Neutrosophic Sets and Systems, Vol. 90, 2025
University of New Mexico

The 2-refined neutrosophic quaternions numbers

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Abstract: This paper explores the concept of 2-refined neutrosophic quaternion numbers by first introducing their formal definition along with the notion of equality between two such numbers. Furthermore, the study develops the foundational algebra of 2-refined neutrosophic quaternions through the operations of addition, product, division, and conjugation. The article also addresses the methods for calculate both the absolute value and the inverse of a 2-refined neutrosophic quaternion number.

Keywords: 2-refined; conjugate; division; neutrosophic quaternions numbers; absolute value; product.

1. Introduction and Preliminaries

In the pursuit of extending classical logic and traditional mathematical frameworks, neutrosophic emerged as a revolutionary theory introduced by Smarandache. Unlike binary or fuzzy systems that operate under limited truth values, neutrosophic offers a more generalized and flexible approach to reasoning, accommodating not only truth and falsity but also indeterminacy. It is designed to mathematically and philosophically handle uncertainty, inconsistency, incompleteness, contradiction, and ambiguity—elements frequently encountered in real-world systems. At the heart of neutrosophic lies neutrosophic logic, which provides the foundation for a wide range of applications, including decision-making, artificial intelligence, data analysis, and more. He extended this logic by defining neutrosophic real numbers [2,4], formulating neutrosophic probabilities [3, 5], and developing the foundation of neutrosophic statistics [4, 6]. Furthermore, Smarandache proposed notions of integration and differentiation within the neutrosophic context [1,8]. The AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala [11] https://fs.unm.edu/NSS/RefinedLiteral21.pdf. In addition, Yaser Alhasan explored the application of neutrosophic concepts within complex numbers [7, 9, 10].

Quaternions play a significant role in various scientific and engineering fields due to their ability to efficiently represent three-dimensional rotations. Unlike traditional rotation methods, quaternions avoid gimbal lock and offer faster, more stable computations, making them essential in computer graphics, robotics, aerospace, and virtual reality. Mathematically, quaternions extend complex numbers and form a non-commutative algebra, providing a powerful tool for modeling spatial transformations and complex systems this is what led us to present the study of the 2-refined neutrosophic quaternions numbers in this paper.

2. Main Discussion

2.1 The 2-refined neutrosophic quaternions numbers

Definition 1

The numbers that take the form:

$$q = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + v = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$$

is the 2-refined neutrosophic quaternions numbers, denoted by symbol: H_{RN}
where: $\acute{r}, \acute{s}, \acute{t}, \acute{r}_1, \acute{s}_1, \acute{t}_1, \acute{r}_2, \acute{s}_2, \acute{t}_2, \acute{r}_3, \acute{s}_3, \acute{t}_3$ are real numbers, while $I_1, I_2 =$ indeterminacy and $\breve{\imath}, \breve{\jmath}, \breve{k}$ are units such that:

$$\breve{\imath}^2 = \breve{\jmath}^2 = k^2 = \breve{\imath}\breve{\jmath}\breve{k} = -1$$
$$\breve{\imath}\breve{\jmath} = \breve{k} = -\breve{\jmath}\breve{\imath}$$
$$\breve{\jmath}\breve{k} = \breve{\imath} = -\breve{k}\breve{\jmath}$$
$$\breve{k}\breve{\imath} = \breve{\jmath} = -\breve{\imath}\breve{k}$$

The 2-refined neutrosophic quaternions number has two parts, a refined neutrosophic real (scalar) part and a refined neutrosophic vector part, where:

$\acute{r} + \acute{s}I_1 + \acute{t}I_2$ is the 2-refined neutrosophic real (scalar) part and $(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$ is the refined neutrosophic vector part

Example 1

1) $q = -5 + 4I_1 + 10I_2 + (7 + I_1 - I_2)\breve{\imath} + (-3 + 7I_1 + I_2)\breve{\jmath} - (1 + 6I_1 - 14I_2)\breve{k}$
2) $q = (7I_1 - I_2)\breve{\imath} + (1 + 2I_1 - 3I_2)\breve{\jmath} + (-1 - 15I_1 - I_2)\breve{k}$
3) $q = 10I_2 + (7I_1 + I_2)\breve{\imath} + (-8 + 2I_1 + 5I_2)\breve{k}$
4) $q = -12 + 6I_1 - 14I_2 + (1 + 11I_1 - 14I_2)\breve{\imath}$

Notes:

