The improper 2-refined neutrosophic integrals

Suliman Sheen[1], Qusay Alhassan[2], Raja Abdullah Abdulfatah[3] and Yaser Ahmad Alhasan[4,*]

[1]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia.; s.almleh@psau.edu.sa
[2]Faculty of Informatics Engineering, Cordoba Private University, Syria; qusayalhassan77@gmail.com
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia; r.abdulfatah@psau.edu.sa
[4]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa

***Corresponding author:** y.alhasan@psau.edu.sa

Abstract: Given the growing body of research surrounding neutrosophic logic and its increasing recognition across various scientific disciplines, it became essential to extend our efforts within this domain—particularly in the area of neutrosophic calculus. This motivation led us to explore improper 2-refined neutrosophic integrals as a continuation of this research trajectory. In this study, we examine both the first and second kinds of improper 2-refined neutrosophic integrals. We also address the notions of convergence and divergence within this context, supported by several illustrative examples that clarify and reinforce the presented concepts.

Keywords: 2-refined neutrosophic, improper integral, divergent integral, converges integral.

1. Introduction and Preliminaries

Unlike traditional logical systems, Smarandache introduced Neutrosophic Logic as a mathematical framework designed to handle various forms of uncertainty, including vagueness, ambiguity, imprecision, incompleteness, inconsistency, redundancy, and contradiction. This logic is part of a broader philosophical approach he called Neutrosophy [4]. Within this framework, he defined the standard form of neutrosophic real numbers [3–5] and developed the foundation for neutrosophic statistics [5–6]. Furthermore, he initiated the concept of a preliminary neutrosophic calculus by introducing novel ideas such as the mereo-limit, mereo-continuity, mereo-derivative, and mereo-integral [1]. Additional advancements in neutrosophic differentiation and integration were contributed by Y. Alhasan, who also formulated the definition and explored the properties of neutrosophic complex numbers, along with their general exponential form [2, 8–11, 13]. Moreover, the AH isometry was employed to investigate a range of mathematical structures, including conic sections, concepts in real analysis, and geometric surfaces [10, 14].

In mathematical analysis, integrals are fundamental tools for measuring quantities such as area, accumulation, and total change. However, many real-world problems involve situations where standard (proper) integrals are insufficient—particularly when dealing with infinite intervals or functions that exhibit discontinuities or singularities. In such cases, improper integrals provide a rigorous framework for evaluating these otherwise non-standard integrals.

The significance of improper integrals extends far beyond pure mathematics. They are widely used in physics, engineering, and probability theory to solve problems involving unbounded domains, singular behavior, and infinite processes. Whether it is modeling the decay of signals, analyzing electric fields, or working with continuous probability distributions, improper integrals offer a powerful method for handling complex scenarios that arise naturally in theoretical and applied contexts this is what led us to present the study of the improper 2-refined neutrosophic integrals.

2. Main Discussion

2.1 The first kind of the improper 2-refined neutrosophic integrals

Definition 1

We will consider the improper 2-refined neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x, I_1, I_2)$ be a function defined on the infinite interval $[r_1 + s_1 I_1 + t_1 I_2, +\infty)$, and integrable over any bounded closed interval $[r_1 + s_1 I_1 + t_1 I_2, k] \subset [r_1 + s_1 I_1 + t_1 I_2, +\infty)$, then we call $f(x, I_1, I_2)$ over the interval $[r + sI_1 + tI_2, +\infty)$ the improper 2-refined neutrosophic integral and write:

$$\int_{r_1+s_1I_1+t_1I_2}^{+\infty} f(x, I_1, I_2)\, dx = \lim_{k\to+\infty} \int_{r_1+s_1I_1+t_1I_2}^{k} f(x, I_1, I_2)\, dx$$

