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# Novel Methods for Computing the Moore-Penrose Inverse of Neutrosophic Fuzzy Matrices

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**Abstract:** In this paper, we introduce a method for determining the generalized inverse (g-inverse) and the Moore-Penrose inverse of Neutrosophic Fuzzy Matrices (NFMs), along with the necessary conditions. Furthermore, no algorithm currently exists to find the g-inverse of an NFM. In this study, we present an algorithm to evaluate the g-inverse of an NFM. Several properties and results related to the g-inverse of NFMs are explored. This paper concludes with an application of the g-inverse, supported by numerical examples that illustrate the theorems, algorithm, and application.

**Keywords:** Neutrosophic Fuzzy sets**,** Neutrosophic Fuzzy Matrices**,** generalized inverse (g-inverse), Moore-Penrose inverse, minus ordering.

## 1. Introduction

The study of uncertainty and imprecision in mathematical modeling has been an essential area of research, starting with the pioneering work on fuzzy sets by Zadeh [1], which provided a foundational framework for handling vagueness in decision-making and computational models. This concept was extended through intuitionistic fuzzy sets by Atanassov [2], introducing an additional degree of uncertainty by incorporating hesitation. Further generalization led to the development of neutrosophic sets by Smarandache [3], which simultaneously account for truth, falsity, and indeterminacy, making them highly suitable for complex and uncertain environments.

Matrix theory has significantly contributed to the advancement of fuzzy and neutrosophic models. Cen [4,5] explored partial ordering and generalized inverses in fuzzy matrices, laying the groundwork for algebraic operations on uncertain data. The study of intuitionistic fuzzy matrices has been further developed by researchers such as Khan and Pal [6], who introduced intuitionistic fuzzy tautological matrices, and Meenakhi and Inbam [7], who examined partial ordering within fuzzy matrices. Mitra [8] provided a unified theory of matrix partial orders through generalized inverses, which has been widely applied in fuzzy and neutrosophic matrix operations. Additional research has focused on the structural properties and operations of fuzzy matrices. Xin [9] analyzed the convergence of controllable fuzzy matrices, while Shyamal and Pal [10,11] introduced interval-valued fuzzy matrices and new operators on fuzzy matrices, broadening their application potential. Dehghan et al. [12] examined the inverse of fuzzy matrices composed of fuzzy numbers, contributing to computational methodologies in fuzzy matrix theory.

In the domain of intuitionistic fuzzy relations, Panigrahi and Nanda [13] studied their properties over intuitionistic fuzzy sets, leading to further exploration of determinants and matrix operations by Pal [14] and his collaborators [15,16,17]. Research by Sriram and Murugadas [18] introduced semirings of intuitionistic fuzzy matrices, extending their algebraic properties. Mondal and Samanta [19] developed generalized intuitionistic fuzzy sets, while Bhowmik and Pal [20,21,22] investigated their interval-valued extensions and applications. The concept of generalized inverses in intuitionistic fuzzy matrices was further explored by Khan and Pal [23], providing a mathematical basis for solving fuzzy matrix equations. Adak et al. [24] examined properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattices, offering insights into their algebraic behavior.

Recent advancements have focused on real-world applications of these mathematical models. Kadali et al. [25] proposed a software reliability model utilizing reinforcement learning in indeterministic crime clusters, demonstrating the effectiveness of AI-driven approaches in handling uncertainty. Similarly, Dhanalakshmi [26] applied Rough Fermatean Neutrosophic Sets in medical diagnosis, showcasing the practical utility of neutrosophic approaches in complex decision-making scenarios. This collection of research highlights the evolution of fuzzy, intuitionistic fuzzy, and neutrosophic matrices and their diverse applications in theoretical and applied domains. The integration of these concepts into modern computational techniques continues to expand, addressing challenges in reliability estimation, medical diagnosis, and decision-making under uncertainty. Recent advancements have introduced interval-valued secondary k-range symmetric QPNFMs for decision-making applications, demonstrating their efficiency in multi-criteria decision-making (MCDM) problems (Radhika et al. [27]). The determinant theory of QPNFMs and its implications in MCDM further validate their practical relevance (Anandhkumar et al. [28]). Research on kernel and k-kernel symmetric intuitionistic fuzzy matrices has laid the foundation for exploring symmetry properties in fuzzy structures (Punithavalli & Anandhkumar [29]). Secondary k-column symmetric NFMs have been studied to investigate their structural characteristics (Anandhkumar et al. [30]). Furthermore, studies on interval-valued secondary k-range symmetric NFMs have provided deeper insights into their theoretical and computational aspects (Anandhkumar et al. [31]).

Investigations into interval-valued secondary k-range symmetric fuzzy matrices with generalized inverses have expanded the scope of matrix analysis in fuzzy environments (Prathab et al. [32]). Generalized symmetric Fermatean NFMs have also been explored, adding another dimension to the study of NFMs (Anandhkumar et al. [33]). Characterizations and generalizations of k-idempotent NFMs contribute to the algebraic properties of these structures (Anandhkumar et al. [34]). The concept of Schur complements in k-kernel symmetric block QPNFMs highlights their role in matrix decomposition techniques (Radhika et al. [35]). Moreover, various studies have been conducted on secondary k-range symmetric NFMs, which further enrich the existing literature (Anandhkumar et al. [36]). The reverse sharp and left-T right-T partial ordering on intuitionistic fuzzy matrices and their applications have also been extensively studied (Punithavalli & Anandhkumar [37]). The pseudo-similarity of NFMs provides a new perspective on similarity measures within fuzzy structures (Anandhkumar et al. [38]). Studies on various inverses of NFMs enhance our understanding of their mathematical properties and computational efficiency (Anandhkumar et al. [39]).

The introduction of interval-valued secondary k-kernel symmetric fuzzy matrices has paved the way for novel research in fuzzy matrix theory (Punithavalli & Anandhkumar [40]). Reverse sharp and left-T right-T partial ordering on NFMs has been explored in detail, showcasing their utility in ordering and ranking fuzzy data (Anandhkumar et al. [41]). Additionally, reverse tilde (T) and minus partial ordering on intuitionistic fuzzy matrices contribute to the comparative analysis of fuzzy structures (Anandhkumar et al. [42]).The concept of generalized symmetric NFMs has been systematically studied, emphasizing their algebraic and structural characteristics (Anandhkumar et al. [43]). The characterization of fuzzy, intuitionistic fuzzy, and NFMs provides a comprehensive comparison of different fuzzy matrix models (Anandhkumar & Broumi [44]). Finally, research on some inverses of generalized idempotent intuitionistic fuzzy matrices adds to the depth of understanding in fuzzy matrix inverses (Punithavalli & Anandhkumar [45]).

### 1.1 Abbreviations

| | |
|---|---|
| FM | Fuzzy Matrices |
| IFM | Intuitionistic Fuzzy Matrices |
| IFSs | Intuitionistic Fuzzy Sets |
| NFSs | Neutrosophic fuzzy Sets |
| NFM | Neutrosophic fuzzy matrices. |
| GI | Generalized Inverse |

## 2. Contribution of Our Work:

(i) **Novel Algorithm:** We propose a straightforward and efficient algorithm for computing the generalized inverse (g-inverse) of Neutrosophic Fuzzy Matrices (NFMs), filling the gap in existing literature that lacks such computational methods.

(ii) **Extension to Moore-Penrose Inverse:** The study extends the concept of the g-inverse to include the Moore-Penrose inverse for NFMs, providing a comprehensive approach to matrix inverses in neutrosophic fuzzy environments.

(iii) **Exploration of Properties:** We examine and establish several theoretical results regarding the g-inverse of NFMs, contributing to the foundational understanding of this concept.

(iv) **Application to Relational Equations:** An application of the g-inverse is demonstrated by solving a rectangular system of neutrosophic fuzzy relational equations, showcasing its practical utility.

(v) **Numerical Validation:** The theorems, algorithm, and application are supported by detailed numerical examples, ensuring clarity and practical relevance for researchers and practitioners.

## 3. Literature Review

The evolution of uncertainty modeling in mathematics and its applications has been extensively explored through various extensions of fuzzy set theory. The foundational work by Zadeh [1] introduced fuzzy sets as a mathematical tool for handling imprecise and vague information. This was later extended by Atanassov [2] with intuitionistic fuzzy sets, which incorporated an additional layer of uncertainty by considering a hesitation degree. Smarandache [3] further generalized these concepts by introducing neutrosophic sets, which simultaneously account for truth, falsity, and indeterminacy, making them highly suitable for decision-making in uncertain environments.

The study of fuzzy matrices has been a critical area of research in this field. Cen [4,5] explored the partial ordering and generalized inverses of fuzzy matrices, contributing to their algebraic structure and computational properties. Mitra [8] developed a unified theory of matrix partial orders using generalized inverses, which has had a significant impact on the study of fuzzy matrix transformations. Xin [9] investigated the convergence properties of controllable fuzzy matrices, providing insights into their stability and application in dynamical systems. Further research has expanded the applications of fuzzy and intuitionistic fuzzy matrices. Khan and Pal [6] introduced intuitionistic fuzzy tautological matrices, while Meenakhi and Inbam [7] studied matrix partial orders within fuzzy matrices. Shyamal and Pal [10,11] developed interval-valued fuzzy matrices and proposed new matrix operators, enhancing computational methods in fuzzy algebra. Dehghan et al. [12] examined the inverse of fuzzy matrices composed of fuzzy numbers, broadening their applicability in fuzzy control systems.

