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Pythagorean neutrosophic ideals in gamma semigroup

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Abstract. Pythagorean neutrosophic ideals in gamma semigroup, Pythagorean neutrosophic bi-ideals in gamma semigroup, and Pythagorean neutrosophic interior ideals in gamma semigroup are the concepts that are defined in this work. We also go through some of its characteristics with appropriate example.

Keywords: Pythagorean; Neutrosophic set; ideals; semigroup

1. Introduction

The study of semigroupsan algebraic structure consisting of a non-empty set equipped with an associative binary operationbegan in earnest in the early 20th century [42]. Semigroups have since become foundational in various areas of mathematics, such as automata theory, combinatorics, coding theory, language theory, and mathematical analysis. Building on this foundational work, Sen and Saha [26] established a crucial relationship between regular -semigroups (Gamma-semigroups) and Γ-groups, thereby introducing and developing the theoretical framework of Gamma-semigroups

. In parallel with the evolution of algebraic structures, the study of uncertainty and vagueness in data led to the development of various generalized set theories. The pioneering concept of fuzzy sets was introduced by Lotfi Zadeh in 1965 [41], which allowed for the representation of data that could not be confined to classical binary logic. Extending this idea, Atanassov [5,6] introduced intuitionistic fuzzy sets (IFS), which include both membership and non-membership degrees for each element, along with a hesitation margin that captures the uncertainty. Further refining this concept, Yager [39,40] proposed the Pythagorean fuzzy set (PFS), wherein the squares of membership and non-membership degrees sum to less than or equal to one. This approach allowed for a broader range of uncertainty modeling compared to IFS. Yager and Abbasov [38] formally introduced the PFS, recognizing its potential as a powerful generalization of intuitionistic fuzzy sets. Following these developments, Chinnadurai [10] explored the concept of fuzzy ideals in algebraic structures, paving the way for more nuanced algebraic modeling under uncertainty. Later, Gun et al. [13] presented novel operations on a newly introduced structure called the spherical fuzzy set, which encapsulates three-dimensional fuzzy information. To address the limitations of prior fuzzy models, the neutrosophic set theory was introduced by Smarandache [27,28], which considers three independent components: truth-membership, indeterminacy, and falsity-membership. This triadic logic framework offers a richer means to express uncertainty. The Pythagorean neutrosophic set, introduced by Jansi et al. [14], extends this idea further by incorporating the Pythagorean structure into the neutrosophic framework. Various scholars [8,9] have also discussed the intuitionistic neutrosophic set, highlighting its value in theoretical and practical applications.

In the context of semigroups, Khan et al. [19] introduced the concept of neutrosophic N-structures and explored their properties. Sardar et al. [25] contributed by studying fuzzy ideals in Gamma-semigroups, bridging fuzzy logic with algebraic structures. Similarly, Uckum et al. [29] extended intuitionistic fuzzy sets into the domain of Gamma-semigroups, enhancing their applicability. In the broader decision-making domain, Rangsuk et al. [24] applied neutrosophic sets to UP-algebra, revealing its relevance in abstract algebraic reasoning. Abdel-Basset et al. made significant contributions to multi-criteria decision-making (MCDM) by integrating fuzzy and neutrosophic environments. They proposed a hybrid MCDM method combining the Analytical Hierarchy Process (AHP) and the Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE-II) to identify optimal locations for offshore wind power plants [1]. Later, they developed a neutrosophic PROMETHEE approach [2] to handle uncertain and imprecise information in MCDM problems. In another application, Abdel-Basset et al. [3] explored how smart Internet of Things (IoT) networks can be employed by clinical staff to mitigate the spread of COVID-19, emphasizing the intersection of health care and decision science. Additionally, their work in [4] applied an MCDM framework to assess the sustainability of various hydrogen production methods, offering insights into renewable energy planning. Focusing on algebraic structures once again, Jun et al. [15,16] examined the properties of intuitionistic fuzzy interior ideals within semigroups and extended their study to the fuzzification of interior ideals. In related research, Kuroki [18] analyzed the behavior of fuzzy sets