We call the improper 2-refined neutrosophic integral convergent when the limit exists, and the 2-refined neutrosophic integral is of first kind, where it has defined limit, it is the value of the 2-refined neutrosophic integral. Otherwise the improper 2-refined neutrosophic integral is divergent (when the improper 2-refined neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 1

$$\int_{1+I_1+I_2}^{+\infty} \frac{2dx}{2x+1+2I_1+I_2} = \lim_{k\to+\infty} \int_{1+I_1+I_2}^{k} \frac{2dx}{2x+1+2I_1+I_2}$$

$$= \lim_{k\to+\infty} ln(2x+1+2I_1+I_2)|_{1+I_1+I_2}^{k}$$

$$= \lim_{k\to+\infty} [ln(2k+1+2I_1+I_2) - ln(3+4I_1+3I_2)]$$

$$= \lim_{k\to+\infty} \big[ln(2k+1) + I_1[ln(2k+4) - ln(2k+2)] + I_2[ln(2k+2) - ln(2k+1)]\big] - \big[ln\,3 + I_1[ln\,10 - ln\,6] - I_2[ln\,6 - ln\,3]\big]$$

$$= \lim_{k\to+\infty} \left[ln(2k+1) + I_1\left[ln\left(\frac{2k+4}{2k+2}\right)\right] + I_2\left[ln\left(\frac{2k+2}{2k+1}\right)\right]\right] - \left[ln\,3 + I_1\,ln\frac{10}{6} + I_2\,ln\frac{6}{3}\right]$$

$$= +\infty + I_1\,ln\,1 + I_2\,ln\,1 - ln\,3 - I_1\,ln\frac{5}{3} - I_2\,ln\,2$$

$$= +\infty - ln\,3 - I_1\,ln\frac{5}{3} - I_2\,ln\,2$$

$$= +\infty - I_1 \ln\frac{5}{3} - I_2 \ln 2$$

$$= \infty_{I_i} ; i = 1,2$$

(the improper 2-refined neutrosophic integral diverges)

Example 2

$$\int_{0+0I_1+0I_2}^{+\infty} e^{-(7+7I_1+7I_2)x}\, dx = \lim_{k\to+\infty} \int_{0+0I_1+0I_2}^{k} e^{-(7+7I_1+7I_2)x}\, dx$$

$$= \lim_{k\to+\infty} \frac{-1}{7+7I_1+7I_2} e^{-(7+7I_1+7I_2)x}\Big|_{0+0I_1+0I_2}^{k}$$

$$= \frac{-1}{7+7I_1+7I_2} \lim_{k\to+\infty}\left[e^{-(7+7I_1+7I_2)k} - 1\right]$$

by applying:

$$f(X) = f(x+yI_1+zI_2) = f(x) + I_1[f(x+y+z) - f(x+z)] + I_2[f(x+z) - f(x)]$$

we get on:

$$\int_{0+0I_1+0I_2}^{+\infty} e^{-(7+7I_1+7I_2)x}\, dx = \frac{-1}{7+7I_1+7I_2} \lim_{k\to+\infty}\left[\left[e^{-7k} + I_1\left[e^{-21k} - e^{-14k}\right] + \left[e^{-14k} - e^{-7k}\right]\right]I_2 - 1\right]$$

$$= \frac{1}{7+7I_1+7I_2} = \frac{1}{7} - \frac{1}{42}I_1 - \frac{1}{14}I_2$$

the improper 2- refined neutrosophic integral converges.

Definition 2

We will consider the improper 2-refined neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x, I_1, I_2)$ be a function defined on the infinite interval $(-\infty, r_2 + s_2I_1 + t_2I_2]$, and integrable over any bounded closed interval $[k, r_2 + s_2I_1 + t_2I_2] \subset (-\infty, r_2 + s_2I_1 + t_2I_2]$, then we call $f(x, I_1, I_2)$ over the interval $(-\infty, r_2 + s_2I_1 + t_2I_2]$ the improper 2-refined neutrosophic integral and write:

$$\int_{-\infty}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)\, dx = \lim_{k\to-\infty} \int_{k}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)\, dx$$

