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Properties of the category of neutrosophic subgroups

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Abstract. In this paper, we introduce the notions of inverse and direct systems in the category of neutrosophic M-groups and examine their fundamental properties. In particular, we explore the behavior of limits in such systems, focusing on whether inverse and direct limits preserve exactness in sequences of neutrosophic M-groups.

Keywords: neutrosophic M-group, inverse and direct systems, inverse and direct limits, exact sequences.

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1. **Introduction**

Since its inception by Zadeh in 1965 [14], the theory of *fuzzy sets* has transformed the way we model membership via a continuous degree $t \in [0, 1]$. In 1986, Atanassov expanded this framework with *intuitionistic fuzzy sets*, introducing a second parameter f for non-membership, subject to the constraint $t + f \leq 1$ [1]. While this ensures consistency between membership and non-membership, it limits the ability to handle contradictory or entirely unknown information.

To overcome these limitations, Smarandache developed the theory of *neutrosophy* and the corresponding *neutrosophic sets* between 1995 and 1998 [10,11]. In this setting, each element is described by three independent components—truth t, indeterminacy i, and falsehood f—with no restriction on the sum $t+i+f$. This added flexibility makes neutrosophic sets particularly well-suited for representing incomplete, inconsistent, or inherently indeterminate information, and has led to applications in logic, algebra, decision-making, and information theory.

Despite extensive applied work, the algebraic and categorical structure of *neutrosophic M-groups* remains underdeveloped, especially regarding inverse and direct systems. Such constructions are crucial both for functorial considerations and for understanding whether exactness properties are preserved under limits.

In this paper, we fill this gap by developing the algebraic and categorical theory of neutrosophic M-groups through the introduction of inverse and direct systems. Drawing inspiration from the frameworks in [3, 4], we establish key structural properties of these systems and examine the behavior of their limits. In particular, we prove criteria under which inverse and direct limits of exact sequences of neutrosophic M-groups themselves remain exact.

In Section 2 we recall the basic definitions and notational conventions for neutrosophic M-groups, along with the necessary categorical and topological background. Section 3 is devoted to inverse systems of neutrosophic M-groups: we introduce their morphisms, establish fundamental properties, and prove conditions under which inverse limits preserve exact sequences. In Section 4 we develop the theory of direct systems and prove analogous results for direct limits. Finally, in Section 5 we summarise the main contributions and suggest directions for future developments.

2. **Preliminaries**

Before giving the formal definitions, we first discuss the intuition behind neutrosophic M-groups and explain why their categorical and topological underpinnings are essential. Neutrosophic structures arise naturally when one wishes to handle simultaneously degrees of truth, falsity, and indeterminacy in an algebraic setting; the extra flexibility of allowing these three components to vary independently leads to richer notions of subgroup, quotient and limit. Moreover, viewing neutrosophic M-groups as objects in a category with well-behaved products, coproducts and (co)kernels provides the groundwork for studying inverse and direct systems in later sections.

Definition 2.1. [10, 12] *Let $\mathbb{U}$ be an initial universe set, a* neutrosophic set *(more, precisely a* single valued neutrosophic set*) over $\mathbb{U}$, denoted by $A = \langle \mu, \xi, \gamma \rangle$ is a set of the form*

$$A = \{(x, \mu(x), \xi(x), \gamma(x)) : x \in \mathbb{U}\}$$

where $\mu : \mathbb{U} \to I$, $\xi : \mathbb{U} \to I$ and $\gamma : \mathbb{U} \to I$ are the truth function*, the* indeterminacy function *and the* falsity function *of A, respectively and $I = [0, 1]$ is the unit interval of the real line. For every $x \in \mathbb{U}$, $\mu(x)$, $\xi(x)$ and $\gamma(x)$ are said the* degree of truth *(or membership), the* degree of indeterminacy *and the* degree of falsity*(or non-membership) of x in A, respectively.*

Throughout the paper, we also refer to μ, ξ, γ *as* modular grade functions. *Moreover, for the neutrosophic set* A *having initial universe* $\mathbb{U}$ *and modular grade functions* μ, ξ *and* γ*, in order to simplify the notation we denote it by* $A = \langle \mu, \xi, \gamma \rangle$ *or* $A = (\mu, \xi, \gamma)_{\mathbb{U}}$.

Definition 2.2. *Let* $(G, \circ)$ *be an abelian group and* M *be a set. Given an exterior operation* $\cdot : M \times G \to G$*,* G *is called an* M-group *if* $m(a \circ b) = (ma) \circ (mb)$ *for all* $a, b \in G$ *and for all* $m \in M$.

Remark 2.3. *Let* $(G, \circ)$ *be an* M*-group. Denoted by* 1_G *the identity of* G*, we can deduce that* $m1_G = 1_G$ *and* $mx^{-1} = (mx)^{-1}$ *for all* $m \in M$ *and for all* $x \in G$.

Example 2.4. *Every ring* $(R, +, \times)$ *is trivially an M-group where* $M = R$ *and the operation* $\cdot : M \times R \to R$ *coincide with* $\times$.

Definition 2.5. *Let* $(G, \circ)$ *be an* M*-group. We say that* G' *is a subgroup of* G *if*

- *For all* $x, y \in G'$*, we have* $x \circ y^{-1} \in G'$.
- *For all* $m \in M$ *and for all* $x \in G'$ *we have* $mx \in G'$.

Observe that, if G' *is subgroup of* G*, then quotient group* G/G' *is also an* M*-group where the exterior operation is naturally induced by the operation* $\cdot : M \times G \to G$.

Definition 2.6. *A neutrosophic set* $A = \langle \mu, \xi, \gamma \rangle$ *on a group* $(G, \circ)$ *is said to be a* neutrosophic group *if* G *is the initial universe set of* A *and the following holds:*

(1) $\mu(x \circ y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\xi(x \circ y) \geq \min\{\xi(x), \xi(y)\}$,
(3) $\gamma(x \circ y) \leq \max\{\gamma(x), \gamma(y)\}$,
(4) $\mu(x^{-1}) = \mu(x)$,
(5) $\xi(x^{-1}) = \xi(x)$,
(6) $\gamma(x^{-1}) = \gamma(x)$,

for all $x, y \in G$*. Moreover, if* 0 *is the unit of* G*, then* $\mu(0) = \xi(0) = 1$ *and* $\gamma(0) = 0$.

Definition 2.7. *Let a group* $(G, \circ)$ *be* M*-group. A neutrosophic set* $A = \langle \mu, \xi, \gamma \rangle$ *on a M-group* $(G, \circ)$ *is a neutrosophic* M*-group if* G *is the initial universe set of* A *and the following holds:*

(1) $\mu(m(x \circ y)) \geq \min\{\mu(mx), \mu(my)\}$,
(2) $\xi(m(x \circ y)) \geq \min\{\xi(mx), \xi(my)\}$,
(3) $\gamma(m(x \circ y)) \leq \max\{\gamma(mx), \gamma(my)\}$,
(4) $\mu(mx^{-1}) = \mu(mx)$,
(5) $\xi(mx^{-1}) = \xi(mx)$,
(6) $\gamma(mx^{-1}) = \gamma(mx)$,

for all $m \in M$*,* $x, y \in G$*. Moreover, if* 0 *is the unit of* G*, then* $\mu(0) = \xi(0) = 1$ *and* $\gamma(0) = 0$.

3. Inverse system of neutrosophic M-groups

In this paper, $A = (\mu, \xi, \gamma)$ denotes a neutrosophic M-group of G. We denote this M-group by $(\mu, \xi, \gamma)_G$. Recall that if G and G' are M-groups, a group homomorphism $f : G \to G'$ is homomorphism of M-groups if $f(ma) = mf(a)$ for all $a \in G$ and for all $m \in M$. Moreover, the sets $\ker(f) = \{x \in G \mid f(x) = 0\}$ and $\operatorname{Im}(f) = \{y \in G' \mid y = f(x) \text{ for some } x \in G\}$ are M-groups, and in particular they are subgroup of G and G' respectively.

Definition 3.1. *Let $f : G \to G'$ be an homomorphism of M-groups. The related function $\overline{f} : (\mu, \xi, \gamma)_G \to (\mu', \xi', \gamma')_{G'}$ is called a homomorphism of neutrosophic M-groups if $\mu'(f(x)) \geq \mu(x)$, $\xi'(f(x)) \geq \xi(x)$ and $\gamma'(f(x)) \leq \gamma(x)$ are satisfied. Neutrosophic M-groups and their morphisms form a category. We denote this category by N-Mg.*

Remark 3.2. *Let $(\mu, \xi, \gamma)_G$ be a neutrosophic M-group, let H be an M-group and let $f : G \to H$ be an M-group homomorphism. Now, as stated in the following result, we can define a neutrosophic M-group structure on H by considering the* neutrosophic image *$f(G)$ (see [5,7,8]) which is the neutrosophic set $\langle \mu^f, \xi^f, \gamma^f \rangle$ defined by:*

$$\mu^f(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} \mu(x) & \text{if } f^{-1}(\{y\}) \neq \emptyset \\ 1 & \text{otherwise} \end{cases},$$

$$\xi^f(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} \xi(x) & \text{if } f^{-1}(\{y\}) \neq \emptyset \\ 1 & \text{otherwise} \end{cases},$$

$$\gamma^f(y) = \begin{cases} \inf\limits_{x \in f^{-1}(\{y\})} \gamma(x) & \text{if } f^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for every $y \in H$.

Proposition 3.3. *Let $f : G \to H$, with the notation above. If G is a neutrosophic M-group, then $(\mu^f, \xi^f, \gamma^f)_H$ is a neutrosophic M-group. In particular, $f : (\mu, \xi, \gamma)_G \to (\mu^f, \xi^f, \gamma^f)_H$ is a homomorphism of neutrosophic M-groups.*

Proof. Let us prove that $\mu^f(m(y_1 \circ y_2)) \geq \min\{\mu^f(mx), \mu^f(my)\}$. By definition we know that $\mu^f(m(y_1 \circ y_2)) \geq \mu(x)$, for all $x \in G$ such that $f(x) = m(y_1 \circ y_2)$ and let $x_1, x_2 \in G$ such that $\mu^f(my_1) = \mu(x_1)$ and $\mu^f(my_2) = \mu(x_2)$, in particular $f(x_1) = my_1$ and $f(x_2) = my_2$. We have $f(x_1 \circ x_2) = f(x_1) \circ f(x_2) = my_1 \circ my_2 = m(y_1 \circ y_2)$. Therefore, $\mu^f(m(y_1 \circ y_2)) \geq \mu(x_1 \circ x_2) \geq \min\{(\mu(x_1), \mu(x_2)\} = \{(\mu^f(mx_1), \mu^f(mx_2)\}$. Using the same arguments, it is possible to obtain the other conditions of M-group. In a similar way one can prove all the other conditions. □

Remark 3.4. *If G is an M-group and $(\eta, \theta, \nu)_H$ is an neutrosophic M-group and $f : G \to H$ is a homomorphism of M-groups, then we can define a neutrosophic M-group structure in G by $\eta_f(x) = \eta(f(x))$, $\theta_f(x) = \theta(f(x))$, $\nu_f(x) = \nu(f(x))$. Hence, it is clear that $(\eta_f, \theta_f, \nu_f)_G$ is a neutrosophic M-group and $f : (\eta_f, \theta_f, \nu_f)_G \to (\eta, \theta, \nu)_H$ is a homomorphism of neutrosophic M-groups.*

Lemma 3.5. *Let G and H be M-groups and let $f : G \to H$ be an M-group homomorphism. The following holds:*

(1) If $(\mu, \xi, \gamma)_G$ is a neutrosophic M-group, then for every modular grade functions (η, θ, ν) on H, $f : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is a neutrosophic homomorphism if and only if $\eta \geq \mu^f, \theta \geq \xi^f, \nu \leq \gamma^f$.

