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Infinitely-Middle State, Probabilistic Health and Interval Energy Forecasting: A Neutrosophic Framework for Electric-Vehicle Electrochemical Energy Storage Technology

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Abstract: This paper presents a new mathematical framework to evaluate the performance of electrochemical storage cells used in electric vehicles. The method applies neutrosophic logic and probability to model uncertainty, inconsistency, and partial truth in cell behavior. Three original components are introduced. First, the law of infinitely many middles is used to describe each battery cell's operational state using neutrosophic triplets. Second, a statistical health model is developed based on neutrosophic means and deviations. Third, a neutrosophic interval-based forecasting model predicts future energy delivery under uncertain conditions. All concepts are defined within a neutrosophic ring structure to ensure mathematical consistency. The paper includes full equations, definitions, and detailed examples with real-number calculations. Tables and case studies are provided to demonstrate the model's validity. This approach offers a structured and realistic tool to support energy reliability decisions in electric vehicle systems.

Keywords: Neutrosophic modeling, battery health, cell uncertainty, electric vehicle storage, energy forecasting, probabilistic reliability, triplet logic.

1. Introduction

Electrochemical energy storage systems are critical to the performance and reliability of electric vehicles (EVs). These systems must provide consistent power under diverse thermal, mechanical, and electrical conditions. However, real-world battery cells exhibit complex behaviors that are difficult to predict with certainty. Factors such as thermal gradients, material aging, and sensor inaccuracies introduce uncertainties that conventional models, relying on binary or probabilistic approaches, often fail to capture adequately [1]. These traditional methods tend to oversimplify the dynamic interplay of efficiency and degradation, neglecting partial states or contradictory behaviors observed during operation.

To address these limitations, this study proposes a mathematical framework based on neutrosophic logic, which extends classical logic by incorporating three independent components: truth (T), indeterminacy (I), and falsehood (F). This approach allows for the representation of ambiguous, uncertain, or conflicting states, making it particularly suitable for electrochemical cells where parameters like state-of-charge (SoC), temperature, or degradation are often imprecise [1]. Unlike classical logic, which assumes mutually exclusive states (e.g., functional or failed), neutrosophic logic accommodates a spectrum of partial states, providing a more realistic depiction of battery behavior [2].

One of the recent developments in neutrosophic logic is the Law of Included Infinitely-Many-Middles, which was introduced by Smarandache in 2023. This law generalizes the classical law of the excluded middle by allowing not just a single neutral value, but an *infinite spectrum* of neutralities between an assertion and its negation, formally represented as (<A>; <neutA1>, <neutA2>, …, <neutA∞>; <antiA>) [16].

The paper introduces three innovative concepts. First, it employs the Law of Infinitely-Many-Middles to model a cell's operating state, recognizing that a cell's condition is not strictly healthy or failed but can exist in a continuum of intermediate states [2]. Second, it develops Neutrosophic Probabilistic Health Modeling, a statistical approach that uses neutrosophic means and deviations to quantify uncertain health parameters, offering a robust method to assess cell reliability [3]. Third, it proposes Neutrosophic Interval Energy Forecasting, a tool that predicts future discharge potential using neutrosophic probability intervals, providing more flexible estimates than fixed predictions [4].

These components are integrated through a neutrosophic ring structure, which provides an algebraic foundation for consistent operations among uncertain quantities [5]. This structure enables the systematic computation of neutrosophic energy metrics, ensuring logical coherence and practical applicability in energy storage systems.

The paper is organized as follows: Section 2 reviews existing literature and highlights research gaps. Section 3 describes the methodology, introducing neutrosophic rings in the context of energy storage. Section 4 presents the proposed model with formal definitions, equations, and numerical examples. Section 5 analyzes the results, Section 6 discusses implications and limitations, and Section 7 concludes with key findings and potential applications.

2. Literature Review

Research on electrochemical energy storage for EVs has progressed significantly over the past two decades, with much of the focus on deterministic models based on thermodynamics, equivalent circuit models, or electrochemical impedance spectroscopy [6]. These models provide valuable insights into cell behavior but rely heavily on precise input data, which is often unavailable in real-world driving conditions [7]. Variability in factors such as temperature, load, and material properties introduces significant uncertainty that deterministic approaches struggle to address.

Statistical methods, such as Monte Carlo simulations and probabilistic degradation models, have been employed to account for uncertainty in battery life predictions [8]. However, these techniques typically assume well-defined probability distributions, limiting their ability to capture partial contradictions or indeterminate states prevalent in complex battery systems [9]. For instance, measurements like internal resistance or temperature may exhibit inconsistencies due to chemical inhomogeneities or sensor errors, which classical statistical models often treat as noise rather than meaningful data [10].

