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Fixed Point Theorems for Multivalued Mappings in Neutrosophic Fuzzy Metric Spaces

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Abstract: For multivalued mappings, Fixed point theorems are examined in this study within the context of Neutrosophic fuzzy metric spaces (NFMS), which are a further period of classical fuzzy metric spaces (FMS) that enable a more sophisticated depiction of indeterminacy, ambiguity, and uncertainty. We define and show three fixed point (fp) theorems under different contractive-type circumstances using the neutrosophic fuzzy Hausdorff metric by generalizing well-known fixed point results from full cone metric spaces. These findings complement and integrate previous research on multivalued contractions in fuzzy and cone metric contexts. A versatile and reliable method for examining the presence of fixed points for multivalued operators under neutrosophic uncertainty is offered by the introduced theorems. Additionally, illustrative examples are provided to show how the primary findings are applicable.

Keywords: Fixed Point, Neutrosophic Fuzzy Metric Space, Convergence

1. Introduction

With a wide range of applications in dynamical systems, differential and integral equations, nonlinear functional analysis, and decision theory, fixed point theory is a key tool in contemporary analysis. Many extensions, such as those in fuzzy metric spaces [George & Veeramani, 1994 [1]], multivalued mappings [Nadler, 1969 [5]], and cone metric spaces [Huang & Zhang, 2007], have been motivated by classical conclusions like Banach's contraction principle. Ghosh [6] prsented NFMS in 2024. Researchers have created frameworks that more effectively manage the inherent uncertainty and indeterminacy of real-world problems thanks to the development of fuzzy set theory and Smarandache's introduction of neutrosophic logic. To model truth, falsity, and indeterminacy components simultaneously, neutrosophic fuzzy metric spaces which were first presented in recent combine fuzzy metrics with neutrosophic logic [1-16].

Numerous applications, including differential inclusions, game theory, and optimization, inherently include multivalued mappings. For such mappings in neutrosophic fuzzy metric spaces, fixed point (fp) theorems are still largely unexplored. This gap motivates us to propose and generalize fixed point findings for multivalued mappings under uncertainty that is neutrosophic.

2. Preliminaries

Definition 2.1. [5] A t-norm stands a binary operation $*: [0,1]^2 \to [0,1]$ filling the following properties for all $a, b, c \in [0,1]$

1. Commutativity: $a * b = b * a$
2. Associativity: $a * (b * c) = (a * b) * c$
3. Monotonicity: If $a \leq c$ and $b \leq d$ then $a * b \leq c * d$
4. Identity: $a * 1 = a$

A t-conorm $\Delta: [0,1]^2 \to [0,1]$ is defined similarly, with $a\Delta 0 = a$ as the identity element.

Definition 2.2 [2] Consider the non-empty set X. A NFMS is a quituple $(X, M, N, *, \Delta)$ where $M: X \times X \times (0, \infty) \to [0,1]$ represents the truth-membership degree
$N: X \times X \times (0, \infty) \to [0,1]$ represents the indetermincy-membership degree.
$*$ is a continous t-norm and Δ is a continuous t-conorm
These satisfy the following conditions for all $\xi, \Gamma, ℶ \in X$ and $s, t > 0$

1. $M(\xi, \Gamma, t) > 0$ and $M(\xi, \Gamma, t) = 1 \Leftrightarrow \xi = \Gamma$
2. $M(\xi, \Gamma, t) = M(\Gamma, \xi, t)$
3. $M(\xi, z, t + s) \geq M(\xi, \Gamma, t) * M(\Gamma, z, s)$
4. $N(\xi, \Gamma, t) < 1$ and $N(\xi, \Gamma, t) = 0 \Rightarrow \xi = \Gamma$
5. $N(\xi, \Gamma, t) = N(\Gamma, \xi, t)$
6. $M(\xi, ℶ, t + s) \geq M(\xi, \Gamma, t) \Delta\, M(\Gamma, ℶ, s)$

Definition 2.3: The convergence of a sequence $\{\xi_n\}$ to ı in X is defined as follows: for any $t > 0$,

$\lim_{n\to\infty} M(\xi_n, \xi, t) = 1$ and $\lim_{n\to\infty} N(\xi_n, \xi, t) = 0$

Definition 2.4: A sequence $\{\xi_n\}$ in X is considered Cauchy, if for every $t > 0$ $\lim_{m,n\to\infty} M(\xi_n, \xi_m, t) = 1$

and $\lim_{m,n\to\infty} N(\xi_n, \xi_m, t) = 0$

Definition 2.5: If every Cauchy sequence converges, the NFMS is complete

Definition 2.6: Let CB(X) denote the collection of all non empty, closed, and bounded subsets of X. A multivalued mapping is a function $T: X \to CB(X)$ that assigns a set $T(\xi)$ to each point $\xi \in X$

Definition 2.7: Let $A, B \in CB(X)$. The Hausdorff neutrosphic fuzzy metric between A and B at time $t > 0$ is defined by

$$H_M(A, B, t) = \min\left\{\inf_{a\in A} \sup_{b\in B} M(a, b, t), \inf_{b\in B} \sup_{a\in A} M(a, b, t)\right\}$$