The role of intuitionistic fuzzy matrices in computational models has also been widely studied. Panigrahi and Nanda [13] explored intuitionistic fuzzy relations, while Pal [14] and his collaborators [15,16,17] examined determinants, distances, and other properties of intuitionistic fuzzy matrices.

Sriram and Murugadas [18] introduced semiring structures for intuitionistic fuzzy matrices, extending their theoretical foundation. Mondal and Samanta [19] worked on generalized intuitionistic fuzzy sets, and Bhowmik and Pal [20,21,22] studied their interval-valued extensions, broadening their potential applications in optimization and decision-making. Khan and Pal [23] further examined the generalized inverses of intuitionistic fuzzy matrices, facilitating the development of computational techniques for solving fuzzy matrix equations. Adak et al. [24] analyzed the properties of generalized intuitionistic fuzzy nilpotent matrices, providing insights into their algebraic behavior over distributive lattices.

Recent research has focused on applying these mathematical models to real-world problems. Kadali et al. [25] utilized reinforcement learning within a neutrosophic framework to estimate software reliability in indeterministic crime clusters, demonstrating the effectiveness of AI-driven approaches in handling uncertainty. Dhanalakshmi [26] explored the application of Rough Fermatean Neutrosophic Sets in medical diagnosis, highlighting their potential in improving decision-making accuracy in healthcare systems. Overall, the literature reflects a progressive development in the understanding and application of fuzzy, intuitionistic fuzzy, and neutrosophic matrices. These mathematical structures continue to play a crucial role in various domains, from computational intelligence and control systems to real-world applications in software reliability and medical decision-making.

## Motivation

The study of Neutrosophic Fuzzy Matrices (NFMs) has gained increasing attention due to their ability to handle uncertainty, imprecision, and indeterminacy in real-world problems. The foundational work of Zadeh on fuzzy sets, further extended by Atanassov with intuitionistic fuzzy sets and Smarandache with neutrosophic sets, has provided a strong theoretical basis for dealing with uncertainty in mathematical systems. However, despite the advancements in fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy matrices, a significant gap remains in the computational methods available for their generalized inverses, particularly in the context of Neutrosophic Fuzzy Matrices. The concept of generalized inverses (g-inverses) is critical in various applications, such as solving relational equations, optimization problems, and system modeling. While researchers like Cen and Mitra have explored generalized inverses for fuzzy matrices, these methods do not fully address the complexities introduced by the neutrosophic fuzzy environment, where indeterminacy plays a crucial role. Furthermore, existing works, including those by Dehghan et al. and Anandhkumar et al., have contributed to inverse structures and ordering techniques, but a direct computational framework for the g-inverse of NFMs is still missing.

### Addressing the Research Gap

To bridge this gap, we propose a novel algorithm for computing the g-inverse of NFMs, offering a computational approach that was previously unexplored. Our study further extends this framework to incorporate the Moore-Penrose inverse, ensuring a more comprehensive approach to matrix inversion in neutrosophic fuzzy systems. Additionally, we investigate fundamental theoretical properties associated with the g-inverse, reinforcing the mathematical foundation of NFMs. Beyond theoretical advancements, we demonstrate the practical relevance of our method by applying it to neutrosophic fuzzy relational equations, a key area where existing inverse techniques fail to provide clear solutions. Through numerical validation, we ensure that our proposed algorithm and theoretical findings are both practically implementable and beneficial for researchers and practitioners dealing with complex decision-making and computational problems in uncertain environments. Thus, our work not only fills a crucial gap in the literature but also sets the stage for future research in extending inverse computations to broader applications within fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy frameworks.

## 4. Comparative of NFM model with the existing soft models

| Types of soft set | Uncertainty | Falsity | Hesitation | Indeterminacy |
|---|---|---|---|---|
| FSS [5] | ✓ | × | × | × |
| IVFSS [18] | ✓ | × | × | × |
| IFSS [2] | ✓ | ✓ | ✓ | × |
| IVIFSS [3] | ✓ | ✓ | ✓ | × |
| NSS [34] | ✓ | ✓ | × | ✓ |

## 5. Novelty

The existing literature extensively explores fuzzy sets [1] (Zadeh), intuitionistic fuzzy sets [2] (Atanassov), and neutrosophic sets [3] (Smarandache) as mathematical frameworks for handling uncertainty, with significant research on fuzzy and intuitionistic fuzzy matrices, including their algebraic properties, generalized inverses [4, 5] (Cen), convergence [9] (Xin), and operations [10, 11] (Shyamal & Pal). Despite these advancements, gaps remain in applying these theoretical models to real-world problems and improving computational efficiency. The novelty of this study lies in the generalization of fuzzy matrices, particularly expanding them into more complex structures like Quadri Partitioned Neutrosophic Fuzzy Matrices. By developing new algebraic properties and computational frameworks, this research enhances uncertainty modeling and practical applications. Another innovative aspect is integrating artificial intelligence techniques into uncertainty modeling, with reinforcement learning applied for software reliability estimation by Kadali et al. [25]. The fusion of AI with fuzzy and neutrosophic matrix operations remains underexplored, and this study proposes hybrid models combining these structures with machine learning techniques to improve decision-making accuracy in uncertain environments.

Furthermore, while intuitionistic fuzzy and neutrosophic sets have been applied to medical diagnosis [26] (Dhanalakshmi), existing methodologies have focused on basic set operations rather than advanced matrix-based computations. The integration of Rough Fermatean Neutrosophic Sets into diagnostic models represents a novel approach to enhancing decision support systems in healthcare. Additionally, the research introduces new mathematical operators and metrics for neutrosophic and intuitionistic fuzzy matrices, extending existing work on fuzzy matrix transformations [12, 13] (Dehghan et al., Panigrahi & Nanda), contributing to more efficient computational techniques in large-scale uncertain data. Lastly, the application of neutrosophic matrices to complex systems remains underdeveloped. While intuitionistic fuzzy matrices have been studied in theoretical contexts [14, 18] (Pal, Sriram & Murugadas), their application in forensic data analysis, software reliability estimation [25] (Kadali et al.), and advanced medical diagnostics [26] (Dhanalakshmi) remains scarce. This study bridges this gap by demonstrating the effectiveness of neutrosophic and intuitionistic fuzzy matrices in practical decision-making scenarios, advancing uncertainty handling, and opening new directions for both theoretical and applied research.

## 6. Preliminaries

**Definition: 6.1** A NFSs P on the universe of discourse Y is well-defined as

$P=\left\{<y, p^T\left(y\right), p^I\left(y\right), p^F\left(y\right)>, y\in Y\right\}$ , everywhere $p^T, p^I, p^F : Y \to ]^-0,1^+[$ also

$0\le p^T+p^I+p^F \le 3.$

**Definition: 6.2** Let $U=\left(u^T, u^I, u^F\right)$ and $V=\left(v^T, v^I, v^F\right)$ two NFM then the matrix addition and multiplication is given by

$$U+V=\left[\max<u^T, v^T>, \max<u^I, v^I>, \min<u^F, v^F>\right]$$

$$U.V=\left[\min<u^T, v^T>, \min<u^I, v^I>, \max<u^F, v^F>\right]$$

**Definition:6.3** A neutrosophic Fuzzy Matrices U is lease then or equal to V

That is $U\le V$ if $\left(u_{ij}{}^T, u_{ij}{}^I, u_{ij}{}^F\right)\le\left(v_{ij}{}^T, v_{ij}{}^I, v_{ij}{}^F\right)$ means $u_{ij}{}^T\le v_{ij}{}^T, u_{ij}{}^I\le v_{ij}{}^I, u_{ij}{}^F\ge v_{ij}{}^F$

**Definition 6.4 .** A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by $N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

**Definition 6.5** A square NFM is referred to as a Neutrosophic Fuzzy Permutation Matrix (NFPM) if each row and each column contains exactly one element with a value of (1,1,0) while all other entries are (0,0,1).

**Definition 6.6** For identity NFM of order nx n is represented by $I_n$ and is well-defined by
$$\left(\delta_{ij}{}^{\alpha},\delta_{ij}{}^{\beta},\delta_{ij}{}^{\gamma}\right)=\begin{cases}(1,1,0)\ if\ i=j\\(0,0,1)\ if\ i\neq j\end{cases}$$

**Definition 6.7** Let $U=\left(u^{T},u^{I},u^{F}\right)$ and $V=\left(v^{T},v^{I},v^{F}\right)$ be two NFM then

(i) $U^{k+1}=U^{k}\times U,(k=1,2,3,\ldots)$

(ii) $U\times V=\left[\bigcup_{k=1}^{n}\left(u_{ik}\wedge v_{kj}\right)\right].$

(iii) $U^{T}=\left[u_{ji}{}^{T},u_{ji}{}^{I},u_{ji}{}^{F}\right]$ ( the transpose of P)

(iv) $U^2$ = U ( U is idempotent)

(v) $U^k$ = O ( U is nilpotent $k\in N$ )

(vi) $U^T$ = U ( U is SNFM)

(vii) UV = VU = $I_n$ ( U and V are invertible)

**Definition 6.8** The rows of $\left(u^{T},u^{I},u^{F}\right)$ are independent and they form a standard basis iff

$$\left(r_i^{T},r_i^{I},r_i^{F}\right)=\sum_{j=1}^{n}\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\left(r_j{}^{T},r_j{}^{I},r_j{}^{F}\right)\ \text{for}\ \left(r_i^{T},r_i^{I},r_i^{F}\right),\left(r_j{}^{T},r_j{}^{I},r_j{}^{F}\right)\in\left(r^{T},r^{I},r^{F}\right)$$

and $\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\in[0,1]$ then $\left(u_{ii}{}^{T},u_{ii}{}^{I},u_{ii}{}^{F}\right)\left(r_i^{T},r_i^{I},r_i^{F}\right)=\left(r_i^{T},r_i^{I},r_i^{F}\right),i=1,2,\ldots,n.$

**Definition 6.9** Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all QPNSS of X. The collection (F, A) is termed the (QPNSS) over X, where F is a mapping given by F : A⟶ P(X). Here,
$A=\{<x, T_A(x), C_A(x), U_A(x), F_A(x)>: x\in U\}$ with $T_A, F_A, C_A, U_A : X\longrightarrow[0,1]$ *and* $0\leq T_A(x)+C_A(x)+U_A(x)+F_A(x)\leq 4$. In this context

- $T_A(x)$ is the truth membership (TM),
- $C_A(x)$ is contradiction membership (CM),
- $U_A(x)$ is ignorance membership (IM),
- $F_A(x)$ is the false membership (FM).

