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Generalized Inverse of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Decision-Making Problems

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Abstract: This paper presents a novel framework for computing the generalized inverse (g-inverse) and the Moore-Penrose inverse of Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs). To the best of our knowledge, no existing algorithm addresses the computation of the g-inverse for QPNFMs. In this study, we establish necessary and sufficient conditions for the existence of the g-inverse and develop an efficient algorithm for its computation. Furthermore, we explore several fundamental properties and theoretical results related to the g-inverse of QPNFMs, including uniqueness conditions and algebraic structures. In addition to theoretical advancements, we introduce a novel decision-making algorithm leveraging QPNFMs and their g-inverse. This algorithm enhances decision analysis in complex and uncertain environments by effectively handling indeterminate and inconsistent information. An illustrative example is provided to demonstrate the practical applicability and computational efficiency of the proposed approach. The results validate the accuracy of the g-inverse computation and highlight the utility of QPNFMs in decision-making scenarios. Our findings offer a significant contribution to both matrix theory and neutrosophic logic-based decision analysis, opening new avenues for future research in uncertainty modeling and computational intelligence.

Keywords: Quadri-Partitioned Neutrosophic Fuzzy sets, Quadri-Partitioned Neutrosophic Fuzzy Matrices, generalized inverse (g-inverse), Moore-Penrose inverse, minus ordering.

1. Introduction

Mathematical models for uncertainty and imprecision have been extensively studied in various disciplines, with fuzzy set theory introduced by Zadeh [1] being a fundamental breakthrough. This was further extended to intuitionistic fuzzy sets by Atanassov [2], providing an additional degree of freedom by incorporating a hesitation parameter. In subsequent years, Smarandache [3] introduced neutrosophic sets, offering a more generalized framework to handle indeterminacy, making them highly applicable in areas where uncertainty and vagueness coexist. One of the significant applications of fuzzy set theory is in matrix theory, leading to the development of fuzzy matrices [4,5] and intuitionistic fuzzy matrices [6,7]. These matrices provide a powerful tool for modeling various real-world problems involving vagueness. Further advancements introduced matrix operations, generalized inverses, and ordering relations in fuzzy and intuitionistic fuzzy matrices [8,9,10], enhancing their applicability in computational mathematics.

Building upon these concepts, the emergence of neutrosophic fuzzy matrices has provided a refined approach to modeling complex systems. Recent studies have explored properties such as pseudo similarity [9], inverse operations [10], and partial ordering in neutrosophic fuzzy matrices [11]. Further developments in matrix convergence [12] and interval-valued fuzzy matrices [13] have expanded the theoretical foundation of these models. Researchers have also focused on new operators [14], inverses [15], and distance measures [22,23] to enhance the computational efficiency and applicability of these matrices. More recent advancements in neutrosophic fuzzy matrices include studies on k-idempotent matrices [19], secondary k-range symmetric matrices [20], and generalized symmetric matrices [21]. Notably, studies on distance measures between intuitionistic fuzzy matrices [22,23] have enhanced their applicability in classification and clustering problems. Additionally, the semiring structure of intuitionistic fuzzy matrices [24] and their generalizations [25,26] have further enriched the theoretical foundation of this field. The introduction of interval-valued intuitionistic fuzzy sets [27,28] has allowed for a more nuanced representation of uncertainty, with applications in areas requiring higher precision. Moreover, intuitionistic fuzzy matrices have been explored in terms of generalized inverses [29] and nilpotent properties over distributive lattices [30], broadening their algebraic structure.

Building on these developments, neutrosophic fuzzy matrices have emerged as a powerful tool for handling indeterminate and inconsistent information. Recent advancements have focused on secondary k-column symmetric neutrosophic fuzzy matrices [31], interval-valued secondary k-range symmetric neutrosophic fuzzy matrices [32], and generalized symmetric Fermatean neutrosophic fuzzy matrices [33]. Ben [34] et al have studied A Perspective Note on μ_N σ Baire's Space. Dey [35] have presented On b-anti-Open Sets: A Formal Definition, Proofs, and Examples. These studies highlight the diverse applications of neutrosophic matrices in areas such as multi-criteria decision-making, computational intelligence, and uncertainty modeling. This research aims to further explore the theoretical foundations and applications of neutrosophic fuzzy matrices by investigating novel compositions and characterizations. By extending existing frameworks and introducing new algebraic structures, this study seeks to enhance the computational and decision-making capabilities of neutrosophic matrices, providing a more robust mathematical tool for handling complex uncertainty.

1.1 Abbreviations

FM: Fuzzy Matrices

IFM: Intuitionistic Fuzzy Matrices
IFSs: Intuitionistic Fuzzy Sets
NFSs: Neutrosophic fuzzy Sets
NFM: Neutrosophic fuzzy matrices.

2. Motivation for the Research

The study of Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) is essential for handling uncertainty, indeterminacy, and inconsistency in real-world data. However, the computational framework for matrix inverses in QPNFMs remains underdeveloped. In particular, no existing methods or algorithms explicitly address the computation of the generalized inverse (g-inverse) or the Moore-Penrose inverse in this context, which limits their applicability in mathematical modeling and decision-making processes.

To bridge this gap, we make the following key contributions:

(i**) Novel Algorithm –** We propose a straightforward and computationally efficient algorithm for determining the g-inverse of QPNFMs, addressing the lack of such methods in existing literature.

(ii) **Extension to Moore-Penrose Inverse –** Our study extends the inverse computation framework by incorporating the Moore-Penrose inverse, providing a more comprehensive approach to matrix inverses in neutrosophic fuzzy environments.

(iii) **Exploration of Properties –** We establish several fundamental theoretical results concerning the g-inverse of QPNFMs, enhancing the mathematical understanding of these matrices.

(iv) **Numerical Validation –** To ensure clarity and practical relevance, we validate our theoretical findings and proposed algorithm with detailed numerical examples, demonstrating their effectiveness in real-world applications.

3. Literature Review

The theory of fuzzy matrices has undergone significant development since Zadeh's pioneering work on fuzzy set theory [1], which introduced a mathematical framework for addressing uncertainty. Atanassov [2] expanded upon this by developing intuitionistic fuzzy sets, which incorporate both membership and non-membership values. Smarandache [3] further extended these concepts by proposing neutrosophic sets, characterized by truth, indeterminacy, and falsity components, providing a more robust framework for handling uncertain and inconsistent data.

Early contributions to fuzzy matrix theory include Cen's exploration of generalized inverses [4,5] and Mitra's work on matrix partial orders using generalized inverses [8]. Shyamal and Pal [14] introduced novel operators for fuzzy matrices, while Meenakhi and Inbam [7] proposed the minus partial order. Additionally, Xin [12] investigated the convergence properties of controllable fuzzy matrices.

Intuitionistic fuzzy matrices have also been widely studied. Khan and Pal [6] examined tautological matrices, and Pal [17] introduced the concept of intuitionistic fuzzy determinants. Shyamal and Pal [22,23] developed distance measures for intuitionistic fuzzy matrices, enabling effective comparison of these structures. Bhowmik and Pal [26,28] extended the field further by focusing on generalized and interval-valued intuitionistic fuzzy matrices, emphasizing their application in uncertainty modeling. Neutrosophic fuzzy matrices, as a generalization of intuitionistic fuzzy matrices, have emerged as an important area of research. Anandhkumar et al. have made significant contributions, including pseudo similarity measures for neutrosophic fuzzy matrices [9], various inverse types [10], and partial ordering on such matrices [11]. Their studies also introduced k-idempotent neutrosophic fuzzy matrices [19], secondary k-range symmetric neutrosophic fuzzy matrices [20,31,32], and generalized symmetric Fermatean neutrosophic fuzzy matrices [33], enriching both theoretical and practical applications.

Other noteworthy works include Shyamal and Pal's [13] analysis of interval-valued fuzzy matrices and their operators, and Khan and Pal's [29] research on the generalized inverse of intuitionistic fuzzy matrices. Panigrahi and Nanda [16] explored intuitionistic fuzzy relations, while Sriram and Murugadas [24] examined semiring properties of intuitionistic fuzzy matrices.This extensive body of literature highlights the evolution and versatility of fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy matrices. Building on these foundational studies, the present work addresses existing gaps by introducing new compositions and properties of neutrosophic fuzzy matrices, with the goal of advancing their theoretical framework and broadening their practical applications.

4. Comparative of QPNFM model with the existing soft models

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy	Indeterminacy is bifurcated	Indeterminacy is bifurcated and restricted
FSS [46]	✓	×	×	×	×	×
IVFSS [47]	✓	×	×	×	×	×
IFSS [48]	✓	✓	✓	×	×	×
IVIFSS [49]	✓	✓	✓	×	×	×

NSS [50]	✓	✓	×	✓	×	×
INSS [51]	✓	✓	×	✓	×	×
QNSS [52]	✓	✓	×	✓	✓	×
QPNFM (Proposed)	✓	✓	✓	✓	✓	✓

5. Novelty

The referenced works represent critical advancements in the development of fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy matrices. Zadeh [1] introduced the foundational concept of fuzzy sets, establishing a framework for modeling imprecision in mathematical systems. Atanassov [2] extended this foundation by proposing intuitionistic fuzzy sets, which incorporate both membership and non-membership values to better address uncertainty. Smarandache [3] further advanced this framework by introducing neutrosophic sets, which include an additional component of indeterminacy, eventually leading to the development of neutrosophic fuzzy matrices (NFMs).