and fuzzy bi-ideals in semigroup environments, providing a foundation for further theoretical development. Jun et al. [17] also investigated the (1,2)-ideals in semigroups and their fuzzification, exploring both structural and operational properties. Recent advancements include the work of Abdel-Monem et al. [43], who introduced a neutrosophic-MARCOS (Measurement of Alternatives and Ranking according to Compromise Solution) method for decision-making problems involving multiple criteria and alternatives. Their model was validated using a detailed numerical case study involving eight criteria and ten alternatives. Ahmed Abdelhafeez et al. [44] proposed a unique MCDM approach that considers the varying expertise levels of multiple decision-makers, adding a realistic dimension to decision modeling.
Finally, Manas Karak et al. [45] addressed the neutrosophic transportation problem (NTP) by introducing a sign distance ranking function. They converted NTPs represented by single-valued neutrosophic numbers (SVNs) into traditional transportation problems with crisp values, demonstrating the method's validity through numerical examples. In This Paper we introduce Pythagorean neutrosophic ideals in gamma semigroup, Pythagorean neutrosophic bi-ideals in gamma semigroup, and Pythagorean neutrosophic interior ideals in gamma semigroup and fundamental Properties are discussed.

2. Preliminaries

Definition 2.1. [39] Let X be a universe of discourse, A **Pythagorean fuzzy set** (PFS) $P = \{z, \vartheta_p(x), \omega_p(x)/z \in X\}$ where $\vartheta : X \to [0,1]$ and $\omega : X \to [0,1]$ represent the degree of membership and non-membership of the object $z \in X$ to the set P subset to the condition $0 \leq (\vartheta_p(z))^2 + (\omega_p(z))^2 \leq 1$ for all $z \in X$. For the sake of simplicity a PFS is denoted as $P = (\vartheta_p(z), \omega_p(z))$.

Definition 2.2. [27] Let X be a universe of discourse, A **Neutrosophic set** (NS) $N = \{z, \vartheta_N(z), \omega_N(z), \psi_N(z)/z \in X\}$ where $\vartheta : X \to [0,1]$, $\omega : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of truth membership, indeterminacy-membership and false-membership of the object $z \in X$ to the set N subset to the condition $0 \leq (\vartheta_N(z))^2 + (\omega_N(z))^2 + (\psi_N(z))^2 \leq 3$ for all $z \in X$. For the sake of simplicity a NS is denoted as $N = (\vartheta_N(z), \omega_N(z), \psi_N(z))$.

Definition 2.3. [14] Let X be a universe of discourse, A **Pythagorean neutrosophic set** (PNS) $P_N = \{z, \mu_p(z), \zeta_p(z), \psi_p(z)/z \in X\}$ where $\mu : X \to [0,1]$, $\zeta : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of membership, non-membership and indeterminacy of the object $z \in X$ to the set P_N subset to the condition $0 \leq (\mu_p(z))^2 + (\zeta_p(z))^2 + (\psi_p(z))^2 \leq 2$ for all $z \in X$. For the sake of simplicity a PNS is denoted as $P_N = (\mu_p(z), \zeta_p(z), \psi_p(z))$.

3. Pythagorean neutrosophic ideals in gamma semigroup

Throughout this paper unless otherwise stated S denote a Γ-semigroup.

Definition 3.1. A non-empty Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ of S is called a Pythagorean neutrosophic subsemigroup of S if it satisfies:
(i) $\mu_{P_N}(x\gamma y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\}$,
(ii) $\zeta_{P_N}(x\gamma y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$,
$(iii)\nu_{P_N}(x\gamma y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Proposition 3.2. *If Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is a Pythagorean neutrosophic subsemigroup of S, then the set $P_N = \{x \in S|\mu_{P_N}(x) = \mu_{P_N}(0), \zeta_{P_N}(x) = \zeta_{P_N}(0), \nu_{P_N}(x) = \nu_{P_N}(0)\}$ is a subsemigroup of S.*

