban, A. (2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers. *Neutrosophic Sets and Systems*, **67**, 169–178.

[22] Abubaker, A. A., Hatamleh, R., Matarneh, K., & Al-Husban, A. (2024). On the Numerical Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials. *International Journal of Neutrosophic Science*, **25**(2), 197–205.

[23] Ahmad, A., Hatamleh, R., Matarneh, K., & Al-Husban, A. (2025). On the Irreversible k-Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs. *International Journal of Neutrosophic Science*, **25**(2), 183–196.

[24] Shihadeh, A., Hatamleh, R., Palanikumar, M., & Al-Husban, A. (2025). New algebraic structures towards different (α, β) intuitionistic fuzzy ideals and its characterization of an ordered ternary semigroup. *Communications on Applied Nonlinear Analysis*, **32**(6), 568–578.

[25] Hatamleh, R., Heilat, A. S., Palanikumar, M., & Al-Husban, A. (2025). Characterization of interaction aggregating operators setting interval-valued Pythagorean neutrosophic set. *Neutrosophic Sets and Systems*, **81**, 285–305.

[26] Hatamleh, R., Al-Husban, A., Zubair, S. A. M., Elamin, M., Saeed, M. M., Abdolmaleki, E., ... & Khattak, A. M. (2025). AI-Assisted Wearable Devices for Promoting Human Health and Strength Using Complex Interval-Valued Picture Fuzzy Soft Relations. *European Journal of Pure and Applied Mathematics*, **18**(1), 5523–5523.

[27] Al-Husban, A., Saeed, M. M., Nordo, G., Fujita, T., Khattak, A. M., Hatamleh, R., ... & Armada, C. L. (2025). A Comprehensive Study of Bipolar Vague Soft Expert P-Open Sets in Bipolar Vague Soft Expert Topological Spaces with Applications to Cancer Diagnosis. *European Journal of Pure and Applied Mathematics*, **18**(2), 5900–5900.

[28] Abobala, M., *On The Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations*, *Journal of Mathematics*, Article ID 5591576, 2021.

[29] Hatamleh, R., Zolotarev, V. A., *On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component*, Journal of Mathematical Physics, Analysis, Geometry, Vol. 11(2), pp. 174–186, 2015.

[30] Hatamleh, R., Zolotarev, V. A., *Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators*, Ukrainian Mathematical Journal, Vol. 68(5), pp. 791–811, 2016.

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