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Directed n-Superhypergraphs Incorporating Bipolar Fuzzy Information: A Multi-Tier Framework for Modeling Bipolar Uncertainty in Complex Networks

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Abstract. Graph theory studies the mathematical structures of vertices and edges to model relationships and connectivity. Hypergraphs extend this framework by allowing hyperedges to connect arbitrarily many vertices at once [1], and Super-HyperGraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical linkages among edges [2, 3]. Bipolar fuzzy directed graphs assign positive and negative membership degrees to directed edges and vertices, and bipolar fuzzy directed hypergraphs extend this assignment to multi-vertex hyperedges. In this paper, we extend directed Super-HyperGraphs by incorporating bipolar fuzzy membership and introduce the Bipolar Fuzzy Directed n-Super-HyperGraph, whose structural properties we investigate.

Keywords: Super-HyperGraph, Hypergraph, Fuzzy Graph, Bipolar Fuzzy Graph, Bipolar Fuzzy HyperGraph

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1. Introduction

1.1. *From Graphs to Hypergraphs and Super-HyperGraphs*

Network structures are often captured by a *graph*, where entities appear as vertices and binary links as edges. To model interactions among groups larger than two, a *hypergraph* allows each hyperedge to connect any nonempty subset of vertices [4]. Extending this further, a *Super-HyperGraph* is obtained by repeatedly applying the powerset operation, thus uncovering nested layers of connectivity that reflect hierarchical relationships [5–9].

Graphs and hypergraphs have proven invaluable across domains such as artificial intelligence, network science, data mining, chemistry, and physics [10]. By introducing multiple lYutakals of edges, Super-HyperGraphs provide a natural framework for representing complex, multi-scale systems.

Model	Notation	Edge/Hyperedge Specification	Construction Principle
Graph	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Simple edges join two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathrm{POW}(V) \setminus \{\emptyset\}$	Hyperedges span any nonempty vertex subset.
Super-HyperGraph	$\mathrm{SupHyG}^{(n)} = (V_0, V, E)$	$V, E \subseteq \mathrm{POW}^n(V_0)$	n-fold powerset rYutakaals nested connectivity.

Table 1. Key distinctions among graph-based models

1.2. *Directed, Bi-directed, and Multi-directed Variants*

Many real-world networks feature intrinsically oriented relations. A *directed graph* assigns an arrow to each edge [11–13]. This idea extends to *directed hypergraphs* [14] and further to *directed Super-HyperGraphs* [15].

In a *bi-directed graph*, each endpoint of an edge carries its own direction indicator [16, 17]. *Multi-directed graphs* allow multiple, possibly conflicting, directed edges between the same vertex pair [18]. These enriched models capture more intricate flow or influence patterns than simple directed networks.

1.3. *Incorporating Uncertainty: Fuzzy, Neutrosophic, and Bipolar Fuzzy Extensions*

To represent imprecise or ambiguous connections, the *fuzzy directed graph* uses membership grades on each arc [19,20]. The *neutrosophic directed graph* further assigns three values—truth, indeterminacy, falsity—to each link for nuanced uncertainty modeling [21, 22].

Analogous concepts apply to hypergraphs. A *fuzzy directed hypergraph* equips each hyperarc with a membership function over multiple vertices [23,24]. Similarly, *single-valued neutrosophic directed hypergraphs* attach a triple of neutrosophic degrees to Yutakary hyperarc, capturing partial truth, indeterminacy, and falsity in group interactions [25, 26].

More recently, *bipolar fuzzy directed graphs* and *bipolar fuzzy directed hypergraphs* have been proposed, allowing both positive and negative membership values for nodes and edges. These provide richer representations of supportive versus opposing influences in directed networks.

1.4. *Contributions of This Work*

In this paper we introduce a novel framework, *Bipolar Fuzzy Directed Super-HyperGraphs*, which combines the hierarchical edge-layering of Super-HyperGraphs with the dual-valued uncertainty of bipolar fuzzy models. We give its formal definition, investigate its key properties, and demonstrate potential applications in decision making and network analysis. In Table 2 we present a concise comparison of bipolar fuzzy directed models. This table further illustrates how our approach can more clearly and intuitively capture uncertain, hierarchical network structures in real-world scenarios.

Model	Domain	Key Constraint
Bipolar Fuzzy Directed Graph	$V,\ E \subseteq V \times V$	$\mu^+ \leq \min \sigma^+,\ \mu^- \geq \max \sigma^-$
Bipolar Fuzzy Directed Hypergraph	$V,\ E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$	$\mathrm{supp}^+(\mathrm{tail}) \cap \mathrm{supp}^+(\mathrm{head}) = \emptyset$
Bipolar Fuzzy Directed n-Superhypergraph	$V \subseteq \mathcal{P}^n(S),\ E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$	same as graph, on n-supervertices

Table 2. Concise comparison of bipolar fuzzy directed models

2. **Preliminaries**

We start by introducing the notation and basic concepts used throughout this paper. Unless noted otherwise, all graphs are simple, finite, and undirected. For more in-depth treatments of specific topics, see the cited references.

2.1. *Super-HyperGraph*

A *hypergraph* extends the classical graph by allowing each hyperedge to join any nonempty subset of vertices simultaneously [1,4]. Building on this, a *Super-HyperGraph* employs repeated powerset operations to capture nested, hierarchical connections among hyperedges—a subject of increasing interest in recent literature [3, 27–29]. Applications include molecular modeling, complex-network analysis, and signal processing [15, 30]. In what follows, n always denotes a nonnegative integer.

Definition 2.1 (Base Set). A *base set* S is the foundational domain for all subsequent constructions:

$$S = \{\, x \mid x \text{ belongs to the chosen universe}\}.$$

Yutakary element of any iterated powerset $\mathrm{POW}^k(S)$ remains an element of S.