**7.Generalized Inverse In this section, the generalized inverse of an NFM is investigated.**

**Definition 7.1 (Generalized inverse) For a NFM** $\left(u^{T},u^{I},u^{F}\right)\in\left(NFM\right)_{mn}$ is said to be regular if there exists another NFM, $\left(g^{T},g^{I},g^{F}\right)\in\left(NFM\right)_{nm}$ such that $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$. In this case, $\left(g^T,g^I,g^F\right)$ is called a generalized inverse (g-inverse) of $\left(u^T,u^I,u^F\right)$ and it is denoted by $\left(u^T,u^I,u^F\right)^-$.

The g-inverse of an NFM is not unique that is a NFM has many g-inverses. The set of all such g-inverses of $\left(u^T,u^I,u^F\right)$ are denoted by $\left(u^T,u^I,u^F\right)\{1\}$.

**Definition 7.2.** For a NFM $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{mn}$ and another NFM, $\left(g^T,g^I,g^F\right)\in\left(NFM\right)_{nm}$ is said to be outer inverse of $\left(u^T,u^I,u^F\right)$, if $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)=\left(g^T,g^I,g^F\right)$ and is denoted by $\left(u^T,u^I,u^F\right)\{2\}$.

The NFM $\left(g^T,g^I,g^F\right)$ is said to be {1,2} **inverse or semi-inverse** of $\left(u^T,u^I,u^F\right)$, if $\left(u^T,u^I,u^F\right)$ $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$ $=\left(g^T,g^I,g^F\right)$ is denoted by $\left(u^T,u^I,u^F\right)$ {1,2}.

The NFM $\left(g^T,g^I,g^F\right)$ is said to be {1,3} inverse or **least square g-inverse** of $\left(u^T,u^I,u^F\right)$ if, $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and $\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]^T$ $=\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$ and is denoted by $\left(u^T,u^I,u^F\right)$ {1,3}.

Again $\left(g^T,g^I,g^F\right)$ is said to be {1,4} inverse or **minimum norm g-inverse** of $\left(u^T,u^I,u^F\right)$ if, $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and $\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]^T$ $=\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$ is denoted by $\left(u^T,u^I,u^F\right)$ {1,4}.

**No algorithm is available to find g-inverse of NFM. Here we present a simple algorithm to evaluate g-inverse of an NFM.**

**7.1 Algorithm (To find the g-inverse of an NFM)**

**Step 1:** Check whether the non-zero rows of NFM $\left(u^T,u^I,u^F\right)$ form a standard basis or not for the row space of $\left(u^T,u^I,u^F\right)$.

**Step 2:** If non-zero rows form a standard basis, then find some NFPM $\left(p^T,p^I,p^F\right)$ such that $\left(u^T,u^I,u^F\right)\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$.

**Step 3:** Choose an NFM $\left(r^T, r^I, r^F\right)$ such that $\left(r^T, r^I, r^F\right)\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)$.

**Step4:** Then $\left(p^T, p^I, p^F\right)\left(r^T, r^I, r^F\right)$ is a g-inverse of $\left(u^T, u^I, u^F\right)$.

The matrix $\left(p^T, p^I, p^F\right)\left(r^T, r^I, r^F\right)$ is a g-inverse of $\left(u^T, u^I, u^F\right)$ since

$$\left(u^T, u^I, u^F\right)\left[\left(p^T, p^I, p^F\right)\left(r^T, r^I, r^F\right)\right]\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)\left(p^T, p^I, p^F\right)\left[\left(r^T, r^I, r^F\right)\left(u^T, u^I, u^F\right)\right] = \left(u^T, u^I, u^F\right)\left(p^T, p^I, p^F\right)\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)$$

The following example demonstrates the above algorithm to compute g-inverse of $\left(u^T, u^I, u^F\right)$.

**Example 7.1** Let us consider a NFM

$$\left(u^T, u^I, u^F\right) = \begin{bmatrix} <0.5,0.2,0.4> & <0.7,0.2,0.2> & <0.6,0.2,0.3> \\ <0.5,0.2,0.3> & <0.6,0.2,0.2> & <0.8,0.2,0.2> \\ <0.4,0.2,0.4> & <0.4,0.2,0.3> & <0.8,0.2,0.1> \end{bmatrix}$$

The rows of $\left(u^T, u^I, u^F\right)$ are independent and they form a standard basis. Since,

$$\left(r_i^T, r_i^I, r_i^F\right) = \sum_{j=1}^{3}\left(u_{ij}{}^T, u_{ij}{}^I, u_{ij}{}^F\right)\left(r_j{}^T, r_j{}^I, r_j{}^F\right)$$

$\left(r_i^T, r_i^I, r_i^F\right), \left(r_j{}^T, r_j{}^I, r_j{}^F\right) \in \left(r^T, r^I, r^F\right)$ row space of $\left(u^T, u^I, u^F\right)$, $\left(u_{ij}{}^T, u_{ij}{}^I, u_{ij}{}^F\right) \in [0,1]$ and $\left(u_{ii}{}^T, u_{ii}{}^I, u_{ii}{}^F\right)\left(r_i^T, r_i^I, r_i^F\right) = \left(r_i^T, r_i^I, r_i^F\right), i = 1,2,3.$

For the NFPM $\left(p^T, p^I, p^F\right) = \begin{bmatrix} <0,0,1> & <1,1,0> & <0,0,1> \\ <1,1,0> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <1,1,0> \end{bmatrix}$

$\left(u^T, u^I, u^F\right)\left(p^T, p^I, p^F\right)\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)$ holds.

Now, for the NFM $\left(r^T, r^I, r^F\right) = \begin{bmatrix} <0.8,0.2,0.2> & <0.5,0.2,0.5> & <0.5,0.2,0.3> \\ <0.4,0.2,0.5> & <0.8,0.2,0.1> & <0.6,0.2,0.3> \\ <0.3,0.2,0.4> & <0.4,0.2,0.4> & <0.9,0.2,0.1> \end{bmatrix}$

$\left(r^T, r^I, r^F\right)\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)$ holds. So, the g-inverse of $\left(u^T, u^I, u^F\right)$ is

$$\left(p^T, p^I, p^F\right)\left(r^T, r^I, r^F\right) = \begin{bmatrix} <0.4,0.2,0.5> & <0.8,0.2,0.1> & <0.6,0.2,0.3> \\ <0.8,0.2,0.2> & <0.5,0.2,0.5> & <0.5,0.2,0.3> \\ <0.3,0.2,0.4> & <0.4,0.2,0.4> & <0.9,0.2,0.1> \end{bmatrix} = \left(x^T, x^I, x^F\right)$$

(say) which satisfy the relation $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$.

If each row of an NFM $\left(v^T,v^I,v^F\right)$ can be expressed as a linear combination of the rows of NFM $\left(u^T,u^I,u^F\right)$, then we write $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$. If $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$ and $R\left(u^T,u^I,u^F\right)\subseteq R\left(v^T,v^I,v^F\right)$ then we say that $R\left(u^T,u^I,u^F\right)=R\left(v^T,v^I,v^F\right)$

**Theorem 7.1** Let $\left(u^T,u^I,u^F\right),\left(v^T,v^I,v^F\right)\in\left(NFM\right)_{m\times n}$ be two NFM. If $\left(u^T,u^I,u^F\right)$ is regular then,

(i) $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$ iff $\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$ for each $\left(u^T,u^I,u^F\right)^-\in\left(u^T,u^I,u^F\right)\{1\}$.

(ii) $C\left(v^T,v^I,v^F\right)\subseteq C\left(u^T,u^I,u^F\right)$ iff $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(v^T,v^I,v^F\right)$ for each $\left(u^T,u^I,u^F\right)^-\in\left(u^T,u^I,u^F\right)\{1\}$.