We call the improper 2-refined neutrosophic integral convergent when the limit exists, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper 2-refined neutrosophic integral is divergent (when the improper 2-refined neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 3

$$\int_{-\infty}^{0+0I_1+0I_2} \sin\left(x + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right) dx = \lim_{k\to-\infty} \int_{k}^{0+0I_1+0I_2} \sin\left(x + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right) dx$$

$$= \lim_{k\to-\infty}\left[-cos\left(x+\frac{\pi}{2}I_1+\frac{\pi}{3}I_2\right)\right]\Big|_k^{0+0I_1+0I_2} = -cos\left(0+\frac{\pi}{2}I_1+\frac{\pi}{3}I_2\right)+\lim_{k\to-\infty}cos\left(k+\frac{\pi}{2}I_1+\frac{\pi}{3}I_2\right)$$

$$= -1+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)I_1-\frac{1}{2}I_2$$
$$+\lim_{k\to-\infty}\left(cos(k)+I_1\left[cos\left(k+\frac{5\pi}{6}\right)-cos\left(k+\frac{\pi}{3}\right)\right]+I_2\left[cos\left(k+\frac{\pi}{3}\right)-cos(k)\right]\right)$$

(Does not exist, so the improper 2-refined neutrosophic integral is divergent)

Example 4

$$\int_{-\infty}^{0+0I_1+0I_2}(6x+6I_1+12I_2)e^x dx = \lim_{k\to-\infty}\int_k^{0+0I_1+0I_2}(6x+6I_1+12I_2)e^x dx$$

$$u = 6x+6I_1+12I_2 \implies du = 6dx$$
$$v\,dx = e^x dx \implies v = e^x$$

then:

$$\int_{-\infty}^{0+0I_1+0I_2}(6x+6I_1+12I_2)e^x dx = \lim_{k\to-\infty}[(6x+6I_1+12I_2)e^x]\Big|_k^{0+0I_1+0I_2} - \lim_{k\to-\infty}\int_{-\infty}^{0+0I_1+0I_2}6e^x dx$$

$$= 6I_1+12I_2-\lim_{k\to-\infty}\left((6k+6I_1+12I_2)e^k\right)-\lim_{k\to-\infty}6e^x\Big|_k^{0+0I_1+0I_2}$$

$$= 6I_1+12I_2-\lim_{k\to-\infty}\left((6ke^k+6e^kI_1+12e^kI_2)\right)-6+\lim_{k\to-\infty}6e^k = -6+6I_1+12I_2$$

the improper 2- refined neutrosophic integral converges.

Theorem 1

$$\int_{1+I_1+I_2}^{+\infty}\frac{dx}{x^q} = \begin{cases}\dfrac{q}{(1+I_1+I_2)^{q-1}} & if \quad q>1 \\ \text{Divergent} & if \quad q\le 1\end{cases}$$

Example 5

$$\int_{1+I_1+I_2}^{+\infty}\frac{-1}{x^2}dx = \lim_{k\to+\infty}\int_{1+I_1+I_2}^{k}\frac{-1}{x^2}dx = \lim_{k\to+\infty}\frac{1}{x}\Big|_{1+I_1+I_2}^{k}$$

$$= \lim_{k\to+\infty}\left[\frac{1}{k}-\frac{1}{1+I_1+I_2}\right] = \frac{-1}{1+I_1+I_2} = -1+\frac{1}{6}I_1+\frac{1}{2}I_2$$

the improper 2- refined neutrosophic integral converges.

Example 6

$$\int_{1+I_1+I_2}^{+\infty}\frac{5}{x^6}dx = \lim_{k\to+\infty}\int_{1+I_1+I_2}^{k}\frac{5}{x^6}dx = \lim_{k\to+\infty}\frac{-1}{x^5}\Big|_{1+I_1+I_2}^{k}$$

$$= \lim_{k\to+\infty}\left[\frac{-1}{k^5}+\frac{1}{(1+I_1+I_2)^5}\right] = \frac{1}{(1+I_1+I_2)^5} = \frac{1}{1+235I_1+7I_2} = 1-\frac{235}{1944}I_1-\frac{7}{8}I_2$$

the improper 2- refined neutrosophic integral converges.