Definition 2.2 (Powerset). The *powerset* of S, written $\mathrm{POW}(S)$, is the set of all subsets of S, including the empty set and S itself:

$$\mathrm{POW}(S) = \{ A \mid A \subseteq S \}.$$

Definition 2.3 (Hypergraph). A *hypergraph* $H = (V(H), E(H))$ comprises

- a finite vertex set $V(H)$, and
- a collection $E(H)$ of nonempty subsets of $V(H)$, called *hyperedges* [1,31].

Hypergraphs naturally model interactions among any number of vertices.

Definition 2.4 (n-th Powerset). The *n-th powerset* of X, denoted $\mathrm{POW}_n(X)$, is defined recursively [32–34]:

$$\mathrm{POW}_1(X) = \mathrm{POW}(X), \quad \mathrm{POW}_{n+1}(X) = \mathrm{POW}\big(\mathrm{POW}_n(X)\big), \quad n \geq 1.$$

The nonempty version $\mathrm{POW}_n^*(X)$ is obtained by removing the empty set at each iteration.

Definition 2.5 (n-Super-HyperGraph). [5,35] Let V_0 be a finite base set. Define

$$\mathrm{POW}^0(V_0) = V_0, \quad \mathrm{POW}^{k+1}(V_0) = \mathrm{POW}\big(\mathrm{POW}^k(V_0)\big).$$

An *n-Super-HyperGraph* is a pair

$$\mathrm{SupHyG}^{(n)} = (V, E), \quad V,\, E \subseteq \mathrm{POW}^n(V_0),$$

where each element of V is called an n-supervertex and each element of E an n-superedge.

Example 2.6 (Cross-Department Collaboration as a 2-Super-HyperGraph). Let the base set of individuals be

$$V_0 \;=\; \{\text{Taro, Bob, Mamoru, Dave, Yutaka, Tae}\}.$$

Compute the first and second iterated powersets:

$$\mathrm{POW}^1(V_0) = \{ \text{Team} \mid \text{Team} \subseteq V_0 \}, \quad \mathrm{POW}^2(V_0) = \{ \mathcal{G} \mid \mathcal{G} \subseteq \mathrm{POW}^1(V_0) \}.$$

Choose three first-lYutakal clusters (project teams):

$$T_1 = \{\text{Taro, Bob, Mamoru}\}, \quad T_2 = \{\text{Dave, Yutaka}\}, \quad T_3 = \{\text{Tae}\}.$$

These lie in $\mathrm{POW}^1(V_0)$. Now form two second-lYutakal clusters (departments) as subsets of teams:

$$D_X = \{T_1, T_2\}, \quad D_Y = \{T_2, T_3\},$$

so each $D_* \in \mathrm{POW}^2(V_0)$. Finally, define the super-hypergraph by

$$V \;=\; \{ D_X, D_Y \} \;\subseteq\; \mathrm{POW}^2(V_0), \qquad E \;=\; \{\{ D_X, D_Y \}\} \;\subseteq\; \mathrm{POW}^2(V_0).$$

Here the single hyperedge $\{D_X, D_Y\}$ represents a cross-department project in which both departments D_X and D_Y collaborate. Thus

$$\mathrm{SupHyG}^{(2)} = (V,\, E)$$

is a 2-Super-HyperGraph encoding the two-tier structure of teams and departments, together with their inter-department collaborations.

2.2. *Directed Super-HyperGraph*

Directed HyperGraphs and Directed Super-HyperGraphs are graph classes that extend HyperGraphs and Super-HyperGraphs, respectively, in a manner analogous to Directed Graphs(cf. [36, 37]) . Below, we present their formal definitions and illustrative examples.

Definition 2.7 (Directed Hypergraph)**.** (cf. [36, 37]) A *directed hypergraph* is a pair

$$H = (V,\, E),$$

where

- V is a finite set of *vertices*.
- E is a finite set of *hyperarcs*, each hyperarc $e \in E$ being an ordered pair

$$e = \bigl(T(e),\, H(e)\bigr) \in \mathrm{POW}(V) \times \mathrm{POW}(V),$$

  with

$$T(e) \subseteq V,\ T(e) \neq \emptyset, \quad H(e) \subseteq V,\ H(e) \neq \emptyset.$$

Intuitively, each $e = (T(e), H(e))$ carries "flow" from all vertices in $T(e)$ (the *tail*) to all vertices in $H(e)$ (the *head*).

Definition 2.8 (Directed n-Super-HyperGraph)**.** (cf. [5, 15, 38]) Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define iterated powersets by

$$\mathrm{POW}^0(S) = S, \quad \mathrm{POW}^{k+1}(S) = \mathrm{POW}\bigl(\mathrm{POW}^k(S)\bigr) \quad (k \geq 0).$$

A *directed n-Super-HyperGraph* is a pair

$$\mathrm{DSupHyG}^{(n)} = (V,\, E),$$

where

$$V \ \subseteq\ \mathrm{POW}^n(S), \quad E \ \subseteq\ \mathrm{POW}^n(S) \times \mathrm{POW}^n(S),$$

and each directed n-superedge $e \in E$ is an ordered pair

$$e = \bigl(\mathrm{Tail}(e),\, \mathrm{Head}(e)\bigr), \quad \mathrm{Tail}(e),\, \mathrm{Head}(e) \subseteq \mathrm{POW}^n(S),$$

typically both nonempty. Such an e carries "flow" from the entire set $\mathrm{Tail}(e)$ of n-supervertices into $\mathrm{Head}(e)$.

