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A certain algebraic structure of bipolar single value neutrosophic subgroups and its essential properties

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Abstract: The idea of bipolar single value neutrosophic set was created as an extension of a single value neutrosophic set when every single value neutrosophic membership function has two poles. In this study, we apply this idea in an algebraic environment when we initiate the novel concept of bipolar single value neutrosophic subgroups and prove that every bipolar single value neutrosophic subgroup generates two bipolar single value neutrosophic subgroups. we explain the level set, support, kernel for bipolar single value neutrosophic set, bipolar single value neutrosophic characteristic function, and bipolar single value neutrosophic point. Then, we illuminate the bipolar single value neutrosophic subgroup, bipolar single value neutrosophic normal subgroup, bipolar single value neutrosophic conjugate, normalizer for bipolar single value neutrosophic subgroup, bipolar single value neutrosophic abelian subgroup, and bipolar single value neutrosophic factor group. Furthermore, we present the linked theorems and examples and prove these theorems. Finally, we discussed the image and pre-image of bipolar single-value neutrosophic subgroups under homomorphism and proved the related theorems.

Keywords: Neutrosophic set; single-valued neutrosophic set; bipolar single-valued neutrosophic; bipolar single-valued neutrosophic group; bipolar single-valued neutrosophic normal group; homomorphism

1. Introduction

Zadeh [1] in 1965 developed the initial results of fuzzy sets (FSs) theory. This theory has been successfully employed in handling uncertainty and ambiguous information in our real world. The mathematical structure of this theory is characterized by the inclusion of a membership function that represents the truth that an object belongs to a set of knowledge called the universal set. For example, FSs provide suitable solutions to many problems related to bioinformatics and computational biology, such as medical image processing, cell reconstruction, protein structure analysis, gene expression analysis, and medical data classification. Atanassov [2] then came up with the idea of the intuitionistic fuzzy set (IFS) to address uncertainty issues that the truth function of the fuzzy set alone cannot handle. Atanassov 's idea is an extension of the mathematical structure of the FS by adding a second membership function called the falsity function. Smarandache [3] introduced the idea of Neutrosophy from a philosophical point of view to handle the indeterminate information that exists commonly in real situations. Neutrosophy idea considers an extension of the mathematical structure when Smarandache adds a third function, named the indeterminacy-membership function, to study the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra that exist commonly in real situations. To adjust Smarandache's idea and put it into practice, Wang et al. [4] new notion called single valued neutrosophic sets (SVNSs), which is more specified from an engineering point of view. Researchers have been working on employing the concept of SVNS in different mathematical directions.

Researchers have been working on employing the concept of SVNS in different mathematical directions. Abuqamar and Hassan [5] defined the notion of neutrosophic normal soft groups under Q effect and discussed some related properties. Al-Masarwah et al. [6] introduced the idea of the direct product of sets that include Fermat neutrosophic (FN). Alqahtani et al. [7] studied neutrosophic graphs under complex numbers and demonstrated their use in the design of hospital infrastructure. Al-Quran et al. [8,9] made remarkable efforts to put out the notion of SVNS in decision-making problems. Palanikumar et al. [10,11] discuss a new aggregation operator in this environment. Abed [12,13] formulates and studies a new property, namely, the indeterminacy of the hollow module. Romdhini et al. [14,15] present the Q-CNSFs and examine their unique algebraic properties. Al-Qudah et al. [16,17] studied this concept in from many mathematical aspects. Bataihah and Azaymeh [18-20] studied topological aspects and presented many important applications. Moreover, Hameed et al.[21] discussed some characterisation SVN subgroups. Çetkin and Aygün [22] examined the subgroup and normal subgroup structures of single-valued neutrosophic sets. Zhang et al. [23] generalized of the notion of the standard neutrosophic triplet group. On the other hand, these modifications of SVNs cannot avail the positive pole and negative opinion of human thinking. To cover this barrier, Deli et al. [24] introduced the concept of bipolar neutrosophic sets (BNSs) by extending the notion of bipolar fuzzy sets (BFSs) [25] where every SVN-membership has two poles, i.e., positive pole and negative pole. This concept has been studied by many researchers, and many results have been presented about it [26 -30]. But until now, the algebraic concepts and properties have not been studied with BNSs, and this prompted us to present this study.

In this work, we initiate an approach to the group structure of bipolar single-valued neutrosophic sets (BSVNSs). We define bipolar neutrosophic normal subgroups and give some properties of these structures. Moreover, we explain the image and preimage of a bipolar single-valued neutrosophic set and examine the homomorphic image and preimage of a bipolar single-valued neutrosophic (normal) subgroup. In this method, we pick up the generalized form of the fuzzy subgroup and the intuitionistic fuzzy subgroups that are defined on a classical group.

The residual paper is organised into five sections: Section 2 presents the preliminary terms and definitions that we used to propose the results in this work. The main results, i.e., the elementary notions of BNSG and its elementary relevant theorems and examples, are present in Section 3. Also, the main results, i.e., the elementary notions of BNNSG and its elementary relevant theorems and examples, are present in Section 4. Section 5 hands over a summary, limitations, and future extent of the results that are present in this work.

2. Preliminaries

Definition 2.1 The following mathematical frame that is known on $\mathbb{X}$

$$\mathcal{A} = \{m, \langle \Pi_{\mathcal{A}}(m), \Xi_{\mathcal{A}}(m), \Sigma_{\mathcal{A}}(m)\rangle, m \in \mathbb{X}\}$$

Is called NS where the three memberships $\Pi_{\mathcal{A}}(m), \Xi_{\mathcal{A}}(m), \Sigma_{\mathcal{A}}(m) \in [0,1]$ denotes to truth-membership function, indeterminacy-membership function, and falsity-membership function respectively for $m \in \mathbb{X}$ with stander condition $0 \leq \Pi_{\mathcal{A}}(m) + \Xi_{\mathcal{A}}(m) + \Sigma_{\mathcal{A}}(m) \leq 3$.