And similarly for the interminancy metric

$$H_N(A, B, t) = \min\left\{\sup_{a\in A} \inf_{b\in B} N(a, b, t), \sup_{b\in B} \inf_{a\in A} N(a, b, t)\right\}$$

3. Main Results

Theorme 3.1: Let $(X, M, N, *, \Delta)$ be a complete NFMS and let $T: X \to CB(X)$ be a multivalued mapping such that for all $\xi, \Gamma \in X$ and for each $t > 0$ the following inequality holds.

$$H_M(T\xi, T\Gamma, t) \geq M(\xi, T\xi, t)^a * M(\Gamma, T\Gamma, t)^a * M(\xi, T\Gamma, t)^b * M(T\xi, \Gamma, t)^b$$

Where $a, b \in \left[0, \frac{1}{2}\right)$ and $a + b < \frac{1}{2}$, Consequenty T reprsents a fixed point, meaning that

$\xi^* \in X$ such that $\xi^* \in T(\xi^*)$

Proof:

Assume that $\xi_0 \in X$ be arbitrary.

$T(\xi_0) \in CB(X)$ there exists $\xi_1 \in T(\xi_0)$

Assume $\xi_n \in X$ is given. Since $T(\xi_n)$ is nonempty, choose $\xi_{n+1} \in T(\xi_n)$

This process defined a sequence $\{\xi_n\} \subset X$ such that

$$\xi_{n+1} \in T(\xi_n), \forall n \in \mathbb{N}$$

From the hypothesis, for each $n \in \mathbb{N}$ and $t > 0$ we apply the inquality to ξ_n and ξ_{n+1}

$$H_M(T\xi_n, T\xi_{n+1}, t) \geq M(\xi_n, T\xi_{n+1}, t)^a * M(\xi_{n+1}, T\xi_{n+1}, t)^a * M(\xi_n, T\xi_{n+1}, t)^b * M(T\xi_n, \xi_{n+1}, t)^b$$

Since $\xi_{n+1} \in T(\xi_n)$ and $\xi_{n+2} \in T(\xi_{n+1})$

We can estimate the Hausdorff distance

$$H_M(T\xi_n, T\xi_{n+1}, t) \leq 1 - \delta_n$$

For some $\delta_n > 0$, Then

$$M(\xi_n, \xi_{n+1}, t) \leq 1 - \delta_n$$

We aim to prove that $\delta_n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} M(\xi_n, \xi_{n+1}, t) = 1$$

The sequence is Cauchy in neutrosophic fuzzy metric space.

By propterties of the t-norm and bounds $a, b \in \left[0, \frac{1}{2}\right)$ there exists $\lambda \in (0,1)$

$$M(\xi_{n+1}, \xi_{n+2}, t) \geq \lambda M(\xi_n, \xi_{n+1}, t)$$

Which leads to a recursive inequality showing that

$M(\xi_n, \xi_{n+k}, t) \to 1$ as $n \to \infty$

Thus $\{\varsigma_n\}$ is a Cauchy sequence in $(X, M, N, *, \Delta)$ is a CNFMS, there exists $\xi^* \in X$ such that

$$\lim_{n\to\infty} M(\xi_n, \xi^*, t) = 1, \forall\, t > 0$$

That is $\xi_n \to \xi^*$ in the neutrosophic fuzzy

We know that $\xi^* \in T(\xi^*)$

$\xi_{n+1} \in T(\xi_n)$

$\xi_n \to \xi^*$, T maps into closed subsets of X

$H_M\big(T(\xi_n), T(\xi^*)\big) \to 0$ by the continuity of T under H_M

Hence form the continuity of the neutrosophic Hausdorff metric, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $H_M\big(T(\xi_n), T(\xi^*)\big) < \varepsilon, \forall n \geq N$

Since $\varsigma_{n+1} \in T(\varsigma_n)$ and $T(\varsigma_n)$ approximates $T(\xi^*)$ the sequence $\{\xi_{n+1}\}$ converging to x^* implies

$$\xi^* \in T(\xi^*)$$

$\therefore$ Fixed point of T is ξ^*.

Example:

Let $X = [0,1]$ be the closed unit intveral.

Define the neutrosophic fuzzy metric (M, N) as follows

For all $\xi, \Gamma \in X$ and $t > 0$

Define $M(\xi, \Gamma, t) = \frac{t}{t+|\xi-\Gamma|}$, $N(\xi, \Gamma, t) = 1 - M(\xi, \Gamma, t)$

Let the t-norm be ordinary multiplication $a * b = ab$

And the s-norm $a\Delta b = min\{a + b, 1\}$

Following choised satisfy the conditions for a neutrosophic metric space

$M(\xi, \xi, t) = 1$

$M(\xi, \Gamma, t) = M(\Gamma, \xi, t)$

$M(\xi, \Gamma, t)$ is non-decreasing in t

The triangular inequality holds under the defined M

Also $(X, M, N,*, \Delta)$ is complete under this metric

Define a multivalued mapping $T: X \to CB(X)$ as

$$T(\xi) = \begin{cases} \left[0, \frac{\pi}{2}\right], if\ \xi > 0 \\ \{0\}, if\ \xi = 0 \end{cases}$$

So $T(\xi)$ is a closed subinterval of $[0,1]$ and $T(\xi) \in CB(X)$ for all $\xi \in X$.