Example 2.9 (Cloud Data Replication as a Directed 2–Super-HyperGraph)**.** Consider a cloud-based data replication system with four server nodes:

$$S \;=\; \{\mathrm{A1},\,\mathrm{A2},\,\mathrm{B1},\,\mathrm{B2}\},\qquad n=2.$$

We form the iterated powersets:

$$\mathrm{POW}^0(S)=S,\quad \mathrm{POW}^1(S)=\mathrm{POW}(S),\quad \mathrm{POW}^2(S)=\mathrm{POW}\big(\mathrm{POW}(S)\big).$$

Choose two first-lYutakal clusters (data-centers) in $\mathrm{POW}^1(S)$:

$$\mathrm{DC}_1=\{\mathrm{A1},\mathrm{A2}\},\quad \mathrm{DC}_2=\{\mathrm{B1},\mathrm{B2}\}.$$

Then form two second-lYutakal clusters (tiers) in $\mathrm{POW}^2(S)$:

$$\mathrm{Tier}_{\mathrm{Primary}}=\{\mathrm{DC}_1,\,\mathrm{DC}_2\},\quad \mathrm{Tier}_{\mathrm{Backup}}=\{\mathrm{DC}_2\}.$$

Thus the vertex set $V\subseteq \mathrm{POW}^2(S)$ is

$$V \;=\; \big\{\mathrm{Tier}_{\mathrm{Primary}},\,\mathrm{Tier}_{\mathrm{Backup}}\big\}.$$

Model the replication flow as a single directed 2–superedge:

$$e=\big(\mathrm{Tier}_{\mathrm{Primary}},\,\mathrm{Tier}_{\mathrm{Backup}}\big),\qquad E=\{\,e\}\;\subseteq\;\mathrm{POW}^2(S)\times\mathrm{POW}^2(S).$$

Here e carries "flow" from Yutakary primary data-center $(\mathrm{DC}_1,\mathrm{DC}_2)$ into the backup data-center (DC_2). Hence

$$\mathrm{DSupHyG}^{(2)} \;=\; \big(V,\,E\big)$$

is a valid directed 2–Super-HyperGraph representing the two-lYutakal replication hierarchy in this cloud system.

2.3. *Fuzzy Directed n-Super-HyperGraph*

We define the *Fuzzy Directed n-Super-HyperGraph* as an extension of the classical *Fuzzy Directed Hypergraph* by incorporating the hierarchical structure of n-Super-HyperGraphs.

Definition 2.10 (Fuzzy Directed n-Super-HyperGraph)**.** Let S be a nonempty *base set* and let $n\geq 0$ be an integer. Define iterated powersets by

$$\mathrm{POW}^0(S)=S,\qquad \mathrm{POW}^{k+1}(S)=\mathrm{POW}\big(\mathrm{POW}^k(S)\big)\quad (k\geq 0).$$

A *directed n-Super-HyperGraph* is a pair $\mathrm{DSupHyG}^{(n)}=(V,E)$ with

$$V\subseteq \mathrm{POW}^n(S),\qquad E\subseteq \mathrm{POW}^n(S)\times \mathrm{POW}^n(S),$$

where each $e\in E$ is of the form $\big(\mathrm{Tail}(e),\mathrm{Head}(e)\big)$. A *fuzzy directed n-Super-HyperGraph* is then the quadruple

$$\big(V,\,E,\,\sigma,\,\mu\big),$$

where

- $\sigma : V \to [0,1]$ assigns to each n-supervertex v a membership degree $\sigma(v)$.
- $\mu : E \to [0,1]$ assigns to each directed n-superedge e a membership degree $\mu(e)$.

These satisfy the *edge-appurtenance constraint*

$$\mu(e) \;\leq\; \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x), \quad \forall\, e \in E.$$

Example 2.11 (Supply-Chain Network as a Fuzzy Directed 1-Super-HyperGraph)**.** Consider a simplified supply-chain consisting of suppliers, factories, and distribution centers:

$$S \;=\; \{\text{SupplierA, SupplierB, FactoryX, FactoryY, DC1, DC2}\}, \qquad n = 1.$$

Since $n = 1$, we have

$$\mathrm{POW}^1(S) \;=\; \mathrm{POW}(S),$$

and we choose three 1-supervertices representing facility clusters:

$$V_1 = \{\mathrm{SupplierA}, \mathrm{SupplierB}\}, \quad V_2 = \{\mathrm{FactoryX}, \mathrm{FactoryY}\}, \quad V_3 = \{\mathrm{DC1}, \mathrm{DC2}\}.$$

Thus

$$V = \{\, V_1,\, V_2,\, V_3\} \;\subseteq\; \mathrm{POW}^1(S).$$

We model two directed 1-superedges corresponding to material flows:

$$e_1 = (V_1,\, V_2), \quad e_2 = (V_2,\, V_3), \qquad E = \{\, e_1,\, e_2\} \;\subseteq\; \mathrm{POW}^1(S) \times \mathrm{POW}^1(S).$$

Assign fuzzy membership degrees:

$$\sigma(V_1) = 0.90, \quad \sigma(V_2) = 0.85, \sigma(V_3) = 0.80,$$

$$\mu(e_1) = 0.80, \quad \mu(e_2) = 0.75.$$

We verify the edge-appurtenance constraint for each e_i:

$$\mu(e_1) = 0.80 \;\leq\; \min\{\sigma(V_1), \sigma(V_2)\} = \min\{0.90, 0.85\} = 0.85,$$

$$\mu(e_2) = 0.75 \;\leq\; \min\{\sigma(V_2), \sigma(V_3)\} = \min\{0.85, 0.80\} = 0.80.$$

Since both inequalities hold, (V, E, σ, μ) is a valid fuzzy directed 1-Super-HyperGraph that models the two-stage flow from suppliers to factories to distribution centers.

Example 2.12 (Corporate Divisional Communication Network as a Fuzzy Directed 2-Super-HyperGraph)**.** Consider a mid-sized company organized into teams, departments, and divisions. We model the two-lYutakal hierarchy of communication as a fuzzy directed 2-Super-HyperGraph.

$$S = \{\text{SN, SS, MO, MF, ST, CS}\}, \qquad n = 2,$$

where

- SN, SS are the two Sales teams (North and South),
- MO, MF are the two Marketing teams (Online and Field),
- ST, CS are the two Support teams (Tech and Customer).