- $0_{H_{RN}} = 0 + 0I_1 + 0I_2 + (0 + 0I_1 + 0I_2)\breve{\imath} + (0 + 0I_1 + 0I_2)\breve{\jmath} + (0 + 0I_1 + 0I_2)\breve{k}$
- $1_{H_{RN}} = 1 + 0I_1 + 0I_2 + (0 + 0I_1 + 0I_2)\breve{\imath} + (0 + 0I_1 + 0I_2)\breve{\jmath} + (0 + 0I_1 + 0I_2)\breve{k}$

Definition 2

Let $q, p \in H_{RN}$ where:

$$p = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + (\acute{a}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{a}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$$

$q = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + v = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$

we say: $p = q$ if and only if

$$\acute{a} = \acute{r}\,, \acute{b} = \acute{s}\,, \acute{c} = \acute{t}\ \ and\ u_I = v_I$$

then:

$$\acute{a}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2 = \acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2 \quad \Rightarrow \quad \acute{a}_1 = \acute{r}_1\,, \acute{b}_1 = \acute{s}_1\ and\ \acute{c}_1 = \acute{t}_1$$

$$\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2 = \acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2 \quad \Rightarrow \quad \acute{a}_2 = \acute{r}_2\,, \acute{b}_2 = \acute{s}_2\ and\ \acute{c}_2 = \acute{t}_2$$

$$\acute{a}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2 = \acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2 \quad \Rightarrow \quad \acute{a}_3 = \acute{r}_3\,, \acute{b}_3 = \acute{s}_3\ and\ \acute{c}_3 = \acute{t}_3$$

2.2 The 2-refined neutrosophic quaternions numbers algebra

2.2.1 Addition of the 2-refined neutrosophic quaternions numbers

Let $q,\ p\ \in H_{RN}$ where:

$p = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + (\acute{a}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{a}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$

$q = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + v = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$

then:

$$p + q = (\acute{a} + \acute{b}I_1 + \acute{c}I_2 + u) + (\acute{r} + \acute{s}I_1 + \acute{t}I_2 + v)$$

$$= (\acute{a} + \acute{r}) + (\acute{b} + \acute{s})I_1 + (\acute{c} + \acute{t})I_2 + \big((\acute{a}_1 + \acute{r}_1) + (\acute{b}_1 + \acute{s}_1)I_1 + (\acute{c}_1 + \acute{t}_1)I_2\big)\breve{\imath} + \big((\acute{a}_2 + \acute{r}_2) + (\acute{b}_2 + \acute{s}_2)I_1 + (\acute{c}_2 + \acute{t}_2)I_2\big)\breve{\jmath} + \big((\acute{a}_3 + \acute{r}_3) + (\acute{b}_3 + \acute{s}_3)I_1 + (\acute{c}_3 + \acute{t}_3)I_2\big)\breve{k}$$

Example 2

Let $p = -2 + I_1 + 11I_2 + (-7 + 6I_1 - 4I_2)\breve{\imath} + (-2 + 4I_1 + 6I_2)\breve{\jmath} - (8 + I_1 - 4I_2)\breve{k}$

$and\ q = 5 + 2I_1 + (3 - 12I_2)\breve{\imath} + (41 + 4I_1 - 3I_2)\breve{\jmath} + (I_1 - 7I_2)\breve{k}$

then:

$$p + q = 3 + 3I_1 + 11I_2 + (-4 + 6I_1 - 16I_2)\breve{\imath} + (39 + 8I_1 + 3I_2)\breve{\jmath} + (-8 - I_1 - 3I_2)\breve{k}$$

Note:

- ✓ Clearly, zero is neutral for addition of the 2-refined neutrosophic quaternions numbers.
- ✓ For every number $q \in H_{RN}$, its additive counterpart of the 2-refined neutrosophic quaternions numbers is:

$$-q = -\acute{r} - \acute{s}I_1 - \acute{t}I_2 - v$$

$$= -\acute{r} - \acute{s}I_1 - \acute{t}I_2 - (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} - (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} - (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$$

2.2.2 Multiplication of the 2-refined neutrosophic quaternions numbers

Let $p,\ q\ \in H_{RN}$ where:

$p = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{a} + \acute{b}I_1 + \acute{c}I_2 + (\acute{a}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{a}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$

$$q=\acute{r}+\acute{s}I_1+\acute{t}I_2+v=\acute{r}+\acute{s}I_1+\acute{t}I_2+(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}+(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}$$