Example 7

$$\int_{1+I_1+I_2}^{+\infty} \frac{-4}{x^5} dx = \lim_{k \to +\infty} \int_{1+I_1+I_2}^{k} \frac{-4}{x^5} dx = \lim_{k \to +\infty} \frac{1}{x^4}\bigg|_{1+I_1+I_2}^{k}$$

$$= \lim_{k \to +\infty} \left[\frac{1}{k^4} - \frac{1}{(1+I_1+I_2)^4}\right] = \frac{-1}{1+77I_1+3I_2} = -1 + \frac{77}{324} I_1 + \frac{3}{4} I_2$$

the improper 2- refined neutrosophic integral converges.

Example 8

$$\int_{1+I_1+I_2}^{+\infty} \frac{-2}{x} dx = \lim_{k \to +\infty} \int_{1+I_1+I_2}^{k} \frac{-2}{x} dx = \lim_{k \to +\infty} -2 ln\, x|_{1+I_1+I_2}^{k}$$

$$= \lim_{k \to +\infty} [-2\, ln\, k + 2\, ln(1+I_1+I_2)]$$

$$= \lim_{k \to +\infty} -2\, ln\, k + 2(ln\, 1 + I_1[ln\, 3 - ln\, 2] + I_2[ln\, 2 - ln\, 1])$$

$$= -\infty - 2\, ln \frac{3}{2} I_1 + 2\, ln\, 2\, I_2$$

$$= \infty_{I_i}\, ; i = 1{,}2$$ (The improper 2- refined neutrosophic integral diverges)

Example 9

$$\int_{4+8I_1+2I_2}^{+\infty} \frac{-1}{x(ln\, x)^2} dx = \lim_{k \to +\infty} \int_{4+8I_1+2I_2}^{k} \frac{-1}{x(ln\, x)^2} dx = \lim_{k \to +\infty} \frac{1}{ln\, x}\bigg|_{4+8I_1+2I_2}^{k}$$

$$= \lim_{k \to +\infty} \left(\frac{1}{ln\, x}\right) - \frac{1}{ln(4+8I_1+2I_2)} = \frac{1}{ln\, 4 + I_1[ln\, 14 - ln\, 6] + I_2[ln\, 6 - ln\, 4]}$$

$$= \frac{1}{ln\, 4 + I_1\, ln \frac{7}{3} + I_2\, ln \frac{3}{2}}$$

$$= \frac{1}{ln\, 4} - \frac{ln \frac{7}{3}}{(ln\, 2 + ln\, 7)(ln\, 2 + ln\, 3)} I_1 - \frac{ln \frac{3}{2}}{ln\, 4\, (ln\, 2 + ln\, 3)} I_2$$

(Then the improper 2- refined neutrosophic integral converges)

Remark 1

The improper neutrosophic integral of $f(x, I_1, I_2)$ over $(-\infty, \infty)$ is define as:

$$\int_{-\infty}^{+\infty} f(x, I_1, I_2) dx = \int_{-\infty}^{c} f(x, I_1, I_2) dx + \int_{c}^{+\infty} f(x, I_1, I_2) dx$$

Example 10

$$\int_{-\infty}^{+\infty} \frac{dx}{4 + 11I_1 + 21I_2 + x^2} = \int_{-\infty}^{0+0I_1+0I_2} \frac{dx}{4 + 11I_1 + 21I_2 + x^2} + \int_{0+0I_1+0I_2}^{+\infty} \frac{dx}{4 + 11I_1 + 21I_2 + x^2}$$

$$= \lim_{k_1 \to -\infty} \int_{k_1}^{0+0I_1+0I_2} \frac{dx}{4 + 11I_1 + 21I_2 + x^2} + \lim_{k_2 \to +\infty} \int_{0+0I_1+0I_2}^{k_2} \frac{dx}{4 + 11I_1 + 21I_2 + x^2}$$