First-lYutakal clusters (departments) in $\mathrm{POW}^1(S)$ are:

$$D_{\text{Sales}} = \{\mathrm{SN}, \mathrm{SS}\}, \quad D_{\text{Marketing}} = \{\mathrm{MO}, \mathrm{MF}\}, \quad D_{\text{Support}} = \{\mathrm{ST}, \mathrm{CS}\}.$$

Second-lYutakal clusters (divisions) in $\mathrm{POW}^2(S)$ are:

$$\mathrm{Div}_A = \{D_{\text{Sales}}, D_{\text{Marketing}}\}, \qquad \mathrm{Div}_B = \{D_{\text{Support}}\}.$$

Thus

$$V = \{\mathrm{Div}_A, \mathrm{Div}_B\} \subseteq \mathrm{POW}^2(S).$$

We define two directed 2-superedges for routine reporting and feedback:

$$e_1 = (\mathrm{Div}_A, \mathrm{Div}_B), \quad e_2 = (\mathrm{Div}_B, \mathrm{Div}_A), \quad E = \{e_1, e_2\} \subseteq \mathrm{POW}^2(S) \times \mathrm{POW}^2(S).$$

Assign fuzzy membership degrees reflecting communication effectiveness:

$$\sigma(\mathrm{Div}_A) = 0.90, \quad \sigma(\mathrm{Div}_B) = 0.85,$$
$$\mu(e_1) = 0.85, \qquad \mu(e_2) = 0.80.$$

We verify the edge-appurtenance constraint:

$$\mu(e_1) = 0.85 \leq \min\{\sigma(\mathrm{Div}_A), \sigma(\mathrm{Div}_B)\} = \min\{0.90, 0.85\} = 0.85,$$

$$\mu(e_2) = 0.80 \leq \min\{\sigma(\mathrm{Div}_B), \sigma(\mathrm{Div}_A)\} = \min\{0.85, 0.90\} = 0.85.$$

Since both inequalities hold, (V, E, σ, μ) is a valid fuzzy directed 2-Super-HyperGraph modeling the two-lYutakal reporting and feedback flows between divisions.

2.4. *Bipolar Fuzzy Directed Hypergraphs*

Bipolar fuzzy directed graphs assign positive and negative membership to directed edges and vertices; hypergraphs extend this to multi-vertex hyperedges.

Definition 2.13 (Bipolar Fuzzy Set)**.** [39–41] Let T be a nonempty finite set. A *bipolar fuzzy subset* X of T is specified by a pair of membership functions

$$\mu_X^+ : T \to [0,1], \quad \mu_X^- : T \to [-1,0],$$

and is denoted

$$X = \{ (v, \mu_X^+(v), \mu_X^-(v)) \mid v \in T \}.$$

Here $\mu_X^+(v)$ and $\mu_X^-(v)$ measure, respectively, the degree to which v belongs positively or negatively to X.

Definition 2.14 (Bipolar Fuzzy Directed Hyperedge)**.** A *bipolar fuzzy directed hyperedge* over a vertex set T is an ordered pair

$$e = \big(t(e),\, h(e)\big),$$

where $t(e)$ (the *tail*) and $h(e)$ (the *head*) are bipolar fuzzy subsets of T satisfying

$$\operatorname{supp} t(e) \;\cap\; \operatorname{supp} h(e) \;=\; \emptyset.$$

Definition 2.15 (Bipolar Fuzzy Directed Hypergraph)**.** (cf. [42]) A *bipolar fuzzy directed hypergraph* is a pair

$$G = \big(T,\, U\big),$$

where

- T is a finite, nonempty set of *vertices*,
- U is a finite collection of bipolar fuzzy directed hyperedges on T.

Each $e \in U$ is a directed hyperedge $(t(e), h(e))$ as above.

Example 2.16 (Humanitarian Aid Distribution Network as a Bipolar Fuzzy Directed Hypergraph)**.** Consider a simplified humanitarian-aid distribution network. We have six facilities:

$$T \;=\; \{\text{WA, WB, DC1, DC2, CX, CY}\},$$

where

- WA, WB are two central Warehouses;
- DC1, DC2 are two Distribution Centers;
- CX, CY are two Field Clinics.

We model two flows of supplies as bipolar fuzzy directed hyperedges:

$$U \;=\; \{\,e_1, e_2\}.$$

Here:

$$\begin{aligned} e_1 :\; \big(t(e_1),\, h(e_1)\big), \quad t(e_1) &= \{(\text{WA}, 0.90, -0.10),\; (\text{WB}, 0.80, -0.20)\}, \\ h(e_1) &= \{(\text{DC1}, 0.85, -0.15),\; (\text{DC2}, 0.80, -0.20)\}, \\[1ex] e_2 :\; \big(t(e_2),\, h(e_2)\big), \quad t(e_2) &= \{(\text{DC1}, 0.90, -0.10),\; (\text{DC2}, 0.75, -0.25)\}, \\ h(e_2) &= \{(\text{CX}, 0.80, -0.20),\; (\text{CY}, 0.70, -0.30)\}. \end{aligned}$$

In each ordered pair $(v, \mu^+(v), \mu^-(v))$, μ^+ measures the reliability of facility v (higher is more reliable) and μ^- measures the operational risk (more negative is higher risk).

One checks immediately that for each e_i:

$$\operatorname{supp} t(e_i) \;=\; \{\, v : \mu^+(v) > 0 \text{ in } t(e_i)\,\}, \quad \operatorname{supp} h(e_i) \;=\; \{\, v : \mu^+(v) > 0 \text{ in } h(e_i)\,\},$$

and

$$\operatorname{supp} t(e_i) \;\cap\; \operatorname{supp} h(e_i) \;=\; \emptyset.$$

Hence (T, U) is a valid bipolar fuzzy directed hypergraph modeling two stages of the aid-distribution process.

3. **Main Results of this paper**

As the principal contribution of this paper, we investigate the structural properties of Bipolar Fuzzy Directed n-Super-HyperGraphs.