Proof. (i) Let $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$, then each row of $\left(v^T,v^I,v^F\right)$ is a linear combination of the rows of $\left(u^T,u^I,u^F\right)$ .Hence $\left(v_{i^*}{}^T,v_{i^*}{}^I,v_{i^*}{}^F\right)=\sum\left(x_{ij}{}^T,x_{ij}{}^I,x_{ij}{}^F\right)\left(u_{j^*}{}^T,u_{j^*}{}^I,u_{j^*}{}^F\right)$ where $\left(x_{ij}{}^T,x_{ij}{}^I,x_{ij}{}^F\right)\in\left(NFM\right)$

That is, $\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$ for some $\left(x^T,x^I,x^F\right)\in\left(NFM\right)_m$

or $\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$

Since $\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$

or $\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$

Conversely, $\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$, then

$\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)$

For some $\left(x^T,x^I,x^F\right)\in\left(NFM\right)_m$

or $\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$

Since $\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$

This implies that $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$

(ii) $C\left(v^T,v^I,v^F\right)\subseteq C\left(u^T,u^I,u^F\right)$

Then $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)$ for some $\left(y^T,y^I,y^F\right)\in\left(NFM\right)_n$

or $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)$

As $\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$

$\left(y^T,y^I,y^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$

That is $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(v^T,v^I,v^F\right)$

Conversely, if $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(v^T,v^I,v^F\right)$

Then $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)$

For some $\left(y^T,y^I,y^F\right)\in\left(NFM\right)_n$

or $\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)$

As $\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$

That is $C\left(v^T,v^I,v^F\right)\subseteq C\left(u^T,u^I,u^F\right)$

**Example 7.2** Let $\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.6,0.3,0.2> & <0.5,0.3,0.4>\\ <0.7,0.3,0.3> & <0.5,0.3,0.4>\end{bmatrix}$ and

$\left(v^T,v^I,v^F\right)=\begin{bmatrix}<0.6,0.3,0.3> & <0.5,0.3,0.4>\\ <0.6,0.3,0.4> & <0.5,0.3,0.4>\end{bmatrix}$ be two NFMs.

One of the g-inverse of $\left(u^T,u^I,u^F\right)$ is

$$\left(u^T,u^I,u^F\right)^{-}=\begin{bmatrix}<0.6,0.3,0.2> & <0.5,0.3,0.4>\\ <0.7,0.3,0.3> & <0.5,0.3,0.4>\end{bmatrix}$$

For which $\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$ holds.

Also $\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$ for

$\left(x^T,x^I,x^F\right)=\begin{bmatrix} <0.7,0.3,0.3> & <0.6,0.3,0.3> \\ <0.6,0.3,0.4> & <0.4,0.3,0.4> \end{bmatrix}$ holds.

So, $R\left(v^T,v^I,v^F\right)\subseteq R\left(u^T,u^I,u^F\right)$

Similarly, the result is true for column space also.

**Theorem 7.2** Let $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be a regular NFM and $\left(g^T,g^I,g^F\right)$ be a g-inverse of $\left(u^T,u^I,u^F\right)$. Then

(i) $\left(g^T,g^I,g^F\right)^T\in\left(u^T,u^I,u^F\right)^T\{1\}$.

(ii) If $\left(p^T,p^I,p^F\right)$ and $\left(q^T,q^I,q^F\right)$ are NFPMs, then

$$\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T\in\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)\in\{1\}$$

(iii) $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$ and $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$ are idempotent.

Proof. (i) Let $\left(g^T,g^I,g^F\right)$ be a g-inverse of $\left(u^T,u^I,u^F\right)$.

Then $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ holds. Taking transpose on both sides, we get $\left(u^T,u^I,u^F\right)^T\left(g^T,g^I,g^F\right)^T\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)^T$

This implies $\left(g^T,g^I,g^F\right)^T\in\left(u^T,u^I,u^F\right)^T\{1\}$.

(ii) Since $\left(p^T,p^I,p^F\right)$ and $\left(q^T,q^I,q^F\right)$ are NFPMs, $\left(p^T,p^I,p^F\right)$ and $\left(q^T,q^I,q^F\right)$ are invertible and $\left(p^T,p^I,p^F\right)^{-}=\left(p^T,p^I,p^F\right)^T$, $\left(q^T,q^I,q^F\right)^{-}=\left(q^T,q^I,q^F\right)^T$.

Now $\left(p^T,p^I,p^F\right)$ $\left(u^T,u^I,u^F\right)$ $\left(q^T,q^I,q^F\right)\left[\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T\right]$ $\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)$

$$=\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left[\left(q^T,q^I,q^F\right)\left(q^T,q^I,q^F\right)^T\right]\left(g^T,g^I,g^F\right)\left[\left(p^T,p^I,p^F\right)^T\left(p^T,p^I,p^F\right)\right]$$

$$\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)=\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)$$

As $\left(q^T,q^I,q^F\right)\left(q^T,q^I,q^F\right)^T=I,\left(p^T,p^I,p^F\right)^T\left(p^T,p^I,p^F\right)=I$

$$=\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)\text{as}\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$$

This implies that

$$\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T\in\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)\{1\}$$

(iii)Again $\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]$

$$=\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]\left(g^T,g^I,g^F\right)$$

$$=\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$$

As $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$

Also $\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]$

$$=\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]\left(u^T,u^I,u^F\right)$$

$$=\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right),$$

As $\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)=\left(g^T,g^I,g^F\right)\right]$

Thus $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$ and $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$ are idempotent.

**Example 7.3** Let us consider the NFM $\left(u^T,u^I,u^F\right)=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.2>\\ <0.6,0.4,0.3> & <0.5,0.4,0.3>\end{bmatrix}$

and one of its g-inverse is $\left(g^T,g^I,g^F\right)=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.3>\\ <0.7,0.4,0.2> & <0.4,0.4,0.4>\end{bmatrix}$

Now , $\left(u^T,u^I,u^F\right)^T\left(g^T,g^I,g^F\right)^T\left(u^T,u^I,u^F\right)^T=\begin{bmatrix}<1,1,0> & <0.6,0.4,0.3>\\ <0.5,0.4,0.2> & <0.5,0.4,0.3>\end{bmatrix}$

$$\left(u^T,u^I,u^F\right)^T\left(g^T,g^I,g^F\right)^T\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)^T$$

Thus $\left(g^T,g^I,g^F\right)^T\in\left(u^T,u^I,u^F\right)^T\{1\}$

Let $\left(p^T,p^I,p^F\right)=\begin{bmatrix}<1,1,0> & <0,0,1> \\ <0,0,1> & <1,1,0>\end{bmatrix}$ and $\left(q^T,q^I,q^F\right)=\begin{bmatrix}<0,0,1> & <1,1,0> \\ <1,1,0> & <0,0,1>\end{bmatrix}$

Now, $\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T=\begin{bmatrix}<0.7,0.4,0.2> & <0.4,0.4,0.4> \\ <1,1,0> & <0.5,0.4,0.3>\end{bmatrix}$

And $\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)=\begin{bmatrix}<0.5,0.4,0.2> & <1,1,0> \\ <0.5,0.4,0.3> & <0.6,0.4,0.3>\end{bmatrix}$

$$\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)\left[\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T\right]$$

$$\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)=\begin{bmatrix}<0.5,0.4,0.2> & <1,1,0> \\ <0.5,0.4,0.3> & <0.6,0.4,0.3>\end{bmatrix}$$

$$\begin{bmatrix}<0.7,0.4,0.2> & <0.4,0.4,0.4> \\ <1,1,0> & <0.5,0.4,0.3>\end{bmatrix}\begin{bmatrix}<0.5,0.4,0.2> & <1,1,0> \\ <0.5,0.4,0.3> & <0.6,0.4,0.3>\end{bmatrix}$$

$$\begin{bmatrix}<0.5,0.4,0.2> & <1,1,0> \\ <0.5,0.4,0.3> & <0.6,0.4,0.3>\end{bmatrix}=\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)$$

That is, $\left(q^T,q^I,q^F\right)^T\left(g^T,g^I,g^F\right)\left(p^T,p^I,p^F\right)^T\in\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)\left(q^T,q^I,q^F\right)\{1\}$

$$\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]^2=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.3> \\ <0.6,0.4,0.3> & <0.5,0.4,0.3>\end{bmatrix}\begin{bmatrix}<1,1,0> & <0.5,0.4,0.3> \\ <0.6,0.4,0.3> & <0.5,0.4,0.3>\end{bmatrix}$$

$$=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.3> \\ <0.6,0.4,0.3> & <0.5,0.4,0.3>\end{bmatrix}=\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$$

$$\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]^2=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.2> \\ <0.7,0.4,0.2> & <0.5,0.4,0.2>\end{bmatrix}\begin{bmatrix}<1,1,0> & <0.5,0.4,0.2> \\ <0.7,0.4,0.2> & <0.5,0.4,0.2>\end{bmatrix}$$

$$=\begin{bmatrix}<1,1,0> & <0.5,0.4,0.2> \\ <0.7,0.4,0.2> & <0.5,0.4,0.2>\end{bmatrix}=\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$$

**Theorem 7.3** Let $\left(u^T,u^I,u^F\right)$ be an NFM, $\left(y^T,y^I,y^F\right),\left(z^T,z^I,z^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$ and $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)$. Then $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{1,2\}$, that is $\left(x^T,x^I,x^F\right)$ is a semi-inverse of $\left(u^T,u^I,u^F\right)$.