Theorme 3.2: Let $(X, M, N,*, \Delta)$ be a CNFMS and $T: X \to CB(X)$ be a multivalued mapping so that for all $\xi, \Gamma \in X$ and for each $t > 0$ the subsequent inequality holds.

$$H_M(T\xi, T\Gamma, t) \geq r * \min\{ M(\xi, \Gamma, t), M(\xi, T\xi, t), M(\Gamma, T\Gamma, t)\}$$

For some constant $r \in (0,1)$. Consequenty T reprsents a fixed point in X.

Where $a, b \in \left[0, \frac{1}{2}\right)$ and $a + b < \frac{1}{2}$, then T has a fixed point that is there exists $\xi^* \in X$ such that $\xi^* \in T(\xi^*)$

Proof:

Assume that $\xi_0 \in X$ be arbitrary.

Since $T(\xi_0) \in CB(X)$ there exists $\xi_1 \in T(\varsigma_0)$

Assume $\xi_n \in X$ is given. Since $T(\xi_n)$ is nonempty, choose $\xi_{n+1} \in T(\xi_n)$

This process defined a sequence $\{\xi_n\} \subset X$ such that

$$\xi_{n+1} \in T(\xi_n), \forall n \in \mathbb{N}$$

Apply the given condition with $\xi = \xi_n, \Gamma = \xi_{n+1}, t > 0$

$$H_M(T\xi_n, T\xi_{n+1}, t) \geq r * \min\{ M(\xi_n, \xi_{n+1}, t), M(\xi_n, T\xi_n, t), M(\xi_{n+1}, T\xi_{n+1}, t)\}$$

Since $\xi_{n+1} \in T(\xi_n)$

$M(\xi_n, T\xi_n, t) \geq M(\xi_n, \xi_{n+1}, t)$ and

$M(\xi_{n+1}, T\xi_{n+1}, t) \geq M(\xi_{n+1}, \xi_{n+2}, t)$

$H_M(T\xi_n, T\xi_{n+1}, t) \geq r * M(\xi_n, \xi_{n+1}, t)$

But $\xi_{n+1} \in T(\xi_n)$ and $\xi_{n+2} \in T(\xi_{n+1})$ then

$M(\xi_{n+1}, \xi_{n+2}, t) \geq H_M(T\xi_n, T\xi_{n+1}, t)$

Combining

$M(\xi_{n+1}, \xi_{n+2}, t) \geq r * M(\xi_n, \xi_{n+1}, t)$

Continuing this process we get

$$M(\xi_{n+k}, \xi_{n+k+1}, t) \geq r^k * M(\xi_n, \xi_{n+1}, t)$$

To Prove $\{\xi_n\}$ is a Cauchy sequence

For any $n > m$ using the neutrosophic fuzzy triangle inequality

$$M(\xi_m, \xi_n, t) \geq M(\xi_m, \xi_{m+1}, t_1) * M(\xi_m, \xi_{m+2}, t_2) * \dots * M(\xi_{n-1}, \xi_n, t_k)$$

Where $\sum t_i = t$

Each term is the prodcut is bounded below by a power of r

$$M(\xi_m, \xi_n, t) \geq r^{n-m} * M(\xi_0, \xi_1, t)$$

As $n - m \to \infty, r^{n-m} \to 0$

So, $M(\xi_m, \xi_n, t) \to 1$ as $m, n \to \infty$

Thus, $\{\xi_n\}$ is a Cauchy sequence in the NFMS.

Since $(X, M, N, *, \Delta)$ is complete, there exists $\xi^* \in X$ such that

$$\lim_{n\to\infty} \xi_n = \xi^* \Rightarrow M(\xi_n, \xi^*, t) \to 1\ \forall t > 0$$

Now we show that $\xi^* \in T(\xi^*)$

We know that $\xi^* \in T(\xi^*)$

$\xi_{n+1} \in T(\xi_n)$

$\xi_n \to \xi^*,$

The graph of T is closed under the Hausdorff metric H_M

Then by the continuity of the neutrosophic fuzzy Hausdorff metric and closedness of the value of T.

We have $\xi_{n+1} \to \xi^*$ and $H_M(T\xi_n, T\xi^*) \to 0 \Rightarrow \xi^* \in T(\xi^*)$

Hence ξ^* is a fixed pint of T.

4. Conclusions

Two important FP theorems for multivalued mappings in the context of CNFMS have been proven in this work. By taking into consideration the fuzziness and indeterminacy present in neutrosophic contexts, as well as multivaluedness and set-valued Hausdorff distances, this result goes beyond traditional fixed point results. All of these findings contribute to the body of knowledge on fixed point theory in fuzzy and neutrosophic environments, especially when it comes to multivalued mappings. They provide opportunities for more study in dynamic systems, decision-making models, and optimization issues where non-uniqueness of solutions, uncertainty, and indeterminacy are significant.

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