3.1. *Bipolar Fuzzy Directed n-Super-HyperGraph*

A Bipolar Fuzzy Directed n-Super-HyperGraph is a hierarchical directed hypergraph with positive, negative membership on multi-lYutakal vertices and directed edges.

Definition 3.1 (Bipolar Fuzzy Directed n-Super-HyperGraph)**.** Let S be a nonempty base set and $n \geq 0$. We write

$$\mathrm{POW}^0(S) = S, \quad \mathrm{POW}^{k+1}(S) = \mathrm{POW}(\mathrm{POW}^k(S)) \quad (k \geq 0).$$

A *bipolar fuzzy directed n-Super-HyperGraph* is a sextuple

$$\mathrm{BFDnSupHyG}^{(n)} = \big(V,\, E,\, \sigma^+,\, \sigma^-,\, \mu^+,\, \mu^-\big),$$

where

- $V \subseteq \mathrm{POW}^n(S)$ is the set of n-supervertices.
- $E \subseteq \mathrm{POW}^n(S) \times \mathrm{POW}^n(S)$ is the set of directed n-superedges.
- $\sigma^+ : V \to [0, 1]$ and $\sigma^- : V \to [-1, 0]$ assign positive and negative membership to each supervertex.
- $\mu^+ : E \to [0, 1]$ and $\mu^- : E \to [-1, 0]$ assign positive and negative membership to each directed superedge.

These functions satisfy, for Yutakary $e = (T(e), H(e)) \in E$:

$$\mu^+(e) \;\leq\; \min_{x \in T(e) \cup H(e)} \sigma^+(x), \quad \mu^-(e) \;\geq\; \max_{x \in T(e) \cup H(e)} \sigma^-(x),$$

and the supports of tail and head in each polarity are disjoint:

$$\mathrm{supp}^+(T(e)) \cap \mathrm{supp}^+(H(e)) = \emptyset, \quad \mathrm{supp}^-(T(e)) \cap \mathrm{supp}^-(H(e)) = \emptyset.$$

Here $\mathrm{supp}^+(X) = \{\, x : \sigma^+(x) > 0\}$ and $\mathrm{supp}^-(X) = \{\, x : \sigma^-(x) < 0\}$.

Example 3.2 (Supply Chain as a Bipolar Fuzzy Directed 1-Super-HyperGraph)**.** Consider a simplified supply-chain network consisting of raw-material suppliers, factories, and distribution centers. We model this as a bipolar fuzzy directed 1-Super-HyperGraph.

$$S = \{\, \text{Supplier A, Supplier B, Factory X, Factory Y, DC1, DC2}\}, \quad n = 1, \quad \mathrm{POW}^1(S) = \mathrm{POW}(S).$$

We choose three 1-supervertices (clusters of facilities):

$$V_1 = \{\text{Supplier A}, \text{Supplier B}\}, \quad V_2 = \{\text{Factory X}, \text{Factory Y}\}, \quad V_3 = \{\text{DC1}, \text{DC2}\}.$$

Thus

$$V = \{ V_1, V_2, V_3\} \subseteq \text{POW}(S).$$

Define two directed 1-superedges representing product flows:

$$e_1 = (V_1, V_2), \quad e_2 = (V_2, V_3), \quad E = \{ e_1, e_2\} \subseteq \text{POW}(S) \times \text{POW}(S).$$

Assign positive and negative membership degrees as follows:

- $\sigma^+(V_1) = 0.90,\ \sigma^-(V_1) = -0.20$ (suppliers are highly reliable but carry moderate risk)
- $\sigma^+(V_2) = 0.85,\ \sigma^-(V_2) = -0.10$ (factories are reliable with low operational risk)
- $\sigma^+(V_3) = 0.80,\ \sigma^-(V_3) = -0.15$ (distribution centers are somewhat reliable but face logistical risk)
- $\mu^+(e_1) = 0.80,\ \mu^-(e_1) = -0.10$ (flow from suppliers to factories is strong with minor concerns)
- $\mu^+(e_2) = 0.75,\ \mu^-(e_2) = -0.12$ (flow from factories to DCs is fairly strong with moderate concerns)

One checks immediately that for each $e_i = (T(e_i), H(e_i))$:

$$\mu^+(e_i) \le \min_{x \in T(e_i) \cup H(e_i)} \sigma^+(x), \quad \mu^-(e_i) \ge \max_{x \in T(e_i) \cup H(e_i)} \sigma^-(x),$$

and that the positive- and negative-support sets of each tail and head are disjoint, since $\{V_1, V_2, V_3\}$ are pairwise distinct. Hence $\left(V, E, \sigma^+, \sigma^-, \mu^+, \mu^-\right)$ is a valid Bipolar Fuzzy Directed 1-Super-HyperGraph modeling this supply chain.

Example 3.3 (Smart-City Sensor Network as Bipolar Fuzzy Directed 2-Super-HyperGraph)**.** We model a simple smart-city deployment of IoT sensors on three floors of a building using a bipolar fuzzy directed 2-Super-HyperGraph.

$$S = \{A, B, C, D, E, F\}, \quad n = 2, \quad \text{POW}^1(S) = \left\{ T_1, T_2, T_3 \right\},$$

where

$$T_1 = \{A, B\}, \quad T_2 = \{C, D\}, \quad T_3 = \{E, F\}.$$

We then form two 2-supervertices, each a set of teams of sensors:

$$G_1 = \{T_1, T_2\}, \quad G_2 = \{T_3\}, \quad V = \{G_1, G_2\} \subseteq \text{POW}^2(S).$$

A single directed 2-superedge represents the data flow from the lower-lYutakal floors to the rooftop cluster:

$$e = \left(G_1, G_2\right), \quad E = \{e\} \subseteq \text{POW}^2(S) \times \text{POW}^2(S).$$