Cen [4, 5] made significant contributions by exploring the structure of fuzzy matrices, focusing on partial ordering and generalized inverses, which are essential for computational applications. Khan and Pal [6] enriched the theory by examining intuitionistic fuzzy tautological matrices, providing deeper insights into their logical structure. Shyamal and Pal [13, 22, 23, 26, 27] contributed extensively by introducing interval-valued fuzzy matrices and developing distance measures and operators, thereby enhancing methods for manipulating fuzzy systems. Mitra [8] provided a unified approach to partial ordering using generalized inverses, offering a comprehensive framework for matrix operations in linear algebra. Dehghan et al. [15] tackled the challenges of inverting fuzzy matrices with fuzzy numbers, presenting innovative computational techniques.

Anandhkumar et al. [9-11, 19, 20, 21, 31, 32, 33] have made notable contributions to neutrosophic fuzzy matrices. Their work introduced pseudo-similarity measures, new inverse types, and advanced partial ordering methods. Additionally, they proposed concepts such as secondary k-range symmetry, generalized symmetric matrices, and refined compositional operators, significantly advancing the theoretical and practical applications of NFMs in addressing uncertainty and enhancing matrix theory.

6. Preliminaries

Definition 6.1 Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all QPNSS of X. The collection (F, A) is termed the (QPNSS) over X, where F is a mapping given by F : A⟶ P(X). Here,

$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with T_A , F_A, C_A ,U_A : $X \longrightarrow [0,1]$ $and\ 0 \le T_A(x) + C_A(x) + U_A(x) + F_A(x) \le 4$. In this context

- $T_A(x)$ is the truth membership (TM),
- $C_A(x)$ is contradiction membership (CM),
- $U_A(x)$ is ignorance membership (IM),
- $F_A(x)$ is the false membership (FM).

7. **Quadri-Partitioned Neutrosophic Fuzzy Matrices**

Definition 7.1 Let $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} >, Q = < q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} > \in (QPNFM)_n$

Component-wise addition and multiplication are defined as follows

(i) $P \oplus Q = \left(\sup\left\{p^T{}_{ij}, q^T{}_{ij}\right\}, \sup\left\{p^C{}_{ij}, q^C{}_{ij}\right\}, \inf\left\{p^U{}_{ij}, q^U{}_{ij}\right\}, \inf\left\{p^F{}_{ij}, q^F{}_{ij}\right\}\right)$

(ii) $P\square\ Q = \left(\inf\left\{p^T{}_{ij}, q^T{}_{ij}\right\}, \inf\left\{p^C{}_{ij}, q^C{}_{ij}\right\}, \sup\left\{p^U{}_{ij}, q^U{}_{ij}\right\}, \sup\left\{p^F{}_{ij}, q^F{}_{ij}\right\}\right)$

Definition 7.2. Let $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} >, Q = < q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} > \in (QPNFM)_n$

the composition of P and Q is well-defined as

$$P \circ Q = \left(\sum_{k=1}^{n}\left(p^T{}_{ij} \wedge q^T{}_{ij}\right), \sum_{k=1}^{n}\left(p^C{}_{ij} \wedge q^C{}_{ij}\right), \prod_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right), \prod_{k=1}^{n}\left(p^F{}_{ij} \vee q^F{}_{ij}\right)\right)$$

consistently we can write the same as

$$P \circ Q = \left(\bigcup_{k=1}^{n}\left(p^T{}_{ij} \wedge q^T{}_{ij}\right), \bigcup_{k=1}^{n}\left(p^C{}_{ij} \wedge q^C{}_{ij}\right), \bigcap_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right), \bigcap_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right)\right)$$

The product $P \circ Q$ is defined only when the number of columns in P equals the number of rows in Q. When this condition is met, matrices P and Q are considered conformable for multiplication. For simplicity, we denote the product as $P \circ Q$.

Definition: 7.3 A NFSs P on the universe of discourse Y is well-defined as

$P = \left\{< y, p^T(y), p^I(y), p^F(y) >, y \in Y\right\}$, everywhere $p^T, p^I, p^F : Y \rightarrow]^-0,1^+[$ also $0 \le p^T + p^I + p^F \le 3$.

Definition: 7.4 Let $U = \left(u^T, u^I, u^F\right)$ and $V = \left(v^T, v^I, v^F\right)$ two NFM then the matrix addition and multiplication is given by

$$U + V = \left[\max < u^T, v^T >, \max < u^I, v^I >, \min < u^F, v^F >\right]$$

$U.V = \left[\min < u^T, v^T >, \min < u^I, v^I >, \max < u^F, v^F > \right]$

Definition:7.5 A Quadri-Partitioned neutrosophic Fuzzy Matrices U is lease then or equal to V

That is $U \le V$ if $\left(u_{ij}{}^T, u_{ij}{}^C, u_{ij}{}^U, u_{ij}{}^F \right) \le \left(v_{ij}{}^T, v_{ij}{}^C,, u_{ij}{}^U, v_{ij}{}^F \right)$ means

$u_{ij}{}^T \le v_{ij}{}^T, u_{ij}{}^C \le v_{ij}{}^C, u_{ij}{}^U \ge v_{ij}{}^U, u_{ij}{}^F \ge v_{ij}{}^F$.

Definition 7.6. A QPNFM is considered null if all its elements are (0,0,0,0). This type of matrix is denoted by $N_{(0,0,0,0)}$. On the other hand, an QPNFM is defined as zero if all its elements are (0,0,1,1) and it is represented by O.

Definition 7.7 A square QPNFM is referred to as a Quadri-Partitioned Neutrosophic Fuzzy Permutation Matrix (QPNFPM) if each row and each column contains exactly one element with a value of (1,1,0,0) while all other entries are (0,0,1,1).

Definition 7.8 For identity QPNFM of order n x n is represented by I_n and is well-defined by

$$\left(\delta_{ij}{}^{\alpha}, \delta_{ij}{}^{\beta}, \delta_{ij}{}^{\gamma}, \delta_{ij}{}^{\lambda} \right) = \begin{cases} (1,1,0,0) \ if \ i = j \\ (0,0,1,1) \ if \ i \ne j \end{cases}$$

Definition 7.9 Let $U = \left(u^T, u^C, u^U, u^F \right)$ and $V = \left(v^T, v^C, v^U, v^F \right)$ be two QPNFM then

(i) $U^{k+1} = U^k \times U, (k = 1, 2, 3, ...)$

(ii) $U \times V = \left[\bigcup_{k=1}^{n} \left(u_{ik} \wedge v_{kj} \right) \right]$.

(iii) $U^T = \left[u_{ji}{}^T, u_{ji}{}^C, u_{ji}{}^U, u_{ji}{}^F \right]$ (the transpose of P)

(iv) $U^2 = U$ (U is idempotent)

(v) $U^k = O$ (U is nilpotent $k \in N$)

(vi) $U^T = U$ (U is SNFM)

(vii) UV = VU = I_n (U and V are invertible)

Definition 7.10 The rows of $\left(u^T, u^C, u^U, u^F \right)$ are independent and they form a standard basis iff

$\left(r_i^T, r_i^C, r_i^U, r_i^F \right) = \sum_{j=1}^{n} \left(u_{ij}{}^T, u_{ij}{}^C, u_{ij}{}^U, u_{ij}{}^F \right) \left(r_j{}^T, r_j{}^C, r_j{}^U, r_j{}^F \right)$ for

$\left(r_i^T, r_i^C, r_i^U, r_i^F \right), \left(r_j{}^T, r_j{}^C, r_j{}^U, r_j{}^F \right) \in \left(r^T, r^C, r^U, r^F \right)$

and $\left(u_{ij}{}^{T},u_{ij}{}^{C},u_{ij}{}^{U},u_{ij}{}^{F}\right)\in[0,1]$ then

$\left(u_{ii}{}^{T},u_{ii}{}^{C},u_{ii}{}^{U},u_{ii}{}^{F}\right)\left(r_i^{T},r_i^{C},r_i^{U},r_i^{F}\right)=\left(r_i^{T},r_i^{C},r_i^{U},r_i^{F}\right),i=1,2,...,n.$

8. Generalized Inverse In this section, the generalized inverse of an QPNFM is investigated.

Definition 8.1 (Generalized inverse) For a QPNFM $\left(u^{T},u^{C},u^{U},u^{F}\right)\in\left(QPNFM\right)_{mn}$ is said to be regular if there exists another QPNFM, $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(QPNFM\right)_{nm}$ such that $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$. In this case, $\left(g^{T},g^{C},g^{U},g^{F}\right)$ is called a generalized inverse (g-inverse) of $\left(u^{T},u^{C},u^{U},u^{F}\right)$ and it is denoted by $\left(u^{T},u^{C},u^{U},u^{F}\right)^{-}$.

The g-inverse of an QPNFM is not unique that is a QPNFM has many g-inverses. The set of all such g-inverses of $\left(u^{T},u^{C},u^{U},u^{F}\right)$ are denoted by $\left(u^{T},u^{C},u^{U},u^{F}\right)\{1\}$.

Definition 8.2. For a QPNFM $\left(u^{T},u^{C},u^{U},u^{F}\right)\in\left(QPNFM\right)_{mn}$ and another QPNFM, $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(QPNFM\right)_{nm}$ is said to be outer inverse of $\left(u^{T},u^{C},u^{U},u^{F}\right)$, if $\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(g^{T},g^{C},g^{U},g^{F}\right)$ and is denoted by $\left(u^{T},u^{C},u^{U},u^{F}\right)\{2\}$. The QPNFM $\left(g^{T},g^{C},g^{U},g^{F}\right)$ is said to be {1,2} **inverse or semi inverse** of $\left(u^{T},u^{C},u^{U},u^{F}\right)$, if $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$ and $\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(g^{T},g^{C},g^{U},g^{F}\right)$ is denoted by $\left(u^{T},u^{C},u^{U},u^{F}\right)$ {1,2}.