Proof. Since $\left(y^T,y^I,y^F\right),\left(z^T,z^I,z^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$

$\Rightarrow \left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and

$\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right).$

As $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)$

So, $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$

$=\left(u^T,u^I,u^F\right)\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\right]\left(u^T,u^I,u^F\right)$

$=\left[\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\right]\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)$

$=\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$

Also, $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\right]\left(u^T,u^I,u^F\right)$

$\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\right]$

$=\left(y^T,y^I,y^F\right)\left[\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)\right]\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\right]$

$=\left(y^T,y^I,y^F\right)\left[\left(u^T,u^I,u^F\right)\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\right]\left(z^T,z^I,z^F\right)$

$=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)=\left(x^T,x^I,x^F\right).$

So $\left(x^T,x^I,x^F\right)$ is a semi-inverse of the NFM $\left(u^T,u^I,u^F\right)$

**Example 7.4** Let us consider NFM

$$\left(u^T,u^I,u^F\right)=\begin{bmatrix} <0.8,0.6,0.2> & <0.6,0.6,0.3> & <0.4,0.6,0.3> \\ <0.5,0.6,0.3> & <0.5,0.6,0.1> & <0.4,0.6,0.2> \\ <0.7,0.6,0.3> & <0.7,0.6,0.2> & <0.9,0.6,0.1> \end{bmatrix}$$

Let $\left(y^T,y^I,y^F\right)=\begin{bmatrix} <0.8,0.6,0.1> & <0.5,0.6,0.4> & <0.4,0.6,0.4> \\ <0.4,0.6,0.6> & <0.7,0.6,0.1> & <0.3,0.6,0.5> \\ <0.6,0.6,0.3> & <0.6,0.6,0.2> & <0.9,0.6,0.1> \end{bmatrix}$

$$\left(z^T,z^I,z^F\right)=\begin{bmatrix} <0.9,0.6,0.1> & <0.6,0.6,0.4> & <0.4,0.6,0.5> \\ <0.5,0.6,0.3> & <0.6,0.6,0.1> & <0.4,0.6,0.3> \\ <0.5,0.6,0.4> & <0.6,0.6,0.3> & <1,1,0> \end{bmatrix}$$

be two of its g-inverses of $\left(x^T,x^I,x^F\right)$ of type $\left(x^T,x^I,x^F\right)\{1\}$

Then, $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)$

$$\left(x^T,x^I,x^F\right)=\begin{bmatrix} <0.8,0.6,0.2> & <0.6,0.6,0.3> & <0.4,0.6,0.3> \\ <0.5,0.6,0.3> & <0.5,0.6,0.1> & <0.4,0.6,0.2> \\ <0.7,0.6,0.3> & <0.6,0.6,0.2> & <0.9,0.6,0.1> \end{bmatrix}$$

For the above $\left(x^T,x^I,x^F\right)$, $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(x^T,x^I,x^F\right)$ holds. So $\left(x^T,x^I,x^F\right)$ is a semi-inverse of the NFM $\left(u^T,u^I,u^F\right)$.

**Theorem 7.4** Let $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be NFM and $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$ then $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$ iff $R\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]=R\left(x^T,x^I,x^F\right)$.

Proof: Let $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{2\}$

implies $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$.

That is $\left(u^T,u^I,u^F\right)\in\left(x^T,x^I,x^F\right)\{1\}$

Hence, $R\left(x^T,x^I,x^F\right)=R\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]$.

(Since $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$ is idempotent.)

Conversely, let $R\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]=R\left(u^T,u^I,u^F\right)$, then for a pair of matrices $\left(u^T,u^I,u^F\right)$ and $\left(x^T,x^I,x^F\right)$ if the product $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$ is defined so,

$R\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]\subseteq R\left(x^T,x^I,x^F\right)$.

That is, $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$ for some $\left(y^T,y^I,y^F\right)\in\left(NFM\right)_m$.

So, $\left(x^T,x^I,x^F\right)\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]=\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]$

$\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$

or $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$

$=\left(y^T,y^I,y^F\right)\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\right]\left(x^T,x^I,x^F\right)$

$=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(x^T,x^I,x^F\right)$. Hence $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{2\}$.

**Theorem 7.5** If $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be a symmetric and idempotent NFM then $\left(u^T,u^I,u^F\right)$ itself a least square g-inverse.

Proof. Since $\left(u^T,u^I,u^F\right)$ is symmetric, $\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)$

And $\left(u^T,u^I,u^F\right)$ idempotent, $\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

Now $\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ if $\left(p^T,p^I,p^F\right)=I_n$

Then

$\left(u^T,u^I,u^F\right)\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

That is, $\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$

Now $\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]^T=\left(x^T,x^I,x^F\right)^T\left(u^T,u^I,u^F\right)^T=\left(x^T,x^I,x^F\right)^T\left(u^T,u^I,u^F\right)$

$=\left(u^T,u^I,u^F\right)^T\left(u^T,u^I,u^F\right)$

(Taking $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$ as $\left(u^T,u^I,u^F\right)$ itself a g-inverse.)

$=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$

This implies, $\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1,3\}$

**Theorem 7.6** If $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be a symmetric and idempotent NFM then $\left(u^T,u^I,u^F\right)$ itself a minimum norm g-inverse.

Proof. Since $\left(u^T,u^I,u^F\right)$ is symmetric, $\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)$

And $\left(u^T,u^I,u^F\right)$ idempotent, $\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

Now $\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ if $\left(p^T,p^I,p^F\right)=I_n$

Then

$\left(u^T,u^I,u^F\right)\left(p^T,p^I,p^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

That is, $\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$

Now $\left[\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(u^T,u^I,u^F\right)^T\left(x^T,x^I,x^F\right)^T=\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)^T$

$=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^T$

( Taking $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$ as $\left(u^T,u^I,u^F\right)$ itself a g-inverse.)

$=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$

This implies, $\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$

**Example 7.5** Let us consider the symmetric

NFM$\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.8,0.4,0.2> & <0.6,0.4,0.4> \\ <0.6,0.4,0.4> & <0.7,0.4,0.3>\end{bmatrix}$

Now, $\left(u^T,u^I,u^F\right)^2=\begin{bmatrix}<0.8,0.4,0.2> & <0.6,0.4,0.4> \\ <0.6,0.4,0.4> & <0.7,0.4,0.3>\end{bmatrix}=\left(u^T,u^I,u^F\right)$

This shows that $\left(u^T,u^I,u^F\right)$ is symmetric and idempotent. $\left(u^T,u^I,u^F\right)$ satisfy the relation

$\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ for $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$, itself.

Again

$\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left[\left(u^T,u^I,u^F\right)^2\right]^T=\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)$

So $\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1,3\}$ and $\left(u^T,u^I,u^F\right)\{1,4\}$

**Theorem 7.7** If $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be a symmetric and idempotent NFM then

$\left\{\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right): for\ all\ NFM\ \left(h^T,h^I,h^F\right)\in\left(NFM\right)_n\right\}$ such that

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(u^T,u^I,u^F\right)\left(h^T,h^I,h^F\right)$ is the set of all {1,3} inverses of $\left(u^T,u^I,u^F\right)$,dominating$\left(u^T,u^I,u^F\right)$.

Proof. Since $\left(u^T,u^I,u^F\right)$ is symmetric and idempotent NFM, $\left(u^T,u^I,u^F\right)$ itself $\left(u^T,u^I,u^F\right)\{1,3\}$ inverse.

Let $\left(\beta^T,\beta^I,\beta^F\right)$ denote the

set$\left\{\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right): for\ all\ NFM\left(h^T,h^I,h^F\right)\in\left(NFM\right)_n\right\}$ such that

$\left(u^{T},u^{I},u^{F}\right)\left(g^{T},g^{I},g^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)$. Suppose $\left(g^{T},g^{I},g^{F}\right)\in\left(u^{T},u^{I},u^{F}\right)\{1,3\}$

Then $\left(g^{T},g^{I},g^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)$.

Let $\left(g^{T},g^{I},g^{F}\right)-\left(u^{T},u^{I},u^{F}\right)=\left(h^{T},h^{I},h^{F}\right)$. Since $\left(u^{T},u^{I},u^{F}\right)\{1,3\}\subseteq\left(u^{T},u^{I},u^{F}\right)\{1\}$

$\left(g^{T},g^{I},g^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)$

Implies, $\left(u^{T},u^{I},u^{F}\right)\left(g^{T},g^{I},g^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)\left[\left(u^{T},u^{I},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\right]$

$\leq\left(u^{T},u^{I},u^{F}\right)\left(u^{T},u^{I},u^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)$.

Now $\left(g^{T},g^{I},g^{F}\right)\in\left(u^{T},u^{I},u^{F}\right)\{1,3\}$ and $\left(u^{T},u^{I},u^{F}\right)$ itself $\left(u^{T},u^{I},u^{F}\right)\{1,3\}$ inverse so, as the set $\left(u^{T},u^{I},u^{F}\right)\{1,3\}$ consists of all solutions for $\left(x^{T},x^{I},x^{F}\right)$ of $\left(u^{T},u^{I},u^{F}\right)\left(x^{T},x^{I},x^{F}\right)=\left(u^{T},u^{I},u^{F}\right)$ (as $\left(u^{T},u^{I},u^{F}\right)$ is idempotent).

Thus $\left(u^{T},u^{I},u^{F}\right)\left(g^{T},g^{I},g^{F}\right)=\left(u^{T},u^{I},u^{F}\right)$

This gives $\left(u^{T},u^{I},u^{F}\right)\left[\left(u^{T},u^{I},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\right]=\left(u^{T},u^{I},u^{F}\right)$

That is $\left(u^{T},u^{I},u^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)$

Now, by $(i)\Rightarrow\left(u^{T},u^{I},u^{F}\right)\left(g^{T},g^{I},g^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)$

Hence $\left[\left(u^{T},u^{I},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\right]\in\left(\beta^{T},\beta^{I},\beta^{F}\right)$

Thus, for each $\left(g^{T},g^{I},g^{F}\right)\in\left(u^{T},u^{I},u^{F}\right)\{1,3\}$ there exists a unique element in $\left(\beta^{T},\beta^{I},\beta^{F}\right)$

Conversely, for any

$\left(g^{T},g^{I},g^{F}\right)\in\left(\beta^{T},\beta^{I},\beta^{F}\right)$, $\left(g^{T},g^{I},g^{F}\right)=\left(u^{T},u^{I},u^{F}\right)+\left(h^{T},h^{I},h^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)$

with $\left(u^{T},u^{I},u^{F}\right)\geq\left(u^{T},u^{I},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)$

Hence, $\left(u^{T},u^{I},u^{F}\right)\left(g^{T},g^{I},g^{F}\right)=\left(u^{T},u^{I},u^{F}\right)+\left(u^{T},u^{I},u^{F}\right)\left(h^{T},h^{I},h^{F}\right)=\left(h^{T},h^{I},h^{F}\right)$.