Assign bipolar fuzzy membership degrees reflecting performance (positive) and risk (negative):

$$\sigma^+(G_1) = 0.90, \qquad \sigma^-(G_1) = -0.20,$$
$$\sigma^+(G_2) = 0.80, \qquad \sigma^-(G_2) = -0.30,$$
$$\mu^+(e) = 0.75, \qquad \mu^-(e) = -0.15.$$

One checks:

$$\mu^+(e) = 0.75 \ \le\ \min\{0.90,\, 0.80\} = 0.80, \quad \mu^-(e) = -0.15 \ \ge\ \max\{-0.20,\, -0.30\} = -0.20,$$

and the positive- and negative-support sets are disjoint:

$$\mathrm{supp}^+(G_1) = \{T_1, T_2\}, \quad \mathrm{supp}^+(G_2) = \{T_3\}, \quad \{T_1, T_2\} \cap \{T_3\} = \emptyset,$$
$$\mathrm{supp}^-(G_1) = \{T_1, T_2\}, \quad \mathrm{supp}^-(G_2) = \{T_3\}, \quad \{T_1, T_2\} \cap \{T_3\} = \emptyset.$$

Hence $\big(V, E, \sigma^+, \sigma^-, \mu^+, \mu^-\big)$ is a valid Bipolar Fuzzy Directed 2-Super-HyperGraph modeling the smart-city sensor network.

Example 3.4 (Cloud-Computing Infrastructure as a Bipolar Fuzzy Directed **3**-Super-HyperGraph)**.** Consider a global cloud-computing network, where individual servers are organized into racks, racks into data centers, and data centers into service regions. We model this as a bipolar fuzzy directed 3-Super-HyperGraph.

$$S \ = \ \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}, \quad n = 3.$$

First-lYutakal clusters (server racks) in $\mathrm{POW}^1(S)$:

$$R_1 = \{s_1, s_2\}, \quad R_2 = \{s_3, s_4\}, \quad R_3 = \{s_5, s_6\}, \quad R_4 = \{s_7, s_8\}.$$

Second-lYutakal clusters (data centers) in $\mathrm{POW}^2(S)$:

$$\mathrm{DC}_A = \{R_1, R_2\}, \quad \mathrm{DC}_B = \{R_3, R_4\}.$$

Third-lYutakal clusters (service regions) in $\mathrm{POW}^3(S)$:

$$\mathrm{Region}_{\mathrm{East}} = \{\mathrm{DC}_A\}, \quad \mathrm{Region}_{\mathrm{West}} = \{\mathrm{DC}_B\}.$$

Thus the set of 3-supervertices is

$$V \ = \ \big\{\mathrm{Region}_{\mathrm{East}}, \mathrm{Region}_{\mathrm{West}}\big\} \ \subseteq\ \mathrm{POW}^3(S).$$

We represent the primary data-flow from East to West as a single directed 3-superedge:

$$e \ = \ \big(\mathrm{Region}_{\mathrm{East}}, \mathrm{Region}_{\mathrm{West}}\big), \quad E = \{\, e\} \ \subseteq\ \mathrm{POW}^3(S) \times \mathrm{POW}^3(S).$$

Assign bipolar fuzzy membership degrees (performance vs. risk):

$$\sigma^+(\mathrm{Region}_{\mathrm{East}}) = 0.92, \quad \sigma^-(\mathrm{Region}_{\mathrm{East}}) = -0.04,$$

$$\sigma^+(\text{Region}_{\text{West}}) = 0.88, \quad \sigma^-(\text{Region}_{\text{West}}) = -0.06,$$

$$\mu^+(e) = 0.85, \quad \mu^-(e) = -0.03.$$

We verify the bipolar fuzzy constraints:

$$\mu^+(e) = 0.85 \ \le \ \min\{0.92, 0.88\} = 0.88, \quad \mu^-(e) = -0.03 \ \ge \ \max\{-0.04, -0.06\} = -0.04,$$

and the supports of tail and head are disjoint:

$$\text{supp}^+(\text{Region}_{\text{East}}) = \{\text{DC}_A\}, \quad \text{supp}^+(\text{Region}_{\text{West}}) = \{\text{DC}_B\}, \quad \{\text{DC}_A\} \cap \{\text{DC}_B\} = \emptyset,$$

similarly for negative–membership supports. Hence

$$\big(V, E, \sigma^+, \sigma^-, \mu^+, \mu^-\big)$$

is a valid Bipolar Fuzzy Directed 3-Super-HyperGraph modeling the three-tier cloud-computing infrastructure.

Theorem 3.5 (Generalization of Classical Structures)**.** *The Bipolar Fuzzy Directed n-Super-HyperGraph* $\text{BFDnSupHyG}^{(n)}$ *simultaneously generalizes the following:*

(1) ***Directed*** n***-Super-HyperGraph:*** *if* $\sigma^+(v) = 1$*,* $\sigma^-(v) = 0$ *for all* $v \in V$*, and* $\mu^+(e) = 1$*,* $\mu^-(e) = 0$ *for all* $e \in E$*, then* $\text{BFDnSupHyG}^{(n)}$ *reduces to a crisp directed* n*-Super-HyperGraph.*

(2) ***Fuzzy Directed*** n***-Super-HyperGraph:*** *if* $\sigma^-(v) = 0$ *and* $\mu^-(e) = 0$*, then it reduces to a fuzzy directed* n*-Super-HyperGraph* $\big(V, E, \sigma^+, \mu^+\big)$*.*

(3) ***Bipolar Fuzzy Directed Hypergraph:*** *when* $n = 0$*, we recover the classical bipolar fuzzy directed hypergraph on* S*.*

Proof. We verify each specialization:

1. (*Crisp case*) Set $\sigma^+(v) = 1$, $\sigma^-(v) = 0$ for Yutakary $v \in V$, and $\mu^+(e) = 1$, $\mu^-(e) = 0$ for Yutakary $e \in E$. Then the membership constraints become $\mu^+(e) \le 1$ and $\mu^-(e) \ge 0$, which hold tautologically, and all vertices and edges are "fully present." Thus the structure is exactly a directed n-Super-HyperGraph $\big(V, E\big)$.
2. (*Fuzzy case*) Further impose $\sigma^-(v) = 0$, $\mu^-(e) = 0$. Then only the positive functions σ^+, μ^+ remain nontrivial, and the definition coincides with that of a fuzzy directed n-Super-HyperGraph $\big(V, E, \sigma^+, \mu^+\big)$ (Definition 2.10).
3. (*Hypergraph case*) Take $n = 0$. Then $\text{POW}^0(S) = S$, so $V \subseteq S$ and $E \subseteq S \times S$. The resulting sextuple $\big(V, E, \sigma^+, \sigma^-, \mu^+, \mu^-\big)$ is precisely the definition of a bipolar fuzzy directed hypergraph on S (see Definition *Bipolar Fuzzy Directed Hypergraph*).