Definition 2.2 Let

$$\mathcal{A} = \{m, \langle \Pi_{\mathcal{A}}(m), \Xi_{\mathcal{A}}(m), \Sigma_{\mathcal{A}}(m)\rangle, m \in \mathbb{X}\}$$

And

$$\mathcal{B} = \{m, \langle \Pi_{\mathcal{B}}(m), \Xi_{\mathcal{B}}(m), \Sigma_{\mathcal{B}}(m)\rangle, m \in \mathbb{X}\}$$

Then the fundamentals set theory on NS given as following:

i. The subset between two NSs denotes as $\mathcal{A} \subseteq \mathcal{B}$ where

$\Pi_{\mathcal{A}}(m) \leq \Pi_{\mathcal{B}}(m)$, $\Xi_{\mathcal{A}}(m) \geq \Xi_{\mathcal{B}}(m)$ and $\Sigma_{\mathcal{A}}(m) \geq \Sigma_{\mathcal{B}}(m)$.

ii. The equals between two NSs denotes as $\mathcal{A} = \mathcal{B}$ where

$\Pi_{\mathcal{A}}(m) = \Pi_{\mathcal{B}}(m)$, $\Xi_{\mathcal{A}}(m) = \Xi_{\mathcal{B}}(m)$ and $\Sigma_{\mathcal{A}}(m) = \Sigma_{\mathcal{B}}(m)$.

iii. The union between two NSs denotes as $\mathcal{A} \cup \mathcal{B}$ where:

$$\mathcal{A} \cup \mathcal{B} = \begin{cases} max\{\Pi_{\mathcal{A}}(m), \Pi_{\mathcal{B}}(m)\} \\ min\{\Xi_{\mathcal{A}}(m), \Xi_{\mathcal{B}}(m)\} \\ min\{\Sigma_{\mathcal{A}}(m), \Sigma_{\mathcal{B}}(m)\} \end{cases}$$

iv. The intersection between two NSs denotes as $\mathcal{A} \cap \mathcal{B}$ where:

$$\mathcal{A} \cap \mathcal{B} = \begin{cases} min\{\Pi_{\mathcal{A}}(m), \Pi_{\mathcal{B}}(m)\} \\ max\{\Xi_{\mathcal{A}}(m), \Xi_{\mathcal{B}}(m)\} \\ max\{\Sigma_{\mathcal{A}}(m), \Sigma_{\mathcal{B}}(m)\} \end{cases}$$

v. The complement of NS denotes as $\mathcal{A}^C$ where:

$$\mathcal{A}^C = \begin{cases} \Sigma_{\mathcal{A}}(m) \\ 1 - \Xi_{\mathcal{A}}(m) \\ \Pi_{\mathcal{A}}(m) \end{cases}$$

Definition 2.3. A NS-Stricture $\mathcal{A}$ given in definition 2.1 on classical group $(\ddot{\mathfrak{G}}, *)$ is called NS-subgroup of $\ddot{\mathfrak{G}}$ if for $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$ then we have,

i. $\Pi_{\mathcal{A}}(m * n) \geq min\{\Pi_{\mathcal{A}}(m), \Pi_{\mathcal{A}}(n)\}$

$\Xi_{\mathcal{A}}(m * n) \geq min\{\Xi_{\mathcal{A}}(m), \Xi_{\mathcal{A}}(n)\}$

$\Sigma_{\mathcal{A}}(m * n) \leq max\{\Sigma_{\mathcal{A}}(m), \Sigma_{\mathcal{A}}(n)\}$

ii. $\Pi_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Pi_{\mathcal{A}}(m)$.

$\Xi_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Xi_{\mathcal{A}}(m)$.

$\Sigma_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Sigma_{\mathcal{A}}(m)$.

The collection of all terms of N-subgroups of $(\ddot{\mathfrak{G}}, *)$ denotes as NS$(\ddot{\mathfrak{G}})$.

Definition 2.4. A NS-Stricture $\mathcal{A}$ given in definition 2.1 on classical group $(\ddot{\mathfrak{G}},*)$ is called NS-subgroup of $\ddot{\mathfrak{G}}$ if for $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$ then we have,

i. $\Pi_{\mathcal{A}}(m * n) \geq min\{\Pi_{\mathcal{A}}(m), \Pi_{\mathcal{A}}(n)\}$
$\Xi_{\mathcal{A}}(m * n) \geq min\{\Xi_{\mathcal{A}}(m), \Xi_{\mathcal{A}}(n)\}$
$\Sigma_{\mathcal{A}}(m * n) \leq max\{\Sigma_{\mathcal{A}}(m), \Sigma_{\mathcal{A}}(n)\}$

ii. $\Pi_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Pi_{\mathcal{A}}(m)$.
$\Xi_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Xi_{\mathcal{A}}(m)$.
$\Sigma_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Sigma_{\mathcal{A}}(m)$.

The collection of all terms of N-subgroups of $(\ddot{\mathfrak{G}},*)$ denotes as $\mathrm{NS}(\ddot{\mathfrak{G}})$.

Definition 2.4 The following mathematical frame that is known on $\mathbb{X}$

$$\mathcal{A} = \left\{m \in \mathbb{X}, \Pi^{+}_{(\mathcal{A})}(m), \Xi^{+}_{(\mathcal{A})}(m), \Sigma^{+}_{(\mathcal{A})}(m), \Pi^{-}_{(\mathcal{A})}(m), \Xi^{-}_{(\mathcal{A})}(m), \Sigma^{-}_{(\mathcal{A})}(m)\right\}$$

Is called BNS where the three memberships $\Pi^{+}_{(\mathcal{A})}(m), \Xi^{+}_{(\mathcal{A})}(m), \Sigma^{+}_{(\mathcal{A})}(m), \Pi^{-}_{(\mathcal{A})}(m), \Xi^{-}_{(\mathcal{A})}(m), \Sigma^{-}_{(\mathcal{A})}(m) \in [-1,1]$ denotes to positive and negative truth-membership function, positive and negative indeterminacy-membership function, and positive and negative falsity-membership function respectively for $m \in \mathbb{X}$.

3. Bipolar single value neutrosophic subgroups

Here in this section, we going to diagnose the mathematical structure of Bipolar single value neutrosophic subgroups and their fundamentals properties.

Definition 3. 1. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic subset of classical group $\ddot{\mathfrak{G}}$ and $\beta \in [1,-1]$. Then

$$(\mathcal{A})_{\beta} = \left\{\ddot{m} \in \ddot{\mathfrak{G}}, \Pi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Xi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Sigma^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Pi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Xi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Sigma^{-}_{(\mathcal{A})_{\beta}}(\ddot{m})\right\}$$

Where,

$\Pi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Pi^{+}_{(\mathcal{A})} \geq \beta, \beta \in [0,1]\}$ and $\Pi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Pi^{-}_{(\mathcal{A})} \leq \beta, \beta \in [-1,0]\}$,

$\Xi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Xi^{+}_{(\mathcal{A})} \geq \beta, \beta \in [0,1]\}$ and $\Xi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Xi^{-}_{(\mathcal{A})} \leq \beta, \beta \in [-1,0]\}$,

$\Sigma^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Sigma^{+}_{(\mathcal{A})} \leq \beta, \beta \in [0,1]\}$ and $\Sigma^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \{\ddot{m} \in \ddot{\mathfrak{G}} | \Sigma^{-}_{(\mathcal{A})} \geq \beta, \beta \in [-1,0]\}$.