Then:

$$p.q=\left(\acute{a}+\acute{b}I_1+\acute{c}I_2+u\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2+v)$$

$$=\left[\acute{a}+\acute{b}I_1+\acute{c}I_2+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}\right]\left[\acute{r}+\acute{s}I_1+\acute{t}I_2 + (\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}+(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}\right]$$

$$\begin{aligned}=&\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)\breve{\imath}\\
&+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r}+\acute{s}I_1+\acute{t}I_2)\\
&+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}\\
&+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r}+\acute{s}I_1+\acute{t}I_2)+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}\end{aligned}$$

$$\begin{aligned}=&\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)\breve{\imath}\\
&-\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r}+\acute{s}I_1+\acute{t}I_2)\\
&+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}-\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\\
&+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}\\
&+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r}+\acute{s}I_1+\acute{t}I_2)+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}\\
&-\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\end{aligned}$$

$$\begin{aligned}=&\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)\\
&-\left[\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\right.\\
&\left.+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\right]\\
&+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)\left[(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\imath}+(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\jmath}+(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{k}\right]\\
&+(\acute{r}+\acute{s}I_1+\acute{t}I_2)\left[\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)\breve{\imath}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)\breve{\jmath}+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)\breve{k}\right]\\
&+\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{k}-\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{\jmath}\\
&-\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{k}+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\breve{\imath}\\
&+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)\breve{\jmath}-\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\breve{\imath}\end{aligned}$$

we can write it by the form:

$$p.q=\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)(\acute{r}+\acute{s}I_1+\acute{t}I_2)-u.v+\left(\acute{a}+\acute{b}I_1+\acute{c}I_2\right)v+(\acute{r}+\acute{s}I_1+\acute{t}I_2)u+u\times v$$

where:

$$\begin{aligned}u.v=&\left(\acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2\right)(\acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2)+\left(\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2\right)(\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2)\\
&+\left(\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\right)(\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2)\end{aligned}$$

$$u\times v=\begin{vmatrix}\breve{\imath}&\breve{\jmath}&\breve{k}\\ \acute{a_1}+\acute{b_1}I_1+\acute{c}_1I_2&\acute{a_2}+\acute{b_2}I_1+\acute{c}_2I_2&\acute{a_3}+\acute{b_3}I_1+\acute{c}_3I_2\\ \acute{r_1}+\acute{s}_1I_1+\acute{t}_1I_2&\acute{r_2}+\acute{s}_2I_1+\acute{t}_2I_2&\acute{r_3}+\acute{s}_3I_1+\acute{t}_3I_2\end{vmatrix}$$

Result 1

Multiplication of the 2-refined neutrosophic quaternions numbers is not commutative because:

$$u\times v\neq v\times u$$

Example 3

Let $p = -2 + I_1 + I_2 + (1 + 2I_1)\breve{\imath} + (I_1 + 3I_2)\breve{\jmath} + (-1 - 4I_2)\breve{k}$
$and\ q = 1 + 2I_2 + (3 - I_2)\breve{\imath} + (1 + I_1 - I_2)\breve{\jmath} + (I_1 + 2I_2)\breve{k}$

Then:

$$p.q = \left[-2 + I_1 + I_2 + (1 + 2I_1)\breve{\imath} + (I_1 + 3I_2)\breve{\jmath} + (-1 - 4I_2)\breve{k}\right]\left[1 + 2I_2 + (3 - I_2)\breve{\imath} + (1 + I_1 - I_2)\breve{\jmath} + (I_1 + 2I_2)\breve{k}\right]$$
$$= -2 + 3I_1 - I_2 - [3 + 4I_1 - I_2 + 4I_1 - 5I_1 - 10I_2] + \left[(-6 + 2I_1 + 4I_2)\breve{\imath} + (-2 - I_1 + 3I_2)\breve{\jmath} + (2I_1 - 2I_2)\breve{k}\right] + \left[(1 + 2I_1 + 4I_2)\breve{\imath} + (3I_1 + 9I_2)\breve{\jmath} + (-1 - 14I_2)\breve{k}\right] + \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ 1 + 2I_1 & I_1 + 3I_2 & -1 - 4I_2 \\ 3 - I_2 & 1 + I_1 - I_2 & I_1 + 2I_2 \end{vmatrix}$$

$$= -5 + 6I_1 - 12I_2 + (-5 + 4I_1 + 8I_2)\breve{\imath} + (-2 + 2I_1 + 12I_2)\breve{\jmath} + (-1 + 2I_1 - 16I_2)\breve{k} + (-1 - 5I_1 + I_2)\breve{\imath} + (3 + 7I_1 + 9I_2)\breve{\jmath} + (1 + 3I_1 - 9I_2)\breve{k}$$

$$= -5 + 6I_1 - 12I_2 + (-6 - I_1 + 9I_2)\breve{\imath} + (1 + 8I_1 + 21I_2)\breve{\jmath} + (5I_1 - 25I_2)\breve{k}$$