In each case the defining inequalities reduce to those of the classical structure, showing that $\text{BFDnSupHyG}^{(n)}$ is a common generalization. □

Theorem 3.6 (Membership Interval Inclusion)**.** *Let* $\mathrm{BFDnSupHyG}^{(n)} = (V, E, \sigma^+, \sigma^-, \mu^+, \mu^-)$ *be a bipolar fuzzy directed n-Super-HyperGraph as in the Definition. Then for Yutakary directed n-superedge* $e = (T(e), H(e)) \in E$*,*

$$\big[\mu^-(e),\, \mu^+(e)\big] \;\subseteq\; \bigcap_{x \in T(e) \cup H(e)} \big[\sigma^-(x),\, \sigma^+(x)\big].$$

Proof. By Definition 3.1, we have

$$\mu^+(e) \;\le\; \min_{x \in T(e) \cup H(e)} \sigma^+(x) \quad \text{and} \quad \mu^-(e) \;\ge\; \max_{x \in T(e) \cup H(e)} \sigma^-(x).$$

Hence for each $x \in T(e) \cup H(e)$ the following two inequalities hold simultaneously:

$$\sigma^-(x) \;\le\; \mu^-(e) \quad \text{and} \quad \mu^+(e) \;\le\; \sigma^+(x).$$

It follows that

$$\mu^-(e) \;\ge\; \sigma^-(x) \quad \Longrightarrow \quad \mu^-(e) \in \big[\sigma^-(x), \sigma^+(x)\big],$$

and

$$\mu^+(e) \;\le\; \sigma^+(x) \quad \Longrightarrow \quad \mu^+(e) \in \big[\sigma^-(x), \sigma^+(x)\big].$$

Since this holds for all $x \in T(e) \cup H(e)$, we conclude $\mu^-(e)$ and $\mu^+(e)$ lie in the intersection of the vertex intervals, proving the claim. □

Theorem 3.7 (Induced Subhypergraph)**.** *Let* $\mathrm{BFDnSupHyG}^{(n)} = (V, E, \sigma^+, \sigma^-, \mu^+, \mu^-)$ *and let* $V' \subseteq V$ *be any nonempty subset of supervertices. Define*

$$E' \;=\; \big\{\, e \in E :\; T(e) \cup H(e) \;\subseteq\; V' \big\}.$$

Then $\mathrm{BFDnSupHyG}' = (V', E', \sigma^+|_{V'}, \sigma^-|_{V'}, \mu^+|_{E'}, \mu^-|_{E'})$ *is again a bipolar fuzzy directed n-Super-HyperGraph.*

Proof. We check each requirement of Definition 3.1 in the restricted structure:

- $V' \subseteq \mathrm{POW}^n(S)$ and $E' \subseteq \mathrm{POW}^n(S) \times \mathrm{POW}^n(S)$ by construction.
- $\sigma^+|_{V'}$ and $\sigma^-|_{V'}$ are well-defined maps $V' \to [0, 1]$ and $V' \to [-1, 0]$, respectively.
- $\mu^+|_{E'}$ and $\mu^-|_{E'}$ are well-defined maps $E' \to [0, 1]$ and $E' \to [-1, 0]$.
- For any $e \in E'$, since $e \in E$, the original inequalities $\mu^+(e) \le \min_{x \in T(e) \cup H(e)} \sigma^+(x)$ and $\mu^-(e) \ge \max_{x \in T(e) \cup H(e)} \sigma^-(x)$ hold. Restricting the domains does not affect their validity.
- Similarly, the disjointness of supports $\mathrm{supp}^+(T(e)) \cap \mathrm{supp}^+(H(e)) = \emptyset$ and $\mathrm{supp}^-(T(e)) \cap \mathrm{supp}^-(H(e)) = \emptyset$ continues to hold, as these conditions refer only to vertices in $T(e) \cup H(e) \subseteq V'$.

Thus all axioms are inherited by the induced substructure, proving it is a valid Bipolar Fuzzy Directed n-Super-HyperGraph. □

Theorem 3.8 (Edge Composition Closure). *Let* $\mathrm{BFDnSupHyG}^{(n)} = (V, E, \sigma^+, \sigma^-, \mu^+, \mu^-)$. *Suppose* $e_1, e_2 \in E$ *satisfy*

$$\mathrm{supp}^+(H(e_1)) \cap \mathrm{supp}^+(T(e_2)) = \emptyset, \quad \mathrm{supp}^-(H(e_1)) \cap \mathrm{supp}^-(T(e_2)) = \emptyset.$$

Define the composed edge

$$e_3 = \big(T(e_1),\, H(e_2)\big),$$

and assign new membership functions

$$\mu_C^+(e_3) = \min\{\mu^+(e_1),\, \mu^+(e_2)\}, \qquad \mu_C^-(e_3) = \max\{\mu^-(e_1),\, \mu^-(e_2)\},$$

while keeping $\sigma^\pm$ *unchanged on* V*. Then* $\big(V, E \cup \{e_3\}, \sigma^+, \sigma^-, \mu_C^+, \mu_C^-\big)$ *again satisfies all axioms of a Bipolar Fuzzy Directed* n*-Super-HyperGraph.*