Here we denotes to $(\mathcal{A})_{\beta}$ as bipolar neutrosophic β-level set.

Remark 3.2. For two bipolar neutrosophic subset of classical group $\ddot{\mathfrak{G}}$ i.e $\mathcal{A},\mathcal{B}$ and if $\mathcal{A} \subseteq \mathcal{B}$ and $\in [1,-1]$ then for positive side $\Pi^{+}_{(\mathcal{A})} \leq \Pi^{+}_{(\mathcal{B})}$, $\Xi^{+}_{(\mathcal{A})} \leq \Xi^{+}_{(\mathcal{B})}$, $\Sigma^{+}_{(\mathcal{A})} \geq \Sigma^{+}_{(\mathcal{B})}$ and for negative side $\Pi^{-}_{(\mathcal{A})} \geq \Pi^{-}_{(\mathcal{B})}$, $\Xi^{-}_{(\mathcal{A})} \geq \Xi^{-}_{(\mathcal{B})}$, $\Sigma^{-}_{(\mathcal{A})} \leq \Sigma^{-}_{(\mathcal{B})}$.

Definition 3. 3. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic subset of classical group $\ddot{\mathfrak{G}}$ and $\beta \in [1,-1]$. Then

$$(\mathcal{A})_{S} = \{\ddot{m} \in \ddot{\mathfrak{G}}, \Pi^{+}_{(\mathcal{A})_{S}}(\ddot{m}), \Xi^{+}_{(\mathcal{A})_{S}}(\ddot{m}), \Sigma^{+}_{(\mathcal{A})_{S}}(\ddot{m}), \Pi^{-}_{(\mathcal{A})_{S}}(\ddot{m}), \Xi^{-}_{(\mathcal{A})_{S}}(\ddot{m}), \Sigma^{-}_{(\mathcal{A})_{S}}(\ddot{m})\}$$

Where,

$\Pi^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Pi^{+}_{(\mathcal{A})} \geq 0\}$ and $\Pi^{-}_{(\mathcal{A})}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Pi^{-}_{(\mathcal{A})} \leq 0\}$,

$\Xi^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Xi^{+}_{(\mathcal{A})} \geq 0\}$ and $\Xi^{-}_{(\mathcal{A})}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Xi^{-}_{(\mathcal{A})} \leq 0\}$,

$\Sigma^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Sigma^{+}_{(\mathcal{A})} \leq 0\}$ and $\Sigma^{-}_{(\mathcal{A})}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Sigma^{-}_{(\mathcal{A})} \geq 0\}$.

Here we denotes to $(\mathcal{A})_S$ as bipolar neutrosophic support set.

Definition 3. 4. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic subset of classical group $\ddot{\mathfrak{G}}$ and $\beta \in [1,-1]$. Then

$$(\mathcal{A})_K = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}}, \Pi^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}), \Xi^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}), \Sigma^{+}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}), \Pi^{-}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}), \Xi^{-}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}}), \Sigma^{-}_{(\mathcal{A})_S}(\ddot{\mathfrak{m}})\}$$

Where,

$\Pi^{+}_{(\mathcal{A})_K}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Pi^{+}_{(K)} = 1\}$ and $\Pi^{-}_{(K)}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Pi^{-}_{(K)} = -1\}$,

$\Xi^{+}_{(\mathcal{A})_K}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Xi^{+}_{(K)} = 1\}$ and $\Xi^{-}_{(K)}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Xi^{-}_{(K)} = -1\}$,

$\Sigma^{+}_{(\mathcal{A})_K}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Sigma^{+}_{(K)} = 1\}$ and $\Sigma^{-}_{(K)}(\ddot{\mathfrak{m}}) = \{\ddot{\mathfrak{m}} \in \ddot{\mathfrak{G}} | \Sigma^{-}_{(K)} = -1\}$.

Here we denotes to $(\mathcal{A})_K$ as Kernel of bipolar neutrosophic subset of classical group $\ddot{\mathfrak{G}}$.

Definition 3. 5. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and for any subset $\mathcal{H}$ of a classical group $\ddot{\mathfrak{G}}$. Then the bipolar characteristic function neutrosophic (BCN) is denotes as

$$(\mathcal{A})_{\mathcal{H}} = \{\Pi_{(\mathcal{A})_{\mathcal{H}}}, \Xi_{(\mathcal{A})_{\mathcal{H}}}, \Sigma_{(\mathcal{A})_{\mathcal{H}}}\}$$

Where

$$\Pi_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = \begin{cases} \Pi^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H} \\ \Pi^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \\ \Pi^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = -1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H}' \\ \Pi^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \end{cases}$$

$$\Xi_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = \begin{cases} \Xi^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H} \\ \Xi^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \\ \Xi^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = -1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H} \\ \Xi^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \end{cases}$$

$$\Sigma_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = \begin{cases} \Sigma^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H} \\ \Sigma^{+}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \\ \Sigma^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = -1 & if \quad \ddot{\mathfrak{m}} \in \mathcal{H} \\ \Sigma^{-}_{(\mathcal{A})_{\mathcal{H}}}(\ddot{\mathfrak{m}}) = 0 & if \quad \ddot{\mathfrak{m}} \notin \mathcal{H} \end{cases}$$

Remark 3.6. Evidently generalize a BCN is a BN-subset of classical group $\ddot{\mathfrak{G}}$.

Definition 3. 7. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and for any $\ddot{\mathfrak{m}}, \ddot{\mathfrak{n}} \in \ddot{\mathfrak{G}}$, $\rho \in [-1,1]$. Then the bipolar neutrosophic subset of $\ddot{\mathfrak{G}}$ given as following structure

$$\mathcal{A} = \{\Pi_{(\mathcal{A})}, \Xi_{(\mathcal{A})}, \Sigma_{(\mathcal{A})}\}$$

Where

$$\Pi_{(\mathcal{A})}(\ddot{m}) = \begin{cases} \Pi^{+}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Pi^{+}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \\ \Pi^{-}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Pi^{-}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \end{cases},$$

$$\Xi_{(\mathcal{A})}(\ddot{m}) = \begin{cases} \Xi^{+}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Xi^{+}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \\ \Xi^{-}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Xi^{-}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \end{cases}$$

$$\Sigma_{(\mathcal{A})}(\ddot{m}) = \begin{cases} \Sigma^{+}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Sigma^{+}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \\ \Sigma^{-}_{(\mathcal{A})}(\ddot{m}) = \rho \ \ if \ \ \ddot{m} = \ddot{n} \\ \Sigma^{-}_{(\mathcal{A})}(\ddot{m}) = 0 \ \ if \ \ \ddot{m} \neq \ddot{n} \end{cases}$$

Is said BN point with value , $\rho \in [-1,1]$ and support $\ddot{m}$.