$$= \lim_{k_1 \to -\infty} \frac{1}{2 + I_1 + 3I_2} tan^{-1}\left(\frac{x}{2 + I_1 + 3I_2}\right)\Big|_{k_1}^{0+0I_1+0I_2} + \lim_{k_2 \to +\infty} \frac{1}{2 + I_1 + 3I_2} tan^{-1}\left(\frac{x}{2 + I_1 + 3I_2}\right)\Big|_{0+0I_1+0I_2}^{k_2}$$

$$= \lim_{k_1 \to -\infty} \left[\frac{1}{2 + I_1 + 3I_2} tan^{-1}(0) - \frac{1}{2 + I_1 + 3I_2} tan^{-1} \frac{k_1}{2 + I_1 + 3I_2}\right] + \lim_{k_2 \to +\infty} \left[\frac{1}{2 + I_1 + 3I_2} tan^{-1} \frac{k_2}{2 + I_1 + 3I_2} - \frac{1}{2 + I_1 + 3I_2} tan^{-1}(0)\right]$$

$$= \lim_{k_1 \to -\infty} \left[-\frac{1}{2 + I_1 + 3I_2} tan^{-1}\left(\frac{1}{2} - \frac{1}{30} I_1 - \frac{3}{10} I_2\right) k_1\right] + \lim_{k_2 \to +\infty} \left[\frac{1}{2 + I_1 + 3I_2} tan^{-1}\left(\frac{1}{2} - \frac{1}{30} I_1 - \frac{3}{10} I_2\right) k_2\right]$$

$$= \lim_{k_1 \to -\infty} \left[-\frac{1}{2 + I_1 + 3I_2}\left(tan^{-1}\left(\frac{1}{2}k_1\right) + I_1\left[tan^{-1}\left(\frac{1}{6}k_1\right) - tan^{-1}\left(\frac{1}{5}k_1\right)\right] + I_2\left[tan^{-1}\left(\frac{1}{5}k_1\right) - tan^{-1}\left(\frac{1}{2}k_1\right)\right]\right)\right] + \lim_{k_2 \to +\infty} \left[\frac{1}{2 + I_1 + 3I_2}\left(tan^{-1}\left(\frac{1}{2}k_2\right) + I_1\left[tan^{-1}\left(\frac{1}{6}k_2\right) - tan^{-1}\left(\frac{1}{5}k_2\right)\right] + I_2\left[tan^{-1}\left(\frac{1}{5}k_2\right) - tan^{-1}\left(\frac{1}{2}k_2\right)\right]\right)\right]$$

$$= -\frac{1}{2 + I_1 + 3I_2}\left(\frac{-\pi}{2} + I_1\left[\frac{-\pi}{2} + \frac{\pi}{2}\right] + I_2\left[\frac{-\pi}{2} + \frac{\pi}{2}\right]\right) + \frac{1}{2 + I_1 + 3I_2}\left(\frac{\pi}{2} + I_1\left[\frac{\pi}{2} - \frac{\pi}{2}\right] + I_2\left[\frac{\pi}{2} - \frac{\pi}{2}\right]\right)$$

$$= \frac{\pi}{2 + I_1 + 3I_2} = \frac{\pi}{2} - \frac{\pi}{30} I_1 - \frac{3\pi}{10} I_2$$

The improper 2- refined neutrosophic integral converges.

2.2 The second kind of the improper 2- refined neutrosophic integrals

Definition 3

If $f(x, I_1, I_2)$ is continuous on the interval $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$ and not defined at $r_2 + s_2I_1 + t_2I_2$, then the improper 2- refined neutrosophic integral of $f(x, I_1, I_2)$ over the interval $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$ is defined as:

$$\int_{r_1+s_1I_1+t_1I_2}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)dx = \lim_{k \to r_2+s_2I_1+t_2I_2^-} \int_{r_1+s_1I_1+t_1I_2}^{k} f(x, I_1, I_2)dx$$