Proof. We must verify the four defining properties for the new edge e_3:

(1) Tail and Head membership bounds. By hypothesis,

$$\mu^+(e_i) \le \min_{x \in T(e_i) \cup H(e_i)} \sigma^+(x), \quad \mu^-(e_i) \ge \max_{x \in T(e_i) \cup H(e_i)} \sigma^-(x), \quad i = 1, 2.$$

Hence

$$\mu_C^+(e_3) = \min\{\mu^+(e_1), \mu^+(e_2)\} \le \min\Big\{\min_{x \in T(e_1) \cup H(e_1)} \sigma^+(x), \min_{y \in T(e_2) \cup H(e_2)} \sigma^+(y)\Big\}.$$

Noting $T(e_3) = T(e_1)$ and $H(e_3) = H(e_2)$, we deduce

$$\mu_C^+(e_3) \le \min_{z \in T(e_3) \cup H(e_3)} \sigma^+(z).$$

An analogous argument using maxima shows

$$\mu_C^-(e_3) = \max\{\mu^-(e_1), \mu^-(e_2)\} \ge \max_{z \in T(e_3) \cup H(e_3)} \sigma^-(z).$$

(2) Disjointness of positive supports. Since $\mathrm{supp}^+(H(e_1)) \cap \mathrm{supp}^+(T(e_2)) = \emptyset$, and $T(e_3) = T(e_1)$, $H(e_3) = H(e_2)$, we have

$$\mathrm{supp}^+(T(e_3)) \cap \mathrm{supp}^+(H(e_3)) = \mathrm{supp}^+(T(e_1)) \cap \mathrm{supp}^+(H(e_2)) = \emptyset.$$

(3) Disjointness of negative supports. Similarly,

$$\mathrm{supp}^-(T(e_3)) \cap \mathrm{supp}^-(H(e_3)) = \mathrm{supp}^-(T(e_1)) \cap \mathrm{supp}^-(H(e_2)) = \emptyset.$$

(4) Domain and range. Obviously $e_3 \in \mathrm{POW}^n(S) \times \mathrm{POW}^n(S)$, so adding e_3 does not violate $E \subseteq \mathrm{POW}^n(S) \times \mathrm{POW}^n(S)$.

Thus all requirements remain satisfied upon adding e_3, proving the composition closure. □

Theorem 3.9 (Order–LYutakal Restriction)**.** *Let* $\mathrm{BFDnSupHyG}^{(n)} = (V_n, E_n, \sigma_n^+, \sigma_n^-, \mu_n^+, \mu_n^-)$. *For any integer* $0 \leq m \leq n$, *define*

$$V_m = V_n \cap \mathrm{POW}^m(S), \quad E_m = E_n \cap \big(\mathrm{POW}^m(S) \times \mathrm{POW}^m(S)\big),$$

with membership functions restricted accordingly: $\sigma_m^{\pm} = \sigma_n^{\pm} |_{V_m}$, $\mu_m^{\pm} = \mu_n^{\pm} |_{E_m}$. *Then* $\mathrm{BFDnSupHyG}^{(m)} = (V_m, E_m, \sigma_m^+, \sigma_m^-, \mu_m^+, \mu_m^-)$ *is a bipolar fuzzy directed* m*-Super-HyperGraph.*

Proof. The proof is analogous to that of Theorem 3.7. Since $\mathrm{POW}^m(S) \subseteq \mathrm{POW}^n(S)$, the restricted sets V_m and E_m lie in the correct domains. All membership inequalities and support-disjointness conditions are inherited verbatim by restriction, because they depend only on the values of $\sigma_n^{\pm}, \mu_n^{\pm}$ on members of V_m and E_m. Thus the axioms persist at lYutakal m. □

Theorem 3.10 (Fuzzy Specialization)**.** *If in* $\mathrm{BFDnSupHyG}^{(n)}$ *one has* $\sigma^-(v) = 0$ *for all* $v \in V$ *and* $\mu^-(e) = 0$ *for all* $e \in E$*, then the structure reduces to a fuzzy directed* n*-Super-HyperGraph* $\big(V, E, \sigma^+, \mu^+\big)$.

Proof. Under the assumption $\sigma^- \equiv 0$ and $\mu^- \equiv 0$, the negative-membership constraints become trivial: $\mu^-(e) \geq \max \sigma^-(x)$ reduces to $0 \geq 0$. The positive-membership constraints $\mu^+(e) \leq \min \sigma^+(x)$ remain exactly those of Definition 2.10. All disjointness conditions on supp^- are vacuous, and supp^+-disjointness is retained. Hence the structure is precisely a fuzzy directed n-Super-HyperGraph. □

Theorem 3.11 (Zero-order Specialization)**.** *If* $n = 0$ *in* $\mathrm{BFDnSupHyG}^{(n)} = (V, E, \sigma^+, \sigma^-, \mu^+, \mu^-)$, *then* $V \subseteq S$ *and* $E \subseteq S \times S$, *and the structure coincides with the classical bipolar fuzzy directed hypergraph on* S.

Proof. By definition $\mathrm{POW}^0(S) = S$. Thus all "supervertices" lie in S and all "superedges" in $S \times S$. The axioms of Definition 3.1 then specialize exactly to those of a bipolar fuzzy directed hypergraph (see Definition "Bipolar Fuzzy Directed Hypergraph"). No further verification is needed. □

4. Conclusion and Future Work

This paper has presented a new graph-based framework—*Bipolar Fuzzy Super-HyperGraphs*. For future work, we plan to extend these constructions by incorporating Neutrosophic Sets [43], hyperneutrosophic sets [44, 45], bipolar neutrosophic sets [46], and Plithogenic Sets [47]. We also aim to explore related generalizations based on bidirected graphs [48] and multidirected graphs [18].

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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Author has declared that no competing interests exist.

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