Definition 3. 8. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic set on classical group $\ddot{\mathfrak{G}}$ and $\beta \in [1,-1]$. Then

$$(\mathcal{A})_{\beta} = \left\{\ddot{m} \in \ddot{\mathfrak{G}}, \Pi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Xi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Sigma^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Pi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Xi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}), \Sigma^{-}_{(\mathcal{A})_{\beta}}(\ddot{m})\right\}$$

Where,
$\Pi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Pi^{+}_{(\mathcal{A})} \geq \beta, \beta \in [0,1]\right\}$ and $\Pi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Pi^{-}_{(\mathcal{A})} \leq \beta, \beta \in [-1,0]\right\}$,
$\Xi^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Xi^{+}_{(\mathcal{A})} \geq \beta, \beta \in [0,1]\right\}$ and $\Xi^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Xi^{-}_{(\mathcal{A})} \leq \beta, \beta \in [-1,0]\right\}$,
$\Sigma^{+}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Sigma^{+}_{(\mathcal{A})} \leq \beta, \beta \in [0,1]\right\}$ and $\Sigma^{-}_{(\mathcal{A})_{\beta}}(\ddot{m}) = \left\{\ddot{m} \in \ddot{\mathfrak{G}} \middle| \Sigma^{-}_{(\mathcal{A})} \geq \beta, \beta \in [-1,0]\right\}$.
Here we denotes to $(\mathcal{A})_{\beta}$ as bipolar neutrosophic β-level set.

Definition 3.9. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic set on classical group $\ddot{\mathfrak{G}}$. Then $\mathcal{A}$ is named bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$ if the following conditions satisfies:

i. $\mathcal{A}(\ddot{m} * \ddot{n}) \geq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$. Such that
$\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Pi^{+}_{\mathcal{A}}(\ddot{m}), \Pi^{+}_{\mathcal{A}}(\ddot{n})\}$,
$\Xi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Xi^{+}_{\mathcal{A}}(\ddot{m}), \Xi^{+}_{\mathcal{A}}(\ddot{n})\}$
$\Sigma^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Sigma^{+}_{\mathcal{A}}(\ddot{m}), \Sigma^{+}_{\mathcal{A}}(\ddot{n})\}$.

ii. $\mathcal{A}(\ddot{m}^{-1}) \geq \mathcal{A}(\ddot{m})$. Such that
$\Pi^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{m})$,
$\Xi^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Xi^{+}_{\mathcal{A}}(\ddot{m})$,
$\Sigma^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Sigma^{+}_{\mathcal{A}}(\ddot{m})$.

iii. $(\ddot{m} * \ddot{n}) \leq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$. Such that
$\Pi^{-}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Pi^{-}_{\mathcal{A}}(\ddot{m}), \Pi^{-}_{\mathcal{A}}(\ddot{n})\}$,
$\Xi^{-}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Xi^{-}_{\mathcal{A}}(\ddot{m}), \Xi^{-}_{\mathcal{A}}(\ddot{n})\}$

$\Sigma^{-}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Sigma^{-}_{\mathcal{A}}(\ddot{m}), \Sigma^{-}_{\mathcal{A}}(\ddot{n})\}$.

iv. $\mathcal{A}(\ddot{m}^{-1}) \leq \mathcal{A}(\ddot{m})$. Such that

$\Pi^{-}_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Pi^{-}_{\mathcal{A}}(\ddot{m})$,

$\Xi^{-}_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Xi^{-}_{\mathcal{A}}(\ddot{m})$,

$\Sigma^{-}_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Sigma^{-}_{\mathcal{A}}(\ddot{m})$.

Here the summation of all bipolar neutrosophic subgroups of $\ddot{\mathfrak{G}}$ indicated as $BNS(\ddot{\mathfrak{G}})$.

Example 3.10. Take $\ddot{\mathfrak{G}} = \mathbb{K}_4 = \{1, \ddot{m}, \ddot{n}, \ddot{m} * \ddot{n}\}$ is classical group with following Cayley table and here *is natural multiplication:

*	**1**	$\ddot{m}$	$\ddot{n}$	$\ddot{m} * \ddot{n}$
1	**1**	$\ddot{m}$	$\ddot{n}$	$\ddot{m} * \ddot{n}$
$\ddot{m}$	$\ddot{m}$	1	$\ddot{m} * \ddot{n}$	$\ddot{n}$
$\ddot{n}$	$\ddot{n}$	$\ddot{m} * \ddot{n}$	1	$\ddot{m}$
$\ddot{m} * \ddot{n}$	$\ddot{m} * \ddot{n}$	$\ddot{n}$	$\ddot{m}$	1

And properties that $\ddot{m}^2 = \ddot{n}^2 = (\ddot{m} * \ddot{n})^2 = 1$ and $\ddot{m} * \ddot{n} = \ddot{n} * \ddot{m}$.

Now take the BN subset values of $\ddot{\mathfrak{G}} = \mathbb{K}_4$ as next

$$\mathcal{A} = \left\{\frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{1}, \frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{\ddot{m}}, \frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{\ddot{n}}, \frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{\ddot{m} * \ddot{n}}\right\}$$

Then we have to investigate:

1. $\mathcal{A}(\ddot{m} * \ddot{n}) \geq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \mathbb{K}_4$. Such that

$\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Pi^{+}_{\mathcal{A}}(\ddot{m}), \Pi^{+}_{\mathcal{A}}(\ddot{n})\}$,

$\Xi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Xi^{+}_{\mathcal{A}}(\ddot{m}), \Xi^{+}_{\mathcal{A}}(\ddot{n})\}$

$\Sigma^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Sigma^{+}_{\mathcal{A}}(\ddot{m}), \Sigma^{+}_{\mathcal{A}}(\ddot{n})\}$.

$\mathcal{A}(\ddot{m}^{-1}) \geq \mathcal{A}(\ddot{m})$. Such that

$\Pi^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{m})$,

$\Xi^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Xi^{+}_{\mathcal{A}}(\ddot{m})$,

$\Sigma^{+}_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Sigma^{+}_{\mathcal{A}}(\ddot{m})$.