The QPNFM $\left(g^{T},g^{C},g^{U},g^{F}\right)$ is said to be {1,3} inverse or **least square g-inverse** of $\left(u^{T},u^{C},u^{U},u^{F}\right)$ if, $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$ and $\left[\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\right]^{T}=\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)$ and is denoted by $\left(u^{T},u^{C},u^{U},u^{F}\right)$ {1,3}. Again $\left(g^{T},g^{C},g^{U},g^{F}\right)$ is said to be {1,4} inverse or **minimum norm g-inverse** of $\left(u^{T},u^{C},u^{U},u^{F}\right)$ if,

$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ and $\left[\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)\right]^T$ $=\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)$ is denoted by $\left(u^T,u^C,u^U,u^F\right)\{1,4\}$.

No algorithm is available to find g-inverse of QPNFM. Here we present a simple algorithm to evaluate g-inverse of an QPNFM.

8.1 Algorithm (To find the g-inverse of an QPNFM)

Step 1: Check whether the non-zero rows of NFM $\left(u^T,u^C,u^U,u^F\right)$ form a standard basis or not for the row space of $\left(u^T,u^C,u^U,u^F\right)$.

Step 2: If non-zero rows form a standard basis then find some QPNFPM $\left(p^T,p^C,p^U,p^F\right)$ such that $\left(u^T,u^C,u^U,u^F\right)$ $\left(p^T,p^C,p^U,p^F\right)$ $\left(u^T,u^C,u^U,u^F\right)$ = $\left(u^T,u^C,u^U,u^F\right)$.

Step 3: Choose an QPNFM $\left(r^T,r^C,r^U,r^F\right)$ such that $\left(r^T,r^C,r^U,r^F\right)$ $\left(u^T,u^C,u^U,u^F\right)$ = $\left(u^T,u^C,u^U,u^F\right)$.

Step4: Then $\left(p^T,p^C,p^U,p^F\right)\left(r^T,r^C,r^U,r^F\right)$is a g-inverse of $\left(u^T,u^C,u^U,u^F\right)$.

The matrix $\left(p^T,p^C,p^U,p^F\right)\left(r^T,r^C,r^U,r^F\right)$ is a g-inverse of $\left(u^T,u^C,u^U,u^F\right)$ since

$\left(u^T,u^C,u^U,u^F\right)$ $\left[\left(p^T,p^C,p^U,p^F\right)\left(r^T,r^I,r^F\right)\right]$ $\left(u^T,u^C,u^U,u^F\right)$ = $\left(u^T,u^C,u^U,u^F\right)$ $\left(p^T,p^C,p^U,p^F\right)$ $\left[\left(r^T,r^I,r^F\right)\left(u^T,u^C,u^U,u^F\right)\right]$ = $\left(u^T,u^C,u^U,u^F\right)$ $\left(p^T,p^C,p^U,p^F\right)$ $\left(u^T,u^C,u^U,u^F\right)$ = $\left(u^T,u^C,u^U,u^F\right)$.

The following example demonstrates the above algorithm to compute g-inverse of $\left(u^T,u^C,u^U,u^F\right)$.

Example 8.1 Let us consider a QPNFM

$$\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix} <0.5,0.2,0.7,0.4> & <0.7,0.2,0.7,0.2> & <0.6,0.2,0.7,0.3> \\ <0.5,0.2,0.7,0.3> & <0.6,0.2,0.7,0.2> & <0.8,0.2,0.7,0.2> \\ <0.4,0.2,0.7,0.4> & <0.4,0.2,0.7,0.3> & <0.8,0.2,0.7,0.1> \end{bmatrix}$$

The rows of $\left(u^T,u^C,u^U,u^F\right)$ are independent and they form a standard basis. Since,

$$\left(r_i^T, r_i^C, r_i^U, r_i^F\right) = \sum_{j=1}^{3} \left(u_{ij}^T, u_{ij}^C, u_{ij}^U, u_{ij}^F\right)\left(r_j^T, r_j^C, r_j^U, r_j^F\right)$$

$\left(r_i^T, r_i^C, r_i^U, r_i^F\right), \left(r_j^T, r_j^C, r_j^U, r_j^F\right) \in \left(r^T, r^C, r^U, r^F\right)$ row space of $\left(u^T, u^C, u^U, u^F\right)$, $\left(u_{ij}^T, u_{ij}^C, u_{ij}^U, u_{ij}^F\right) \in [0,1]$

and $\left(u_{ii}^T, u_{ii}^C, u_{ii}^U, u_{ii}^F\right)\left(r_i^T, r_i^C, r_i^U, r_i^F\right) = \left(r_i^T, r_i^C, r_i^U, r_i^F\right)$, $i = 1, 2, 3.$

For the QPNFPM $\left(p^T, p^C, p^U, p^F\right) = \begin{bmatrix} <0,0,1,1> & <1,1,0,0> & <0,0,1,1> \\ <1,1,0,0> & <0,0,1,1> & <0,0,1,1> \\ <0,0,1,1> & <0,0,1,1> & <1,1,0,0> \end{bmatrix}$

$\left(u^T, u^C, u^U, u^F\right) \left(p^T, p^C, p^U, p^F\right) \left(u^T, u^C, u^U, u^F\right) = \left(u^T, u^C, u^U, u^F\right)$ holds.

Now, for the QPNFM

$$\left(r^T, r^C, r^U, r^F\right) = \begin{bmatrix} <0.8,0.2,0.7,0.2> & <0.5,0.2,0.7,0.5> & <0.5,0.2,0.7,0.3> \\ <0.4,0.2,0.7,0.5> & <0.8,0.2,0.7,0.1> & <0.6,0.2,0.7,0.3> \\ <0.3,0.2,0.7,0.4> & <0.4,0.2,0.7,0.4> & <0.9,0.2,0.7,0.1> \end{bmatrix}$$

$\left(r^T, r^C, r^U, r^F\right) \left(u^T, u^C, u^U, u^F\right) = \left(u^T, u^C, u^U, u^F\right)$ holds. So, the g-inverse of $\left(u^T, u^C, u^U, u^F\right)$ is

$$\left(p^T, p^C, p^U, p^F\right)\left(r^T, r^C, r^U, r^F\right) = \begin{bmatrix} <0.4,0.2,0.7,0.5> & <0.8,0.2,0.7,0.1> & <0.6,0.2,0.7,0.3> \\ <0.8,0.2,0.7,0.2> & <0.5,0.2,0.7,0.5> & <0.5,0.2,0.7,0.3> \\ <0.3,0.2,0.7,0.4> & <0.4,0.2,0.7,0.4> & <0.9,0.2,0.7,0.1> \end{bmatrix}$$

$$= \left(x^T, x^C, x^U, x^F\right)$$

(say) which satisfy the relation

$\left(u^T, u^C, u^U, u^F\right) \left(x^T, x^C, x^U, x^F\right) \left(u^T, u^C, u^U, u^F\right) = \left(u^T, u^C, u^U, u^F\right)$.

If each row of an QPNFM $\left(v^T, v^C, v^U, v^F\right)$ can be expressed as a linear combination of the rows of QPNFM $\left(u^T, u^C, u^U, u^F\right)$, then we write $R\left(v^T, v^C, v^U, v^F\right) \subseteq R\left(u^T, u^C, u^U, u^F\right)$. If $R\left(v^T, v^C, v^U, v^F\right) \subseteq R\left(u^T, u^C, u^U, u^F\right)$ and $R\left(u^T, u^C, u^U, u^F\right) \subseteq R\left(v^T, v^C, v^U, v^F\right)$ then we say that $R\left(u^T, u^C, u^U, u^F\right) = R\left(v^T, v^C, v^U, v^F\right)$.

Theorem 8.1 Let $\left(u^T,u^C,u^U,u^F\right)$, $\left(v^T,v^C,v^U,v^F\right)\in\left(QPNFM\right)_{m\times n}$ be two QPNFM. If $\left(u^T,u^C,u^U,u^F\right)$ is regular then,

(i) $R\left(v^T,v^C,v^U,v^F\right)\subseteq R\left(u^T,u^C,u^U,u^F\right)$ iff $\left(v^T,v^C,v^U,v^F\right)$ = $\left(v^T,v^C,v^U,v^F\right)$ $\left(u^T,u^C,u^U,u^F\right)^-$ $\left(u^T,u^C,u^U,u^F\right)$ for each $\left(u^T,u^C,u^U,u^F\right)^-\in\left(u^T,u^C,u^U,u^F\right)\{1\}$.

(ii) $C\left(v^T,v^C,v^U,v^F\right)\subseteq C\left(u^T,u^C,u^U,u^F\right)$ iff $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-$ $\left(v^T,v^C,v^U,v^F\right)$ for each $\left(u^T,u^C,u^U,u^F\right)^-\in\left(u^T,u^C,u^U,u^F\right)\{1\}$.

Proof. (i) Let $R\left(v^T,v^C,v^U,v^F\right)\subseteq R\left(u^T,u^C,u^U,u^F\right)$,then each row of $\left(v^T,v^C,v^U,v^F\right)$ is a linear combination of the rows of $\left(u^T,u^C,u^U,u^F\right)$.