Result 2

1) The 2-refined neutrosophic quaternions numbers H_{RN} is closed in relation to the addition operation, as the product of adding two 2-refined neutrosophic quaternions numbers is a 2-refined neutrosophic quaternions numbers, its real part is $(\acute{\alpha} + \acute{r}) + (\acute{b} + \acute{s})I_1 + (\acute{c} + \acute{t})I_2$, and its vector part is:
$\left((\acute{\alpha}_1 + \acute{r}_1) + (\acute{b}_1 + \acute{s}_1)I_1 + (\acute{c}_1 + \acute{t}_1)I_2\right)\breve{\imath} + \left((\acute{\alpha}_2 + \acute{r}_2) + (\acute{b}_2 + \acute{s}_2)I_1 + (\acute{c}_2 + \acute{t}_2)I_2\right)\breve{\jmath} + \left((\acute{\alpha}_3 + \acute{r}_3) + (\acute{b}_3 + \acute{s}_3)I_1 + (\acute{c}_3 + \acute{t}_3)I_2\right)\breve{k}$.

2) The 2-refined neutrosophic quaternions numbers H_{RN} is closed in relation to the multiplication operation, as the product of multiple two 2-refined neutrosophic quaternions numbers is a 2-refined neutrosophic quaternions numbers, its real part is $(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - u.v$, and its vector part is $(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)v + (\acute{r} + \acute{s}I_1 + \acute{t}I_2)u + u \times v$.

3) Multiplication accepts distribution on addition from the right and the left, so if we have three 2-refined neutrosophic quaternions numbers $p, q, r \in H_{RN}$, then:

$$q(p + r) = qp + qr$$

$$(p + r)q = pq + rq$$

4) The neutrality of multiplying 2-refined neutrosophic quaternions numbers is $1 + 0I_1 + 0I_2$

2.3 The 2-refined neutrosophic quaternions numbers conjugate

Definition 3

Let $p \in H_{RN}$, where: $p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)\breve{k}$. The 2-refined neutrosophic quaternions number conjugate define by the following form:

$\bar{p} = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} - (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$. .

Example 4

1) $p = -1 + 7I_1 - 2I_2 - (-16 - 12I_1 + I_2)\breve{\imath} + (1 + I_1 + 11I_2)\breve{\jmath} + (8 - 5I_1 - 5I_2)\breve{k}$
   $\Longrightarrow\ \bar{p} = -1 + 7I_1 - 2I_2 + (-16 - 12I_1 + I_2)\breve{\imath} - (1 + I_1 + 11I_2)\breve{\jmath} - (8 - 5I_1 - 5I_2)\breve{k}$

2) $p = (-I_1 + 3I_2)\breve{\imath} - (11 + 10I_1 + 14I_2)\breve{\jmath} - (15 - I_1 + 6I_2)\breve{k}$
   $\Longrightarrow\ \bar{p} = -(-I_1 + 3I_2)\breve{\imath} + (11 + 10I_1 + 14I_2)\breve{\jmath} + (15 - I_1 + 6I_2)\breve{k}$

Result 3

1. The 2-refined neutrosophic quaternions number conjugate of $\bar{p}$ is the same the 2-refined neutrosophic quaternions number p.

$$\overline{(\bar{p})} = p$$

Proof:

Let $p \in H_{RN}$, where $p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u$, then:

$$\bar{p} = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u$$

$$\overline{(\bar{p})} = \overline{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u)} = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = p$$

2. If $p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$

then:

- $p + \bar{p} = 2(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2) = Re(p)$
- $p - \bar{p} = 2u = 2(\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + 2(\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + 2(\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k} = V(p)$

where $Re(p)$ is the 2-refined neutrosophic real part (scalar) of the 2-refined complex number and $V(p)$ is the 2-refined neutrosophic vector part.

3. The 2-refined neutrosophic quaternions number is real (scalar) if and only if: $= \bar{p}$, and it is vector if and only if $p = -\bar{p}$.