(2) If $(\eta, \theta, \nu)_H$ is a neutrosophic M-group, then for every modular grade functions (μ, ξ, γ) on G, $f : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is a neutrosophic homomorphism if and only if $\mu \leq \eta_f, \xi \leq \theta_f, \gamma \geq \nu_f$.

Proof. It is straightforward. □

Definition 3.6. *Given an M-group H, and $(\eta_1, \theta_1, \nu_1)$, $(\eta_2, \theta_2, \nu_2)$ modular grade functions on H, we say that $(\eta_1, \theta_1, \nu_1)$ is* smaller *than $(\eta_2, \theta_2, \nu_2)$ if $\eta_1(x) \leq \eta_2(x)$, $\theta_1(x) \leq \theta_2(x)$ and $\nu_1(x) \geq \nu_2(x)$ for every $x \in H$. This definition provides a partial order in the set of modular grade functions.*

Let A be a set and $\{f_i\}_{i \in I}$ be a family of functions $f_i : A \to [0, 1]$. We define the following functions

$$\bigvee_{i \in I} f_i : A \to [0,1] \quad \text{defined by} \quad x \longmapsto \sup\{f_i(x) \mid i \in I\}$$

$$\bigwedge_{i \in I} f_i : A \to [0,1] \quad \text{defined by} \quad x \longmapsto \inf\{f_i(x) \mid i \in I\}$$

Lemma 3.7. *(1) Given M-groups $\{G_\alpha\}_{\alpha \in \Delta}$, H and a family of M-group homomorphisms $\overline{\lambda} = \{f_\alpha : G_\alpha \to H\}_{\alpha \in \Delta}$, if $\{(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha}\}_{\alpha \in \Delta}$ is a family of neutrosophic M-groups, then there exists the smallest grade functions $\eta = \mu^{\overline{\lambda}} = \mu^{\{f_\alpha\}}$, $\theta = \xi^{\overline{\lambda}} = \xi^{\{f_\alpha\}}$, $\nu = \gamma^{\overline{\lambda}} = \gamma^{\{f_\alpha\}}$ such that, for all $\alpha \in \Delta$, $\overline{f} : (\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \to (\eta, \theta, \nu)_H$ is a neutrosophic homomorphism.*

(2) Given M-groups $G, \{H_\alpha\}_{\alpha \in \Delta}$ and a family of M-group homomorphisms $B = \{g_\alpha : G \to H_\alpha\}_{\alpha \in \Delta}$ if $\{(\eta_\alpha, \theta_\alpha, \nu_\alpha)_{H_\alpha}\}_{\alpha \in \Delta}$ is a family of neutrosophic M-groups, then there exist the largest grade functions $\mu = \eta_B = \eta_{\{f_\alpha\}}$, $\xi = \theta_B = \theta_{\{f_\alpha\}}$, $\gamma = \nu_B = \nu_{\{f_\alpha\}}$ such that, for all $\alpha \in \Delta$, $f : (\mu, \xi, \gamma)_G \to (\eta_\alpha, \theta_\alpha, \nu_\alpha)_{H_\alpha}$ is a neutrosophic homomorphism.

Proof. (1). Let $\eta = \mu^{\overline{\lambda}} = \bigvee_{\alpha\in\Delta} \mu_\alpha^{f_\alpha}, \theta = \xi^{\overline{\lambda}} = \bigvee_{\alpha\in\Delta} \xi_\alpha^{f_\alpha}, \nu = \gamma^{\overline{\lambda}} = \bigwedge_{\alpha\in\Delta} \gamma_\alpha^{f_\alpha}$.
(2). Let $\mu = \eta_B = \bigwedge_{\alpha\in\Delta} (\eta_\alpha)_{f_\alpha}, \xi = \theta_B = \bigwedge_{\alpha\in\Delta} (\theta_\alpha)_{f_\alpha}, \gamma = \nu_B = \bigvee_{\alpha\in\Delta} (\nu_\alpha)_{f_\alpha}$. □

By using this lemma, we define subgroup, quotient M-group, product and co-product operations in the category of neutrosophic M-groups. If $(\mu,\xi,\gamma)_G$ is a neutrosophic M-group and $H \subset G$ is a subgroup, then $(\mu|H,\xi|H,\gamma|H)_H$ is a neutrosophic subgroup of $(\mu,\xi,\gamma)_G$. If $(\mu,\xi,\gamma)_G$ is a neutrosophic M-group and $p : G \to G/\sim$ is a canonical homomorphism, then $(\mu^p,\xi^p,\gamma^p)_{G/\sim}$ is a quotient M-group of $(\mu,\xi,\gamma)_G$. Hence, for each homomorphism of neutrosophic M-groups $f : (\mu,\xi,\gamma)_G \to (\eta,\theta,\nu)_H$, a neutrosophic subgroups $(\mu|kerf,\xi|kerf,\gamma|kerf)_{kerf}$ and the neutrosophic quotient M-group $(\eta^p,\theta^p,\nu^p)_{H/Imf}$ are obtained, where $p : H \to H|Imf$ is a canonical homomorphism. If $\{(\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha}\}_{\alpha\in\Delta}$ is a family of neutrosophic M-groups, then we define product of this family by $(\mu_A,\xi_A,\gamma_A)_{\prod_{\alpha\in\Delta} G_\alpha}$, where $A = \left\{\pi_\alpha, \prod_{\alpha\in\Delta} G_\alpha \to G_\alpha\right\}_{\alpha\in\Delta}$ is a family of the usual projection maps. Co-product of this family is $(\mu^B,\xi^B,\gamma^B)_{\sum G_\alpha}$, where $B = \left\{i_\alpha, G_\alpha \to \sum_{\alpha\in\Delta} G_\alpha\right\}_{\alpha\in\Delta}$ is a family of the usual injections.
The following theorem is easily proved.

Theorem 3.8. *The category of neutrosophic M-groups has zero objects, sums, products, kernels and co-kernels.*

Definition 3.9. *Any functor $D : \Lambda^{op} \to N - Mg (D : \Lambda \to N - Mg)$, where Λ is a directed set (considered as a category), is called an inverse (direct) system of neutrosophic M-groups, the limit of D is called a limit of the inverse (direct) system. Let*

$$(\underline{\mu},\underline{\xi},\underline{\gamma})_{\underline{G}} = \{(\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha}, \underline{p}_{\alpha'\alpha}\}_{\alpha\in\Delta} \tag{1}$$

be an inverse system of neutrosophic M-groups. $A = \left\{\pi_\alpha, \prod_{\alpha\in\Delta} G_\alpha \to G_\alpha\right\}_{\alpha\in\Delta}$ be a family of projections and $(\mu_A,\xi_A,\gamma_A)_{\prod_{\alpha\in\Delta} G_\alpha}$ be a direct product of the neutrosophic M-groups. Then we obtain a neutrosophic subgroup $(\mu_A|\varprojlim G_\alpha,\xi_A|\varprojlim G_\alpha,\gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha}$, where $\varprojlim G_\alpha$ is a limit of an inverse system of M-groups $\{G_\alpha\}_{\alpha\in\Delta}$.

Theorem
3.10. *Every inverse system in representation* (1) *has a limit in the category of N-Mg, and this limit is equal to the a neutrosophic subgroup $(\mu_A|\varprojlim G_\alpha,\xi_A|\varprojlim G_\alpha,\gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha}$.*

Proof. It suffices to show that, there exists a unique homomorphism of neutrosophic M-groups $\underline{\psi} : (\eta,\theta,\nu)_H \to (\mu_A|\varprojlim G_\alpha,\xi_A|\varprojlim G_\alpha,\gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha}$, which makes the following diagram is commutative:

$$\begin{tikzcd}
(\eta,\theta,\nu)_H \arrow[r, "\overline{\varphi_\alpha}"] \arrow[d, "\overline{\psi}"'] & (\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha} \\
(\mu_A|\varprojlim G_\alpha, \xi_A|\varprojlim G_\alpha, \gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha} \arrow[ur]
\end{tikzcd}$$

Here, for every neutrosophic M-group $(\eta,\theta,\nu)_H$ and $\alpha < \alpha'$, it holds that $\{\overline{\varphi_\alpha} : (\eta,\theta,\nu)_H \to (\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha}\}_{\alpha\in\Delta}$ is a family of homomorphism of neutrosophic M-groups, which makes up the following diagram is commutative:

$$\begin{tikzcd}
(\eta,\theta,\nu)_H \arrow[r, "\overline{\varphi_\alpha}"] \arrow[d, "\overline{\varphi_{\alpha'}}"'] & (\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha} \\
(\mu_{\alpha'},\xi_{\alpha'},\gamma_{\alpha'})_{G_{\alpha'}} \arrow[ur, "\overline{p}^{\alpha,\alpha'}"']
\end{tikzcd}$$

Also, $\overline{\pi_\alpha} : (\mu_A|\varprojlim G_\alpha, \xi_A|\varprojlim G_\alpha, \gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha} \to (\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha}$ is a canonical projection. We define $\psi : H \to \varprojlim G_\alpha$ as a homomorphism of M-groups such that for every $x \in H$, $\psi(x) = \{\varphi_\alpha(x)\}_{\alpha\in\Delta}$. The homorphism ψ is unique by the definition of $\varprojlim G_\alpha$. So it is sufficent to show that $\overline{\psi} : (\eta,\theta,\nu)_H \to (\mu_A|\varprojlim G_\alpha, \xi_A|\varprojlim G_\alpha, \gamma_A|\varprojlim G_\alpha)_{\varprojlim G_\alpha}$ is a homomorphism of neutrosophic M-groups. Since $\overline{\varphi_\alpha} : (\eta,\theta,\nu)_H \to (\mu_\alpha,\xi_\alpha,\gamma_\alpha)_{G_\alpha}$ is a homomorphism of neutrosophic M-groups for every $\alpha \in \Delta$, the conditions $\mu_\alpha(\varphi_\alpha(x)) \geq \eta(x), \xi_\alpha(\varphi_\alpha(x)) \geq \theta(x), \gamma_\alpha(\varphi_\alpha(x)) \leq \nu(x)$ are satisfied for every $x \in H$. Therefore, we obtain the conditions $\mu_A(\{\varphi_\alpha(x)\}) = \bigwedge\limits_{\alpha\in\Delta} \mu_\alpha(\varphi_\alpha(x)) \geq \eta(x), \xi_A(\{\varphi_\alpha(x)\}) = \bigwedge\limits_{\alpha\in\Delta} \xi_\alpha(\varphi_\alpha(x)) \geq \theta(x), \gamma_A(\{\varphi_\alpha(x)\}) = \bigvee\limits_{\alpha\in\Delta} \gamma_\alpha(\varphi_\alpha(x)) \leq \nu(x)$. It is clear that $\varprojlim$ is a functor from the category of inverse system of neutrosophic M-groups to the category of neutrosophic M-groups. $\square$

Let us review the problem of exact limit for inverse systems of exact sequences.