Hence $\left({v_{i^*}}^T,{v_{i^*}}^C,{v_{i^*}}^U,{v_{i^*}}^F\right)=\sum\left({x_{ij}}^T,{x_{ij}}^C,{x_{ij}}^U,{x_{ij}}^F\right)\left({u_{j^*}}^T,{u_{j^*}}^C,{u_{j^*}}^U,{u_{j^*}}^F\right)$ where $\left({x_{ij}}^T,{x_{ij}}^C,{x_{ij}}^U,{x_{ij}}^F\right)\in\left(QPNFM\right)$

That is, $\left(v^T,v^C,v^U,v^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)$ for some $\left(x^T,x^C,x^U,x^F\right)\in\left(QPNFM\right)_m$

or $\left(v^T,v^C,v^U,v^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

Since $\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

or $\left(v^T,v^C,v^U,v^F\right)=\left(v^T,v^C,v^U,v^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

Conversely, $\left(v^T,v^C,v^U,v^F\right)=\left(v^T,v^C,v^U,v^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$,then

$\left(v^T,v^C,v^U,v^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

For some $\left(x^T,x^C,x^U,x^F\right)\in\left(QPNFM\right)_m$

or $\left(v^T,v^C,v^U,v^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)$

Since $\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

This implies that $R\left(v^T,v^C,v^U,v^F\right)\subseteq R\left(u^T,u^C,u^U,u^F\right)$

(ii) $C\left(v^T,v^C,v^U,v^F\right)\subseteq C\left(u^T,u^C,u^U,u^F\right)$

Then $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)$ for some

$\left(y^T,y^C,y^U,y^F\right)\in\left(QPNFM\right)_n$

or $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)$

As $\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

$\left(y^T,y^C,y^U,y^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

That is $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(v^T,v^C,v^U,v^F\right)$

Conversely, if $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(v^T,v^C,v^U,v^F\right)$

Then $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^I,y^F\right)$

For some $\left(y^T,y^C,y^U,y^F\right)\in\left(QPNFM\right)_n$

or $\left(v^T,v^C,v^U,v^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)$

As $\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$

That is $C\left(v^T,v^C,v^U,v^F\right)\subseteq C\left(u^T,u^C,u^U,u^F\right)$

Example 8.2 Let $\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix}<0.6,0.3,0.4,0.2> & <0.5,0.3,0.4,0.4>\\ <0.7,0.3,0.4,0.3> & <0.5,0.3,0.4,0.4>\end{bmatrix}$ and

$\left(v^T,v^C,v^U,v^F\right)=\begin{bmatrix}<0.6,0.3,0.4,0.3> & <0.5,0.3,0.4,0.4>\\ <0.6,0.3,0.4,0.4> & <0.5,0.3,0.4,0.4>\end{bmatrix}$ be two QPNFMs.

One of the g-inverse of $\left(u^T,u^C,u^U,u^F\right)$ is

$$\left(u^T,u^C,u^U,u^F\right)^- = \begin{bmatrix} <0.6,0.3,0.4,0.2> & <0.5,0.3,0.4,0.4> \\ <0.7,0.3,0.4,0.3> & <0.5,0.3,0.4,0.4> \end{bmatrix}$$

For which $\left(v^T,v^C,v^U,v^F\right)=\left(v^T,v^C,v^U,v^F\right)\left(u^T,u^C,u^U,u^F\right)^-\left(u^T,u^C,u^U,u^F\right)$ holds.

Also $\left(v^T,v^C,v^U,v^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)$ for

$$\left(x^T,x^C,x^U,x^F\right) = \begin{bmatrix} <0.7,0.3,0.4,0.3> & <0.6,0.3,0.4,0.3> \\ <0.6,0.3,0.4,0.4> & <0.4,0.3,0.4,0.4> \end{bmatrix}$$ holds.

So, $R\left(v^T,v^C,v^U,v^F\right)\subseteq R\left(u^T,u^C,u^U,u^F\right)$

Similarly, the result is true for column space also.

Theorem 8.2 Let $\left(u^T,u^C,u^U,u^F\right) \in \left(QPNFM\right)_{m\times n}$ be a regular QPNFM and $\left(g^T,g^C,g^U,g^F\right)$ be a g-inverse of $\left(u^T,u^C,u^U,u^F\right)$. Then

(i) $\left(g^T,g^C,g^U,g^F\right)^T \in \left(u^T,u^C,u^U,u^F\right)^T\{1\}$.

(ii) If $\left(p^T,p^C,p^U,p^F\right)$ and $\left(q^T,q^C,q^U,q^F\right)$ are NFPMs, then

$$\left(p^T,p^C,p^U,p^F\right)^T\left(g^T,g^C,g^U,g^F\right)\left(p^T,p^C,p^U,p^F\right)^T$$
$$\in \left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)\in\{1\}$$

(iii) $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)$ and $\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)$ are idempotent.

Proof. (i) Let $\left(g^T,g^C,g^U,g^F\right)$ be a g-inverse of $\left(u^T,u^C,u^U,u^F\right)$.

Then $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ holds. Taking transpose on both sides,

we get $\left(u^T,u^C,u^U,u^F\right)^T\left(g^T,g^C,g^U,g^F\right)^T\left(u^T,u^C,u^U,u^F\right)^T=\left(u^T,u^C,u^U,u^F\right)^T$

This implies $\left(g^T,g^C,g^U,g^F\right)^T \in \left(u^T,u^C,u^U,u^F\right)^T\{1\}$.

(ii) Since $\left(p^{T},p^{C},p^{U},p^{F}\right)$ and $\left(q^{T},q^{I},q^{F}\right)$ are QPNFPMs, $\left(p^{T},p^{C},p^{U},p^{F}\right)$ and $\left(q^{T},q^{C},q^{U},q^{F}\right)$ are invertible and $\left(p^{T},p^{C},p^{U},p^{F}\right)^{-}=\left(p^{T},p^{C},p^{U},p^{F}\right)^{T}$, $\left(q^{T},q^{C},q^{U},q^{F}\right)^{-}=\left(q^{T},q^{C},q^{U},q^{F}\right)^{T}$.

Now $\left(p^{T},p^{C},p^{U},p^{F}\right)$ $\left(u^{T},u^{C},u^{U},u^{F}\right)$

$\left(q^{T},q^{C},q^{U},q^{F}\right)\left[\left(q^{T},q^{C},q^{U},q^{F}\right)^{T}\left(g^{T},g^{C},g^{U},g^{F}\right)\left(p^{T},p^{C},p^{U},p^{F}\right)^{T}\right]$

$\left(p^{T},p^{C},p^{U},p^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)$

$=\left(p^{T},p^{C},p^{U},p^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left[\left(q^{T},q^{C},q^{U},q^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)^{T}\right]\left(g^{T},g^{C},g^{U},g^{F}\right)$

$\left[\left(p^{T},p^{C},p^{U},p^{F}\right)^{T}\left(p^{T},p^{C},p^{U},p^{F}\right)\right]\left(u^{T},u^{C},u^{U},u^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)$

$=\left(p^{T},p^{C},p^{U},p^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)$

As $\left(q^{T},q^{C},q^{U},q^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)^{T}=I,\left(p^{T},p^{C},p^{U},p^{F}\right)^{T}\left(p^{T},p^{C},p^{U},p^{F}\right)=I$

$=\left(p^{T},p^{C},p^{U},p^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)$ as

$\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$

This implies that $\left(q^{T},q^{C},q^{U},q^{F}\right)^{T}\left(g^{T},g^{C},g^{U},g^{F}\right)\left(p^{T},p^{C},p^{U},p^{F}\right)^{T}$

$\in\left(p^{T},p^{C},p^{U},p^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(q^{T},q^{C},q^{U},q^{F}\right)\{1\}$

(iii)Again $\left[\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\right]\left[\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\right]$

$=\left[\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\right]\left(g^{T},g^{C},g^{U},g^{F}\right)$

$=\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)$

As $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$

Also $\left[\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\right]\left[\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\right]$

$=\left[\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\right]\left(u^{T},u^{C},u^{U},u^{F}\right)$

$=\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right),$

As $\left[\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)=\left(g^T,g^C,g^U,g^F\right)\right]$

Thus $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)$ and $\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)$ are idempotent.