Remarks1

$$\overline{p_1 + p_2} = \overline{p_1} + \overline{p_2}$$

Proof:

Let $p_1, p_2 \in H_{RN}$, where

$$p_1 = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$$
$$p_2 = a + bI_1 + cI_2 + (\alpha_1 + b_1 I_1 + c_1 I_2)\breve{\imath} + (\alpha_2 + b_2 I_1 + c_2 I_2)\breve{\jmath} + (\alpha_3 + b_3 I_1 + c_3 I_2)\breve{k}$$

then:

$$p_1 + p_2 = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + a + bI_1 + cI_2) + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2 + \alpha_1 + b_1 I_1 + c_1 I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2 + \alpha_2 + b_2 I_1 + c_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2 + \alpha_3 + b_3 I_1 + c_3 I_2)\breve{k}$$

$$\overline{p_1 + p_2} = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + a + bI_1 + cI_2) - (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2 + \alpha_1 + b_1 I_1 + c_1 I_2)\breve{\imath} - (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2 + \alpha_2 + b_2 I_1 + c_2 I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2 + \alpha_3 + b_3 I_1 + c_3 I_2)\breve{k}$$

$$= \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} - (\acute{a}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k} + a + bI_1 + cI_2 - (\alpha_1 + b_1 I_1 + c_1 I_2)\breve{\imath} - (\alpha_2 + b_2 I_1 + c_2 I_2)\breve{\jmath} - (\alpha_3 + b_3 I_1 + c_3 I_2)\breve{k}$$

$$= \overline{p_1} + \overline{p_2}$$

Theorem 1

The conjugate of the product of two 2-refined neutrosophic quaternion numbers is equal to the product of their individual conjugates.

$$\overline{p.q} = \bar{q}.\bar{p}$$

where $p, q \in H_{RN}$

Proof:

Let $p,\ q \in H_{RN}$ where:

$$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$$

$$q = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + v = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$$

Then:

$$p.q = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u)(\acute{r} + \acute{s}I_1 + \acute{t}I_2 + v)$$

$$= [\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}][\acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}]$$

$$\begin{aligned}
= (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)
&- [(\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2) + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)(\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2) \\
&+ (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)(\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)] \\
&+ (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)[(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}] \\
&+ (\acute{r} + \acute{s}I_1 + \acute{t}I_2)[(\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}] \\
&+ (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)(\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{k} - (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)(\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{\jmath} \\
&- (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{k} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)(\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{\imath} \\
&+ (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)(\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\imath}
\end{aligned}$$

we can write it by the form:

$$p.q = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - u.v + (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)v + (\acute{r} + \acute{s}I_1 + \acute{t}I_2)u + u \times v$$

where:

$$u.v = (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)(\acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2) + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)(\acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2) + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)(\acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2)$$

$$u \times v = \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ \acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2 & \acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2 & \acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2 \\ \acute{r}_1 + \acute{s}_1 I_1 + \acute{t}_1 I_2 & \acute{r}_2 + \acute{s}_2 I_1 + \acute{t}_2 I_2 & \acute{r}_3 + \acute{s}_3 I_1 + \acute{t}_3 I_2 \end{vmatrix}$$

then:

$$\overline{p.q} = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - u.v - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)v - (\acute{r} + \acute{s}I_1 + \acute{t}I_2)u - u \times v$$

$$\bar{q}.\bar{p} = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u)(\acute{r} + \acute{s}I_1 + \acute{t}I_2 - v)$$

$$= [\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)\breve{\imath} - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)\breve{k}][\acute{r} + \acute{s}I_1 + \acute{t}I_2 - (\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{\imath} - (\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{\jmath} - (\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{k}]$$

$$\begin{aligned} = {} & (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{\imath} - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{\jmath} \\ & - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{k} - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{\imath} \\ & - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2) + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{k} \\ & - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{\jmath} - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{\jmath} \\ & - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{k} - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2) \\ & - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{\imath} - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{k} \\ & + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{\imath} \\ & - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2) \end{aligned}$$

$$\begin{aligned} = {} & (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2) \\ & - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2) - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2) \\ & - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{\imath} - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{\jmath} \\ & - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{k} - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{\imath} \\ & - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2)\breve{k} \\ & + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{k} - (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{\jmath} \\ & - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{k} - (\acute{\alpha}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)(\acute{r}_3 + \acute{s}_3I_1 + \acute{t}_3I_2)\breve{\imath} \\ & + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_1 + \acute{s}_1I_1 + \acute{t}_1I_2)\breve{\jmath} - (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)(\acute{r}_2 + \acute{s}_2I_1 + \acute{t}_2I_2)\breve{\imath} \end{aligned}$$

$$= (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - u.v - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)v - (\acute{r} + \acute{s}I_1 + \acute{t}I_2)u - u \times v$$

$$\Rightarrow \quad \overline{p.q} = \bar{q}.\bar{p}$$

2.4 The absolute value of a 2-refined neutrosophic quaternions number

Let $p \in H_{RN}$, where:

$$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)\breve{\imath} + (\acute{a}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)\breve{k}$$

then the absolute value of a 2-refined neutrosophic quaternions numbers defined by form:

$$|p| = \sqrt{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)^2 + (\acute{a}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)^2}$$