The improper 2- refined neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper 2- refined neutrosophic integral diverges (when the improper 2- refined neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 11

$$\int_{0+0I_1+0I_2}^{2+3I_1+4I_2} \frac{1}{2+3I_1+4I_2-x}\,dx = \lim_{k\to 2+3I_1+4I_2^{-}} \int_{0+0I_1+0I_2}^{k} \frac{1}{2+3I_1+4I_2-x}\,dx$$

$$= \lim_{k\to 2+3I_1+4I_2^{-}} - ln(2+3I_1+4I_2-x)|_{0+0I_1+0I_2}^{k}$$

$$= \lim_{k\to 2+3I_1+4I_2^{-}} - ln(2+3I_1+4I_2-k) + ln(2+3I_1+4I_2)$$

$$= +\infty + ln\,2 + I_1(ln\,9 - ln\,6) + I_2(ln\,6 - 2)$$

$$= +\infty_{I_i}\,;\, i = 1,2$$

So, the improper 2- refined neutrosophic integral diverges
Example 12

$$\int_{0+0I_1+0I_2}^{3+4I_1+I_2} \frac{dx}{\sqrt{9-9I_1+16I_2-x^2}} = \lim_{k\to 3+4I_1+I_2^{-}} \int_{0+0I_1+0I_2}^{k} \frac{dx}{\sqrt{9-9I_1+16I_2-x^2}}$$

$$= \lim_{k\to 3+4I_1+I_2^{-}} sin^{-1}\left(\frac{x}{3+4I_1+I_2}\right)\Big|_{0+0I_1+0I_2}^{k} = \lim_{k\to 3+4I_1+I_2^{-}} sin^{-1}\left(\frac{k}{3+4I_1+I_2}\right) - sin^{-1}(0)$$

$$= sin^{-1}\left(\frac{3+4I_1+I_2}{3+4I_1+I_2}\right) = \frac{\pi}{2}$$

The improper 2- refined neutrosophic integral converges.

Definition 4

If $f(x, I_1, I_2)$ is continuous on the interval $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$, except for a singular point (discontinuity) $r_1 + s_1I_1 + t_1I_2$, then the improper 2- refined neutrosophic integral of $f(x, I_1, I_2)$ over the interval $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$ is defined as:

$$\int_{r_1+s_1I_1+t_1I_2}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)dx = \lim_{k\to r_1+s_1I_1+t_1I_2^{+}} \int_{k}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)dx$$

The improper 2- refined neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper 2- refined neutrosophic integral diverges (when the improper 2- refined neutrosophic integral doesn't exist or equal to $\pm\infty$).

Remark 2

If$f(x, I_1, I_2)$ is a continuous neutrosophic function over $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$ except value on in $(r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2)$, then the improper 2- refined neutrosophic integral of$f(x, I_1, I_2)$ over interval $[r_1 + s_1I_1 + t_1I_2, r_2 + s_2I_1 + t_2I_2]$ is defined as:

$$\int_{r_1+s_1I_1+t_1I_2}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)dx = \int_{r_1+s_1I_1+t_1I_2}^{k} f(x, I_1, I_2)dx + \int_{k}^{r_2+s_2I_1+t_2I_2} f(x, I_1, I_2)dx$$

Where k is any 2- refined number. The improper 2- refined neutrosophic integral is convergent if the first term and second term on the right side of the various integral (both of them) are convergent, while the integral is divergent if either term (at least one or both) are divergent.

Example 13

$$\int_{1+I_1+I_2}^{7+6I_1+8I_2} \frac{2dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}}$$

Clear that the 2- refined neutrosophic integral is improper and the integrand neutrosophic function is not defined at $3+I_1+2I_2$, where $1+I_1+I_2 < 3+I_1+2I_2 < 7+6I_1+8I_2$, then:

$$\int_{1+I_1+I_2}^{7+6I_1+8I_2} \frac{dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}} = \int_{1+I_1+I_2}^{3+I_1+2I_2} \frac{dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}} + \int_{3+I_1+2I_2}^{7+6I_1+8I_2} \frac{dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}}$$

$$= \lim_{k\to 3+I_1+2I_2^-} \int_{1+I_1+I_2}^{k} \frac{dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}} + \lim_{k\to 3+I_1+2I_2^+} \int_{k}^{7+6I_1+8I_2} \frac{dx}{\sqrt[3]{(2x-6-2I_1-4I_2)^2}}$$