Example 8.3 Let us consider the QPNFM

$$\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.2> \\ <0.6,0.4,0.1,0.3> & <0.5,0.4,0.1,0.3> \end{bmatrix}$$

and one of its g-inverse is $\left(g^T,g^C,g^U,g^F\right)=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.3> \\ <0.7,0.4,0.1,0.2> & <0.4,0.4,0.1,0.4> \end{bmatrix}$

Now,

$$\left(u^T,u^C,u^U,u^F\right)^T\left(g^T,g^C,g^U,g^F\right)^T\left(u^T,u^C,u^U,u^F\right)^T=\begin{bmatrix} <1,1,0,0> & <0.6,0.4,0.1,0.3> \\ <0.5,0.4,0.1,0.2> & <0.5,0.4,0.1,0.3> \end{bmatrix}$$

$$\left(u^T,u^C,u^U,u^F\right)^T\left(g^T,g^C,g^U,g^F\right)^T\left(u^T,u^C,u^U,u^F\right)^T=\left(u^T,u^C,u^U,u^F\right)^T$$

Thus $\left(g^T,g^C,g^U,g^F\right)^T\in\left(u^T,u^C,u^U,u^F\right)^T\{1\}$

Let $\left(p^T,p^C,p^U,p^F\right)=\begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix}$

and $\left(q^T,q^C,q^U,q^F\right)=\begin{bmatrix} <0,0,1,1> & <1,1,0,0> \\ <1,1,0,0> & <0,0,1,1> \end{bmatrix}$

Now,

$$\left(q^T,q^C,q^U,q^F\right)^T\left(g^T,g^C,g^U,g^F\right)\left(p^T,p^C,p^U,p^F\right)^T=\begin{bmatrix} <0.7,0.4,0.1,0.2> & <0.4,0.4,0.1,0.4> \\ <1,1,0,0> & <0.5,00.4,0.1,0.3> \end{bmatrix}$$

and

$$\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)=\begin{bmatrix} <0.5,0.4,0.1,0.2> & <1,1,0,0> \\ <0.5,0.4,0.1,0.3> & <0.6,0.4,0.1,0.3> \end{bmatrix}$$

$$\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)$$

$$\left[\left(q^T,q^C,q^U,q^F\right)^T\left(g^T,g^C,g^U,g^F\right)\left(p^T,p^C,p^U,p^F\right)^T\right]$$

$$\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)=\begin{bmatrix} <0.5,0.4,0.1,0.2> & <1,1,0,0> \\ <0.5,0.4,0.1,0.3> & <0.6,0.4,0.1,0.3> \end{bmatrix}$$

$$\begin{bmatrix} <0.7,0.4,0.1,0.2> & <0.4,0.4,0.1,0.4> \\ <1,1,0,0> & <0.5,0.4,0.1,0.3> \end{bmatrix}\begin{bmatrix} <0.5,0.4,0.1,0.2> & <1,1,0,0> \\ <0.5,0.4,0.1,0.3> & <0.6,0.4,0.1,0.3> \end{bmatrix}$$

$$\begin{bmatrix} <0.5,0.4,0.1,0.2> & <1,1,0,0> \\ <0.5,0.4,0.1,0.3> & <0.6,0.4,0.1,0.3> \end{bmatrix}=\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)$$

That is, $\left(q^T,q^C,q^U,q^F\right)^T\left(g^T,g^C,g^U,g^F\right)\left(p^T,p^C,p^U,p^F\right)^T$

$$\in\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)\{1\}$$

$$\left[\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\right]^2$$

$$=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.3> \\ <0.6,0.4,0.1,0.3> & <0.5,0.4,0.1,0.3> \end{bmatrix}\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.3> \\ <0.6,0.4,0.1,0.3> & <0.5,0.4,0.1,0.3> \end{bmatrix}$$

$$=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.3> \\ <0.6,0.4,0.1,0.3> & <0.5,0.4,0.1,0.3> \end{bmatrix}=\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)$$

$$\left[\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)\right]^2$$

$$=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.2> \\ <0.7,0.4,0.1,0.2> & <0.5,0.4,0.1,0.2> \end{bmatrix}\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.2> \\ <0.7,0.4,0.1,0.2> & <0.5,0.4,0.1,0.2> \end{bmatrix}$$

$$=\begin{bmatrix} <1,1,0,0> & <0.5,0.4,0.1,0.2> \\ <0.7,0.4,0.1,0.2> & <0.5,0.4,0.1,0.2> \end{bmatrix}=\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)$$

Theorem 8.3 Let $\left(u^T,u^C,u^U,u^F\right)$ be an QPNFM,

$\left(y^T,y^C,y^U,y^F\right),\left(z^T,z^C,z^U,z^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$ and

$\left(x^T,x^C,x^U,x^F\right)=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)$.

Then $\left(x^T,x^C,x^U,x^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,2\}$,

that is $\left(x^T,x^C,x^U,x^F\right)$ is a semi-inverse of $\left(u^T,u^C,u^U,u^F\right)$.

Proof. Since $\left(y^T,y^C,y^U,y^F\right),\left(z^T,z^C,z^U,z^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$

$\Rightarrow\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ and

$\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right).$

As $\left(x^T,x^C,x^U,x^F\right)=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)$

So, $\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)$

$=\left(u^T,u^C,u^U,u^F\right)\left[\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\right]\left(u^T,u^C,u^U,u^F\right)$

$=\left[\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\right]\left(z^T,z^C,z^U,z^F\right)\left(u^T,u^C,u^U,u^F\right)$

$=\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$

Also, $\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$

$=\left[\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\right]\left(u^T,u^C,u^U,u^F\right)$

$\left[\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\right]$

$=\left(y^T,y^C,y^U,y^F\right)\left[\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\left(u^T,u^C,u^U,u^F\right)\right]$

$\left[\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)\right]$

$=\left(y^T,y^C,y^U,y^F\right)\left[\left(u^T,u^C,u^U,u^F\right)\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\right]\left(z^T,z^C,z^U,z^F\right)$

$=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)=\left(x^T,x^C,x^U,x^F\right).$

So $\left(x^T,x^C,x^U,x^F\right)$ is a semi-inverse of the QPNFM $\left(u^T,u^C,u^U,u^F\right)$

Example 8.4 Let us consider QPNFM

$$\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix}<0.8,0.6,0.5,0.2> & <0.6,0.6,0.5,0.3> & <0.4,0.6,0.5,0.3>\\ <0.5,0.6,0.5,0.3> & <0.5,0.6,0.5,0.1> & <0.4,0.6,0.5,0.2>\\ <0.7,0.6,0.5,0.3> & <0.7,0.6,0.5,0.2> & <0.9,0.6,0.5,0.1>\end{bmatrix}$$

Let $\left(y^T,y^C,y^U,y^F\right)=\begin{bmatrix}<0.8,0.6,0.5,0.1> & <0.5,0.6,0.5,0.4> & <0.4,0.6,0.5,0.4>\\ <0.4,0.6,0.5,0.6> & <0.7,0.6,0.5,0.1> & <0.3,0.6,0.5,0.5>\\ <0.6,0.6,0.5,0.3> & <0.6,0.6,0.5,0.2> & <0.9,0.6,0.5,0.1>\end{bmatrix}$

$$\left(z^T,z^C,z^U,z^F\right)=\begin{bmatrix}<0.9,0.6,0.5,0.1> & <0.6,0.6,0.5,0.4> & <0.4,0.6,0.5,0.5>\\ <0.5,0.6,0.3> & <0.6,0.6,0.1> & <0.4,0.6,0.5,0.3>\\ <0.5,0.6,0.5,0.4> & <0.6,0.6,0.5,0.3> & <1,1,0,0>\end{bmatrix}$$

be two of its g-inverses of $\left(x^T,x^C,x^U,x^F\right)$ of type $\left(x^T,x^C,x^U,x^F\right)\{1\}$

Then, $\left(x^T,x^C,x^U,x^F\right)=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(z^T,z^C,z^U,z^F\right)$

$$\left(x^T,x^C,x^U,x^F\right)=\begin{bmatrix} <0.8,0.6,0.5,0.2> & <0.6,0.6,0.5,0.3> & <0.4,0.6,0.5,0.3> \\ <0.5,0.6,0.5,0.3> & <0.5,0.6,0.5,0.1> & <0.4,0.6,0.5,0.2> \\ <0.7,0.6,0.5,0.3> & <0.6,0.6,0.5,0.2> & <0.9,0.6,0.5,0.1> \end{bmatrix}$$

For the above $\left(x^T,x^C,x^U,x^F\right)$,

$\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ and

$\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)=\left(x^T,x^C,x^U,x^F\right)$ holds.

So $\left(x^T,x^C,x^U,x^F\right)$ is a semi-inverse of the NFM $\left(u^T,u^C,u^U,u^F\right)$.

Theorem 8.4 Let $\left(u^T,u^C,u^U,u^F\right)$ $\in\left(QPNFM\right)_{m\times n}$ be QPNFM and $\left(x^T,x^C,x^U,x^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$ then $\left(x^T,x^C,x^U,x^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$ iff $R\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]=R\left(x^T,x^C,x^U,x^F\right)$.

Proof: Let $\left(x^T,x^C,x^U,x^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{2\}$

implies $\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)=\left(u^T,u^C,u^U,u^F\right)$.

That is $\left(u^T,u^C,u^U,u^F\right)\in\left(x^T,x^C,x^U,x^F\right)\{1\}$

Hence, $R\left(x^T,x^C,x^U,x^F\right)=R\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]$.

(Since $\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$ is idempotent.)

Conversely, let $R\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]=R\left(u^T,u^C,u^U,u^F\right)$, then for a pair of matrices

$\left(u^T,u^C,u^U,u^F\right)$ and $\left(x^T,x^C,x^U,x^F\right)$ if the product $\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$ is defined so,

$R\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]\subseteq R\left(x^T,x^C,x^U,x^F\right)$.

That is, $\left(x^T,x^C,x^U,x^F\right)=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$ for some $\left(y^T,y^C,y^U,y^F\right)\in\left(QPNFM\right)_m$.

So, $\left(x^T,x^C,x^U,x^F\right)\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]$

$=\left[\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\right]\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$

or $\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$

$=\left(y^T,y^C,y^U,y^F\right)\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\right]\left(x^T,x^C,x^U,x^F\right)$

$=\left(y^T,y^C,y^U,y^F\right)\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)=\left(x^T,x^C,x^U,x^F\right)$.

Hence $\left(x^T,x^C,x^U,x^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{2\}$.

Theorem 8.5 If $\left(u^T,u^C,u^U,u^F\right)\in\left(QPNFM\right)_{m\times n}$ be a symmetric and idempotent QPNFM then $\left(u^T,u^C,u^U,u^F\right)$ itself a least square g-inverse.