Example 5

Let $p = -2 + I_1 + I_2 + (1 + 2I_1)\breve{\imath} + (I_1 + 3I_2)\breve{\jmath} + (1 - I_2)\breve{k}$, then:

$$|p| = \sqrt{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1I_1 + \acute{c}_1I_2)^2 + (\acute{a}_2 + \acute{b}_2I_1 + \acute{c}_2I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3I_1 + \acute{c}_3I_2)^2}$$

$$= \sqrt{(-2 + I_1 + I_2)^2 + (1 + 2I_1)^2 + (I_1 + 3I_2)^2 + (1 - I_2)^2}$$

$$= \sqrt{4 - 4I_1 + I_1 - 4I_2 + 2I_1 + I_2 + 1 + 4I_1 + 4I_1 + I_1 + 6I_1 + 9I_2 + 1 - 2I_2 + I_2}$$

$$= \sqrt{6 + 14I_1 + 5I_2}$$

$$= \sqrt{6} + \left[\sqrt{6 + 14 + 5} - \sqrt{6 + 5}\right]I_1 + \left[\sqrt{6 + 5} - \sqrt{6}\right]I_2$$

$$\Rightarrow \quad |p| = \sqrt{6} + \left[5 - \sqrt{11}\right]I_1 + \left[\sqrt{11} - \sqrt{6}\right]I_2$$

Theorem2

Let $p \in H_{RN}$, where:
$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha_1} + \acute{b_1}I_1 + \acute{c}_1I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b_2}I_1 + \acute{c}_2I_2)\breve{\jmath} + (\acute{\alpha_3} + \acute{b_3}I_1 + \acute{c}_3I_2)\breve{k}$
Then product the absolute value of 2-refined neutrosophic quaternions number p by its conjugate equals to square of the absolute value of p.

$$p.\bar{p} = |p|^2$$

Proof:

$$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u$$
$$\Rightarrow \quad \bar{p} = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u$$

$$p.\bar{p} = (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u)(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 - u)$$
$$= (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 - (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)u + (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)u - u.u$$
$$= (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 - u.u$$
$$= (\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha_1} + \acute{b_1}I_1 + \acute{c}_1I_2)^2 + (\acute{\alpha}_2 + \acute{b_2}I_1 + \acute{c}_2I_2)^2 + (\acute{\alpha_3} + \acute{b_3}I_1 + \acute{c}_3I_2)^2 = |p|^2$$

$$\Rightarrow \quad p.\bar{p} = |p|^2$$

Example 6

Let $p = 3 + 2I_2 + (1 + 2I_1 + I_2)\breve{\imath} - 4I_2\breve{\jmath} + 2I_1\breve{k}$, then:

$$p.\bar{p} = |p|^2$$

$$= (3 + 2I_2)^2 + (1 + 2I_1 + I_2)^2 + 16I_2 + 4I_1$$

$$= 9 + 12I_2 + 4I_2 + (1 + 4I_1 + 4I_1 + 2I_2 + 4I_1 + I_2) + 16I_2 + 4I_1$$

$$= 10 + 16I_1 + 35I_2$$

Remarks 2

Let $p, q \in H_{RN}$, then:

1) $|p| = |p| = |-p|$
2) $|pq| = |p|.|q|$

Proof (2):

$$|p.q|^2 = p.q\overline{(p.q)} = p.q.\bar{q}.\bar{p} = p.|q|^2.\bar{p} = p.\bar{p}.|q|^2 = |p|^2.|q|^2$$

2.5 Division of 2-refined neutrosophic quaternions numbers

Let $p, q \in H_{RN}$ where:

$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha_1} + \acute{b_1}I_1 + \acute{c}_1I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b_2}I_1 + \acute{c}_2I_2)\breve{\jmath} + (\acute{\alpha_3} + \acute{b_3}I_1 + \acute{c}_3I_2)\breve{k}$