Fuzzy logic has emerged as a promising approach to handle uncertainty in battery systems, with fuzzy membership functions used to represent parameters like SoC, state of health (SoH), and thermal safety margins [11]. Despite their advantages, fuzzy systems assume a continuous transition between extremes (e.g., fully charged vs. depleted) and fail to distinguish between true indeterminacy and incomplete knowledge [12]. They also cannot model simultaneous degrees of truth and falsehood, which are critical when diagnosing conflicting signals from multiple subsystems, such as self-heating effects or sensor inaccuracies [13].

Neutrosophic logic addresses these shortcomings by introducing a three-valued structure (T, I, F), allowing for a more nuanced representation of uncertainty [1]. The Law of Infinitely-Many-Middles, a key principle in neutrosophic logic, posits that system states exist on a continuous spectrum between full functionality and complete failure, aligning closely with the observed behavior of battery cells in degraded or ambiguous conditions [2]. Neutrosophic statistics further enhance this framework by defining means, variances, and distributions that preserve inconsistencies rather than dismissing them as outliers, making them highly relevant for battery monitoring [3].

Despite these advancements, the application of neutrosophic logic and statistics to electrochemical energy storage remains underexplored. Previous studies have applied neutrosophic concepts to fields like decision-making and reliability engineering [14], but none have directly addressed battery systems in EVs. Furthermore, the use of neutrosophic rings, algebraic structures that support operations among neutrosophic values, has not been explored in energy systems engineering [15]. This study bridges these gaps by developing a cohesive neutrosophic framework that integrates logic, probability, and algebra to model electrochemical cell behavior with greater accuracy and realism.

3. Methodology

This section introduces the mathematical framework used to represent, analyze, and compute the reliability and energy behavior of electrochemical cells in electric vehicles using neutrosophic structures. The proposed methodology is built on three key components:

1. Neutrosophic logic for describing uncertain and contradictory information.
2. Neutrosophic rings as an algebraic structure for modeling cell behavior.

3. Neutrosophic statistics for aggregating and analyzing cell health.

We start by defining all core elements and operations, then demonstrate their application with full numerical examples and an interpreted dataset.

3.1 Neutrosophic Triplet Logic

Each real-world phenomenon, such as the state of a battery cell, is rarely absolutely true or false. Neutrosophic logic models such states as triplets:

$$C = (T, I, F)$$

Where:

$T \in [0,1]$: degree of truth (confirmed operation),

$I \in [0,1]$: degree of indeterminacy (uncertainty).

$F \in [0,1]$: degree of falsehood (failure tendency).

There is no constraint on $T + I + F$, and each component is independent, allowing simultaneous partial truth and falsehood.

3.2 Neutrosophic Rings

We now define the structure in which electrochemical cell computations occur.

Definition 1: Neutrosophic Ring

Let $\mathbb{Z}_n$ be the ring of integers modulo n. A neutrosophic ring is:

$$R = (\mathbb{Z}_n(I))_{f_n}$$

Where:

I is the indeterminate symbol with property $I^2 = I$.

Elements of R are of the form $a + bI$, where $a, b \in \mathbb{Z}_\mathrm{n}$.

f_I is a neutrosophic fuzzification function assigning indeterminacy weights.

This ring allows combination of certain and uncertain values within the same algebraic space.

3.3 Cell Representation in Neutrosophic Rings

Definition 2: Cell as Ring Element

A battery cell C_k is expressed in the neutrosophic ring as:

$$C_k = T_k + I_k \cdot I + F_k \cdot I^2$$

Applying the identity $I^2 = I$, this simplifies to:

$$C_k = T_k + (I_k + F_k) \cdot I$$

Where $T_k, I_k, F_k \in [0,1]$ are the components of the neutrosophic triplet.

3.4 Arithmetic Operations

Let $C_1 = T_1 + (I_1 + F_1)I$ and $C_2 = T_2 + (I_2 + F_2)I$.

Addition:

$$C_1 + C_2 = (T_1 + T_2) + [(I_1 + F_1) + (I_2 + F_2)]I$$

Multiplication (Expanded Form):

$$C_1 \cdot C_2 = T_1T_2 + [T_1(I_2 + F_2) + T_2(I_1 + F_1) + (I_1 + F_1)(I_2 + F_2)] \cdot I$$

This multiplication supports interactions under indeterminacy, such as energy demand/load relationships between cells.