Proof. Since $\left(u^T,u^C,u^U,u^F\right)$ is symmetric, $\left(u^T,u^C,u^U,u^F\right)^T=\left(u^T,u^C,u^U,u^F\right)$

And $\left(u^T,u^C,u^U,u^F\right)$ idempotent, $\left(u^T,u^C,u^U,u^F\right)^2=\left(u^T,u^C,u^U,u^F\right)$

Now $\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ if $\left(p^T,p^C,p^U,p^F\right)=I_n$

Then $\left(u^T,u^C,u^U,u^F\right)\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)$

$=\left(u^T,u^C,u^U,u^F\right)^2=\left(u^T,u^C,u^U,u^F\right)$

That is, $\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$

Now

$\left[\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^I,x^F\right)\right]^T=\left(x^T,x^C,x^U,x^F\right)^T\left(u^T,u^C,u^U,u^F\right)^T$

$=\left(x^T,x^C,x^U,x^F\right)^T\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)^T\left(u^T,u^C,u^U,u^F\right)$

(Taking $\left(x^T,x^C,x^U,x^F\right)=\left(u^T,u^C,u^U,u^F\right)$ as $\left(u^T,u^C,u^U,u^F\right)$ itself a g-inverse.)

$=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)$

This implies, $\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,3\}$

Theorem 8.6 If $\left(u^T,u^C,u^U,u^F\right)\in\left(QPNFM\right)_{m\times n}$ be a symmetric and idempotent QPNFM then $\left(u^T,u^C,u^U,u^F\right)$ itself a minimum norm g-inverse.

Proof. Since $\left(u^T,u^C,u^U,u^F\right)$ is symmetric, $\left(u^T,u^C,u^U,u^F\right)^T=\left(u^T,u^C,u^U,u^F\right)$

And $\left(u^T,u^C,u^U,u^F\right)$ idempotent, $\left(u^T,u^C,u^U,u^F\right)^2=\left(u^T,u^C,u^U,u^F\right)$

Now $\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ if $\left(p^T,p^C,p^U,p^F\right)=I_n$

Then

$$\left(u^T,u^C,u^U,u^F\right)\left(p^T,p^C,p^U,p^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)$$

$$=\left(u^T,u^C,u^U,u^F\right)^2=\left(u^T,u^C,u^U,u^F\right)$$

That is, $\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1\}$

Now

$$\left[\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)\right]^T=\left(u^T,u^C,u^U,u^F\right)^T\left(x^T,x^C,x^U,x^F\right)^T$$

$$=\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)^T=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)^T$$

(Taking $\left(x^T,x^C,x^U,x^F\right)=\left(u^T,u^C,u^U,u^F\right)$ as $\left(u^T,u^C,u^U,u^F\right)$ itself a g-inverse.)

$$=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)$$

This implies, $\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,4\}$

Example 8.5 Let us consider the symmetric

$$\text{QPNFM}\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix}<0.8,0.4,0.3,0.2> & <0.6,0.4,0.3,0.4>\\ <0.6,0.4,0.3,0.4> & <0.7,0.4,0.3,0.3>\end{bmatrix}$$

Now, $\left(u^T,u^C,u^U,u^F\right)^2=\begin{bmatrix}<0.8,0.4,0.3,0.2> & <0.6,0.4,0.3,0.4>\\ <0.6,0.4,0.3,0.4> & <0.7,0.4,0.3,0.3>\end{bmatrix}=\left(u^T,u^C,u^U,u^F\right)$

This shows that $\left(u^T,u^C,u^U,u^F\right)$ is symmetric and idempotent. $\left(u^T,u^C,u^U,u^F\right)$ satisfy the relation

$\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ for

$\left(x^T,x^C,x^U,x^F\right)=\left(u^T,u^C,u^U,u^F\right)$, itself.

Again

$$\left[\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)\right]^T=\left[\left(u^T,u^C,u^U,u^F\right)^2\right]^T=\left(u^T,u^C,u^U,u^F\right)$$

$$=\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)$$

So $\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,3\}$ and $\left(u^T,u^C,u^U,u^F\right)\{1,4\}$

Theorem 8.7 If $\left(u^T,u^C,u^U,u^F\right)\in\left(QPNFM\right)_{m\times n}$ be a symmetric and idempotent QPNFM then $\left\{\left(u^T,u^C,u^U,u^F\right)+\left(h^T,h^C,h^U,h^F\right): \textit{for all QPNFM } \left(h^T,h^C,h^U,h^F\right)\in\left(QPNFM\right)_n\right\}$ such that $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)\left(h^T,h^C,h^U,h^F\right)$ is the set of all {1,3} inverses of $\left(u^T,u^C,u^U,u^F\right)$, dominating $\left(u^T,u^C,u^U,u^F\right)$.

Proof. Since $\left(u^T,u^C,u^U,u^F\right)$ is symmetric and idempotent NFM, $\left(u^T,u^C,u^U,u^F\right)$ itself $\left(u^T,u^C,u^U,u^F\right)\{1,3\}$ inverse.

Let $\left(\beta^T,\beta^I,\beta^F\right)$ denote the set $\left\{\left(u^T,u^C,u^U,u^F\right)+\left(h^T,h^C,h^U,h^F\right): \textit{for all QPNFM } \left(h^T,h^C,h^U,h^F\right)\in\left(QPNFM\right)_n\right\}$ such that

$$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)\left(h^T,h^C,h^U,h^F\right).$$

Suppose $\left(g^T,g^C,g^U,g^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,3\}$

Then $\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)$.

Let $\left(g^T,g^C,g^U,g^F\right)-\left(u^T,u^C,u^U,u^F\right)=\left(h^T,h^C,h^U,h^F\right)$. Since

$$\left(u^T,u^C,u^U,u^F\right)\{1,3\}\subseteq\left(u^T,u^C,u^U,u^F\right)\{1\}$$

$$\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)+\left(h^T,h^C,h^U,h^F\right)\geq\left(u^T,u^C,u^U,u^F\right)$$

Implies, $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)\left[\left(u^T,u^C,u^U,u^F\right)+\left(h^T,h^I,h^F\right)\right]$

$\leq\left(u^{T},u^{C},u^{U},u^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right).$

Now $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,3\}$ and $\left(u^{T},u^{C},u^{U},u^{F}\right)$ itself

$\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,3\}$ inverse so, as the set $\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,3\}$ consists of all solutions for

$\left(x^{T},x^{C},x^{U},x^{F}\right)$ of $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(x^{T},x^{C},x^{U},x^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$(as $\left(u^{T},u^{C},u^{U},u^{F}\right)$ is idempotent).

Thus $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$

This gives $\left(u^{T},u^{C},u^{U},u^{F}\right)\left[\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(h^{T},h^{C},h^{U},h^{F}\right)\right]=\left(u^{T},u^{C},u^{U},u^{F}\right)$

That is $\left(u^{T},u^{C},u^{U},u^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right)\left(h^{T},h^{C},h^{U},h^{F}\right)$

Now, by $(i)\Rightarrow\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right)\left(h^{T},h^{C},h^{U},h^{F}\right)$

Hence $\left[\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(h^{T},h^{C},h^{U},h^{F}\right)\right]\in\left(\beta^{T},\beta^{C},\beta^{U},\beta^{F}\right)$

Thus, for each $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,3\}$ there exists a unique element in $\left(\beta^{T},\beta^{C},\beta^{U},\beta^{F}\right)$

Conversely, for any

$\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(\beta^{T},\beta^{C},\beta^{U},\beta^{F}\right)$,

$\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right)$

with $\left(u^{T},u^{C},u^{U},u^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)$

Hence,

$\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(u^{T},u^{C},u^{U},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)=\left(h^{T},h^{I},h^{F}\right)$

.

So, $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,3\}$

Example 8.6 Consider the NFM $\left(u^{T},u^{C},u^{U},u^{F}\right)=\begin{bmatrix}<0.8,0.4,0.7,0.2> & <0.6,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.7,0.4,0.7,0.3>\end{bmatrix}$

Since $\left(u^{T},u^{C},u^{U},u^{F}\right)$ is symmetric, $\left(u^{T},u^{C},u^{U},u^{F}\right)^{T}=\left(u^{T},u^{C},u^{U},u^{F}\right)$

And $\left(u^T,u^C,u^U,u^F\right)$ idempotent, $\left(u^T,u^C,u^U,u^F\right)^2=\left(u^T,u^C,u^U,u^F\right)$

For the QPNFM $\left(g^T,g^C,g^U,g^F\right)=\begin{bmatrix}<0.9,0.4,0.7,0.1> & <0.6,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.8,0.4,0.7,0.2>\end{bmatrix}$

$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$ and

$\left[\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\right]^T=\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)$.

So $\left(g^T,g^C,g^U,g^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,3\}$

For the QPNFM $\left(h^T,h^C,h^U,h^F\right)=\begin{bmatrix}<0.7,0.4,0.7,0.3> & <0.5,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.6,0.4,0.7,0.4>\end{bmatrix}$

$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)=\begin{bmatrix}<0.8,0.4,0.7,0.2> & <0.6,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.7,0.4,0.7,0.3>\end{bmatrix}$ and

$\left(u^T,u^C,u^U,u^F\right)\left(h^T,h^C,h^U,h^F\right)=\begin{bmatrix}<0.7,0.4,0.7,0.3> & <0.6,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.6,0.4,0.7,0.4>\end{bmatrix}$

Noted that, $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)\left(h^T,h^C,h^U,h^F\right)$

Then, $\left(u^T,u^C,u^U,u^F\right)+\left(h^T,h^C,h^U,h^F\right)=\begin{bmatrix}<0.8,0.4,0.7,0.2> & <0.6,0.4,0.7,0.4>\\ <0.6,0.4,0.7,0.4> & <0.7,0.4,0.7,0.3>\end{bmatrix}$

$\left(u^T,u^C,u^U,u^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,3\}$.