$q = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + v = \acute{r} + \acute{s}I_1 + \acute{t}I_2 + (\acute{r_1} + \acute{s}_1 I_1 + \acute{t}_1 I_2)\breve{\imath} + (\acute{r_2} + \acute{s}_2 I_1 + \acute{t}_2 I_2)\breve{\jmath} + (\acute{r_3} + \acute{s}_3 I_1 + \acute{t}_3 I_2)\breve{k}$

then:

$$\frac{p}{q} = \frac{\acute{a} + \acute{b}I_1 + \acute{c}I_2 + u}{\acute{r} + \acute{s}I_1 + \acute{t}I_2 + v}$$

multiply the numerator and denominator by conjugate of q we get:

$$\frac{p}{q} = \frac{(\acute{a} + \acute{b}I_1 + \acute{c}I_2 + u)(\acute{r} + \acute{s}I_1 + \acute{t}I_2 - v)}{(\acute{r} + \acute{s}I_1 + \acute{t}I_2 + v)(\acute{r} + \acute{s}I_1 + \acute{t}I_2 - v)}$$

$$= \frac{(\acute{a} + \acute{b}I_1 + \acute{c}I_2 + u)(\acute{r} + \acute{s}I_1 + \acute{t}I_2 - v)}{(\acute{r} + \acute{s}I_1 + \acute{t}I_2)^2 - (v)^2}$$

$$= \frac{(\acute{a} + \acute{b}I_1 + \acute{c}I_2)(\acute{r} + \acute{s}I_1 + \acute{t}I_2) - u.v - (\acute{a} + \acute{b}I_1 + \acute{c}I_2)v + (\acute{r} + \acute{s}I_1 + \acute{t}I_2)u - u \times v}{(\acute{r} + \acute{s}I_1 + \acute{t}I_2)^2 - (v)^2}$$

where:

$$u.v = (\acute{a_1} + \acute{b_1}I_1 + \acute{c}_1 I_2)(\acute{r_1} + \acute{s}_1 I_1 + \acute{t}_1 I_2) + (\acute{a_2} + \acute{b_2}I_1 + \acute{c}_2 I_2)(\acute{r_2} + \acute{s}_2 I_1 + \acute{t}_2 I_2) + (\acute{a_3} + \acute{b_3}I_1 + \acute{c}_3 I_2)(\acute{r_3} + \acute{s}_3 I_1 + \acute{t}_3 I_2)$$

$$u \times v = \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ \acute{a_1} + \acute{b_1}I_1 + \acute{c}_1 I_2 & \acute{a_2} + \acute{b_2}I_1 + \acute{c}_2 I_2 & \acute{a_3} + \acute{b_3}I_1 + \acute{c}_3 I_2 \\ \acute{r_1} + \acute{s}_1 I_1 + \acute{t}_1 I_2 & \acute{r_2} + \acute{s}_2 I_1 + \acute{t}_2 I_2 & \acute{r_3} + \acute{s}_3 I_1 + \acute{t}_3 I_2 \end{vmatrix}$$

And: $(\acute{r} + \acute{s}I_1 + \acute{t}I_2)^2 - (v)^2 = (\acute{r} + \acute{s}I_1 + \acute{t}I_2)^2 + (\acute{r_1} + \acute{s}_1 I_1 + \acute{t}_1 I_2)^2 + (\acute{r_2} + \acute{s}_2 I_1 + \acute{t}_2 I_2)^2 + (\acute{r_3} + \acute{s}_3 I_1 + \acute{t}_3 I_2)^2$

Example 7

Let $p = 3 + \breve{\imath} + I_2\breve{k}$ *and* $q = 3 + 2I_2 + (1 + 2I_1 + I_2)\breve{\imath} - 4I_2\breve{\jmath} + 2I_1\breve{k}$

then:

$$\frac{p}{q} = \frac{3 + \breve{\imath} + I_2\breve{k}}{3 + 2I_2 + (1 + 2I_1 + I_2)\breve{\imath} - 4I_2\breve{\jmath} + 2I_1\breve{k}}$$

$$= \frac{(3 + \breve{\imath} + I_2\breve{k})(3 + 2I_2 - (1 + 2I_1 + I_2)\breve{\imath} + 4I_2\breve{\jmath} - 2I_1\breve{k})}{(3 + 2I_2 + (1 + 2I_1 + I_2)\breve{\imath} - 4I_2\breve{\jmath} + 2I_1\breve{k})(3 + 2I_2 - (1 + 2I_1 + I_2)\breve{\imath} + 4I_2\breve{\jmath} - 2I_1\breve{k})}$$

$$= \frac{10 + 2I_1 + 9I_2 + (6I_1 + I_2)\breve{\imath} + (-4I_1 + 10I_2)\breve{\jmath} + (-6I_1 + 9I_2)\breve{k}}{10 + 16I_1 + 35I_2}$$