Proof. Let $x, y \in S, \gamma \in \Gamma$. Then $\mu_{P_N}(x) = \mu_{P_N}(y) = \mu_{P_N}(0)$, $\zeta_{P_N}(x) = \zeta_{P_N}(y) = \zeta_{P_N}(0)$ and $\nu_{P_N}(x) = \nu_{P_N}(y) = \nu_{P_N}(0)$. Since $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is Pythagorean neutrosophic subsemigroup of S, follows that $\mu_{P_N}(x\gamma y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\} = \mu_{P_N}(0)$, $\zeta_{P_N}(x\gamma y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\} = \zeta_{P_N}(0)$, $\nu_{P_N}(x\gamma y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\} = \nu_{P_N}(0)$, so that $\mu_{P_N}(x\gamma y) = \mu_{P_N}(0)$, $\zeta_{P_N}(x\gamma y) = \zeta_{P_N}(0)$ and $\nu_{P_N}(x\gamma y) = \nu_{P_N}(0)$. Thus $x\gamma y \in P_N$, and consequently P_N is a subsemigroup of S. Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic subsemigroup in S and $a + b + c \in [0, 1]$ be such that $a + b + c \leq 1$.
Then we define the set $P_N^{a,b,c} = \{x \in S|\mu_{P_N}(x) \geq a, \zeta_{P_N}(x) \leq b, \nu_{P_N}(x) \leq c\}$. □

Theorem 3.3. *Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic subsemigroup of S. Then $P_N^{a,b,c}$ is a subsemigroup of semigroup S for every $(a, b, c) \in Im(\mu_{P_N}) \times Im(\zeta_{P_N}) \times Im(\nu_{P_N})$ with $a + b + c \leq 1$.*

Proof. Let $x, y \in P_N^{a,b,c}, \gamma \in \Gamma$. Then $\mu_{P_N}(x) \geq a, \zeta_{P_N}(x) \leq b, \nu_{P_N}(x) \leq c, \mu_{P_N}(y) \geq a, \zeta_{P_N}(y) \leq b, \nu_{P_N}(y) \leq c$ which implies that
$\mu_{P_N}(x\gamma y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\} \geq a$
$\zeta_{P_N}(x\gamma y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\} \leq a$
$\nu_{P_N}(x\gamma y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\} \leq a$.
Thus $x - y \in P_N^{a,b,c}$. Therefore $P_N^{a,b,c}$ is a subsemigroup of semigroup S.
A semigroup S is said to be a monoid if there exists an identity element $e \in S$ such that $xe = ex = x$ for all $x \in S$. □

Note that every Pythagorean neutrosophic left(right) ideal of S is a Pythagorean neutrosophic subsemigroup of S. But the converse is not true.

Definition 3.4. A non-empty Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ of S is called a Pythagorean neutrosophic left ideal of S if it satisfies:
(i) $\mu_{P_N}(x\gamma y) \geq \mu_{P_N}(y)$,

(ii) $\zeta_{P_N}(x\gamma y) \leq \zeta_{P_N}(y)$,
(iii) $\nu_{P_N}(x\gamma y) \leq \nu_{P_N}(y)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 3.5. A non-empty Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ of S is called a Pythagorean neutrosophic right ideal of S if it satisfies:
(i) $\mu_{P_N}(x\gamma y) \geq \mu_{P_N}(x)$,
(ii) $\zeta_{P_N}(x\gamma y) \leq \zeta_{P_N}(x)$,
(iii) $\nu_{P_N}(x\gamma y) \leq \nu_{P_N}(x)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Lemma 3.6. *Let Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic subgroup of S such that $\mu_{P_N}(x) \geq \mu_{P_N}(y)(or(\mu_{P_N}(y) \geq \mu_{P_N}(x)))$, $\zeta_{P_N}(x) \leq \zeta_{P_N}(y)(or(\zeta_{P_N}(y) \leq \zeta_{P_N}(x)))$ and $\nu_{P_N}(x) \leq \nu_{P_N}(y)(or(\nu_{P_N}(y) \leq \nu_{P_N}(x)))$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is a Pythagorean neutrosophic left(right) ideal of S.*