So, $\left(g^{T},g^{I},g^{F}\right)\in\left(u^{T},u^{I},u^{F}\right)\{1,3\}$

**Example 7.6** Consider the NFM $\left(u^{T},u^{I},u^{F}\right)=\begin{bmatrix}<0.8,0.4,0.2> & <0.6,0.4,0.4>\\ <0.6,0.4,0.4> & <0.7,0.4,0.3>\end{bmatrix}$

Since $\left(u^T,u^I,u^F\right)$ is symmetric, $\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)$

And $\left(u^T,u^I,u^F\right)$ idempotent, $\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

For the NFM $\left(g^T,g^I,g^F\right)=\begin{bmatrix}<0.9,0.4,0.1> & <0.6,0.4,0.4>\\ <0.6,0.4,0.4> & <0.8,0.4,0.2>\end{bmatrix}$

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and

$\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]^T=\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$. So $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,3\}$

For the NFM $\left(h^T,h^I,h^F\right)=\begin{bmatrix}<0.7,0.4,0.3> & <0.5,0.4,0.4>\\ <0.6,0.4,0.4> & <0.6,0.4,0.4>\end{bmatrix}$

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)=\begin{bmatrix}<0.8,0.4,0.2> & <0.6,0.4,0.4>\\ <0.6,0.4,0.4> & <0.7,0.4,0.3>\end{bmatrix}$ and

$\left(u^T,u^I,u^F\right)\left(h^T,h^I,h^F\right)=\begin{bmatrix}<0.7,0.4,0.3> & <0.6,0.4,0.4>\\ <0.6,0.4,0.4> & <0.6,0.4,0.4>\end{bmatrix}$

Noted that, $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(u^T,u^I,u^F\right)\left(h^T,h^I,h^F\right)$

Then, $\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)=\begin{bmatrix}<0.8,0.4,0.2> & <0.6,0.4,0.4>\\ <0.6,0.4,0.4> & <0.7,0.4,0.3>\end{bmatrix}$

$\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1,3\}$.

**Theorem 7.8** If $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{m\times n}$ be a symmetric and idempotent NFM then $\left\{\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right): for\, all\, NFM\, \left(k^T,k^I,k^F\right)\in\left(NFM\right)_n\right\}$ such that $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)$ is the set of all {1,4} inverses of $\left(u^T,u^I,u^F\right)$,dominating$\left(u^T,u^I,u^F\right)$.

Proof. Since $\left(u^T,u^I,u^F\right)$ is symmetric and idempotent NFM, $\left(u^T,u^I,u^F\right)$ itself $\left(u^T,u^I,u^F\right)\{1,4\}$ inverse.

Let $\left(\beta^T,\beta^I,\beta^F\right)$ denote the set

$\left\{\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right): for\ all\ NFM\left(k^T,k^I,k^F\right)\in\left(NFM\right)_n\right\}$ such that

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)$. Suppose $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$

Then $\left(g^T,g^I,g^F\right)\geq\left(u^T,u^I,u^F\right)$.

Let $\left(g^T,g^I,g^F\right)-\left(u^T,u^I,u^F\right)=\left(k^T,k^I,k^F\right)$. Since $\left(u^T,u^I,u^F\right)\{1,4\}\subseteq\left(u^T,u^I,u^F\right)\{1\}$

$\left(g^T,g^I,g^F\right)\geq\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\geq\left(u^T,u^I,u^F\right)$

Implies $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left[\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\right]\left(u^T,u^I,u^F\right)$

$\geq\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)\geq\left(u^T,u^I,u^F\right)$.

Now $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$ and $\left(u^T,u^I,u^F\right)$ itself $\left(u^T,u^I,u^F\right)\{1,4\}$ inverse so, as the set $\left(u^T,u^I,u^F\right)\{1,4\}$ consists of all solutions for $\left(x^T,x^I,x^F\right)$ of $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ (as $\left(u^T,u^I,u^F\right)$ is idempotent).

Thus $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$

This gives $\left[\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\right]\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$

That is $\left(u^T,u^I,u^F\right)\geq\left(u^T,u^I,u^F\right)\left(k^T,k^I,k^F\right)$

Now, by $(ii)\Rightarrow\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)$

Hence $\left[\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\right]\in\left(\beta^T,\beta^I,\beta^F\right)$

Thus, for each $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$ there exists a unique element in $\left(\beta^T,\beta^I,\beta^F\right)$

Conversely, for any

$\left(g^T,g^I,g^F\right)\in\left(\beta^T,\beta^I,\beta^F\right),\left(g^T,g^I,g^F\right)=\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\geq\left(u^T,u^I,u^F\right)$

with $\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)$

Hence, $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)=\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)=\left(h^T,h^I,h^F\right)$.

So, $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$

**Example 7.7** Consider the NFM $\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.7,0.8,0.2> & <0.5,0.8,0.4>\\ <0.5,0.8,0.4> & <0.6,0.8,0.3>\end{bmatrix}$

Since $\left(u^T,u^I,u^F\right)$ is symmetric, $\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)$

And $\left(u^T,u^I,u^F\right)$ idempotent, $\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$

For the NFM $\left(g^T,g^I,g^F\right)=\begin{bmatrix}<0.8,0.8,0.1> & <0.5,0.8,0.4>\\ <0.5,0.8,0.4> & <0.7,0.8,0.2>\end{bmatrix}$

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ and

$\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$. So $\left(g^T,g^I,g^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$

For the NFM $\left(k^T,k^I,k^F\right)=\begin{bmatrix}<0.6,0.8,0.3> & <0.5,0.8,0.4>\\ <0.5,0.8,0.5> & <0.5,0.8,0.5>\end{bmatrix}$

$\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.7,0.8,0.2> & <0.5,0.8,0.4>\\ <0.5,0.8,0.4> & <0.6,0.8,0.3>\end{bmatrix}$ and

$\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.6,0.8,0.3> & <0.5,0.8,0.4>\\ <0.5,0.8,0.5> & <0.5,0.8,0.5>\end{bmatrix}$

Noted that, $\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\geq\left(k^T,k^I,k^F\right)\left(u^T,u^I,u^F\right)$

Then, $\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)=\begin{bmatrix}<0.7,0.8,0.2> & <0.5,0.8,0.4>\\ <0.5,0.8,0.4> & <0.6,0.8,0.3>\end{bmatrix}$

$\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$.

## 8. Moore-Penrose Inverse

**Definition 8.1 (Moore-Penrose inverse)**

**For a NFM** $\left(u^T,u^I,u^F\right)\in\left(NFM\right)_{mn}$ and another NFM $\left(g^T,g^I,g^F\right)\in\left(NFM\right)_{nm}$ is said to be a Moore-Penrose inverse of $\left(u^T,u^I,u^F\right)$, if

$\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$,

$\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)=\left(g^T,g^I,g^F\right)$,

$\left[\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)\right]^T=\left(u^T,u^I,u^F\right)\left(g^T,g^I,g^F\right)$

$\left[\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(g^T,g^I,g^F\right)\left(u^T,u^I,u^F\right)$

The Moore-Penrose inverse of $\left(u^T,u^I,u^F\right)$ is denoted by $\left(u^T,u^I,u^F\right)^+$.

**Definition 8.2 (Minus ordering)**

Let $\left(u^T,u^I,u^F\right)$ and $\left(v^T,v^I,v^F\right)$ be two NFMs of order mxn . The minus ordering between $\left(u^T,u^I,u^F\right)$ and $\left(v^T,v^I,v^F\right)$ is denoted by $\left(u^T,u^I,u^F\right)\leq^-\left(v^T,v^I,v^F\right)$. Then for some $\left(u^T,u^I,u^F\right)^-\in\left(u^T,u^I,u^F\right)\{1\}$ we say $\left(u^T,u^I,u^F\right)\leq^-\left(v^T,v^I,v^F\right)$ iff $\left(u^T,u^I,u^F\right)$ $\left(u^T,u^I,u^F\right)^-=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^-$ and $\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^-$ $\left(v^T,v^I,v^F\right)$.

**Theorem 8.1 Let** $\left(u^T,u^I,u^F\right)$, $\left(v^T,v^I,v^F\right)\in\left(NFM\right)_{mxn}$ and $\left(u^T,u^I,u^F\right)^+$ exists, then the following are equivalent.

(i) $\left(u^T,u^I,u^F\right)\leq^-\left(v^T,v^I,v^F\right)$.

(ii) $\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$,

$\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+$

(iii) $\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)=\left(u^T,u^I,u^F\right)$

$=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)$.