$$= \frac{10 + 2I_1 + 9I_2}{10 + 16I_1 + 35I_2} + \frac{6I_1 + I_2}{10 + 16I_1 + 35I_2}\breve{\imath} + \frac{-4I_1 + 10I_2}{10 + 16I_1 + 35I_2}\breve{\jmath} + \frac{-6I_1 + 9I_2}{10 + 16I_1 + 35I_2}\breve{k}$$

$$= 1 - \frac{214}{2745}I_1 - \frac{26}{45}I_2 + \left(\frac{254}{2745}I_1 + \frac{1}{45}I_2\right)\breve{\imath} + \left(-\frac{68}{549}I_1 + \frac{2}{9}I_2\right)\breve{\jmath} + \left(-\frac{46}{305}I_1 + \frac{1}{5}I_1\right)\breve{k}$$

2.6 Inverted 2-refined neutrosophic quaternions numbers

Definition4

We define Inverted of $p \in H_{RN}$ as $p^{-1} \in H_{RN}$, whereas:

$$\mathrm{p}.p^{-1} = \mathrm{p}^{-1}.\mathrm{p} = 1_{H_{RN}}$$

whereas: $p \neq 0_{H_{RN}}$

Remark 3

$$|p|^2 = p.\bar{p} \quad \Rightarrow \quad p = \frac{|p|^2}{\bar{p}} \quad \Rightarrow \quad p^{-1} = \frac{\bar{p}}{|p|^2}$$

Proof:

Let $p \in H_{RN}$ where:

$$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + u = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}$$

then:

$$p^{-1} = \frac{1}{p} = \frac{1}{\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)\breve{k}}$$

$$= \frac{\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2}{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)^2 + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)^2}$$
$$- \frac{(\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)}{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)^2 + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)^2}\breve{\imath}$$
$$- \frac{(\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)}{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)^2 + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)^2}\breve{\jmath}$$
$$- \frac{(\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)}{(\acute{\alpha} + \acute{b}I_1 + \acute{c}I_2)^2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)^2 + (\acute{\alpha}_2 + \acute{b}_2 I_1 + \acute{c}_2 I_2)^2 + (\acute{\alpha}_3 + \acute{b}_3 I_1 + \acute{c}_3 I_2)^2}\breve{k}$$
$$= \frac{\bar{p}}{|p|^2}$$

Example 8

$$\frac{1}{3 + 2I_2 + (1 + 2I_1 + I_2)\breve{\imath} - 4I_2\breve{\jmath} + 2I_1\breve{k}}$$

$$= \frac{3 + 2I_2}{10 + 16I_1 + 35I_2} - \frac{(1 + 2I_1 + I_2)}{10 + 16I_1 + 35I_2}\breve{\imath} + \frac{4I_2}{10 + 16I_1 + 35I_2}\breve{\jmath} - \frac{2I_1}{10 + 16I_1 + 35I_2}\breve{k}$$

$$= \frac{3}{10} - \frac{16}{549}I_1 - \frac{17}{90}I_2 + \left(-\frac{1}{10} - \frac{58}{2745}I_1 + \frac{1}{18}I_2\right)\breve{\imath} + \left(-\frac{16}{675}I_1 + \frac{4}{45}I_2\right)\breve{\jmath} + \left(-\frac{2}{61}I_1\right)\breve{k}$$

Remark 4

$$(p.q)^{-1} = q^{-1}.p^{-1} \qquad \text{, whereas: } p.q \neq 0_{H_{RN}}$$

Remark 5

Since any 2-refined neutrosophic complex number $p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath}$ can be written in the form:

$$p = \acute{\alpha} + \acute{b}I_1 + \acute{c}I_2 + (\acute{\alpha}_1 + \acute{b}_1 I_1 + \acute{c}_1 I_2)\breve{\imath} + 0\breve{\jmath} + 0\breve{k}$$

then:

$$R_N \subseteq C_N \subseteq H_{RN}$$

3. Conclusions

Quaternions are widely recognized for their effectiveness in representing rotations in three-dimensional space, which makes them fundamental in numerous scientific and engineering disciplines. In this work, we introduced the concept of 2-refined neutrosophic quaternion numbers and examine their algebraic structure, including operations such as addition, multiplication, conjugation, and inversion. We also investigated how to compute the absolute value and the inverse of a 2-refined neutrosophic quaternion. we obtained accurate calculation results by providing appropriate examples for each case.

Acknowledgments "This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2025/R/1446)".

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Received: Jan. 5, 2025. Accepted: July 6, 2025