$$= \lim_{k\to 3+I_1+2I_2^-} \sqrt[3]{2x-6-2I_1-4I_2}\Big|_{1+I_1+I_2}^{k} + \lim_{k\to 3+I_1+2I_2^+} \sqrt[3]{2x-6-2I_1-4I_2}\Big|_{k}^{7+6I_1+8I_2}$$

$$= \lim_{k\to 3+I_1+2I_2^-} \sqrt[3]{2k-6-2I_1-4I_2} - \sqrt[3]{2+2I_1+2I_2-6-2I_1-4I_2}$$
$$+ \sqrt[3]{14+12I_1+16I_2-6-2I_1-4I_2} - \lim_{k\to 3+I_1+2I_2^+} \sqrt[3]{2k-6-2I_1-4I_2}$$

$$= \sqrt[3]{4+0I_1+2I_2} + \sqrt[3]{8+10I_1+12I_2}$$

$$= \sqrt[3]{4} + I_1\left[\sqrt[3]{6}-\sqrt[3]{6}\right] + I_2\left[\sqrt[3]{6}-\sqrt[3]{4}\right] + 2 + I_1\left[\sqrt[3]{30}-\sqrt[3]{20}\right] + I_2\left[\sqrt[3]{20}-2\right]$$

$$= 2 + \sqrt[3]{4} + I_1\left[\sqrt[3]{30}-\sqrt[3]{20}\right] + I_2\left[\sqrt[3]{20}+\sqrt[3]{6}-\sqrt[3]{4}-2\right]$$

The improper 2- refined neutrosophic integral converges.

Example 14

$$\int_{-4-I_1-2I_2}^{4+I_1+2I_2} \frac{dx}{\sqrt{16+15I_1+18I_2-x^2}}$$

$$= \int_{-4-I_1-2I_2}^{0+0I_1+0I_2} \frac{dx}{\sqrt{16+13I_1+20I_2-x^2}} + \int_{0+0I_1+0I_2}^{4+I_1+2I_2} \frac{dx}{\sqrt{16+13I_1+20I_2-x^2}}$$

$$= \lim_{k_1\to -4-I_1-2I_2^+} \int_{-4-I_1-2I_2}^{0+0I_1+0I_2} \frac{dx}{\sqrt{16+13I_1+20I_2-x^2}} + \lim_{k_2\to 4+I_1+2I_2^-} \int_{0+0I_1+0I_2}^{4+I_1+2I_2} \frac{dx}{\sqrt{16+13I_1+20I_2-x^2}}$$

$$= \lim_{k_1\to -4-I_1-2I_2^+} sin^{-1}\left(\frac{x}{4+I_1+2I_2}\right)\Big|_{k_1}^{0+0I_1+0I_2} + \lim_{k_2\to 4+I_1+2I_2^-} sin^{-1}\left(\frac{x}{4+I_1+2I_2}\right)\Big|_{0+0I_1+0I_2}^{k_2}$$

$$= sin^{-1}(0) - \lim_{k_1 \to -4-I_1-2I_2^{+}} sin^{-1}\left(\frac{x}{4+I_1+2I_2}\right) + \lim_{k_2 \to 4+I_1+2I_2^{-}} sin^{-1}\left(\frac{x}{4+I_1+2I_2}\right) - sin^{-1}(0)$$

$$= -sin^{-1}\left(\frac{-4-I_1-2I_2}{4+I_1+2I_2}\right) + sin^{-1}\left(\frac{4+I_1+2I_2}{4+I_1+2I_2}\right) = -sin^{-1}(-1) + sin^{-1}(1) = \pi$$

The improper 2- refined neutrosophic integral converges.

3. Conclusions

In this paper, we presented the concept of improper 2-refined neutrosophic integrals by introducing several formal definitions that establish a clear foundation for understanding this type of integral. Through our analysis, we derived a number of key results related to their behavior. Notably, we showed that an improper 2-refined neutrosophic integral may exhibit either convergence or divergence, depending on the nature of the function and the domain involved.

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