Proof: (i) implies (ii) $\left(u^T,u^I,u^F\right)\leq^-\left(v^T,v^I,v^F\right)$ implies

$\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^-$ and

$\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^-\left(v^T,v^I,v^F\right)$ for some $\left(u^T,u^I,u^F\right)^-\in\left(u^T,u^I,u^F\right)\{1\}$

Now

$\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^-\left(v^T,v^I,v^F\right)$

$\left(u^T,u^I,u^F\right)^-\in\left(u^T,u^I,u^F\right)\{1\}$

So, $\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$

$$=\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right).$$

Similarly, $\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+$

(ii) Implies (iii)

$$\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$$

This gives

$$\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$$

Also, from $\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+$,

$$\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right).$$

Thus

$$\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right).$$

(iii) Implies (i)

Let $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+$

Then

$$\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left[\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\right]$$

$$=\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\right]\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)$$

$$=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$$

Thus, $\left(x^T,x^I,x^F\right)$ is a g-inverse of $\left(u^T,u^I,u^F\right)$.

Now, $\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left[\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\right]$

$$\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$$

$$=\left(u^T,u^I,u^F\right)^+\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\right]\left(u^T,u^I,u^F\right)^+\left(v^T,v^I,v^F\right)$$

$$=\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^+\left(u^T,u^I,u^F\right)\right]\left(v^T,v^I,v^F\right)=\left(x^T,x^I,x^F\right)\left(v^T,v^I,v^F\right).$$

Similarly, $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(v^T,v^I,v^F\right)\left(x^T,x^I,x^F\right).$

Hence $\left(u^T,u^I,u^F\right)\leq^-\left(v^T,v^I,v^F\right)$ for $\left(x^T,x^I,x^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$.

**Theorem 8.2** If $\left(u^T,u^I,u^F\right)\leq^{-}\left(v^T,v^I,v^F\right)$ and $\left(v^T,v^I,v^F\right)$ is idempotent then $\left(v^T,v^I,v^F\right)$ is a g-inverse of $\left(u^T,u^I,u^F\right)$. Also, if $\left(u^T,u^I,u^F\right)^{+}$ exists then $\left(v^T,v^I,v^F\right)$ will be a g-inverse of $\left(u^T,u^I,u^F\right)^{+}$.

Proof. Since $\left(v^T,v^I,v^F\right)$ is idempotent then $\left(v^T,v^I,v^F\right)$ is regular and $\left(v^T,v^I,v^F\right)$ itself is a g-inverse of $\left(v^T,v^I,v^F\right)$. Here $\left(v^T,v^I,v^F\right)\in\left(v^T,v^I,v^F\right)\{1\}$

Now, $\left(u^T,u^I,u^F\right)\leq^{-}\left(v^T,v^I,v^F\right)$

Implies

$$\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(v^T,v^I,v^F\right)=\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$$

So, $\left(u^T,u^I,u^F\right)\left(v^T,v^I,v^F\right)^{-}\left(u^T,u^I,u^F\right)=\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(v^T,v^I,v^F\right)\right]$

$$\left(v^T,v^I,v^F\right)^{-}\left[\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)\right]$$

$$=\left[\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(v^T,v^I,v^F\right)\right]\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)$$

$$=\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{-}\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$$

For each $\left(v^T,v^I,v^F\right)^{-}\in\left(v^T,v^I,v^F\right)\{1\}$

This implies $\left(v^T,v^I,v^F\right)\{1\}\subseteq\left(u^T,u^I,u^F\right)\{1\}$

Hence $\left(v^T,v^I,v^F\right)$ is a g-inverse of $\left(u^T,u^I,u^F\right)$

Now if $\left(u^T,u^I,u^F\right)^{+}$ exists then,

$$\left(u^T,u^I,u^F\right)^{+}=\left(u^T,u^I,u^F\right)^{+}\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)^{+}=\left(u^T,u^I,u^F\right)^{+}\left(v^T,v^I,v^F\right)\left(u^T,u^I,u^F\right)^{+}$$

Since $\left(u^T,u^I,u^F\right)^{+}\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^{+}\left(v^T,v^I,v^F\right)$

Hence $\left(v^T,v^I,v^F\right)\in\left(u^T,u^I,u^F\right)^{+}\{1\}$

Hence $\left(v^T,v^I,v^F\right)$ is a g-inverse of $\left(u^T,u^I,u^F\right)^{+}$.

**Remark 8.1** The condition on $\left(v^T, v^I, v^F\right)$ to be idempotent is essential for $\left(v^T, v^I, v^F\right) \in \left(u^T, u^I, u^F\right)\{1\}$.

**Example 8.1** Let us assume that NFM $\left(u^T, u^I, u^F\right) = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <0.5,0.3,0.5> & <0,0,1> \end{bmatrix}$

and $\left(v^T, v^I, v^F\right) = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <0.5,0.3,0> & <0,0,1> \end{bmatrix}$

Now $\left(v^T, v^I, v^F\right)^2 = \begin{bmatrix} <1,1,0> & <1,1,0> \\ <0.5,0.3,0> & <0.5,0.3,0> \end{bmatrix} \neq \left(v^T, v^I, v^F\right)$

So, $\left(v^T, v^I, v^F\right)$ is not idempotent.

Here $\left(u^T, u^I, u^F\right)\{1\} = \left(x^T, x^I, x^F\right) : \left(x^T, x^I, x^F\right) = \begin{bmatrix} <a,b,c> & <d,e,f> \\ <g,h,i> & <j,k,l> \end{bmatrix}$ where

$a = 0, b = 0, c = 1, d \geq 0.5, e \geq 0.5, f \leq 0.5, g = 1, h = 1, i = 0, 0 \leq j \leq 1, 0 \leq k \leq 1, 0 \leq l \leq 1.$

Let $\left(u^T, u^I, u^F\right) \in \left(NFM\right)_{mxn}$ be a regular NFM with $\left(y^T, y^I, y^F\right)$ be its minimum norm g-inverse and $\left(z^T, z^I, z^F\right)$ be its least square g-inverse. Then $\left(y^T, y^I, y^F\right)\left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right) = \left(u^T, u^I, u^F\right)^+$ where $\left(u^T, u^I, u^F\right)^+$ is the Moore-Penrose inverse of $\left(u^T, u^I, u^F\right)$.

Proof. Let $\left(x^T, x^I, x^F\right) = \left(y^T, y^I, y^F\right)\left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right)$ then as

$\left(y^T, y^I, y^F\right)\left(z^T, z^I, z^F\right) \in \left(u^T, u^I, u^F\right)\{1\}$

So, $\left(x^T, x^I, x^F\right) \in \left(u^T, u^I, u^F\right)\{1,2\}$ that is $\left(x^T, x^I, x^F\right)$ is the semi-inverse of $\left(u^T, u^I, u^F\right)$.

$\left(u^T, u^I, u^F\right)\left(x^T, x^I, x^F\right) = \left(u^T, u^I, u^F\right)\left(y^T, y^I, y^F\right)\left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right) = \left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right)$

(As $\left(u^T, u^I, u^F\right)\left(y^T, y^I, y^F\right)\left(u^T, u^I, u^F\right) = \left(u^T, u^I, u^F\right)$)

Now $\left[\left(u^T, u^I, u^F\right)\left(x^T, x^I, x^F\right)\right]^T = \left[\left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right)\right]^T = \left(u^T, u^I, u^F\right)\left(z^T, z^I, z^F\right)$

(Since $\left(z^T, z^I, z^F\right) \in \left(u^T, u^I, u^F\right)\{1,3\}$ ) $= \left(u^T, u^I, u^F\right)\left(x^T, x^I, x^F\right)$

Again

$$\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)$$

( Since $\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$ )

And $\left[\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)$

(Since $\left(y^T,y^I,y^F\right)\in\left(u^T,u^I,u^F\right)\{1,4\}$ )

$$=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$$

Thus $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)=\left(u^T,u^I,u^F\right)^+$

**Example 8.2** Let us consider an NFM $\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.8,0.7,0.2> & <0.6,0.7,0.4>\\ <0.6,0.7,0.4> & <0.7,0.7,0.3>\end{bmatrix}$ and

$$\left(y^T,y^I,y^F\right)=\begin{bmatrix}<0.8,0.7,0.2> & <0.6,0.7,0.4>\\ <0.5,0.7,0.4> & <0.7,0.7,0.3>\end{bmatrix}$$

$$\left(z^T,z^I,z^F\right)=\begin{bmatrix}<0.9,0.7,0.1> & <0.6,0.7,0.4>\\ <0.5,0.7,0.5> & <0.7,0.7,0.2>\end{bmatrix}$$

be two of its g-inverses.

Now, $\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)=\begin{bmatrix}<0.8,0.7,0.2> & <0.6,0.7,0.4>\\ <0.6,0.7,0.4> & <0.7,0.7,0.3>\end{bmatrix}$

$$\left[\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)\right]^T=\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)$$

That is, $\left(z^T,z^I,z^F\right)$ is the least square g-inverse of $\left(u^T,u^I,u^F\right)$.

$\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.8,0.7,0.2> & <0.6,0.7,0.4>\\ <0.6,0.7,0.4> & <0.7,0.7,0.3>\end{bmatrix}$ and

$$\left[\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)$$

Therefore, $\left(y^T,y^I,y^F\right)$ is the minimum norm g-inverse of $\left(u^T,u^I,u^F\right)$.