$(\ddot{m} * \ddot{n}) \leq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \mathbb{K}_4$. Such that

$\Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{n})\}$,

$\Xi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Xi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{n})\}$

$\Sigma^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{n})\}$.

$\mathcal{A}(\ddot{m}^{-1}) \leq \mathcal{A}(\ddot{m})$. Such that

$\Pi^-_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Pi^-_{\mathcal{A}}(\ddot{m})$,

$\Xi^-_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Xi^-_{\mathcal{A}}(\ddot{m})$,

$\Sigma^-_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Sigma^-_{\mathcal{A}}(\ddot{m})$.

2. Let $\ddot{m} = 1$ and $\ddot{n} = \ddot{m}$

$\mathcal{A}(\ddot{m} * \ddot{n}) \geq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \mathbb{K}_4$. Such that

$\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{n})\}$,

$\Xi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Xi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{n})\}$

$\Sigma^+_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{n})\}$.

$\mathcal{A}(\ddot{m}^{-1}) \geq \mathcal{A}(\ddot{m})$. Such that

$\Pi^+_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Pi^+_{\mathcal{A}}(\ddot{m})$,

$\Xi^+_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Xi^+_{\mathcal{A}}(\ddot{m})$,

$\Sigma^+_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Sigma^+_{\mathcal{A}}(\ddot{m})$.

$(\ddot{m} * \ddot{n}) \leq \min\{\mathcal{A}(\ddot{m}), \mathcal{A}(\ddot{n})\}$ for all $\ddot{m}, \ddot{n} \in \mathbb{K}_4$. Such that

$\Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{n})\}$,

$\Xi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \leq \max\{\Xi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{n})\}$

$\Sigma^-_{\mathcal{A}}(\ddot{m} * \ddot{n}) \geq \min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{n})\}$.

$\mathcal{A}(\ddot{m}^{-1}) \leq \mathcal{A}(\ddot{m})$. Such that

$\Pi^-_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Pi^-_{\mathcal{A}}(\ddot{m})$,

$\Xi^-_{\mathcal{A}}(\ddot{m}^{-1}) \leq \Xi^-_{\mathcal{A}}(\ddot{m})$,

$\Sigma^-_{\mathcal{A}}(\ddot{m}^{-1}) \geq \Sigma^-_{\mathcal{A}}(\ddot{m})$.

And so on for the rest of the elements in $\mathbb{K}_4$, thus BN subset $\mathcal{A}$ of $\mathbb{K}_4$ is BNSG.

Theorem 3.11. Assume that $\ddot{\mathfrak{G}}$ is a traditional group and $\mathcal{A}$ is a bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$. Then, the subsequent properties are fulfilled:

i. $\mathcal{A}(\ddot{e}) \geq \mathcal{A}(\ddot{m})$ where $\ddot{e}$ is the identity element in group $\ddot{\mathfrak{G}}$ and $\ddot{m} \in \ddot{\mathfrak{G}}$.

ii. $\mathcal{A}(\ddot{m}^{-1}) \geq \mathcal{A}(\ddot{m})$, where $\ddot{m}^{-1}$is the inverse element in group $\ddot{\mathfrak{G}}$ and $\ddot{m} \in \ddot{\mathfrak{G}}$.

Proof (***i***) Suppose that $\ddot{e}$ is the identity element in group $\ddot{\mathfrak{G}}$ and $\ddot{m} \in \ddot{\mathfrak{G}}$.

Then by definition 3.1, we get for positive side:

$\Pi^+_{\mathcal{A}}(\ddot{e}) = \Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{m}^{-1})$

$\geq \min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{m}^{-1})\}$

$\geq \min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{m})\} = \Pi^+_{\mathcal{A}}(\ddot{m})$

$\Sigma^+_{\mathcal{A}}(\ddot{e}) = \Sigma^+_{\mathcal{A}}(\ddot{m} * \ddot{m}^{-1})$

$\leq \max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{m}^{-1})\}$

$\leq \max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{m})\} = \Sigma^+_{\mathcal{A}}(\ddot{m})$

From similar steps, it is clearly shown that $\Xi^+_{\mathcal{A}}(\ddot{e}) \geq \Xi^+_{\mathcal{A}}(\ddot{m})$.

Now for negative side, we get:

$\Pi^-_{\mathcal{A}}(\ddot{e}) = \Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{m}^{-1})$

$\leq \max\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{m}^{-1})\}$

$\leq \max\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{m})\} = \Pi^-_{\mathcal{A}}(\ddot{m})$

$\Sigma^-_{\mathcal{A}}(\ddot{e}) = \Sigma^-_{\mathcal{A}}(\ddot{m} * \ddot{m}^{-1})$

$\geq \min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{m}^{-1})\}$

$\geq \min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{m})\} = \Sigma^-_{\mathcal{A}}(\ddot{m})$

From similar steps, it is clearly shown that $\Xi^-_{\mathcal{A}}(\ddot{e}) \leq \Xi^-_{\mathcal{A}}(\ddot{m})$. Hence, we get the desired inequality in (i) $\mathcal{A}(\ddot{e}) \geq \mathcal{A}(\ddot{m})$.

Proof $(\boldsymbol{ii.})$ Suppose that $\ddot{m} \in \ddot{\mathfrak{G}}$ and $\ddot{m}^{-1}$ is the inverse element of $\ddot{m} \in \ddot{\mathfrak{G}}$. Again, by applying (ii) in definition 3.1 and utilizing the group properties of $\ddot{\mathfrak{G}}$, the other side of the inequality is proved as next:

$\Pi^+_{\mathcal{A}}(\ddot{m}) = \Pi^+_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \geq \Pi^+_{\mathcal{A}}(\ddot{m}^{-1})$ and $\Pi^-_{\mathcal{A}}(\ddot{m}) = \Pi^-_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \leq \Pi^-_{\mathcal{A}}(\ddot{m}^{-1})$,

$\Xi^+_{\mathcal{A}}(\ddot{m}) = \Xi^+_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \geq \Xi^+_{\mathcal{A}}(\ddot{m}^{-1})$ and $\Xi^-_{\mathcal{A}}(\ddot{m}) = \Xi^-_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \leq \Xi^-_{\mathcal{A}}(\ddot{m}^{-1})$,

$\Sigma^+_{\mathcal{A}}(\ddot{m}) = \Sigma^+_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \leq \Sigma^+_{\mathcal{A}}(\ddot{m}^{-1})$ and $\Sigma^-_{\mathcal{A}}(\ddot{m}) = \Sigma^-_{\mathcal{A}}(\ddot{m}^{-1})^{-1} \geq \Sigma^-_{\mathcal{A}}(\ddot{m}^{-1})$,