3.5 Dataset and Neutrosophic Triplet

We present real-like values in Table 1 for four battery cells to demonstrate computations.

Table 1. Neutrosophic Triplets for Sample EV Cells

Cell ID	T (Truth - Reliability)	1 (Indeterminacy Uncertainty)	F (Falsehood - Degradation)
C1	0.92	0.04	0.04
C2	0.85	0.10	0.05
C3	0.70	0.15	0.15
C4	0.55	0.20	0.25

Table 1 represents a sample of cells in different health conditions. These triplets form the basis for the following statistical calculations.

3.6 Neutrosophic Mean

Definition 3: Neutrosophic Mean

For N cells, the mean triplet is:

$$\bar{C} = \left(\frac{1}{N}\sum_{i=1}^{N} T_i, \frac{1}{N}\sum_{i=1}^{N} I_i, \frac{1}{N}\sum_{i=1}^{N} F_i\right)$$

Using Table 1 values:

$$\bar{T} = \frac{0.92 + 0.85 + 0.70 + 0.55}{4} = 0.755$$

$$\bar{I} = \frac{0.04 + 0.10 + 0.15 + 0.20}{4} = 0.1225$$

$$\bar{F} = \frac{0.04 + 0.05 + 0.15 \div 0.25}{4} = 0.1225$$

$$\bar{C} = (0.755, 0.1225, 0.1225)$$

This triplet summarizes the average performance of the battery module.

3.7 Neutrosophic Variance and Standard Deviation

Definition 4: Neutrosophic Variance

$$\sigma_T^2 = \frac{1}{N}\sum (T_i - \bar{T})^2, \sigma_I^2 = \frac{1}{N}\sum (I_i - \bar{I})^2, \sigma_F^2 = \frac{1}{N}\sum (F_i - \bar{F})^2$$

Definition 5: Neutrosophic Standard Deviation

$$\sigma = \left(\sqrt{\sigma_T^2}, \sqrt{\sigma_I^2}, \sqrt{\sigma_F^2}\right)$$

Using the data from Table 1:

$$\sigma_T^2 = \frac{(0.92 - 0.755)^2 + (0.85 - 0.755)^2 + (0.70 - 0.755)^2 + (0.55 - 0.755)^2}{4} = 0.023625$$

$$\sigma_I^2 = \frac{(0.04 - 0.1225)^2 + (0.10 - 0.1225)^2 + (0.15 - 0.1225)^2 + (0.20 - 0.1225)^2}{4} = 0.003125$$

$$\sigma_F^2 = \frac{(0.04 - 0.1225)^2 + (0.05 - 0.1225)^2 + (0.15 - 0.1225)^2 + (0.25 - 0.1225)^2}{4} = 0.007775$$

Taking square roots:

$\sigma_T = \sqrt{0.023625} \approx 0.1538$
$\sigma_I = \sqrt{0.003125} \approx 0.0559$
$\sigma_F = \sqrt{0.007775} \approx 0.0882$

$$\sigma = (0.1538, 0.0559, 0.0882)$$

This reflects moderate variability in truth, low variability in uncertainty, and noticeable variability in degradation.

3.8 Group Interaction Example

Add two cells:

1. $C_1 = 0.70 + (0.15 + 0.15)I = 0.70 + 0.30I$
2. $C_2 = 0.55 + (0.20 + 0.25)I = 0.55 + 0.45I$

Addition:

$$C_1 + C_2 = (0.70 + 0.55) + (0.30 + 0.45)I = 1.25 + 0.75I$$

This combined state shows strong certainty (1.25) but significant indeterminacy (0.75), indicating potential risk in parallel discharge.

4. Proposed Model

This section introduces three original modeling frameworks that address uncertainty and degradation in electric vehicle battery cells. Each model operates independently but shares a neutrosophic mathematical foundation. These models enable:

1. Real-time interpretation of gradual performance loss using the Infinitely-Middle State Model.
2. Health scoring of cells through Neutrosophic Statistical Estimation.
3. Energy output prediction using Interval-Based Neutrosophic Forecasting.

4.1. Infinitely-Middle State Model

This model represents a cell's condition at any moment as a non-binary logic triplet. It does not assume clear transitions between "healthy" and "failing" states. Instead, it allows a smooth trajectory within an infinite spectrum of intermediate conditions.

$$S(t) = (T(t), I(t), F(t))$$

Where:

$T(t)$: degree of confident operation,
$I(t)$: measure of uncertainty in the state,
$F(t)$: estimated likelihood of functional failure.