Proof. Let $\mu_{P_N}(x) \geq \mu_{P_N}(y)$, $\zeta_{P_N}(x) \leq \zeta_{P_N}(y)$ and $\nu_{P_N}(x) \leq \nu_{P_N}(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$.
Then we have
$\mu_{P_N}(x\gamma y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\} = \mu_{P_N}(y)$, $\zeta_{P_N}(x\gamma y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\} = \zeta_{P_N}(y)$, $\nu_{P_N}(x\gamma y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\} = \nu_{P_N}(y)$. Hence $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is a Pythagorean neutrosophic left ideal of S. Similarly if we take $\mu_{P_N}(y) \geq \mu_{P_N}(x)$, $\zeta_{P_N}(y) \leq \zeta_{P_N}(x)$ and $\nu_{P_N}(y) \leq \nu_{P_N}(x)$ for all $x, y \in S$ and $\gamma \in \Gamma$, then prove that $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is a Pythagorean neutrosophic right ideal of S. □

Definition 3.7. A Pythagorean neutrosophic subsemigroup $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ of S is called a Pythagorean neutrosophic bi-ideal of S if it satisfies:
(i)$\mu_{P_N}(x\gamma a\beta y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\}$,
(ii) $\zeta_{P_N}(x\gamma a\beta y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$,
(iii) $\nu_{P_N}(x\gamma a\beta y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$, for all $x, y \in S$ and $\gamma, \beta \in \Gamma$.

Example 3.8. Let $P_N = \{0, a, b, c\}$ and $\Gamma = \{\gamma, \beta\}$be non-empty set of binary operations defined as follows.

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

and

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Clearly S is a Γ-semigroup. A Pythagorean neutrosophic set $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ where $\mu_{P_N} : S \to [0,1]$ by $\mu_{P_N}(0) = 0.8, \mu_{P_N}(a) = 0.6, \mu_{P_N}(b) = 0.3 = \mu_{P_N}(c)$, $\zeta_{P_N} : S \to [0,1]$ by $\zeta_{P_N}(0) = 0.4, \zeta_{P_N}(a) = 0.6, \zeta_{P_N}(b) = 0.7 = \zeta_{P_N}(c)$ and $\nu_{P_N} : S \to [0,1]$ by $\nu_{P_N}(0) = 0.3, \nu_{P_N}(a) = 0.5, \nu_{P_N}(b) = 0.6 = \nu_{P_N}(c)$.
Thus $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ is a Pythagorean neutrosophic bi-ideal of S.

Theorem 3.9. *Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic ideal of S. If S is an intra-regular, then $P_N(a) = P_N(a\beta a)$ for all $a \in S, \beta \in \Gamma$.*

Proof. Let a be any element of S. Then since S is an intra-regular, there exists $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = x\alpha a\beta a\gamma y$. Hence $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic ideal, $\mu_{P_N}(a) = \mu_{P_N}(x\alpha a\beta a\gamma y) \geq \mu_{P_N}(x\alpha a\beta a) \geq \mu_{P_N}(a\beta a) =\geq \mu_{P_N}(a)$,
$\zeta_{P_N}(a) = \zeta_{P_N}(x\alpha a\beta a\gamma y) \leq \zeta_{P_N}(x\alpha a\beta a) \leq \zeta_{P_N}(a\beta a) =\leq \zeta_{P_N}(a)$,
$\nu_{P_N}(a) = \nu_{P_N}(x\alpha a\beta a\gamma y) \leq \nu_{P_N}(x\alpha a\beta a) \leq \nu_{P_N}(a\beta a) =\leq \nu_{P_N}(a)$.
Hence we have $\mu_{P_N}(a) = \mu_{P_N}(a\beta a)$, $\zeta_{P_N}(a) = \zeta_{P_N}(a\beta a)$ and $\nu_{P_N}(a) = \nu_{P_N}(a\beta a)$.
Therefore $P_N(a) = P_N(a\beta a)$ for all $a \in S, \beta \in \Gamma$. □

Theorem 3.10. *Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic ideal of S is an inter-regular, then $P_N(a\beta b) = P_N(b\beta a)$ for all $a, b \in S, \beta \in \Gamma$.*