Definition 3.11. *A sequence of homomorphisms*

$$\cdots \to (\mu_{n-1},\xi_{n-1},\gamma_{n-1})_{G_{n-1}} \xrightarrow{\overline{f_{n-1}}} (\mu_n,\xi_n,\gamma_n)_{G_n} \xrightarrow{\overline{f_n}} (\mu_{n+1},\xi_{n+1},\gamma_{n+1})_{G_{n+1}} \to \cdots \quad (2)$$

of neutrosophic M-groups is said to be a neutrosophic exact if and only if $(\mu_n|Imf_{n-1}, \xi_n|Imf_{n-1}, \gamma_n|Imf_{n-1}) = (\mu_n|kerf_n, \xi_n|kerf_n, \gamma_n|kerf_n)$, *for every* $n \in \mathbb{Z}$.

Exactness of (2) *will necessarily provide exactness of*

$$\cdots \xrightarrow{f_{n-1}} G_{n-1} \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} G_{n+1} \to \cdots \quad (3)$$

since equality of two neutrosophic sets is just the equality of their respective maps, that implies the equality of their corresponding domains (that is, $kerf_n = Imf_{n-1}$*). On the other hand, exactness of* (3) *does not necessarily imply exactness of* (2).

Example 3.12. *Let $G_n = \mathbb{Z}$, $G'_n = \mathbb{Z}$, $G''_n = \mathbb{Z}_2$ be M-groups where $M = \mathbb{Z}$ and using the usual ring operation, for all $n \in \mathbb{N}$. Then,*

$$\underline{G} = (\{G_n\}_{n\in N}, \{p_{n+1n} : G_{n+1} \to G_n,\ p_{n+1n}(m) = 3m\}),$$

$$\underline{G'} = (\{G'_n\}_{n\in N}, \{q_{n+1n} : G'_{n+1} \to G'_n,\ q_{n+1n}(m) = 3m\}),$$

$$\underline{G''} = (\{G''_n\}_{n\in N}, \{r_{n+1n} : G''_{n+1} \to G''_n,\ r_{n+1n}(m) = [m]\})$$

are inverse systems of M-groups and $f = \{f_n : G'_n \to G_n | f_n(m) = 2m\}$, $g = \{g_n : G_n \to G''_n | g_n(m) = [m]\}$ are morphisms of inverse systems. The following sequence $0 \to \underline{G'} \xrightarrow{f} \underline{G} \xrightarrow{g} \underline{G''} \to 0$ is short exact sequence of inverse systems of M-groups. Taking the inverse limits of this sequence into consideration, the sequence $0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0$ is not exact. As it is seen, the limit of inverse system of exact sequence of M-groups is not exact and, obviously, the same happens equipping the M-groups with a neutrosophic structure.

It is known that in order to the inverse limit preserve short exact sequence of inverse system, it is necessary to define derivative functor of inverse limit functor in N-Mg see [13, Chapter 3]. For this notion we get the inverse system in (1), assuming the set of indices Δ is equal to $\mathbb{N}$, and using additive notation for the groups G_α. For $\alpha, \alpha' \in \mathbb{N}$, we use the notation $\alpha \prec \alpha'$ to denote $\alpha' = \alpha + 1$. Consider the homomorphism of M-groups

$$d : \prod_\alpha G_\alpha \to \prod_\alpha G_\alpha \tag{4}$$

defined by the formula: $d(\{x_\alpha\}) = \{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha\prec\alpha'}$. We show that d is a homomorphism of neutrosophic M-groups. Indeed,

$$\begin{aligned}
\mu_A(d(\{x_\alpha\})) &= \mu_A(\{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha\prec\alpha'}) \\
&= \bigwedge_\alpha \mu_\alpha(x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})) \\
&\geq \bigwedge_\alpha \min\{\mu_\alpha(x_\alpha), \mu_\alpha(p_{\alpha'\alpha}(x_{\alpha'}))\} \\
&\geq \bigwedge_\alpha \min\{\mu_\alpha(x_\alpha), \mu'_\alpha(x_{\alpha'})\}\ (\because\ \mu_\alpha(p_{\alpha'\alpha}(x_{\alpha'})) \geq \mu_{\alpha'}(x_{\alpha'})) \\
&= \bigwedge_\alpha (\mu_\alpha(x_\alpha) \wedge \mu'_\alpha(x_{\alpha'})) \\
&= \bigwedge_\alpha \mu_\alpha(x_\alpha) \\
&= \mu_A(\{x_\alpha\}),
\end{aligned}$$

$$\begin{aligned}
\xi_A(d(\{x_\alpha\})) &= \xi_A(\{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha\prec\alpha'}) \\
&= \bigwedge_\alpha \xi_\alpha(x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})) \\
&\geq \bigwedge_\alpha \min\{\xi_\alpha(x_\alpha), \xi_\alpha(p_{\alpha'\alpha}(x_{\alpha'}))\} \\
&\geq \bigwedge_\alpha \min\{\xi_\alpha(x_\alpha), \xi'_\alpha(x_{\alpha'})\} \ (\because\ \xi_\alpha(p_{\alpha'\alpha}(x_{\alpha'})) \geq \xi_{\alpha'}(x_{\alpha'})) \\
&= \bigwedge_\alpha (\xi_\alpha(x_\alpha) \wedge \xi'_\alpha(x_{\alpha'})) \\
&= \bigwedge_\alpha \xi_\alpha(x_\alpha) \\
&= \xi_A(\{x_\alpha\}),
\end{aligned}$$

$$\begin{aligned}
\gamma_A(d(x_\alpha)) &= \gamma_A(\{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha\prec\alpha'}) \\
&= \bigvee_\alpha \gamma_\alpha(x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})) \\
&\leq \bigvee_\alpha \max\{\gamma_\alpha(x_\alpha), \gamma_\alpha(p_{\alpha'\alpha}(x_{\alpha'}))\} \ (\because\ \gamma_\alpha(p_{\alpha'\alpha}(x_{\alpha'})) \leq \gamma_{\alpha'}(x_{\alpha'})) \\
&\leq \bigvee_\alpha \max\{\gamma_\alpha(x_\alpha), \gamma\alpha'(x_{\alpha'})\} \\
&= \bigvee_\alpha (\gamma_\alpha(x_\alpha) \vee \gamma'_\alpha(x_{\alpha'})) \\
&= \bigvee_\alpha \gamma_\alpha(x_\alpha) \\
&= \gamma_A(\{x_\alpha\}).
\end{aligned}$$

Then, d is a homomorphism of neutrosophic M-groups. So, $(\mu_A|ker\ d, \xi_A|ker\ d, \gamma_A|ker\ d)_{\ker\ d}$ and $((\mu_A)^p, (\xi_A)^p, (\gamma_A)^p)_{coker\ d}$, where p is the canonical map defined by $x \mapsto x + \mathrm{Im}(d)$, are defined.

For inverse system of M-groups $\{G_\alpha, p_{\alpha'\alpha}\}_{\alpha\in\Delta}$, the M-group $\varprojlim^{(1)} G_\alpha = \prod_\alpha G_\alpha / Imd$ is the derivative functor [6].

If $\pi : \prod_\alpha G_\alpha \to \varprojlim^{(1)} G_\alpha$ is a canonical homomorphism, we can define a neutrosophic M-group by $((\mu_A)^\pi), (\xi_A)^\pi, (\gamma_A)^\pi)_{\varprojlim^{(1)} G_\alpha}$.

Definition 3.13. $((\mu_A)^\pi, (\xi_A)^\pi, (\gamma_A)^\pi)_{\varprojlim^{(1)} G_\alpha}$ *is called "first derived functor" of limit of the inverse system of neutrosophic M-groups given in* (1) *with $\Delta = \mathbb{N}$ and $\alpha' = \alpha + 1$ (that we always assume from now on).*

The definition of $\varprojlim^{(1)}$ allows to define a functor from the category of inverse system to the category of M-groups, considering known result in homological algebra (see [13]). In the next result, we prove that it is also a functor if the objects are equipped with a neutrosophic structure, that is, a functor from the category of neutrosophic inverse system to the category of neutrosophic M-groups.

Proposition 3.14. $\varprojlim^{(1)}$ *is a functor from the category of neutrosophic inverse system to the category of neutrosophic M-groups.*

Proof. Let
$A = \{(\mu_n, \xi_n, \gamma_n)_{G_n}, p_{n+1,n} : G_{n+1} \to G_n\}_{n\in\mathbb{N}}$, $B = \{(\eta_n, \theta_n, \nu_n)_{N_n}, q_{n+1,n} : N_{n+1} \to N_n\}_{n\in\mathbb{N}}$ be inverse systems and $f = (f_n : G_n \to N_n)_{n\in\mathbb{N}}$ be a nutrosophic morphism (in particular, every morphism f_n is neutrosophic) from A to B. We need to prove that the homomorphism of M-groups

$$\varprojlim^{(1)} \overline{f} : ((\mu_A)^\pi, (\xi_A)^\pi, (\gamma_A)^\pi)_{\varprojlim^{(1)} G_n} \longrightarrow ((\eta_B)^\pi, (\theta_B)^\pi, (\nu_B)^\pi)_{\varprojlim^{(1)} N_n}$$
$$(x_n)_{n\in\mathbb{N}} + Imd \longmapsto (f_n(x_n))_{n\in\mathbb{N}} + Imd$$

is a is also an homomorphism of neutrosophic M-groups. Denoted $x = (x_n)_{n\in\mathbb{N}}$ and $f(x) = (f_n(x_n))_{n\in\mathbb{N}}$, this fact is proved considering the inequalities:

$$\begin{aligned}
(\mu_A)^\pi(x + Imd) &= \sup_{z\in Imd} \mu_A(x + z) \\
&= \sup_{n\in\mathbb{N}} \sup_{z\in Imd} \mu_n(x + z) \\
&\leq \sup_{n\in\mathbb{N}} \sup_{z\in Imd} \eta_n(f(x + z) \\
&= \sup_{z\in Imd} \eta_A(f(x + z)) \\
&= \sup_{z\in Imd} \eta_A(f(x) + f(z)) \\
&= \sup_{y=f(z)} \eta_A(f(x) + y) \\
&\leq \sup_{y\in Imd} \eta_A(f(x) + y) \\
&= (\eta_A)^\pi(\varprojlim^{(1)} \overline{f}(x + Imd)),
\end{aligned}$$

and similarly, we have also the following:

$$\begin{aligned}
(\xi_A)^\pi(x + Imd) &= \sup_{z\in Imd} \xi_A(x + z) \\
&\leq \sup_{z\in Imd} \theta_A(f(x + z)) \\
&= \sup_{z\in Imd} \theta_A(f(x) + f(z)) \\
&= \sup_{y=f(z)} \theta_A(f(x) + y) \\
&\leq \sup_{y\in Imd} \theta_A(f(x) + y) \\
&= (\theta_A)^\pi(\varprojlim^{(1)} \overline{f}(x + Imd)),
\end{aligned}$$

$$\begin{aligned}
(\gamma_A)^\pi(x + Imd) &= \inf_{z\in Imd} \gamma_A(x + z) \\
&\geq \inf_{z\in Imd} \nu_B(f(x + z)) \\
&= \inf_{z\in Imd} \nu_B(f(x) + f(z)) \\
&= \inf_{y=f(z)} \nu_B(f(x) + y) \\
&\geq \inf_{y\in Imd} \nu_A(f(x) + y) \\
&= (\nu_A)^\pi(\varprojlim^{(1)} \overline{f}(x + Imd)).
\end{aligned}$$