Therefor,

$$\begin{aligned}\mathcal{A}(\ddot{m}^{-1}) &= \left(\Pi^+_{\mathcal{A}}(\ddot{m}^{-1}), \Xi^+_{\mathcal{A}}(\ddot{m}^{-1}), \Sigma^+_{\mathcal{A}}(\ddot{m}^{-1}), \Pi^-_{\mathcal{A}}(\ddot{m}^{-1}), \Xi^-_{\mathcal{A}}(\ddot{m}^{-1}), \Sigma^-_{\mathcal{A}}(\ddot{m}^{-1})\right)\\ &= \left(\Pi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{m})\right)\\ &= \mathcal{A}(\ddot{m}).\end{aligned}$$

Theorem 3.12. Assume that $\ddot{\mathfrak{G}}$ is a traditional group and $\mathcal{A}$ is a bipolar neutrosophic set of $\ddot{\mathfrak{G}}$. Then $\mathcal{A} \in BNS(\ddot{\mathfrak{G}})$ if and only if $\mathcal{A}^+(\ddot{m} * \ddot{n}^{-1}) \geq \min\{\mathcal{A}^+(\ddot{m}), \mathcal{A}^+(\ddot{n}^{-1})\} = \min\{\mathcal{A}^+(\ddot{m}), \mathcal{A}^+(\ddot{n})\}$ and $\mathcal{A}^-(\ddot{m} * \ddot{n}^{-1}) \leq \max\{\mathcal{A}^-(\ddot{m}), \mathcal{A}^-(\ddot{n}^{-1})\} = \max\{\mathcal{A}^-(\ddot{m}), \mathcal{A}^-(\ddot{n})\}$ where each $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$.

Proof. Suppose that $\mathcal{A}$ is a bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$ and for $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$.

Then it's clear based on above discation we get

$\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \geq \min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{n}^{-1})\} = \min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{n})\}$ and

$\Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \leq \min\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{n}^{-1})\} = \min\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{n})\}$

Thus for other terms,

$\Xi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \geq \min\{\Xi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{n}^{-1})\} = \min\{\Xi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{n})\}$ and

$\Xi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \leq \min\{\Xi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{n}^{-1})\} = \min\{\Xi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{n})\}$,

$\Sigma^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \leq \max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{n}^{-1})\} = \max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{n})\}$ and

$\Sigma^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) \geq \min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{n}^{-1})\} = \max\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{n})\}$,

Hence

$\mathcal{A}(\ddot{m} * \ddot{n}^{-1}) = \left(\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Xi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Sigma^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Xi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Sigma^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1})\right)$

$\geq (min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{n})\}, min\{\Xi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{n})\}, max\{\Sigma^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{n})\}$,

$max\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Pi^-_{\mathcal{A}}(\ddot{n})\}, max\{\Xi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{n})\}, min\{\Sigma^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{n})\})$

$= min(\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Xi^+_{\mathcal{A}}(\ddot{m}), \Sigma^+_{\mathcal{A}}(\ddot{m})\}, \{\Pi^+_{\mathcal{A}}(\ddot{n}), \Xi^+_{\mathcal{A}}(\ddot{n}), \Sigma^+_{\mathcal{A}}(\ddot{n})\})$,

$max(\{\Pi^-_{\mathcal{A}}(\ddot{m}), \Xi^-_{\mathcal{A}}(\ddot{m}), \Sigma^-_{\mathcal{A}}(\ddot{m})\}, \{\Pi^-_{\mathcal{A}}(\ddot{n}), \Xi^-_{\mathcal{A}}(\ddot{n}), \Sigma^-_{\mathcal{A}}(\ddot{n})\})$

$= \min\{\mathcal{A}^+(\ddot{m}), \mathcal{A}^+(\ddot{n})\}$ and $\max\{\mathcal{A}^-(\ddot{m}), \mathcal{A}^-(\ddot{n})\}$

Conversely, let $\ddot{e}$ be the identity element of classical group $\ddot{\mathfrak{G}}$.Then

$\Xi^+_{\mathcal{A}}(\ddot{n}^{-1}) = \Xi^+_{\mathcal{A}}(\ddot{e} * \ddot{n}^{-1}) \geq \min\{\Xi^+_{\mathcal{A}}(\ddot{e}), \Xi^+_{\mathcal{A}}(\ddot{n}^{-1})\} = \min\{\Xi^+_{\mathcal{A}}(\ddot{n} * \ddot{n}^{-1}), \Xi^+_{\mathcal{A}}(\ddot{n}^{-1})\}$

$\geq \min\{\Xi^+_{\mathcal{A}}(\ddot{n}), \Xi^+_{\mathcal{A}}(\ddot{n}), \Xi^+_{\mathcal{A}}(\ddot{n})\} = \Xi^+_{\mathcal{A}}(\ddot{n})$.

And with similar steps with the rest of the borders we get $\mathcal{A} \in BNS(\ddot{\mathfrak{G}})$.

Theorem 3.13. Assume that $\ddot{\mathfrak{G}}$ is a traditional group and $\mathcal{A}$ and $\mathcal{B}$ are two bipolar neutrosophic set on $\ddot{\mathfrak{G}}$, if $\mathcal{A}$ and $\mathcal{B}$ are two bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$, then $\mathcal{A}\cap\mathcal{B}$ is also bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$.

Proof. Let $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$ and by theorem 3.4, we have ,

$\mathcal{A}\cap\mathcal{B}(\ddot{m} * \ddot{n}^{-1}) \geq min\{\mathcal{A}\cap\mathcal{B}(\ddot{m}), \mathcal{A}\cap\mathcal{B}(\ddot{n})\}$, such that

$\Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n}^{-1}) \geq min\{\Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m}), \Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})\}$,

$\Xi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n}^{-1}) \geq min\{\Xi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m}), \Xi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})\}$,

$\Sigma^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n}^{-1}) \leq max\{\Sigma^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m}), \Sigma^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})\}$.

Now, reflect on the positive truth membership degree of the intersection as follows, and the other inequalities are likewise proved.

$$\begin{aligned}\Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n}^{-1}) &= min\{\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Pi^+_{\mathcal{B}}(\ddot{m} * \ddot{n}^{-1})\}\\ &\geq \left(min\left\{min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{A}}(\ddot{n})\}, min\{\Pi^+_{\mathcal{B}}(\ddot{m}), \Pi^+_{\mathcal{B}}(\ddot{n})\}\right\}\right)\\ &= \left(min\left\{min\{\Pi^+_{\mathcal{A}}(\ddot{m}), \Pi^+_{\mathcal{B}}(\ddot{m})\}, min\{\Pi^+_{\mathcal{A}}(\ddot{n}), \Pi^+_{\mathcal{B}}(\ddot{n})\}\right\}\right)\\ &= min\{\Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{m}), \Pi^+_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})\}.\end{aligned}$$

Therefor $\mathcal{A}\cap\mathcal{B} \in BNS$.