Proof. Let $a, b \in S$ and $\beta \in \Gamma$. Then Theorem 3.7 we have

$$\begin{aligned}\mu_{P_N}(a\beta b) &= \mu_{P_N}(a\beta b\beta a\beta b)\\ &= \mu_{P_N}(a\beta(b\beta a)\beta b)\\ &\geq \mu_{P_N}(b\beta a)\\ &= \mu_{P_N}(b\beta a\beta b\beta a)\\ &= \mu_{P_N}(b\beta(a\beta b)\beta a)\\ &\geq \mu_{P_N}(a\beta b),\end{aligned}$$

$$\begin{aligned}\zeta_{P_N}(a\beta b) &= \zeta_{P_N}(a\beta b\beta a\beta b)\\ &= \zeta_{P_N}(a\beta(b\beta a)\beta b)\\ &\leq \zeta_{P_N}(b\beta a)\\ &= \zeta_{P_N}(b\beta a\beta b\beta a)\\ &= \zeta_{P_N}(b\beta(a\beta b)\beta a)\\ &\leq \zeta_{P_N}(a\beta b)\end{aligned}$$

and

$$\begin{aligned}\nu_{P_N}(a\beta b) &= \nu_{P_N}(a\beta b\beta a\beta b)\\ &= \nu_{P_N}(a\beta(b\beta a)\beta b)\\ &\leq \nu_{P_N}(b\beta a)\\ &= \nu_{P_N}(b\beta a\beta b\beta a)\end{aligned}$$

$= \nu_{P_N}(b\beta(a\beta b)\beta a)$
$\leq \nu_{P_N}(a\beta b).$
Hence we have $\mu_{P_N}(a\beta b) = \mu_{P_N}(b\beta a)$, $\zeta_{P_N}(a\beta b) = \zeta_{P_N}(b\beta a)$ and $\nu_{P_N}(a\beta b) = \nu_{P_N}(b\beta a)$. Therefore $P_N(a\beta b) = P_N(b\beta a)$ for all $a, b \in S, \beta \in \Gamma$. □

Theorem 3.11. *Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic bi-ideal of S if and only if the fuzzy set $\mu_{P_N}, \overline{\zeta_{P_N}}$ and $\overline{\nu_{P_N}}$ are fuzzy bi-ideals of S.*

Proof. Let $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ be a Pythagorean neutrosophic bi-ideal of S. Then clearly μ_{P_N} is a fuzzy bi-ideal of S. Let $x, a, y \in S, \alpha, \beta \in \Gamma$. Then

$\overline{\zeta_{P_N}}(x\alpha y) = 1 - \zeta_{P_N}(x\alpha y)$
$\geq 1 - max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$
$= \min\{1 - \zeta_{P_N}(x), 1 - \zeta_{P_N}(y)\}$
$= \min\{\overline{\zeta_{P_N}}(x), \overline{\zeta_{P_N}}(y)\},$

$\overline{\nu_{P_N}}(x\alpha y) = 1 - \nu_{P_N}(x\alpha y)$
$\geq 1 - max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$
$= \min\{1 - \nu_{P_N}(x), 1 - \nu_{P_N}(y)\}$
$= \min\{\overline{\nu_{P_N}}(x), \overline{\nu_{P_N}}(y)\},$

and

$\overline{\zeta_{P_N}}(x\alpha a\beta y) = 1 - \zeta_{P_N}(x\alpha a\beta y)$
$\geq 1 - max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$
$= \min\{1 - \zeta_{P_N}(x), 1 - \zeta_{P_N}(y)\}$
$= \min\{\overline{\zeta_{P_N}}(x), \overline{\zeta_{P_N}}(y)\},$

$\overline{\nu_{P_N}}(x\alpha a\beta y) = 1 - \nu_{P_N}(x\alpha a\beta y)$
$\geq 1 - max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$
$= \min\{1 - \nu_{P_N}(x), 1 - \nu_{P_N}(y)\}$
$= \min\{\overline{\nu_{P_N}}(x), \overline{\nu_{P_N}}(y)\}.$

Hence $\overline{\zeta_{P_N}}, \overline{\nu_{P_N}}$ are fuzzy bi-ideal of S. Conversely, suppose that μ_{P_N}, ζ_{P_N} and ν_{P_N} are fuzzy bi-ideal of S. Let $a, x, y \in S, \alpha, \beta \in \Gamma$. Then