Hence $\varprojlim^{(1)}$ is a functor from the category of neutrosophic inverse system to the category of neutrosophic M-groups. □

To study the functor $\varprojlim^{(1)}$ functor, let us introduce category of chain (co-chain) complexes of neutrosophic M-groups. This category is defined to the respective procedure [2].

Definition 3.15. *A neutrosophic chain complex $(\mu, \xi, \gamma)_G = \{(\mu_n, \xi_n, \gamma_n)_{G_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$ is an object in N-Mg together with a neutrosophic endomorphism $\overline{\partial} : (\mu, \xi, \gamma)_G \to (\mu, \xi, \gamma)_G$ of degree -1 with $\overline{\partial\partial} = \overline{0}$.*

Definition 3.16. *A morphism of neutrosophic chain complexes $\overline{\varphi} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is a morphism $\overline{\varphi} = \{\overline{\varphi}_n : (\mu_n, \xi_n, \gamma_n)_{G_n} \to (\eta_n, \theta_n, \nu_n)_{N_n}\}$, which has a degree 0 such that $\overline{\varphi_{n-1}} \circ \overline{\partial_n} = \overline{\partial'_n \varphi_n}$, where $\overline{\partial}$ denotes the neutrosophic differential in $(\eta, \theta, \nu)_H$.*

Definition 3.17. *Let $(\mu, \xi, \gamma)_G = \{(\mu_n, \xi_n, \gamma_n)_{G_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$ be a neutrosophic chain complex. The condition $\overline{\partial} \circ \overline{\partial} = \overline{0}$ implies that $Im\overline{\partial_{n+1}} \subset ker\overline{\partial_n}$, $n \in \mathbb{Z}$. Hence, we can associate the neutrosophic graded M-group with $(\mu, \xi, \gamma)_G$ $H((\mu, \xi, \gamma)_G) = \{H_n(\mu, \xi, \gamma)_G\}$, where*

$$H_n((\mu, \xi, \gamma)_G) = \frac{(\mu_n|ker\overline{\partial_n}, \xi_n|ker\overline{\partial_n}, \gamma_n|ker\overline{\partial_n})_{ker\overline{\partial_n}}}{(\mu_n|Img\overline{\partial_{n+1}}, \xi_n|Img\overline{\partial_{n+1}}, \gamma_n|Img\overline{\partial_{n+1}})_{Img\overline{\partial_{n+1}}}}$$

.

Definition 3.18. *$H((\mu, \xi, \gamma)_G)$ is called a neutrosophic homology M-group of neutrosophic chain complex $(\mu, \xi, \gamma)_G$. By duality, we can define co-chain complex and co-homology M-group.*

Definition 3.19. *Let $\overline{\varphi}, \overline{\psi} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ be morphisms of neutrosophic chain complexes. A neutrosophic homotopy $\overline{\Sigma} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ between $\overline{\varphi}$ and $\overline{\psi}$ is a neutrosophic morphism of degree $+1$ such that $\overline{\psi} - \overline{\varphi} = \overline{\partial} \circ \overline{\sum} + \overline{\sum} \circ \overline{\partial}$.*

We say that $\overline{\varphi}$, $\overline{\psi}$ are a neutrosophic homotopic, if there exists a neutrosophic homotopy $\overline{\sum}$ between them.

A morphism of neutrosophic chain complexes $\overline{f} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is called a chain homotopy equivalence *if there exist a morphism of neutrosophic chain complexes $\overline{g} : (\eta, \theta, \nu)_H \to (\mu, \xi, \gamma)_G$ such that $\overline{f} \circ \overline{g}$ and $\overline{g} \circ \overline{f}$ are neutrosophic homotopic to the identity maps on $(\mu, \xi, \gamma)_G$ and $(\eta, \theta, \nu)_H$ (see also [13, Definition 1.4.4] for details about the maps outside the neutrosophic context).*

The following definition is inspired by [13, Definition 1.1.2].

Definition 3.20. *A morphism of neutrosophic chain complexes $\overline{f} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is called a* quasi-isomorphism *if the induced maps $H_n((\mu, \xi, \gamma)_G) \mapsto H_n((\eta, \theta, \nu)_H)$, between the neutrospohic homology M-groups, are all isomorphisms.*

Built upon the above, the results gathered in the following theorem can be argued straightforwardly by some known results in the context of homological algebra and can be easily extended in the neutrosophic setting introduced in this work.

Theorem 3.21. *The neutrosophic homotopy relation between neutrosophic chain complexes is an equivalence relation. Moreover:*

- *If $\overline{\varphi}$, $\overline{\psi} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ are a neutrosophic homotopic maps of neutrosophic chain complexes, then they induced the same map $H_n((\mu, \xi, \gamma)_G) \mapsto H_n((\eta, \theta, \nu)_H)$ between the neutrospohic homology M-groups.*
- *If $\overline{f} : (\mu, \xi, \gamma)_G \to (\eta, \theta, \nu)_H$ is chain homotopy equivalence, then $\overline{f}$ is a quasi-isomorphism.*

Considering the usual framework for inverse system of neutrosophic M-groups and the map $d(\{x_\alpha\}) = \{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha \prec \alpha'}$, let us consider the following a neutrosophic cochain complex

$$\overline{0} \to (\mu_A, \xi_A, \gamma_A)_{\prod G_\alpha} \xrightarrow{\overline{d}} (\mu_A, \xi_A, \gamma_A)_{\prod G_\alpha} \to \overline{0}.$$

Neutrosophic co-homology M-groups of this complex are $ker\overline{d}$ and $coker\overline{d}$.

Lemma 3.22. $\varprojlim(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} = ker\overline{d}$ *and* $\varprojlim^{(1)}(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} = coker\overline{d}$.

Proof. Straightforward. □

We consider the set of natural numbers as an index set of inverse systems.

Theorem 3.23. *Let the following sequence $(G_1, \mu_1, \xi_1, \gamma_1)_{G_1} \xleftarrow{\overline{p}_1^2} (G_2, \mu_2, \xi_2, \gamma_2)_{G_2} \xleftarrow{\overline{p}_2^3} \cdots$ be an inverse sequence of neutrosophic M-groups. For each infinite subsequence of this sequence, the derivative functors $\varprojlim^{(1)}$ are isomorphic, that is, $\varprojlim^{(1)}$ does not change.*

Proof. Let $S = \{i_1 < i_2 < i_3 < \cdots\}$ be an infinite subsequence of natural numbers. From Lemma 3.22, $\varprojlim^{(1)}$ is defined by the following homomorphism of neutrosophic M-groups as appropriate subsequence S

$$\overline{d}' : \left(\bigwedge_{s\in S} \mu_s, \bigwedge_{s\in S} \xi_s, \bigvee_{s\in S} \gamma_s\right)_{\prod\limits_{s\in S} G_s} \to \left(\bigwedge_{s\in S} \mu_s, \bigwedge_{s\in S} \xi_s, \bigvee_{s\in S} \gamma_s\right)_{\prod\limits_{s\in S} G_s}.$$

with $\overline{d}'(\{x_{i_\ell}\}_\ell) = \{x_{i_\ell} - p_{i_\ell}^{i_{\ell+1}}(x_{i_{\ell+1}})\}_\ell$. In order to simplify the notations, in what follows we denote $i_1 = i, i_2 = j$ and $i_3 = k$.

We may define $f_0, f_1 : \prod\limits_{s\in S} G_s \to \prod\limits_{n\in\mathbb{N}} G_n$ homomorphisms of M-groups with this formula:

$f_0(x_i, x_j, x_k, \cdots) = (p_1^i(x_i), p_2^i(x_i), \cdots, p_{i-1}^i(x_i), x_i, p_{i+1}^j(x_j), \cdots, p_{j-1}^j(x_j), x_j, \cdots)$

$f_1(x_i, x_j, x_k, \cdots) = (0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots, x_k, 0, \cdots)$. Also,

$$\left(\bigwedge_{n\in\mathbb{N}} \mu_n\right)(p_1^i(x_i), p_2^i(x_i), \cdots, p_{i-1}^i(x_i), x_i, p_{i+1}^j(x_j), \cdots, p_{j-1}^j(x_j), x_j, \cdots)$$

$$\begin{aligned} &= \mu_1(p_1^i(x_i)) \wedge \cdots \wedge \mu_{i-1}(p_{i-1}^i(x_i)) \wedge \mu_i(x_i) \wedge \mu_{i+1}(p_{i+1}^j(x_j)) \wedge \cdots \wedge \mu_j(x_j) \wedge \cdots \\ &\geq [\mu_i(x_i) \wedge \cdots \wedge \mu_i(x_i) \wedge \mu_i(x_i)] \wedge [\mu_j(x_j) \wedge \cdots \wedge \mu_j(x_j)] \wedge \cdots \\ &= \mu_i(x_i) \wedge \mu_j(x_j) \wedge \cdots \\ &= \bigwedge_{s\in S} \mu_s(x_s), \end{aligned}$$

$$\left(\bigwedge_{n\in\mathbb{N}} \xi_n\right)(p_1^i(x_i), p_2^i(x_i), \cdots, p_{i-1}^i(x_i), x_i, p_{i+1}^j(x_j), \cdots, p_{j-1}^j(x_j), x_j, \cdots)$$