Proposition 3.14. Let $\ddot{\mathfrak{G}}$ be a traditional group and $\mathcal{A}$ be a bipolar neutrosophic set of $\ddot{\mathfrak{G}}$ if and only if for all $\beta \in [-1,1]$, $\beta -$levelset of $\mathcal{A}$ $\Pi^+_{(\mathcal{A})_\beta}$,$\Pi^-_{(\mathcal{A})_\beta}$, $\Xi^+_{(\mathcal{A})_\beta}$, $\Xi^-_{(\mathcal{A})_\beta}$, $\Sigma^+_{(\mathcal{A})_\beta}$, $\Sigma^-_{(\mathcal{A})_\beta}$ are classical subgroups of traditional group $\ddot{\mathfrak{G}}$.

Proof. In this proof we will only deal with the positive and negative true membership side and the rest of the sides the proof is similar based on above definitions. Therefor,

Let $\mathcal{A}$ be a bipolar neutrosophic subgroup of $\ddot{\mathfrak{G}}$, $\beta \in [-1,1]$ and for $\ddot{m}, \ddot{n} \in \Pi^+_{(\mathcal{A})_\beta}$. Then, by the assumption, we have:

$$\begin{aligned}\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) &\geq min\{\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Pi^+_{\mathcal{B}}(\ddot{m} * \ddot{n}^{-1})\}\\ &\geq min\{\beta, \beta\} = \beta\end{aligned}$$

And

$$\begin{aligned}\Pi^-_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}) &\leq max\{\Pi^+_{\mathcal{A}}(\ddot{m} * \ddot{n}^{-1}), \Pi^+_{\mathcal{B}}(\ddot{m} * \ddot{n}^{-1})\}\\ &\leq max\{\beta, \beta\} = \beta\end{aligned}$$

Hence, we conclude $\Pi^+_{(\mathcal{A})_\beta}$, $\Pi^-_{(\mathcal{A})_\beta}$are a classical subgroups of $\ddot{\mathfrak{G}}$ for each $\beta \in [-1,1]$.

Conversely, Assume that $\Pi^+_{(\mathcal{A})_\beta}$, $\Pi^-_{(\mathcal{A})_\beta}$ are a classical subgroups of $\ddot{\mathfrak{G}}$ for each $\beta \in [-1,1]$. Let $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$, $\beta = \min\left\{\Pi^+_{(\mathcal{A})_\beta}(\ddot{m}), \Pi^+_{(\mathcal{A})_\beta}(\ddot{n})\right\}$ and $\delta = \Pi^+_{(\mathcal{A})}(\ddot{m})$.

Since $\Pi^+_{(\mathcal{A})_\beta}$ and $\Pi^+_{(\mathcal{A})_\delta}$ are a classical subgroups of $\ddot{\mathfrak{G}}$, then $\ddot{m} * \ddot{n} \in \Pi^+_{(\mathcal{A})_\beta}$ and $\ddot{m}^{-1} \in \Pi^+_{(\mathcal{A})_\delta}$.

Thus, $\Pi^+_{(\mathcal{A})}(\ddot{m} * \ddot{n}) \geq \beta = \min\left\{\Pi^+_{(\mathcal{A})_\beta}(\ddot{m}), \Pi^+_{(\mathcal{A})_\beta}(\ddot{n})\right\}$ and $\Pi^+_{(\mathcal{A})}(\ddot{m}^{-1}) \geq \beta = \Pi^+_{(\mathcal{A})}(\ddot{m})$.

Hence, the conditions given in Definition 3.1 are satisfied.

Theorem 3.15. Let $\mathcal{F}$ be a homomorphism mapping from classical groups $\ddot{\mathfrak{G}}_1$ to classical groups $\ddot{\mathfrak{G}}_2$ and $\mathcal{A}$ be a BN-subgroup of $\ddot{\mathfrak{G}}_1$, then the image of $\mathcal{A}$,i.e $\mathcal{F}(\ddot{\mathfrak{G}}_1)$ is a BN-subgroup of $\ddot{\mathfrak{G}}_2$.

Proof. In this proof we will only deal with the positive memberships side and the rest of the sides the proof is similar based on above definitions. Therefor,

Assume that $\ddot{m}_1$, $\ddot{m}_2 \in \ddot{\mathfrak{G}}_1$ such that $\mathcal{F}(\ddot{m}_1) = \ddot{n}_1$ and $\mathcal{F}(\ddot{m}_2) = \ddot{n}_2$. Since $\mathcal{F}$ be a group homomorphism, then:

$$\mathcal{F}(\Pi^+_{(\mathcal{A})})(\ddot{n} * \ddot{n}^{-1}) = \min_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Pi^+_{(\mathcal{A})}(\ddot{m}) \geq \Pi^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1}),$$

$$\mathcal{F}(\Xi^+_{(\mathcal{A})})(\ddot{n} * \ddot{n}^{-1}) = \min_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Xi^+_{(\mathcal{A})}(\ddot{m}) \geq \Xi^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1}),$$

$$\mathcal{F}(\Sigma^+_{(\mathcal{A})})(\ddot{n} * \ddot{n}^{-1}) = \max_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Sigma^+_{(\mathcal{A})}(\ddot{m}) \leq \Sigma^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1})$$

Now we will work to prove that $\mathcal{F}(\mathcal{A})(\ddot{n}_1 * \ddot{n}_2{}^{-1}) \geq min\{\mathcal{F}(\mathcal{A})(\ddot{n}_1), \mathcal{F}(\mathcal{A})(\ddot{n}_2)\}$,

$$\mathcal{F}(\mathcal{A})(\ddot{n}_1 * \ddot{n}_2{}^{-1}) = \left(\mathcal{F}(\Pi^+_{(\mathcal{A})})(\ddot{n}_1 * \ddot{n}_2{}^{-1}), \mathcal{F}(\Xi^+_{(\mathcal{A})})(\ddot{n}_1 * \ddot{n}_2{}^{-1}), \mathcal{F}(\Sigma^+_{(\mathcal{A})})(\ddot{n}_1 * \ddot{n}_2{}^{-1})\right)$$

$$= \left(\min_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Pi^+_{(\mathcal{A})}(\ddot{m}), \min_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Xi^+_{(\mathcal{A})}(\ddot{m}), \max_{\ddot{n}*\ddot{n}^{-1}=\mathcal{F}(\ddot{m})} \Sigma^+_{(\mathcal{A})}(\ddot{m})\right)$$