$1 - \zeta_{P_N}(x\alpha y) = \overline{\zeta_{P_N}}(x\alpha y)$
$\geq \min\{\overline{\zeta_{P_N}}(x), \overline{\zeta_{P_N}}(y)\}$
$= \min\{1 - \zeta_{P_N}(x), 1 - \zeta_{P_N}(y)\}$
$= \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$

and

$1 - \zeta_{P_N}(x\alpha a\beta y) = \overline{\zeta_{P_N}}(x\alpha a\beta y)$
$\geq \min\{\overline{\zeta_{P_N}}(x), \overline{\zeta_{P_N}}(y)\}$

$= 1 - \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\},$

$1 - \nu_{P_N}(x\alpha y) = \overline{\nu_{P_N}}(x\alpha y)$

$\geq \min\{\overline{\nu_{P_N}}(x), \overline{\nu_{P_N}}(y)\}$

$= \min\{1 - \nu_{P_N}(x), 1 - \nu_{P_N}(y)\}$

$= \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$

and

$1 - \nu_{P_N}(x\alpha a\beta y) = \overline{\nu_{P_N}}(x\alpha a\beta y)$

$\geq \min\{\overline{\nu_{P_N}}(x), \overline{\nu_{P_N}}(y)\}$

$= 1 - \max\{\nu_{P_N}(x), \nu_{P_N}(y)\},$

which implies that $\zeta_{P_N}(x\alpha y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$, $\nu_{P_N}(x\alpha y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$ and $\zeta_{P_N}(x\alpha a\beta y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$, $\nu_{P_N}(x\alpha a\beta y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$. □

Definition 3.12. A Pythagorean neutrosophic subsemigroup $P_N = (\mu_{P_N}, \zeta_{P_N}, \nu_{P_N})$ of S is called a Pythagorean neutrosophic interior ideal of S if it satisfies:

(i) $\mu_{P_N}(x\gamma a\beta y) \geq \mu_{P_N}(a)$,

(ii) $\zeta_{P_N}(x\gamma a\beta y) \leq \zeta_{P_N}(a)$,

(iii) $\nu_{P_N}(x\gamma a\beta y) \leq \nu_{P_N}(a)$, for all $x, y \in S$ and $\gamma, \beta \in \Gamma$.

Proposition 3.13. *Let P_N be a Pythagorean neutrosophic ideal of S. Then P_N is a Pythagorean neutrosophic interior ideal of S.*

Proof. Since P_N is a Pythagorean neutrosophic ideal of S, for any $x, y \in S$ and $\gamma \in \Gamma$, $\mu_{P_N}(x\gamma y) \geq \mu_{P_N}(x)$, $\zeta_{P_N}(x\gamma y) \leq \zeta_{P_N}(x)$, $\nu_{P_N}(x\gamma y) \leq \nu_{P_N}(x)$ are Pythagorean neutrosophic left ideals of S and $\mu_{P_N}(x\gamma y) \geq \mu_{P_N}(y)$, $\zeta_{P_N}(x\gamma y) \leq \zeta_{P_N}(y)$, $\nu_{P_N}(x\gamma y) \leq \nu_{P_N}(y)$ are Pythagorean neutrosophic right ideal of S, which implies that $\mu_{P_N}(x\gamma y) \geq \min\{\mu_{P_N}(x), \mu_{P_N}(y)\}$,
$\zeta_{P_N}(x\gamma y) \leq \max\{\zeta_{P_N}(x), \zeta_{P_N}(y)\}$, $\nu_{P_N}(x\gamma y) \leq \max\{\nu_{P_N}(x), \nu_{P_N}(y)\}$. Hence P_N is a Pythagorean neutrosophic sub-semigroup of S. Now let $x, a, y \in S$ and $\alpha, \beta \in \Gamma$,
$\mu_{P_N}(x\gamma a\beta y) = \mu_{P_N}(x\gamma(a\beta y)) \geq \mu_{P_N}(a\beta y) \geq \mu_{P_N}(a)$. $\zeta_{P_N}(x\gamma a\beta y) = \zeta_{P_N}(x\gamma(a\beta y)) \leq \zeta_{P_N}(a\beta y) \leq \zeta_{P_N}(a)$. $\nu_{P_N}(x\gamma a\beta y) = \nu_{P_N}(x\gamma(a\beta y)) \leq \nu_{P_N}(a\beta y) \leq \nu_{P_N}(a)$. Consequently, P_N is a Pythagorean neutrosophic interior ideal of S. □