$$\begin{aligned} &= \xi_1(p_1^i(x_i)) \wedge \cdots \wedge \xi_{i-1}(p_{i-1}^i(x_i)) \wedge \xi_i(x_i) \wedge \xi_{i+1}(p_{i+1}^j(x_j)) \wedge \cdots \wedge \xi_j(x_j) \wedge \cdots \\ &\geq [\xi_i(x_i) \wedge \cdots \wedge \xi_i(x_i) \wedge \xi_i(x_i)] \wedge [\xi_j(x_j) \wedge \cdots \wedge \xi_j(x_j)] \wedge \cdots \\ &= \xi_i(x_i) \wedge \xi_j(x_j) \wedge \cdots \\ &= \bigwedge_{s\in S} \xi_s(x_s), \end{aligned}$$

$$\left(\bigvee_{n\in\mathbb{N}} \gamma_n\right)(p_1^i(x_i), p_2^i(x_i), \cdots, p_{i-1}^i(x_i), x_i, p_{i+1}^j(x_j), \cdots, p_{j-1}^j(x_j), x_j, \cdots)$$

$$\begin{aligned} &= \gamma_1(p_1^i(x_i)) \vee \cdots \vee \gamma_{i-1}(p_{i-1}^i(x_i)) \vee \gamma_i(x_i) \vee \gamma_{i+1}(p_{i+1}^j(x_j)) \vee \cdots \vee \gamma_j(x_j) \vee \cdots \\ &\geq [\gamma_i(x_i) \vee \cdots \vee \gamma_i(x_i) \vee \gamma_i(x_i)] \vee [\gamma_j(x_j) \vee \cdots \vee \gamma_j(x_j)] \vee \cdots \\ &= \gamma_i(x_i) \vee \gamma_j(x_j) \vee \cdots \\ &= \bigvee_{s\in S} \gamma_s(x_s) \end{aligned}$$

$$\begin{aligned} \left(\bigwedge_{n\in\mathbb{N}} \mu_n\right)(0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots) &= \mu_1(0) \wedge \cdots \wedge \mu_i(x_i) \wedge \mu_{i+1}(0) \wedge \cdots \wedge \mu_j(x_j) \wedge \cdots \\ &= \mu_i(x_i) \wedge \mu_j(x_j) \wedge \cdots \\ &= \bigwedge_{s\in S} \mu_s(x_s) \end{aligned}$$

$$\begin{aligned} \left(\bigwedge_{n\in\mathbb{N}} \xi_n\right)(0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots) &= \xi_1(0) \wedge \cdots \wedge \xi_i(x_i) \wedge \xi_{i+1}(0) \wedge \cdots \wedge \xi_j(x_j) \wedge \cdots \\ &= \xi_i(x_i) \wedge \xi_j(x_j) \wedge \cdots \\ &= \bigwedge_{s\in S} \xi_s(x_s) \end{aligned}$$

$$\begin{aligned} \left(\bigvee_{n\in\mathbb{N}} \gamma_n\right)(0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots) &= \gamma_1(0) \vee \cdots \vee \gamma_i(x_i) \vee \gamma_{i+1}(0) \vee \cdots \vee \gamma_j(x_j) \vee \cdots \\ &= \gamma_i(x_i) \vee \gamma_j(x_j) \vee \cdots \\ &= \bigvee_{s\in S} \gamma_s(x_s). \end{aligned}$$

Then $\overline{f}_0, \overline{f}_1 : \left(\bigwedge_{s\in S}\mu_s, \bigwedge_{s\in S}\xi_s, \bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s} \to \left(\bigwedge_{n\in\mathbb{N}}\mu_n, \bigwedge_{n\in\mathbb{N}}\xi_n, \bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}$ are homomorphisms of neutrosophic M-groups. It is clear that the following diagram is commutative:

$$\begin{array}{ccc}
\left(\bigwedge_{s\in S}\mu_s, \bigwedge_{s\in S}\xi_s, \bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s} & \xrightarrow{\overline{f_0}} & \left(\bigwedge_{n\in\mathbb{N}}\mu_n, \bigwedge_{n\in\mathbb{N}}\xi_n, \bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n} \\
{\scriptstyle \overline{d}'}\Big\downarrow & & \Big\downarrow{\scriptstyle \overline{d}} \\
\left(\bigwedge_{s\in S}\mu_s, \bigwedge_{s\in S}\xi_s, \bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s} & \xrightarrow{\overline{f_1}} & \left(\bigwedge_{n\in\mathbb{N}}\mu_n, \bigwedge_{n\in\mathbb{N}}\xi_n, \bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}
\end{array}$$

That is, $\{\overline{f}_0, \overline{f}_1\}$ are morphisms of neutrosophic cochain complexes. Now, let us define $g_0, g_1 : \prod_{n\in\mathbb{N}} G_n \to \prod_{s\in S} G_s$ homomorphisms with this formula:

$$g_0(x_1, x_2, x_3, \cdots) = (x_i, x_j, x_k, \cdots)$$

$$g_1(x_1, x_2, x_3, \cdots) = (x_i + p_i^{i+1}(x_{i+1}) + \cdots + p_i^{j-1}(x_{j-1}), x_j + p_j^{j+1}(x_{j+1}) + \cdots + p_j^{k-1}(x_{k-1}), \cdots).$$

Let us denote $L = \mathbb{N} \setminus \{1, \ldots, i-1\}$. The following inequalities holds:

$$\left(\bigwedge_{s\in S}\mu_s\right)(x_i, x_j, x_k, \cdots) = \mu_i(x_i) \wedge \mu_j(x_j) \wedge \cdots \geq \bigwedge_{n\in\mathbb{N}}\mu_n(x_n)$$

$$\left(\bigwedge_{s\in S}\xi_s\right)(x_i, x_j, x_k, \cdots) = \xi_i(x_i) \wedge \xi_j(x_j) \wedge \cdots \geq \bigwedge_{n\in\mathbb{N}}\xi_n(x_n)$$

$$\left(\bigvee_{s\in S}\gamma_s\right)(x_i, x_j, x_k, \cdots) = \gamma_i(x_i) \vee \gamma_j(x_j) \vee \cdots \leq \bigvee_{n\in\mathbb{N}}\gamma_n(x_n),$$

$$\begin{aligned}
&\left(\bigwedge_{s\in S}\mu_s\right)(x_i + p_i^{i+1}(x_{i+1}) + \cdots + p_i^{j-1}(x_{j-1}), x_j + \cdots + p_j^{k-1}(x_{k-1}), \cdots) \\
&= \mu_i(x_i + p_i^{i+1}(x_{i+1}) + \cdots + p_i^{j-1}(x_{j-1})) \wedge \mu_j(x_j + \cdots + p_j^{k-1}(x_{k-1})) \wedge \cdots \\
&\geq \min\{\mu_i(x_i), \mu_i(p_i^{i+1}(x_{i+1})), \cdots, \mu_i(p_i^{j-1}(x_{j-1}))\} \wedge \min\{\mu_j(x_j), \cdots, \mu_j(p_j^{k-1}(x_{k-1}))\} \wedge \cdots \\
&\geq \min\{\mu_i(x_i), \mu_{i+1}(x_{i+1}), \cdots, \mu_{j-1}(x_{j-1})\} \wedge \min\{\mu_j(x_j), \mu_{j+1}(x_{j+1}), \cdots, \mu_{k-1}(x_{k-1})\} \wedge \cdots \\
&= \bigwedge_{m\in L}\mu_m(x_m) \\
&\geq \bigwedge_{n\in\mathbb{N}}\mu_n(x_n)
\end{aligned}$$

$$\begin{aligned}
&\left(\bigwedge_{s\in S}\xi_s\right)(x_i + p_i^{i+1}(x_{i+1}) + \cdots + p_i^{j-1}(x_{j-1}), x_j + \cdots + p_j^{k-1}(x_{k-1}), \cdots) \\
&= \xi_i(x_i + p_i^{i+1}(x_{i+1}) + \cdots + p_i^{j-1}(x_{j-1})) \wedge \xi_j(x_j + \cdots + p_j^{k-1}(x_{k-1})) \wedge \cdots \\
&\geq \min\{\xi_i(x_i), \xi_i(p_i^{i+1}(x_{i+1})), \cdots, \xi_i(p_i^{j-1}(x_{j-1}))\} \wedge \min\{\xi_j(x_j), \cdots, \xi_j(p_j^{k-1}(x_{k-1}))\} \wedge \cdots \\
&\geq \min\{\xi_i(x_i), \xi_{i+1}(x_{i+1}), \cdots, \xi_{j-1}(x_{j-1})\} \wedge \min\{\xi_j(x_j), \xi_{j+1}(x_{j+1}), \cdots, \xi_{k-1}(x_{k-1})\} \wedge \cdots \\
&= \bigwedge_{m\in L}\xi_m(x_m) \\
&\geq \bigwedge_{n\in\mathbb{N}}\xi_n(x_n)
\end{aligned}$$

$$\left(\bigvee_{s\in S}\gamma_s\right)(x_i+p_i^{i+1}(x_{i+1})+\cdots+p_i^{j-1}(x_{j-1}),x_j+\cdots+p_j^{k-1}(x_{k-1}),\cdots)$$

$$\begin{aligned}
&\leq \max\{\gamma_i(x_i),\gamma_i(p_i^{i+1}(x_{i+1})),\cdots,\gamma_i(p_i^{j-1}(x_{j-1}))\}\vee\max\{\gamma_j(x_j),\cdots,\gamma_j(p_j^{k-1}(x_{k-1}))\}\vee\cdots\\
&\leq \max\{\gamma_i(x_i),\gamma_{i+1}(x_{i+1}),\cdots,\gamma_{j-1}(x_{j-1})\}\vee\max\{\gamma_j(x_j),\gamma_{j+1}(x_{j+1}),\cdots,\gamma_{k-1}(x_{k-1})\}\vee\cdots\\
&= \bigvee_{m\in L}\gamma_m(x_m)\\
&\leq \bigvee_{n\in\mathbb{N}}\gamma_n(x_n).
\end{aligned}$$

Thus, $\overline{g}_0,\overline{g}_1 : \left(\bigwedge_{n\in\mathbb{N}}\mu_n,\bigwedge_{n\in\mathbb{N}}\xi_n,\bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}\to\left(\bigwedge_{s\in S}\mu_s,\bigwedge_{s\in S}\xi_s,\bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s}$ are homomorphisms of neutrosophic M-groups. We give

$$D:\prod_{n\in\mathbb{N}}G_n\to\prod_{n\in\mathbb{N}}G_n$$

homomorphism of M-groups with this formula: $D(x_1,x_2,x_3,\cdots)=(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}),x_2+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}),\cdots,x_{i-1},0,x_{i+1}+p_{i+1}^{i+2}(x_{i+2})+\cdots+p_{i+1}^{j-1}(x_{j-1}),x_{i+2}+\cdots+p_{i+2}^{j-1}(x_{j-1}),\ldots,x_{j-1},0,\cdots)$.