These values are collected from sensors and diagnostic signals. Below is a tracking example.

Table 2. Neutrosophic State Trajectory Over Time

Time (min)	$T(t)$	$I(t)$	$F(t)$
0	0.98	0.01	0.01
5	0.95	0.03	0.02
10	0.82	0.10	0.08
15	0.70	0.15	0.15
20	0.60	0.20	0.20

The cell begins in an excellent state. Over time, its reliability declines, while both uncertainty and failure likelihood increase. At $t = 20$, the cell is still active but shows signs of instability.

Computation of Relative Instability Index (RII):
To quantify instability between two timestamps:

$$\mathrm{R\Pi}(t) = \left|\frac{F(t) - T(t)}{T(t)}\right|$$

At $t = 10$:

$$\mathrm{R\Pi}(10) = \left|\frac{0.08 - 0.82}{0.82}\right| = 0.902$$

At $t = 20$:

$$\mathrm{R\Pi}(20) = \left|\frac{0.20 - 0.60}{0.60}\right| = 0.667$$

Although reliability drops at $t = 20$, relative instability is slightly lower due to proportional change.

4.2. Neutrosophic Statistical Health Estimation

This model summarizes the health of each battery cell over time using neutrosophic statistical aggregates.

Each cell is described by a tuple:

$$H_k = \left(\mu_T^{(k)}, \mu_I^{(k)}, \mu_F^{(k)}; \sigma_T^{(k)}, \sigma_I^{(k)}, \sigma_F^{(k)}\right)$$

Where:
μ : mean values across observed periods,
σ : standard deviations indicating variability.

Table 3. Neutrosophic Health Parameters for EV Cells

Cell	μ^T	μI	μF	σ_T	σ_I	σ_F
C1	0.91	0.05	0.04	0.04	0.01	0.01
C2	0.80	0.12	0.08	0.06	0.03	0.02
C3	0.65	0.18	0.17	0.08	0.04	0.05

From Table 3:
1. C1 is highly stable.
2. C3 has high mean failure and uncertainty, with wide standard deviations, signaling rapid degradation and instability.

Calculation of Neutrosophic Health Score (NHS):

$$\mathrm{NHS}_k = \mu_T^{(k)} - \mu_F^{(k)} - \sigma_F^{(k)}$$

For C2:

$$\mathrm{NHS}_2 = 0.80 - 0.08 - 0.02 = 0.70$$

For C3:

$$\mathrm{NHS}_3 = 0.65 - 0.17 - 0.05 = 0.43$$

A lower NHS implies lower confidence in the cell's long-term viability.

4.3. Neutrosophic Interval-Based Forecasting

This model predicts how much usable energy will likely be delivered over future intervals. Instead of a single estimate, a neutrosophic triplet is produced for each time window:

$$E(t) = (T_E(t), I_E(t), F_E(t))$$

Where:

$T_E(t)$: expected energy fulfillment ratio,

$I_E(t)$: energy prediction uncertainty.

$F_E(t)$: probability of underperformance or energy dropout.

Table 4. Neutrosophic Forecasting of Energy Delivery

Interval (min)	T_E	I_E	F_E
0-10	0.90	0.05	0.05
10-20	0.78	0.12	0.10
20-30	0.65	0.18	0.17

From Table 4:

1. The first interval has a high expected output and low uncertainty.
2. As time progresses, both uncertainty and failure probability increase, making energy prediction riskier.

Computation of Discharge Approval Metric (DAM):

Define:

$$\mathrm{DAM}(t) = T_E(t) - [I_E(t) + F_E(t)]$$

For $0 - 10$ min :

$$\mathrm{DAM}(0 - 10) = 0.90 - (0.05 + 0.05) = 0.80$$

For $20 - 30$ min :

$$\mathrm{DAM}(20 - 30) = 0.65 - (0.18 + 0.17) = 0.30$$

Assuming a minimum DAM threshold of 0.60 for stable operation:

$0 - 10$ min → Approved

$20 - 30$ min → Not approved (high failure risk)

5. Results and Evaluation

This section analyzes the outputs from the three neutrosophic models introduced in Section 4. Using the datasets and computed metrics from Tables 2, 3, and 4, we assess how each model captures battery behavior under uncertainty and degradation. The results are interpreted both mathematically and operationally.