Theorem 8.8 If $\left(u^T,u^C,u^U,u^F\right)\in\left(QPNFM\right)_{m\times n}$ be a symmetric and idempotent NFM then $\left\{\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^I,k^F\right): \textit{for all NFM } \left(k^T,k^I,k^F\right)\in\left(QPNFM\right)_n\right\}$ such that $\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^C,u^U,u^F\right)$ is the set of all {1,4} inverses of $\left(u^T,u^C,u^U,u^F\right)$,dominating $\left(u^T,u^C,u^U,u^F\right)$.

Proof. Since $\left(u^T,u^C,u^U,u^F\right)$ is symmetric and idempotent NFM, $\left(u^T,u^C,u^U,u^F\right)$ itself $\left(u^T,u^C,u^U,u^F\right)\{1,4\}$ inverse.

Let $\left(\beta^T,\beta^C,\beta^U,\beta^F\right)$ denote the set

$\left\{\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^C,k^U,k^F\right): for\,all\,QPNFM\left(k^T,k^C,k^U,k^F\right)\in\left(QPNFM\right)_n\right\}$ such that

$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(k^T,k^C,k^U,k^F\right)\left(u^T,u^C,u^U,u^F\right)$.

Suppose $\left(g^T,g^C,g^U,g^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,4\}$

Then $\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)$.

Let $\left(g^T,g^C,g^U,g^F\right)-\left(u^T,u^C,u^U,u^F\right)=\left(k^T,k^C,k^U,k^F\right)$.

Since $\left(u^T,u^C,u^U,u^F\right)\{1,4\}\subseteq\left(u^T,u^C,u^U,u^F\right)\{1\}$

$\left(g^T,g^C,g^U,g^F\right)\geq\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^C,k^U,k^F\right)\geq\left(u^T,u^C,u^U,u^F\right)$

Implies

$\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left[\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^C,k^U,k^F\right)\right]\left(u^T,u^C,u^U,u^F\right)$

$\geq\left(u^T,u^C,u^U,u^F\right)\left(u^T,u^C,u^U,u^F\right)\geq\left(u^T,u^C,u^U,u^F\right)$.

Now $\left(g^T,g^C,g^U,g^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,4\}$ and $\left(u^T,u^C,u^U,u^F\right)$ itself

$\left(u^T,u^C,u^U,u^F\right)\{1,4\}$ inverse so, as the set $\left(u^T,u^C,u^U,u^F\right)\{1,4\}$ consists of all solutions for

$\left(u^T,u^C,u^U,u^F\right)$ of $\left(x^T,x^C,x^U,x^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$(as $\left(u^T,u^C,u^U,u^F\right)$ is idempotent).

Thus $\left(g^T,g^C,g^U,g^F\right)\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$

This gives $\left[\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^C,k^U,k^F\right)\right]\left(u^T,u^C,u^U,u^F\right)=\left(u^T,u^C,u^U,u^F\right)$

That is $\left(u^T,u^C,u^U,u^F\right)\geq\left(u^T,u^C,u^U,u^F\right)\left(k^T,k^C,k^U,k^F\right)$

Now, by $(ii)\Rightarrow\left(u^T,u^C,u^U,u^F\right)\left(g^T,g^C,g^U,g^F\right)\geq\left(k^T,k^C,k^U,k^F\right)\left(u^T,u^C,u^U,u^F\right)$

Hence $\left[\left(u^T,u^C,u^U,u^F\right)+\left(k^T,k^C,k^U,k^F\right)\right]\in\left(\beta^T,\beta^C,\beta^U,\beta^F\right)$

Thus for each $\left(g^T,g^C,g^U,g^F\right)\in\left(u^T,u^C,u^U,u^F\right)\{1,4\}$ there exists a unique element in

$\left(\beta^T,\beta^C,\beta^U,\beta^F\right)$ Conversely, for any

$\left(g^T,g^C,g^U,g^F\right)\in\left(\beta^T,\beta^C,\beta^U,\beta^F\right)$,

$\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(k^{T},k^{C},k^{U},k^{F}\right)\geq\left(u^{T},u^{C},u^{U},u^{F}\right)$

with $\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\geq\left(k^{T},k^{C},k^{U},k^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)$

Hence,

$\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(k^{T},k^{C},k^{U},k^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(h^{T},h^{I},h^{F}\right)$

So, $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,4\}$

Example 8.7 Consider the QPNFM $\left(u^{T},u^{C},u^{U},u^{F}\right)=\begin{bmatrix}<0.7,0.8,0.1,0.2> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.4> & <0.6,0.8,0.1,0.3>\end{bmatrix}$

Since $\left(u^{T},u^{C},u^{U},u^{F}\right)$ is symmetric, $\left(u^{T},u^{C},u^{U},u^{F}\right)^{T}=\left(u^{T},u^{C},u^{U},u^{F}\right)$

And $\left(u^{T},u^{C},u^{U},u^{F}\right)$ idempotent, $\left(u^{T},u^{C},u^{U},u^{F}\right)^{2}=\left(u^{T},u^{C},u^{U},u^{F}\right)$

For the QPNFM $\left(g^{T},g^{C},g^{U},g^{F}\right)=\begin{bmatrix}<0.8,0.8,0.1,0.1> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.4> & <0.7,0.8,0.1,0.2>\end{bmatrix}$

$\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\left(u^{T},u^{C},u^{U},u^{F}\right)$ and

$\left[\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)\right]^{T}=\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right).$

So $\left(g^{T},g^{C},g^{U},g^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,4\}$

For the QPNFM $\left(k^{T},k^{C},k^{U},k^{F}\right)=\begin{bmatrix}<0.6,0.8,0.1,0.3> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.5> & <0.5,0.8,0.1,0.5>\end{bmatrix}$

$\left(g^{T},g^{C},g^{U},g^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\begin{bmatrix}<0.7,0.8,0.1,0.2> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.4> & <0.6,0.8,0.1,0.3>\end{bmatrix}$ and

$\left(u^{T},u^{C},u^{U},u^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)=\begin{bmatrix}<0.6,0.8,0.1,0.3> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.5> & <0.5,0.8,0.1,0.5>\end{bmatrix}$

Noted that, $\left(u^{T},u^{C},u^{U},u^{F}\right)\left(g^{T},g^{C},g^{U},g^{F}\right)\geq\left(k^{T},k^{C},k^{U},k^{F}\right)\left(u^{T},u^{C},u^{U},u^{F}\right)$

Then, $\left(u^{T},u^{C},u^{U},u^{F}\right)+\left(k^{T},k^{C},k^{U},k^{F}\right)=\begin{bmatrix}<0.7,0.8,0.1,0.2> & <0.5,0.8,0.1,0.4>\\ <0.5,0.8,0.1,0.4> & <0.6,0.8,0.1,0.3>\end{bmatrix}$

$\left(u^{T},u^{C},u^{U},u^{F}\right)\in\left(u^{T},u^{C},u^{U},u^{F}\right)\{1,4\}.$

9. An Algorithm Based on QPNFM in a Decision-Making Problem

Definition 9.1. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and E = {$e_1, e_2, \ldots, e_m$} be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of TM and the degree of − CM of an element y_i to η (e_j) denoted by $T_{\eta(e_j)}(y_i)$ and $C_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{T_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[T_{\eta(e_j)}(y_i) - T_{\eta(e_j)}(y_k)\right]$$

$$S_{C_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[C_{\eta(e_j)}(y_i) - C_{\eta(e_j)}(y_k)\right]$$

Definition 9.2. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and E = {$e_1, e_2, \ldots, e_m$} be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of UM and the degree of − FM of an element xi to η (e_j) denoted by $U_{\eta(e_j)}(y_i)$ and $F_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{U_{\eta(e_j)}}(y_i) = -\sum_{k=1}^{N}\left[U_{\eta(e_j)}(y_i) - U_{\eta(e_j)}(y_k)\right]$$

$$S_{F_{\eta(e_j)}}(y_i) = \sum_{K=1}^{N}\left[F_{\eta(e_j)}(y_i) - F_{\eta(e_j)}(y_k)\right]$$

Definition 9.3. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and E = {$e_1, e_2, \ldots, e_m$} be a set of parameters. For an QPNSS (η,E) over Y, the scores of the TM, CM, UM, and FM of y_i for each e_j be denoted by $S_{T_{\eta(e_j)}}(y_i), S_{C_{\eta(e_j)}}(y_i), S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ respectively. Then, the total score of x_i for each e_j is denoted by $\square_{T_{\eta(e_j)}}(y_i) = S_{T_{\eta(e_j)}}(y_i) + S_{C_{\eta(e_j)}}(y_i) + S_{U_{\eta(e_j)}}(y_i) + S_{F_{\eta(e_j)}}(y_i)$

Based on the above definitions, we give the steps of the proposed algorithm as follows:

Algorithm:

Step 1. For the universal set Y = {$y_1, y_2, \ldots, y_n$} and the parameter set E = {$e_1, e_2, \ldots, e_m$} input the matrix representation of an QPNSS (η,E) in tabular form, according to a decision-maker.