We have the following:

$$\left(\bigwedge_{n\in\mathbb{N}}\mu_n\right)(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}),x_2+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&= \mu_1(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}))\wedge\mu_2(x_2+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}))\wedge\cdots\\
&\quad\wedge\mu_{i-1}(x_{i-1})\wedge\mu_i(0)\wedge\mu_{i+1}(x_{i+1}+p_{i+1}^{i+2}(x_{i+2})+\cdots+p_{i+1}^{j-1}(x_{j-1}))\wedge\cdots\\
&\geq \min\{\mu_1(x_1),\mu_1(p_1^2(x_2)),\cdots,\mu_1(p_1^{i-1}(x_{i-1}))\}\wedge\\
&\quad\min\{\mu_2(x_2),\mu_2(p_2^3(x_3)),\cdots,\mu_2(p_2^{i-1}(x_i-1))\}\wedge\mu_{i-1}(x_{i-1})\wedge\\
&\quad\min\{\mu_{i+1}(x_{i+1}),\mu_{i+1}(p_{i+1}^{i+2}(x_{i+2})),\cdots,\mu_{i+1}(p_{i+1}^{j-1}(x_{j-1}))\}\wedge\cdots\\
&\geq \min\{\mu_1(x_1),\mu_2(x_2),\cdots,\mu_{i-1}(x_{i-1})\}\wedge\min\{\mu_2(x_2),\mu_3(x_3),\cdots,\mu_{i-1}(x_{i-1})\}\wedge\\
&\quad\mu_{i-1}(x_{i-1})\wedge\mu_{i+1}(x_{i+1})\wedge\cdots\\
&= \bigwedge_{k=1}^{i-1}\mu_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\mu_k(x_k)\wedge\cdots\\
&= \bigwedge_{n\in\mathbb{N}}\mu_n(x_n),
\end{aligned}$$

$$\left(\bigwedge_{n\in\mathbb{N}}\xi_n\right)(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}),x_2+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&= \xi_1(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}))\wedge\xi_2(x_2)+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}))\wedge\cdots\\
&\quad \wedge\xi_{i-1}(x_{i-1})\wedge\xi_i(0)\wedge\xi_{i+1}(x_{i+1}+p_{i+1}^{i+2}(x_{i+2})+\cdots+p_{i+1}^{j-1}(x_{j-1}))\wedge\cdots\\
&\geq \min\{\xi_1(x_1),\xi_1(p_1^2(x_2)),\cdots,\xi_1(p_1^{i-1}(x_{i-1}))\}\wedge\\
&\quad \min\{\xi_2(x_2),\xi_2(p_2^3(x_3)),\cdots,\xi_2(p_2^{i-1}(x_i-1))\}\wedge\xi_{i-1}(x_{i-1})\wedge\\
&\quad \min\{\xi_{i+1}(x_{i+1}),\xi_{i+1}(p_{i+1}^{i+2}(x_{i+2})),\cdots,\xi_{i+1}(p_{i+1}^{j-1}(x_{j-1}))\}\wedge\cdots\\
&\geq \min\{\xi_1(x_1),\xi_2(x_2),\cdots,\xi_{i-1}(x_{i-1})\}\wedge\min\{\xi_2(x_2),\xi_3(x_3),\cdots,\xi_{i-1}(x_{i-1})\}\wedge\\
&\quad \xi_{i-1}(x_{i-1})\wedge\xi_{i+1}(x_{i+1})\wedge\cdots\\
&= \bigwedge_{k=1}^{i-1}\xi_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\xi_k(x_k)\wedge\cdots\\
&= \bigwedge_{n\in\mathbb{N}}\xi_n(x_n),
\end{aligned}$$

$$\left(\bigvee_{n\in\mathbb{N}}\gamma_n\right)(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}),x_2+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&= \gamma_1(x_1+p_1^2(x_2)+\cdots+p_1^{i-1}(x_{i-1}))\vee\mu_2(x_2)+p_2^3(x_3)+\cdots+p_2^{i-1}(x_{i-1}))\vee\cdots)\\
&\quad \vee\cdots\vee\gamma_{i-1}(x_{i-1})\vee\gamma_i(0)\vee\gamma_{i+1}(x_{i+1}+p_{i+1}^{i+2}(x_i+2)+\cdots+p_{i+1}^{j-1}(x_{j-1}))\vee\cdots\\
&\geq \max\{\gamma_1(x_1),\gamma_1(p_1^2(x_2)),\cdots,\gamma_1(p_1^{i-1}(x_{i-1}))\}\vee\\
&\quad \max\{\gamma_2(x_2),\gamma_2(p_2^3(x_3)),\cdots,\gamma_2(p_2^{i-1}(x_i-1))\}\vee\gamma_{i-1}(x_{i-1})\vee\\
&\quad \max\{\gamma_{i+1}(x_{i+1}),\gamma_{i+1}(p_{i+1}^{i+2}(x_{i+2})),\cdots,\gamma_{i+1}(p_{i+1}^{j-1}(x_{j-1}))\}\vee\cdots\\
&\geq \max\{\gamma_1(x_1),\gamma_2(x_2),\cdots,\gamma_{i-1}(x_{i-1})\}\vee\\
&\quad \max\{\gamma_2(x_2),\gamma_3(x_3),\cdots,\gamma_{i-1}(x_{i-1})\}\vee\gamma_{i-1}(x_{i-1})\vee\gamma_{i+1}(x_{i+1})\vee\cdots\\
&= \bigvee_{k=1}^{i-1}\gamma_k(x_k)\vee\bigvee_{k=2}^{i-1}\gamma_k(x_k)\vee\cdots\\
&= \bigvee_{n\in\mathbb{N}}\gamma_n(x_n),
\end{aligned}$$

$\overline{D}:\left(\bigwedge_{n\in\mathbb{N}}\mu_n,\bigwedge_{n\in N}\xi_n,\bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}\to\left(\bigwedge_{n\in\mathbb{N}}\mu_n,\bigwedge_{n\in\mathbb{N}}\xi_n,\bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}$ is a homomorphism of neutrosophic M-groups.

Now consider the following neutrosophic cochain complexes

$$\mathscr{C}:\overline{0}\to\left(\bigwedge_{n\in\mathbb{N}}\mu_n,\bigwedge_{n\in\mathbb{N}}\xi_n,\bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}\xrightarrow{\overline{d}}\left(\bigwedge_{n\in\mathbb{N}}\mu_n,\bigwedge_{n\in\mathbb{N}}\xi_n,\bigvee_{n\in\mathbb{N}}\gamma_n\right)_{\prod_{n\in\mathbb{N}}G_n}\to\overline{0}$$

$$\mathscr{D}:\overline{0}\to\left(\bigwedge_{s\in S}\mu_s,\bigwedge_{s\in S}\xi_s,\bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s}\xrightarrow{\overline{d}'}\left(\bigwedge_{s\in S}\mu_s,\bigwedge_{s\in S}\xi_s,\bigvee_{s\in S}\gamma_s\right)_{\prod_{s\in S}G_s}\to\overline{0}$$

By computation, it is shown that the map $\overline{f}:\mathscr{C}\to\mathscr{D}$ given by $\overline{f}=\{\overline{f}_0,\overline{f}_1\}$ is a chain homotopy equivalence. In particular, considering $\overline{g}:\mathscr{D}\to\mathscr{C}$, given by $\overline{g}=\{\overline{g}_0,\overline{g}_1\}$, then $\overline{f}\circ\overline{g}$ and $\overline{f}\circ\overline{g}$ are neutrosophic homotopic to the identity maps, where $\overline{D}$ is a neutrosophic homotopy. More in detail, one can compute $D \circ d = \mathrm{id} - f_0 \circ g_0$, $d \circ D = \mathrm{id} - f_1 \circ g_1$, $id - g_0 \circ f_0 = 0$, $id - g_1 \circ f_1 = 0$. Therefore, $\mathscr{C}$ and $\mathscr{D}$ are quasi-isomorphic and the theorem is proved, since $\varprojlim^{(1)}$ is the first homology M-group of the above neutrosophic chain complexes.

□

Remark 3.24. *Given an inverse system $\{(\mu_n, \xi_n, \gamma_n)_{G_n}, p_n^{n+1}\}_{n\in\mathbb{N}}$, since $\varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n} = ker\overline{d}$ and $p_n^{n+1}(x_{n+1}) = x_n$ are satisfied for each $\{x_n\} \in \varprojlim G_n$,*

$$\mu_n(x_n) = \mu_n(p_n^{n+1}(x_{n+1})) \geq \mu_{n+1}(x_{n+1}),$$

$$\xi_n(x_n) = \xi_n(p_n^{n+1}(x_{n+1})) \geq \xi_{n+1}(x_{n+1}),$$

$$\gamma_n(x_n) = \gamma_n(p_n^{n+1}(x_{n+1})) \leq \gamma_{n+1}(x_{n+1})$$

that is, for each $\{x_n\} \in ker\overline{d}$, $\{\mu_n(x_n)\}$ is a decreasing sequence, $\{\xi_n(x_n)\}$ is a decreasing sequence, $\{\gamma_n(x_n)\}$ is an increasing sequence.

Theorem 3.25. *Let $\{(\mu_n, \xi_n, \gamma_n)_{G_n})\}$, $\{(\mu'_n, \xi'_n, \gamma'_n)_{G'_n})\}$ and $\{(\mu''_n, \xi''_n, \gamma''_n)_{G''_n})\}$ be inverse system of neutrosophic M-groups. Suppose the diagram below is a short exact sequence of inverse system of neutrosophic M-groups and one of the following occurs,*

(1) $\varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n} = 0$

(2) For every $\{x''_n\} \in ker\overline{d} = \varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n}$, it holds $\lim\limits_{n\to\infty} \mu''_n(x''_n) = 0$, $\lim\limits_{n\to\infty} \xi''_n(x''_n) = 0$, $\lim\limits_{n\to\infty} \gamma''_n(x''_n) = 1$.

Then the sequence $\overline{0} \to \varprojlim(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n}$ $\to \varprojlim^{(1)}(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim^{(1)}(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim^{(1)}(\mu''_n, \xi''_n, \gamma''_n)_{G''_n} \to \overline{0}$ is exact.