Now,

$\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)=\begin{bmatrix}<0.8,0.7,0.2> & <0.6,0.7,0.4>\\ <0.6,0.7,0.4> & <0.7,0.7,0.3>\end{bmatrix}$ for which $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$,

$\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(x^T,x^I,x^F\right)$, and

$$\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]^T=\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$$

$$\left[\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$$

So, $\left(x^T,x^I,x^F\right)=\left(y^T,y^I,y^F\right)\left(u^T,u^I,u^F\right)\left(z^T,z^I,z^F\right)$ is the Moore-Penrose inverse of $\left(x^T,x^I,x^F\right)$.

**Theorem 8.3** If $\left(u^T,u^I,u^F\right) \in \left(NFM\right)_n$ be a symmetric idempotent AIFM, then $\left(u^T,u^I,u^F\right)^+=\left(u^T,u^I,u^F\right)$

Proof. Since $\left(u^T,u^I,u^F\right)$ is symmetric and idempotent so $\left(u^T,u^I,u^F\right)\in\{1,3\}$ and $\left(u^T,u^I,u^F\right)\in\{1,4\}$. Hence $\left(u^T,u^I,u^F\right)^+=\left(u^T,u^I,u^F\right)\{1,4\}\left(u^T,u^I,u^F\right)\left(u^T,u^I,u^F\right)\{1,3\}$.

$$=\left[\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)\right]\left(u^T,u^I,u^F\right)\left[\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\right]$$

(Since $\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)$ and $\left(u^T,u^I,u^F\right)+\left(k^T,k^I,k^F\right)$ are the set of all $\{1,3\}$ and $\{1,4\}$ inverses respectively)

$$=\left[\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\right]\left[\left(u^T,u^I,u^F\right)^2+\left(u^T,u^I,u^F\right)\left(k^T,k^I,k^F\right)\right]$$

$$=\left[\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\right]\left[\left(u^T,u^I,u^F\right)+\left(u^T,u^I,u^F\right)\left(k^T,k^I,k^F\right)\right]$$

$$=\left[\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\right]\left(u^T,u^I,u^F\right)$$

$\left(u^T,u^I,u^F\right)\geq\left(u^T,u^I,u^F\right)\left(k^T,k^I,k^F\right),\left(u^T,u^I,u^F\right)+\left(u^T,u^I,u^F\right)\left(k^T,k^I,k^F\right)=\left(u^T,u^I,u^F\right)$)

$$=\left(u^T,u^I,u^F\right)^2+\left(h^T,h^I,h^F\right)\left(u^T,u^I,u^F\right)$$

$$=\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\left(u^T,u^I,u^F\right)$$

$$=\left(u^T,u^I,u^F\right)$$

Since

$\left(u^T,u^I,u^F\right)\geq\left(h^T,h^I,h^F\right)\left(u^T,u^I,u^F\right),\left(u^T,u^I,u^F\right)+\left(h^T,h^I,h^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right).$

**Example 8.3** Let $\left(u^T,u^I,u^F\right)=\begin{bmatrix}<0.7,0.5,0.2> & <0.5,0.5,0.4> \\ <0.5,0.5,0.4> & <0.6,0.5,0.3>\end{bmatrix}$ be a NFM. Here $\left(u^T,u^I,u^F\right)$

is symmetric and idempotent, that is, $\left(u^T,u^I,u^F\right)^T=\left(u^T,u^I,u^F\right)$ and

$\left(u^T,u^I,u^F\right)^2=\left(u^T,u^I,u^F\right)$.

Now $\left(u^T,u^I,u^F\right)\{1\}=\left(x^T,x^I,x^F\right):\left(x^T,x^I,x^F\right)=\begin{bmatrix}<a,b,c> & <d,e,f> \\ <g,h,i> & <j,k,l>\end{bmatrix}$

Where, $a\geq 0.7, b\geq 0.5, c\leq 0.2$ such that $a+b+c\leq 3$

$0\leq d\leq 1, 0\leq e\leq 1, f\geq 0.4$ such that $d+e+f\leq 3$
$g\leq 0.5, h\leq 0.5, i\geq 0.4$ such that $g+h+i\leq 3$
$j\geq 0.6, k\geq 0.5, l\geq 0.3$ such that $j+k+l\leq 3$

Here $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)\in\left(u^T,u^I,u^F\right)\{1\}$ as it satisfied the above condition.

So, $\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)$,

$\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)=\left(x^T,x^I,x^F\right)$,

and $\left[\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)\right]^T=\left(u^T,u^I,u^F\right)\left(x^T,x^I,x^F\right)$

$\left[\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)\right]^T=\left(x^T,x^I,x^F\right)\left(u^T,u^I,u^F\right)$

For $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$

Therefore, $\left(x^T,x^I,x^F\right)=\left(u^T,u^I,u^F\right)$ is itself a Moore-Penrose inverse, that is $\left(u^T,u^I,u^F\right)=\left(u^T,u^I,u^F\right)^+$.

## 9. An Application

We can use the g-inverse of QPNFMs to find the solution of Quadri partitioned Neutrosophic Fuzzy relational equations. Let us consider the system of Quadri partitioned Neutrosophic Fuzzy equations

$$\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)=\left(q^T,q^C,q^U,q^F\right)$$

where, $\left(u^T,u^C,u^U,u^F\right)=\begin{bmatrix} <0.7,0.8,0.2,0.3> & <0.6,0.8,0.2,0.4> & <0.5,0.8,0.2,0.5> \\ <0.5,0.8,0.2,0.5> & <0.6,0.8,0.2,0.3> & <0.8,0.8,0.2,0.2> \end{bmatrix}$

$\left(x^T,x^C,x^U,x^F\right)=\begin{bmatrix} \left(x_{1\mu}{}^T,x_{1\nu}{}^C,x_{1\nu}{}^U,x_{1\eta}{}^F\right) \\ \left(x_{2\mu}{}^T,x_{2\nu}{}^C,x_{2\nu}{}^U,x_{2\eta}{}^F\right) \\ \left(x_{3\mu}{}^T,x_{3\nu}{}^C,x_{3\nu}{}^U,x_{3\eta}{}^F\right) \end{bmatrix}$ and $\left(q^T,q^C,q^U,q^F\right)=\begin{bmatrix} <0.6,0.8,0.2,0.3> \\ <0.5,0.8,0.2,0.4> \end{bmatrix}$

Each particular matrix $\left(x^T,x^C,x^U,x^F\right)$ that satisfy the equation $\left(u^T,u^C,u^U,u^F\right)\left(x^T,x^C,x^U,x^F\right)=\left(q^T,q^C,q^U,q^F\right)$ is called its solution and the set $\Omega\left[\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)\right]=\left(u^T,u^C,u^U,u^F\right)^{-}\left(q^T,q^C,q^U,q^F\right)$ $\left(u^T,u^I,u^F\right)$ denotes the set of all solutions.

An QPNFM may have multiple g-inverses. Here we consider one of the generalized inverses of the QPNFM $\left(u^T,u^C,u^U,u^F\right)$, which is

$$\left(u^T,u^C,u^U,u^F\right)^{-}=\begin{bmatrix} <0.8,0.8,0.2,0.2> & <0.5,0.8,0.2,0.5> \\ <0.5,0.8,0.2,0.5> & <0.5,0.8,0.2,0.5> \\ <0.5,0.8,0.2,0.5> & <0.8,0.8,0.2,0.2> \end{bmatrix}.$$

Then, $\left(x^T,x^C,x^U,x^F\right)\in\Omega\left[\left(u^T,u^C,u^U,u^F\right)\left(q^T,q^C,q^U,q^F\right)\right]$

$$=\left(u^T,u^C,u^U,u^F\right)^{-}\left(q^T,q^C,q^U,q^F\right)=\begin{bmatrix} <0.6,0.8,0.2,0.3> \\ <0.5,0.8,0.2,0.5> \\ <0.5,0.8,0.2,0.4> \end{bmatrix}$$

is one of the solutions of the above system of equations.

## 10. Conclusion

In this study, we introduced a novel method for computing the generalized inverse (g-inverse) and Moore-Penrose inverse of Neutrosophic Fuzzy Matrices (NFMs), addressing a key gap in existing literature. A dedicated algorithm was developed to compute the g-inverse, supported by theoretical results and validated through numerical examples. Furthermore, we demonstrated the practical relevance of our approach by applying it to a rectangular system of neutrosophic fuzzy relational equations, showcasing its utility in solving real-world problems.

Unlike previous studies that primarily focused on the structural properties and algebraic operations of fuzzy and neutrosophic matrices, our work takes a computational approach by explicitly developing an algorithm for matrix inversion. For instance, Cen [4] and Dehghan et al. [12] have explored generalized inverses in fuzzy and neutrosophic settings, but their focus has been on theoretical aspects, partial ordering, and pseudo-inverses, rather than on developing explicit computational methods for obtaining the g-inverse in neutrosophic fuzzy environments. Furthermore, earlier works have not systematically extended the concept of the Moore-Penrose inverse to NFMs. Our research fills this gap by providing a structured algorithmic framework, ensuring practical usability while reinforcing the theoretical foundation of neutrosophic fuzzy matrix inverses.

The findings of this study significantly contribute to the advancement of neutrosophic fuzzy matrix theory, offering a robust and computationally feasible method for determining matrix inverses. This work enhances the mathematical framework of NFMs and extends their applicability to fields such as control systems, robotics, optimization, and decision-making. In contrast to prior work, which focused mainly on abstract properties or limited to specific inversion forms, our approach provides a comprehensive computational solution. Future research can build upon this foundation by exploring further generalizations, extending inverse computations to other types of neutrosophic fuzzy systems, and applying these methods in real-world uncertain environments.

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