5.1 Infinitely-Middle State Model Results

Using the data from Table 2, we observed that the neutrosophic truth value $T(t)$ of the battery cell decreased from 0.98 to 0.60 over 20 minutes. At the same time:
$I(t)$ rose from 0.01 to 0.20
$F(t)$ rose from 0.01 to 0.20
This shows not only performance degradation but increasing ambiguity and failure risk, which are not detectable in classical binary models. This means:

1. Classical systems would only respond to sharp failures.
2. Here, the Relative Instability Index (RII) showed a critical spike at $t = 10(0.902)$, far before reliability dropped to dangerous levels.
3. This model therefore predicts behavioral instability earlier, making it more effective for preemptive actions.

5.2 Health Estimation Analysis

From Table 3, we compared neutrosophic health scores (NHS) across three cells. Key findings are presented in Table 5.

Table 5. Key findings

Cell	NHS
C1	0.86
C2	0.70
C3	0.43

Explanation from Table 5:

1. C1 is healthy and stable: high truth means, low failure.
2. C2 shows moderate uncertainty but is operational.
3. C3 is unstable due to both high failure mean and variance.

This type of scoring offers quantitative differentiation between cells with the same overall output but very different risks - a key advantage in real-time battery management systems (BMS).

5.3 Forecasting Output Analysis

Using Table 4, the Discharge Approval Metric (DAM) provides interval-specific energy confidence presented in Table 6.

Table 6. DAM

Interval	DAM	Decision
0-10	0.80	Approve
10-20	0.56	Reject
20-30	0.30	Reject

From Table 6:

1. The first interval meets discharge thresholds.
2. Later intervals fail due to compounded uncertainty and risk.

3. Classical energy forecast models would simply average output, potentially approving intervals with hidden failure potential.

5.4 General Remarks

1. Neutrosophic structures allow multiple truth evaluations to co-exist. A cell may be partially healthy and partially failing at the same time.
2. Indeterminacy is treated as a measurable property rather than error or noise. This is critical in physical systems like batteries, where ambiguity is natural.
3. Metrics like RII, NHS, and DAM offer clear decision rules while preserving the complexity of the system.

6. Discussion

This section explains what the results mean and how they affect electric vehicle (EV) performance, without repeating earlier equations or tables.

6.1 A New Way to See Battery Behavior

Most current battery monitoring systems rely on fixed rules: a cell is either working or not. But the models in this study show that a battery can be in many possible states: slightly working, mostly working, partially failing, and even uncertain. This "in-between" understanding gives engineers a clearer picture of what's happening inside the cells.
For example, a cell might show good output while quietly building up heat or stress. With neutrosophic modeling, such behavior appears early, even when failure hasn't yet occurred. This can prevent unexpected breakdowns and make vehicle systems safer.

6.2 Practical Insights for EV Design

The results also show how battery data can be used in smarter ways:

1. The state model tells when a cell is starting to behave strangely, even if it still works.
2. The health model helps rank all cells from best to worst, so weak cells can be isolated or replaced.
3. The forecast model ensures energy is drawn only from safe cells, reducing load on risky ones.

Together, these models allow fine control over how energy flows in the battery. This is very important for EVs, where both safety and energy efficiency are critical.

6.3 Interpreting Uncertainty as Information

In most systems, uncertainty is treated as a problem or ignored. But in this research, uncertainty is treated as useful information. For example:

I. A small increase in uncertainty may show sensor drift.
II. A sharp rise in uncertainty may show thermal instability.

By including uncertainty directly in calculations, we get a more realistic view of what's happening inside the battery, especially under load or in bad weather.

6.4 Beyond Electric Vehicles

Although this model was built for EV batteries, the same approach can be used in other fields:

1. In renewable energy systems, to manage unpredictable solar or wind inputs.
2. In medical devices, to monitor implants with uncertain sensor data.
3. In aerospace, where system health must be monitored under extreme conditions.

Anywhere uncertainty matters, neutrosophic models provide a powerful tool.

7. Conclusion

This paper introduces three new models that help us understand and manage battery cells in electric vehicles more accurately. Each model uses neutrosophic logic, which allows us to include uncertainty and partial information directly in the analysis.

We showed how to:

1. Track changes in a cell's behavior over time, even when those changes are unclear.
2. Measure the overall health of each cell using smart statistics.
3. Predict future energy output while accounting for risks and unknowns.

These models work together to give a full view of the battery system, helping engineers make safer and smarter decisions. The results confirm that this method is useful, reliable, and ready to be tested in real EV systems or other energy technologies.

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