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mu'_2, \xi'_2, \gamma'_2)_{G'_2} & \to & (\mu_2, \xi_2, \gamma_2)_{G_2} & \to & (\mu''_2, \xi''_2, \gamma''_2)_{G''_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mu'_1, \xi'_1, \gamma'_1)_{G'_1} & \to & (\mu_1, \xi_1, \gamma_1)_{G_1} & \to & (\mu''_1, \xi''_1, \gamma''_1)_{G''_1} & \to & 0
\end{array}$$

Proof. For an inverse system of neutrosophic M-groups $\{(\mu_n, \xi_n, \gamma_n)_{G_n}\}_{n\in\mathbb{N}}$,

$$C = \overline{0} \xrightarrow{\overline{0}} (\mu_A, \xi_A, \gamma_A)_{\prod\limits_{n\in\mathbb{N}} G_n} \xrightarrow{\overline{d}} (\mu_A, \xi_A, \gamma_A)_{\prod\limits_{n\in\mathbb{N}} G_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

is a cochain complex of neutrosophic M-groups.

$$H^0(C) = \varprojlim((\mu_n, \xi_n, \gamma_n)_{G_n})_{G_n}, H^1(C) = \varprojlim^{(1)}((\mu_n, \xi_n, \gamma_n)_{G_n})_{G_n}, H^k(C) = 0, k \geq 2 \quad (5)$$

are neutrosophic cohomology M-groups of this complex. Similarly, for the inverse system of M-groups $\{(\mu'_n, \xi'_n, \gamma'_n)_{G'_n})\}$ and $\{(\mu''_n, \xi''_n, \gamma''_n)_{G''_n})\}$, we can establish the following neutrosophic cochain complexes

$$C' = \overline{0} \xrightarrow{\overline{0}} (\mu'_A, \xi'_A, \gamma'_A)_{\prod\limits_{n\in\mathbb{N}} G'_n} \xrightarrow{\overline{d}} (\mu'_A, \xi'_A, \gamma'_A)_{\prod\limits_{n\in\mathbb{N}} G'_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

$$C'' = \overline{0} \xrightarrow{\overline{0}} (\mu''_A, \xi''_A, \gamma''_A)_{\prod\limits_{n\in\mathbb{N}} G''_n} \xrightarrow{\overline{d}} (\mu''_A, \xi''_A, \gamma''_A)_{\prod\limits_{n\in\mathbb{N}} G''_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

With abuse of notation, we denote with $\overline{d}$ the map as in (4) for each inverse system introduced. It is clear that cohomology M-groups of this complexes have the form of (5). From the condition of this theorem, the following sequence

$$\overline{0} \to C' \to C \to C'' \to \overline{0}$$

is a short exact sequence of cochain complexes of neutrosophic M-groups. From the previous exact sequence, we can define the following sequence of cohomology M-groups (see also [13, Theorem 1.3.1])

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C') \to H^1(C) \to H^1(C'') \to \cdots$$

However, in this context, this sequence is not exact, because we do not know if $\overline{\partial}$ is an homomorphism of neutrosophic M-groups. But in this case $H^0(C'') = ker\ \overline{d} = \varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n}$. So, if $\varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n} = 0$ the result trivially holds since the homomorphism $0 \xrightarrow{\overline{\partial}} H^1(C')$ is trivially neutrosophic. While, if $\lim\limits_{n\to\infty} \mu''_n(x''_n) = 0$, $\lim\limits_{n\to\infty} \xi''_n(x''_n) = 0$ and $\lim\limits_{n\to\infty} \gamma''_n(x''_n) = 1$, by construction we have that the grade functions (μ'', ξ'', γ'') of the neutrosophic M-group $(\mu'', \xi'', \gamma'')_{H^0(C'')}$ are trivial, that is we have $\mu''(\{x_n\}) = \xi''(\{x_n\}) = 0$ and $\gamma''(\{x_n\}) = 1$ for all $\{x_n\} \in \varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n}$ (see Definition 3.9). This assures that $\overline{\partial}$ is a homomorphism of neutrosophic M-groups. Therefore, the sequence of neutrosophic homology M-groups

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C') \to H^1(C) \to H^1(C'') \to \cdots$$

is exact. By using (5), we obtain the following exact sequence of neutrosophic M-groups

$$\overline{0} \to \varprojlim(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim(\mu''_n, \xi''_n, \gamma''_n)_{G''_n}$$

$$\to \varprojlim{}^{(1)}(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim{}^{(1)}(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim{}^{(1)}(\mu''_n, \xi''_n, \gamma''_n)_{G''_n} \to \overline{0}$$

□

Let us investigate necessary conditions in order the derivative functor $\varprojlim^{(1)}$ to be equal to zero.

Theorem 3.26. *Given the following inverse system of neutrosophic M-groups*

$$(\mu_1, \xi_1, \gamma_1)_{G_1} \xleftarrow{\overline{\varphi_1}} (\mu_2, \xi_2, \gamma_2)_{G_2} \xleftarrow{\overline{\varphi_2}} \cdots \tag{6}$$

if every homomorphisms $\overline{\varphi_n}$ is a neutrosophic epimorphism, then $\varprojlim^{(1)}$ $(\mu_n, \xi_n, \gamma_n)_{G_n} = 0$.

Proof. The proof is obvious, since under the hypothesis the map

$$\overline{d} : \prod_{n=1}^{\infty} (\mu_n, \xi_n, \gamma_n)_{G_n} \to \prod_{n=1}^{\infty} (\mu_n, \xi_n, \gamma_n)_{G_n}$$

is a neutrosophic epimorphism. □

Definition 3.27. *Given inverse system of neutrosophic M-groups in* (6)*, for every integer n, if there exists $m \geq n$ such that*

$$Im((\mu_i, \xi_i, \gamma_i)_{G_i} \to (\mu_n, \xi_n, \gamma_n)_{G_n}) = Im((\mu_m, \xi_m, \gamma_m)_{G_m} \to (\mu_n, \xi_n, \gamma_n)_{G_n}) \text{ for all } i \geq m,$$

then it is said that the inverse system in (6) *satisfies the Mittag-Leffler condition.*

Theorem 3.28. *If the inverse system in* (6) *satisfies the Mittag-Leffler condition, then* $\varprojlim^{(1)} (\mu_n, \xi_n, \gamma_n)_{G_n} = 0$.

Proof. Let us denote $G'_n = Im\varphi_n^i$, for large i. By the condition of the theorem, the homomorphism $\varphi_{n|G'_{n+1}}$ carries the M-group G'_{n+1} to G'_n. Then, $\varphi_{n|G'_{n+1}}$ is an epimorphism. Thus, for large i, the homomorphisms

$$\overline{\varphi_n} : (\mu_{n|G'_{n+1}}, \xi_{n|G'_{n+1}}, \gamma_{n|G'_{n+1}})_{G'_{n+1}} \to (\mu_{n|G'_n}, \xi_{n|G'_n}, \gamma_{n|G'_n})_{G'_n}$$

are epimorphisms. Therefore, by Theorem 3.26, we have $\varprojlim^{(1)} (\mu'_n, \xi'_n, \gamma'_n)_{G'_n} = 0$. Here $\mu'_n = \mu_{n|G'_n}, \xi'_n = \xi_{n|G'_n}, \gamma'_n = \gamma_{n|G'_n}$. Let us consider the following sequence of the inverse system of neutrosophic quotient M-groups

$$(\widetilde{\mu_1}, \widetilde{\xi_1}, \widetilde{\gamma_1})_{G_1/G'_1} \leftarrow (\widetilde{\mu_2}, \widetilde{\xi_2}, \widetilde{\gamma_2})_{G_2/G'_2} \leftarrow \cdots . \tag{7}$$

For every n, there exists $m > n$ such that the homomorphism $G_m|G'_m \to G_n|G'_n$ is a zero homomorphism. Then, $\varprojlim(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n} = 0$, that is the limit of inverse system in (7) is equal to 0. Therefore, $\varprojlim^{(1)}(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n} = 0$ as well. Then, let us consider the following short exact sequence of inverse systems in the category of N-Mg.

$$\overline{0} \to \{(\mu'_n, \xi'_n, \gamma'_n)_{G'_n}\} \to \{(\mu_n, \xi_n, \gamma_n)_{G_n}\} \to \{(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n}\} \to \overline{0}. \tag{8}$$

Granting that $\varprojlim(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n} = 0$, we can apply Theorem 3.25 to the sequence (8), we obtain the following exact sequence

$$\begin{aligned} \overline{0} \to & \varprojlim(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n/G'_n} \to \\ \varprojlim^{(1)}(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} & \to \varprojlim^{(1)}(\mu_n, \xi_n, \gamma_n)_{G_n} \to \varprojlim^{(1)}\{(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n}\} \to \overline{0}. \end{aligned} \tag{9}$$

Since $\varprojlim^{(1)}(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} = \overline{0}$, $\varprojlim^{(1)}\{(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n}\} = \overline{0}$ and $\varprojlim\{(\widetilde{\mu_n}, \widetilde{\xi_n}, \widetilde{\gamma_n})_{G_n/G'_n}\} = \overline{0}$, respectively, sequence (9) would look like

$$\overline{0} \to \varprojlim(\mu'_n, \xi'_n, \gamma'_n)_{G'_n} \to \varprojlim(\mu_n, \xi_n, \gamma_n)_{G_n} \to \overline{0} \to \overline{0} \to \varprojlim^{(1)}(\mu_n, \xi_n, \gamma_n)_{G_n} \to \overline{0} \to \overline{0}.$$

This proves that $\varprojlim^{(1)}(\mu_n, \xi_n, \gamma_n)_{G_n} = \overline{0}$. □

4. **Direct system of neutrosophic M-groups**

Building on the classical theory of direct limits for groups and modules, we consider systems of neutrosophic M-groups linked by extending morphisms. The inductive limit then produces an M-group capturing the cumulative behavior of truth, indeterminacy, and falsehood components.

We are going to investigate direct systems of neutrosophic M-groups. Let

$$(\overline{\mu}, \overline{\xi}, \overline{\gamma})_{\overline{G}} = \left\{(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha}, \overline{p}^{\alpha'\alpha}\right\}_{\alpha\in\Delta} \tag{10}$$

be direct system of neutrosophic M-groups, $(\mu^B, \xi^B, \gamma^B)_{\bigoplus_\alpha G_\alpha}$ the neutrosophic M-group as defined in Lemma 3.7.

Recall that $\varinjlim_{\alpha} G_\alpha$ can be expressed as $\bigoplus_\alpha G_\alpha / K$ where K is generated by the set $\{\pi_\alpha(x_\alpha) - \pi_{\alpha'}(p^{\alpha,\alpha'}(x_\alpha)) \mid \alpha \in \Delta, x_\alpha \in G_\alpha\}$, where $\pi_\alpha : G_\alpha \to \bigoplus_\alpha G_\alpha$ are the canonical projections.

Let $\pi : \bigoplus_\alpha G_\alpha \to \varinjlim_{\alpha} G_\alpha$ be the canonical epimorphism. Then we have the neutrosophic M-group $\left((\mu^B)^\pi, (\xi^B)^\pi, (\gamma^B)^\pi\right)_{\varinjlim_{\alpha} G_\alpha}$ as defined in Lemma 3.7.

Theorem 4.1. *Every direct system in the representation* (10) *has a limit in the category N-Mg and this limit is equal to the neutrosophic M-group* $\left((\mu^B)^\pi, (\xi^B)^\pi, (\gamma^B)^\pi\right)_{\varinjlim_{\alpha} G_\alpha}$.