Proposition 3.14. *If $\{P_{N_i}\}_{i\in I}$ is a family of Pythagorean neutrosophic interior ideals of S, then so is $\bigcap_{i\in I} \mu_{P_{N_i}}(x) = \inf\{\mu_{P_{N_i}}(x) : i \in I, x \in S\}$, $\bigcap_{i\in I} \zeta_{P_{N_i}}(x) = \sup\{\zeta_{P_{N_i}}(x) : i \in I, x \in S\}$, $\bigcap_{i\in I} \nu_{P_{N_i}}(x) = \sup\{\nu_{P_{N_i}}(x) : i \in I, x \in S\}$, provided it is non-empty.*

Proof. Let $x, a, y \in S$ and $\alpha, \beta \in \Gamma$. Then,

$\bigcap_{i\in I} \mu_{P_{N_i}}(x\gamma y) = \inf\{\mu_{P_{N_i}}(x\gamma y) : i \in I\}$

$\geq \inf\{min\{\mu_{P_{N_i}}(x), \mu_{P_{N_i}}(y)\} : i \in I\}$
$= min[\inf\{\mu_{P_{N_i}}(x) : i \in I\}, \inf\{\mu_{P_{N_i}}(y) : i \in I\}]$
$= min\{\bigcap \mu_{P_{N_i}}(x), \bigcap \mu_{P_{N_i}}(y)\}.$
$\bigcap_{i\in I} \zeta_{P_{N_i}}(x\gamma y) = \sup\{\zeta_{P_{N_i}}(x\gamma y) : i \in I\}$
$\leq \sup\{max\{\zeta_{P_{N_i}}(x), \zeta_{P_{N_i}}(y)\} : i \in I\}$
$= max[\sup\{\zeta_{P_{N_i}}(x) : i \in I\}, \sup\{\zeta_{P_{N_i}}(y) : i \in I\}]$
$= max\{\bigcap \zeta_{P_{N_i}}(x), \bigcap \zeta_{P_{N_i}}(y)\}.$
$\bigcap_{i\in I} \nu_{P_{N_i}}(x\gamma y) = \sup\{\nu_{P_{N_i}}(x\gamma y) : i \in I\}$
$\leq \sup\{max\{\nu_{P_{N_i}}(x), \nu_{P_{N_i}}(y)\} : i \in I\}$
$= max[\sup\{\nu_{P_{N_i}}(x) : i \in I\}, \sup\{\nu_{P_{N_i}}(y) : i \in I\}]$
$= max\{\bigcap \nu_{P_{N_i}}(x), \bigcap \nu_{P_{N_i}}(y)\}.$

Hence $\bigcap P_{N_i}$ is a Pythagorean neutrosophic subsemigroup of S.

Now $\bigcap_{i\in I} \mu_{P_{N_i}}(x\alpha a\beta y) = \inf\{\mu_{P_{N_i}}(x\alpha a\beta y) : i \in I\} \inf\{\mu_{P_{N_i}}(a) : i \in I\} = \bigcap \mu_{P_{N_i}}(a)$

$\bigcap_{i\in I} \zeta_{P_{N_i}}(x\alpha a\beta y) = \sup\{\zeta_{P_{N_i}}(x\alpha a\beta y) : i \in I\} \sup\{\zeta_{P_{N_i}}(a) : i \in I\} = \bigcap \zeta_{P_{N_i}}(a)$

$$\bigcap_{i\in I} \nu_{P_{N_i}}(x\alpha a\beta y) = \sup\{\nu_{P_{N_i}}(x\alpha a\beta y) : i \in I\} \sup\{\nu_{P_{N_i}}(a) : i \in I\} = \bigcap \nu_{P_{N_i}}(a).$$

Consequently, $\bigcap P_{N_i}$ is a Pythagorean neutrosophic interior ideal of S. □

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