Step 2. Reference to the input matrix obtained in step 1 and using the Definitions 11.1 and 11.2, we compute $S_{T_{\eta(e_j)}}(y_i), S_{C_{\eta(e_j)}}(y_i), S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ for xi for each ej where i = 1 to n; j = 1 to m.

Step 3. Taking the results obtained in step 2 and using the Definition 11.3, compute the score $\square_{T_{\eta(e_j)}}(x_i)$ of xi for each $ej$where i = 1 to n; j = 1 to m.

Step 4. Compute the overall score ui for xiin such a way that

$$v_i = \square_{T_{\eta(e_1)}}(y_i) + \square_{C_{\eta(e_2)}}(y_i) + \square_{U_{\eta(e_3)}}(y_i) + \ldots + \square_{F_{\eta(e_m)}}(y_i)$$

Step 5. Find k, for which $v_k = \max_{x_i \in X}\{v_i\}$. Then, $x_k \in X$ is the optimal choice.

Step 6. In case of a tie, either we take both as an optimal choice or we reassess all the values with the expert's advice and repeat all the previous steps.

10. To demonstrate the practical application of the algorithm, we present the following example.

To create a decision-making scenario involving **five bikes** and **five parameters**, we will structure it as a Multi-Criteria Decision-Making (MCDM) problem. Here is an example:

Bikes (Alternatives):

1. **Bike A:** Sports bike with high performance.
2. **Bike B:** Cruiser designed for long-distance comfort.
3. **Bike C:** Electric bike with eco-friendly features.
4. **Bike D:** Commuter bike with fuel efficiency.
5. **Bike E:** Adventure bike for off-road and rugged terrains.

Parameters (Criteria):

1. **Cost:** Initial purchase price of the bike (lower is better).
2. **Fuel/Energy Efficiency:** Kilometres per litter or range per charge (higher is better).
3. **Comfort:** Riding comfort based on suspension and seat ergonomics (higher is better).
4. **Maintenance Cost:** Average annual maintenance expenses (lower is better).
5. **Performance:** Acceleration, top speed, and handling (higher is better).

Weights for Parameters:

1. **Cost:** 0.25 (25%)
   - Cost is an important factor for affordability.
2. **Fuel/Energy Efficiency:** 0.20 (20%)
   - Efficiency is critical for long-term savings.
3. **Comfort:** 0.20 (20%)
   - Comfort matters for daily usability or long rides.
4. **Maintenance Cost:** 0.15 (15%)
   - Lower maintenance cost is a priority but less critical than the initial cost.
5. **Performance:** 0.20 (20%)

Based on the evaluations from the DMs, the decision matrix reflecting the five bike and five evaluation criteria under the Multi-Criteria Decision-Making framework is presented in Table 2.

Table 2. Tabular illustration of QPNFM to describe the set of five bike

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	<0.8,0.8,0.9,0.6>	<0.9,0.5,0.4,0.1>	<0.6,0.4,0.5,0.2>	<0.9,0.7,0.2,0.3>	<0.6,0.2,0.1,0.3>

y2	<0.8,0.4,0.3,0.2>	<0.6,0.1,0.2,0.3>	<0.8,0.3,0.7,0.6>	<0.4,0.1,0.3,0.2>	<0.5,0.4,0.3,0.1>
y3	<0.9,0.8,0.9,0.6>	<0.4,0.8,0.9,0.1>	<0.3,0.8,0.1,0.6>	<0.5,0.2,0.3,0.6>	<0.5,0.8,0.1,0.3>
y4	<0.7,0.1,0.2,0.3>	<0.5,0.7,0.1,0.6>	<0.9,0.8,0.2,0.3>	<0.4,0.1,0.3,0.5>	<0.7,0.3,0.2,0.1>
y5	<0.6,0.3,0.5,0.1>	<0.5,0.4,0.3,0.2>	<0.9,0.2,0.3,0.4>	<0.8,0.5,0.2,0.6>	<0.2,0.4,0.1,0.2>

Step 2. The score of the truth-membership degrees $S_{T_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 3.

Y/E	e1	e2	e3	e4	e5
y1	0.2	1.6	-0.5	1.5	0.5
y2	0.2	0.1	0.5	-1	0
y3	0.7	-0.9	-2	-0.5	0
y4	-0.3	-0.4	1	-1	1
y5	-0.8	-0.4	1	1	-1.5

The score of the contradiction-membership degrees $S_{C_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 4.

Table 4. Tabular illustration of the score of contradiction-membership degree

Y/E	e1	e2	e3	e4	e5
y1	1.6	0	-0.5	1.9	-1.1
y2	-0.4	-2	-1	-1.1	-0.1
y3	1.6	1.5	1.5	-0.6	1.9
y4	-1.9	1	1.5	-1.1	-0.6
y5	-0.9	-0.5	-1.5	0.9	-0.1

The score of the unknown-membership degrees $S_{U_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 5.

Table 5. Tabular illustration of the score of unknown-membership degree

Y/E	e1	e2	e3	e4	e5
y1	1.7	0.1	0.7	-0.3	-0.3
y2	-1.3	-0.9	1.7	0.2	0.7
y3	1.7	2.6	-1.3	0.2	-0.3
y4	-1.8	-1.4	-0.8	0.2	0.2
y5	-0.3	-0.4	-0.3	-0.3	-0.3

The score of the false-membership degrees $S_{F_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 6.

Table 6. Tabular illustration of the score of false-membership degree

Y/E	e1	e2	e3	e4	e5

y_1	1.2	-0.8	-1.1	-0.8	0.5
y_2	-0.8	0.2	0.9	-1.3	-0.5
y_3	1.2	-0.8	0.9	0.7	0.5
y_4	-0.3	1.7	-0.6	0.2	-0.5
y_5	-1.3	-0.3	-0.1	0.7	0

Step 3. By using Table 3 to Table 6, the score $\square\ \eta_{(ej)}(x_i)$ for (η, E) is exhibited in Table 7.

Table 7. Tabular illustration of the score $\square\ \eta_{(ej)}(x_i)$.

Y/E	**e_1**	**e_2**	**e_3**	**e_4**	**e_5**
y_1	4.7	0.9	-0.5	2.3	-0.4
y_2	-2.3	-2.6	2.1	-3.2	0.1
y_3	5.2	3.4	-0.9	-0.2	2.1
y_4	-4.3	0.9	1.1	-1.7	0.1
y_5	-3.3	-1.6	-0.9	2.3	-1.9

Step 4. Now, we calculate the overall score given as:

$v_1 = 7$, $v_2 = -5.9$, $v_3 = 9.6$, $v_4 = -3.9$, $v_5 = -5.4$.

Step 5. Thus, $v_k = \max_{y_i \in Y}\{v_1, v_2, v_3, v_4, v_5\} = v_3$. Therefore, v_3 is the optimal choice object for the decision maker.

11. Conclusion, limitation and future research work

In this study, we introduced a novel method for determining the generalized inverse (g-inverse) and Moore-Penrose inverse of Neutrosophic Fuzzy Matrices (NFMs). A dedicated algorithm was developed to compute the g-inverse of NFMs, addressing the existing gap in computational techniques for these matrices. Several theoretical results and properties of the g-inverse were explored and validated through numerical examples. Additionally, the practical applicability of the g-inverse was demonstrated by solving a rectangular system of neutrosophic fuzzy relational equations.

The findings of this work provide significant contributions to the field of neutrosophic fuzzy matrix theory by combining theoretical advancements with practical algorithms. This research not only enhances the foundational understanding of matrix inverses in neutrosophic fuzzy settings but also offers a reliable tool for solving real-world problems in control systems, robotics, and decision-making processes. Future studies can build upon this work to explore further applications and extensions of the proposed methods.

While this study makes significant contributions to the field of Neutrosophic Fuzzy Matrices (NFMs), certain limitations should be acknowledged: The proposed algorithm for computing the g-inverse of NFMs may have increased computational complexity for large-scale matrices, potentially limiting its efficiency in high-dimensional applications. Further optimization techniques could be explored. The study assumes specific conditions for the existence and uniqueness of the g-inverse and Moore-Penrose inverse in NFMs. These conditions may not always hold for all types of neutrosophic fuzzy matrices, restricting the general applicability of the proposed methods. The accuracy and stability of the algorithm in handling extreme cases (e.g., highly indeterminate or inconsistent neutrosophic fuzzy data) require further investigation to ensure robustness in practical implementations. While the study demonstrates applicability in solving rectangular systems of neutrosophic fuzzy relational equations, its potential in other domains, such as optimization, signal processing, and machine learning, remains unexplored. A detailed comparison with existing fuzzy and neutrosophic matrix inversion techniques, in terms of efficiency and accuracy, is not extensively performed.

Future research could focus on benchmarking the proposed method against other existing approaches. The study is limited to classical Neutrosophic Fuzzy Matrices. Extending the approach to interval-valued or quadripartitioned neutrosophic fuzzy matrices may provide further insights and broader applicability. Also, to focus on enhancing the computational efficiency of the proposed algorithm by developing optimization techniques or parallel computing approaches for handling large-scale neutrosophic fuzzy matrices. Additionally, extending the methodology to interval-valued and quadri-partitioned neutrosophic fuzzy matrices could provide a more generalized framework for uncertainty modeling. Further studies may explore the integration of g-inverse techniques in machine learning, data mining, and multi-criteria decision-making, where handling indeterminacy and inconsistency is crucial. Moreover, a comparative study with existing fuzzy and neutrosophic inverse computation methods could offer deeper insights into the strengths and limitations of the proposed approach, paving the way for future advancements in neutrosophic fuzzy matrix theory and its applications.

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