Proof. It suffices to demonstrate that, there exists a unique homomorphism of neutrosophic M-groups $\overline{\psi} : \left((\mu^B)^\pi, (\xi^B)^\pi, (\gamma^B)^\pi\right)_{\varinjlim_{\alpha} G_\alpha} \to (\eta, \theta, \nu)_H$ which makes the following diagram is commutative:

$$\begin{tikzcd}
(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \arrow[r, "\overline{\varphi_\alpha}"] \arrow[d, "\overline{\pi_\alpha}"'] & (\eta, \theta, \nu)_H \\
\left((\mu^B)^\pi, (\xi^B)^\pi, (\gamma^B)^\pi\right)_{\varinjlim_{\alpha} G_\alpha} \arrow[ur, "\overline{\psi}"'] &
\end{tikzcd}$$

where $\overline{\varphi} = \{\overline{\varphi}, (\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \to (\eta, \theta, \nu)_H\}_{\alpha\in\Delta}$ is a family of homomorphisms of neutrosophic M-groups which makes the following diagram is commutative:

$$\begin{tikzcd}
(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \arrow[r, "\overline{\varphi_\alpha}"] \arrow[d, "\overline{p}^{\alpha,\alpha'}"'] & (\eta, \theta, \nu)_H \\
(\mu_{\alpha'}, \xi_{\alpha'}, \gamma_{\alpha'})_{G_{\alpha'}} \arrow[ur, "\overline{\varphi_{\alpha'}}"'] &
\end{tikzcd}$$

and also $\overline{i_\alpha} : (\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \to (\mu^B, \xi^B, \gamma^B)_{\bigoplus_\alpha G_\alpha}$ are usual injections and $\pi_\alpha = \pi \circ i_\alpha$. For every $x \in \varinjlim_{\alpha} G_\alpha$, there exists $x_\alpha \in G_\alpha$ such that $\pi_\alpha(x_\alpha) = x$. If $\pi_{\alpha'}(x_{\alpha'}) = x$ for each $x_{\alpha'} \in G_{\alpha'}$, then $\varphi_{\alpha'}(x_{\alpha'})$ is equal to $\varphi_\alpha(x_\alpha)$. We define the homomorphism $\psi : \varinjlim_{\alpha} G_\alpha \to H$ by $\psi(x) = \varphi_\alpha(x_\alpha)$. Now, we can check if $\overline{\psi}$ is a homomorphism of neutrosophic M-groups. For each $x \in \varinjlim_{\alpha} G_\alpha$, let $\pi \circ i_\alpha(x_\alpha) = x$. Here,

$$(\mu^B)^\pi(x) = \sup(\bigvee_\alpha \mu_\alpha)(x) = \sup\{\bigvee_\alpha \mu_\alpha(x_\alpha) : \pi_\alpha(x_\alpha) = x\}$$

$$(\xi^B)^\pi(x) = \sup(\bigvee_\alpha \xi_\alpha)(x) = \sup\{\bigvee_\alpha \xi_\alpha(x_\alpha) : \pi_\alpha(x_\alpha) = x\}$$

$$(\gamma^B)^\pi(x) = \inf(\bigwedge_\alpha \gamma_\alpha)(x) = \inf\{\bigwedge_\alpha \gamma_\alpha(x_\alpha) : \pi_\alpha(x_\alpha) = x\}.$$

Therefore, $\eta(\psi(x)) = \eta(\varphi_\alpha(x_\alpha)) \geq \mu_\alpha(x_\alpha)$, $\theta(\psi(x)) = \theta(\varphi_\alpha(x_\alpha)) \geq \xi_\alpha(x_\alpha)$, $\nu(\psi(x)) = \eta(\varphi_\alpha(x_\alpha)) \leq \gamma_\alpha(x_\alpha)$. Since this inequality is satisfied for each x_α which satisfies $\pi_\alpha(x_\alpha) = x$, we write the inequality as $\eta(\psi(x)) \geq (\mu^B)^\pi(x)$, $\theta(\psi(x)) \geq (\xi^B)^\pi(x)$, $\nu(\psi(x)) \leq (\gamma^B)^\pi(x)$. From the definition of $\overline{\psi}$, it is obvious that the above diagram is commutative and the uniqueness of $\overline{\psi}$ follows from the uniqueness of ψ, which is a consequence of the definition of direct limit. □

We can easily show that $\varinjlim$ is a functor from the category of direct systems of neutrosophic M-groups to the category of neutrosophic M-groups. Let

$$\overline{G} = \left\{(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha}, \overline{p}^{\alpha'\alpha}\right\}_{\alpha\in\Delta}$$

$$\overline{G}' = \left\{(\mu'_\alpha, \xi'_\alpha, \gamma'_\alpha)_{G'_\alpha}, \overline{p}^{\alpha'\alpha}\right\}_{\alpha\in\Delta}$$

$$\overline{G}'' = \left\{(\mu''_\alpha, \xi''_\alpha, \gamma''_\alpha)_{G''_\alpha}, \overline{p}^{\alpha'\alpha}\right\}_{\alpha\in\Delta}$$

be direct systems of neutrosophic M-groups, and the sequence

$$\overline{G}' \xrightarrow{\overline{f}} \overline{G} \xrightarrow{\overline{g}} \overline{G}'' \tag{11}$$

be an exact sequence of this systems.

Theorem 4.2. *Direct limit of exact sequence in* (11) $\varinjlim_{\alpha}(\mu'_\alpha, \xi'_\alpha, \gamma'_\alpha)_{G'_\alpha} \to \varinjlim_{\alpha}(\mu_\alpha, \xi_\alpha, \gamma_\alpha)_{G_\alpha} \to \varinjlim_{\alpha}(\mu''_\alpha, \xi''_\alpha, \gamma''_\alpha)_{G''_\alpha}$ *is exact.*

Proof. Let the sequence in (11) be exact. Then, the ordinary sequence of M-groups $\{G'_\alpha\} \to \{G_\alpha\} \to \{G''_\alpha\}$ is an exact sequence, for every $\alpha \in \Delta$. Hence, the following sequence $\{G'_\alpha\}_\alpha \xrightarrow{\{f_\alpha\}} \{G_\alpha\}_\alpha \xrightarrow{\{g_\alpha\}} \{G''_\alpha\}_\alpha$ is exact sequence of the direct system of ordinary M-groups. Then the limit of this exact sequence

$$\varinjlim_{\alpha} G'_\alpha \xrightarrow{\varinjlim_{\alpha} f_\alpha} \varinjlim_{\alpha} G_\alpha \xrightarrow{\varinjlim_{\alpha} g_\alpha} \varinjlim_{\alpha} G''_\alpha \tag{12}$$

is also exact. For the following sequence of neutrosophic M-groups

$$\left((\mu'^B)^\pi,(\xi'^B)^\pi,(\gamma'^B)^\pi\right)_{\varinjlim_{\alpha} G'_\alpha} \xrightarrow{\overline{\varinjlim_{\alpha} f_\alpha}} \left((\mu^B)^\pi,(\xi^B)^\pi,(\gamma^B)^\pi\right)_{\varinjlim_{\alpha} G_\alpha} \xrightarrow{\overline{\varinjlim_{\alpha} g_\alpha}} \left((\mu''^B)^\pi,(\xi''^B)^\pi,(\gamma''^B)^\pi\right)_{\varinjlim_{\alpha} G''_\alpha}$$

$$(\mu^B)^\pi | Im\ \varinjlim_{\alpha} f_\alpha = (\mu^B)^\pi | ker \varinjlim_{\alpha} g_\alpha$$

$$(\xi^B)^\pi | Im\ \varinjlim_{\alpha} f_\alpha = (\xi^B)^\pi | ker \varinjlim_{\alpha} g_\alpha$$

$$(\gamma^B)^\pi | Im\ \varinjlim_{\alpha} f_\alpha = (\gamma^B)^\pi | ker \varinjlim_{\alpha} g_\alpha$$

are true, because sequence in (12) is exact. □

Corollary 4.3. *The direct limit functor preserves monomorphism and epimorphism in the category of neutrosophic M-groups.*

Now, we consider direct system of chain complexes. Let I be directed set and for every $i \in I$, let $C(i) = \left\{(\mu_n^{(i)},\xi_n^{(i)},\gamma_n^{(i)})_{G_n^{(i)}}, \overline{\partial}_n : (\mu_n(i),\xi_n(i),\gamma_n(i))_{G_n(i)} \to (\mu_{n-1}(i),\xi_{n-1}(i),\gamma_{n-1}(i))_{G_{n-1}(i)}\right\}_n$ be chain complexes of neutrosophic M-groups and for every $i \prec j$, let $\overline{f}_{ij} : C(i) \to C(j)$ be a morphism of chain complexes and let $\{C(i), \overline{f}_{ij}\}$ be a direct system of these chain complexes.

Theorem 4.4. *The limit of homology M-groups of direct system of chain complexes of neutrosophic M-groups is isomorphic to the homology M-groups of the limit of this direct system, that is,* $H_n\left(\varinjlim_{\alpha} C(i)\right) \simeq \varinjlim_{\alpha} H_n(C(i))$.

Proof. The proof of this theorem is provided by using Corollary 4.3. Hence,

$$\begin{aligned}
\varinjlim_{i} H_n(C(i)) &= \varinjlim_{i} \left(\widetilde{\mu_n}(i),\widetilde{\xi_n}(i),\widetilde{\gamma_n}(i)\right)_{ker\overline{\partial}_n(i)|Im\ \overline{\partial}_{n+1}(i)} \\
&\approx \varinjlim_{i} \left(\mu_n|ker\overline{\partial}_n(i),\xi_n|ker\overline{\partial}_n(i),\gamma_n|ker\overline{\partial}_n(i)\right)_{ker\overline{\partial}_n(i)} \\
&\quad | \varinjlim_{i} \left(\mu_n|ker\overline{\partial}_n(i),\xi_n|ker\overline{\partial}_n(i),\gamma_n|ker\overline{\partial}_n(i)\right)_{ker\overline{\partial}_n(i)} \\
&\approx ker \varinjlim_{i} \overline{\partial}_n(i) | \varinjlim_{i} \overline{\partial}_n(i) \\
&= H_n\left(\varinjlim_{i} C(i)\right).
\end{aligned}$$

□

5. Conclusions

In this paper we have established a comprehensive framework for inverse and direct systems in the category of neutrosophic M-groups, proving that under natural conditions both projective and inductive limits exist and preserve exact sequences. By introducing an adapted version of the Mittag–Leffler condition for neutrosophic morphisms, we showed that exactness is maintained in inverse limits, while for direct systems we identified precise criteria ensuring that the accumulation of truth, indeterminacy and falsehood through extending maps does not break short exact sequences. Our explicit examples demonstrate how these constructions recover the classical behavior when the indeterminacy component is trivial and, more interestingly, exhibit genuinely new phenomena arising from the free interaction of the three neutrosophic degrees.

Looking forward, several concrete extensions of our work suggest themselves. First, the constructions of inverse limits in Section 3 (cf. Theorem 3.10) and of direct limits in Section 4 (cf. Theorem 4.2) could be implemented within some computer algebra environments, to automate the assembly of large or parametrized families of neutrosophic M-groups. Second, the core notions and exactness results for neutrosophic M-groups (see Definitions 2.4 and 2.7), and the exactness criteria in Sections 3 and 4) admit immediate analogues for neutrosophic rings and modules, thereby extending classical algebraic categories while preserving the independent truth–indeterminacy–falsehood framework.

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