$$\geq \left(\Pi^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1}), \Xi^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1}), \Sigma^+_{(\mathcal{A})}(\ddot{m} * \ddot{m}^{-1})\right)$$

$$\geq \left(min\{\Pi^+_{(\mathcal{A})}(\ddot{m}_1), \Pi^+_{(\mathcal{A})}(\ddot{m}_2)\}, min\{\Xi^+_{(\mathcal{A})}(\ddot{m}_1), \Xi^+_{(\mathcal{A})}(\ddot{m}_2)\}, max\{\Sigma^+_{(\mathcal{A})}(\ddot{m}_1), \Sigma^+_{(\mathcal{A})}(\ddot{m}_2)\}\right)$$

$$= \left(min\left\{\left(\Pi^+_{(\mathcal{A})}(\ddot{m}_1), \Xi^+_{(\mathcal{A})}(\ddot{m}_1), \Sigma^+_{(\mathcal{A})}(\ddot{m}_1)\right), \left(\Pi^+_{(\mathcal{A})}(\ddot{m}_2), \Xi^+_{(\mathcal{A})}(\ddot{m}_2), \Sigma^+_{(\mathcal{A})}(\ddot{m}_2)\right)\right\}\right)$$

This fluffed for each $\ddot{m}_1$, $\ddot{m}_2 \in \ddot{\mathfrak{G}}_1$ with $\mathcal{F}(\ddot{m}_1) = \ddot{n}_1$ and $\mathcal{F}(\ddot{m}_2) = \ddot{n}_2$. Then we get the following:

$$\mathcal{F}(\mathcal{A})(\ddot{n}_1 * \ddot{n}_2{}^{-1}) \geq min\left\{\left(\bigvee_{\mathcal{F}(\ddot{m}_1)=\ddot{n}_1} \Pi^+_{(\mathcal{A})}(\ddot{m}_1), \bigvee_{\mathcal{F}(\ddot{m}_1)=\ddot{n}_1} \Xi^+_{(\mathcal{A})}(\ddot{m}_1), \bigwedge_{\mathcal{F}(\ddot{m}_1)=\ddot{n}_1} \Xi^+_{(\mathcal{A})}(\ddot{m}_1)\right) \left(\bigvee_{\mathcal{F}(\ddot{m}_2)=\ddot{n}_2} \Pi^+_{(\mathcal{A})}(\ddot{m}_2), \bigvee_{\mathcal{F}(\ddot{m}_2)=\ddot{n}_2} \Xi^+_{(\mathcal{A})}(\ddot{m}_2), \bigwedge_{\mathcal{F}(\ddot{m}_2)=\ddot{n}_2} \Xi^+_{(\mathcal{A})}(\ddot{m}_2)\right)\right\}$$

$$= \left(min\left\{\left(\Pi^+_{(\mathcal{A})}(\ddot{n}_1), \Xi^+_{(\mathcal{A})}(\ddot{n}_1), \Sigma^+_{(\mathcal{A})}(\ddot{n}_1)\right), \left(\Pi^+_{(\mathcal{A})}(\ddot{n}_2), \Xi^+_{(\mathcal{A})}(\ddot{n}_2), \Sigma^+_{(\mathcal{A})}(\ddot{n}_2)\right)\right\}\right)$$

$$= \{\mathcal{F}(\mathcal{A})(\ddot{n}_1), \mathcal{F}(\mathcal{A})(\ddot{n}_2)\}.$$

Hence, the image of a BN-subgroup is also a BN-subgroup.

Theorem 3.16. Let $\mathcal{F}$ be a homomorphism mapping from classical groups $\ddot{\mathfrak{G}}_1$ to classical groups $\ddot{\mathfrak{G}}_2$ and $\mathcal{B}$ be a BN-subgroup of $\ddot{\mathfrak{G}}_2$, then the preimage of $\mathcal{A}$,i.e $\mathcal{F}^{-1}(\ddot{\mathfrak{G}}_2)$ is a BN-subgroup of $\ddot{\mathfrak{G}}_1$.

Proof. In this proof we will only deal with the positive memberships side and the rest of the sides the proof is similar based on above definitions. Therefor,

Assume that $\ddot{m}_1$, $\ddot{m}_2 \in \ddot{\mathfrak{G}}_1$ such that $\mathcal{F}(\ddot{m}_1) = \ddot{n}_1$ and $\mathcal{F}(\ddot{m}_2) = \ddot{n}_2$. Since $\mathcal{F}$ be a group homomorphism, then:

$$\mathcal{F}^{\ 1}(\mathcal{B})\left(\mathfrak{m}_1 * \mathfrak{m}_2^{\ 1}\right)$$

$$= \left(\Pi^{+}_{(\mathcal{B})}\left(\mathcal{F}\left(\ddot{\mathfrak{m}}_1 * \ddot{\mathfrak{m}}_2{}^{-1}\right)\right), \Xi^{+}_{(\mathcal{B})}\left(\mathcal{F}\left(\ddot{\mathfrak{m}}_1 * \ddot{\mathfrak{m}}_2{}^{-1}\right)\right), \Sigma^{+}_{(\mathcal{B})}\left(\mathcal{F}\left(\ddot{\mathfrak{m}}_1 * \ddot{\mathfrak{m}}_2{}^{-1}\right)\right)\right)$$

$$= \left(\Pi^{+}_{(\mathcal{B})}\left(\mathcal{F}(\ddot{\mathfrak{m}}_1) * \mathcal{F}\left(\ddot{\mathfrak{m}}_2{}^{-1}\right)\right), \Xi^{+}_{(\mathcal{B})}\left(\mathcal{F}(\ddot{\mathfrak{m}}_1) * \mathcal{F}\left(\ddot{\mathfrak{m}}_2{}^{-1}\right)\right),\right.$$

$$\left.\Sigma^{+}_{(\mathcal{B})}\left(\mathcal{F}(\ddot{\mathfrak{m}}_1) * \mathcal{F}\left(\ddot{\mathfrak{m}}_2{}^{-1}\right)\right)\right)$$

$$\geq \left(min\{\Pi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Pi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2))\}, min\{\Xi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Xi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2))\},\right.$$
$$\left. max\{\Sigma^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Sigma^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2))\}\right)$$

$$= min\left(\Pi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Xi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Sigma^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_1)), \Pi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2)), \Xi^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2)), \Sigma^{+}_{(\mathcal{B})}(\mathcal{F}(\ddot{\mathfrak{m}}_2))\right)$$

$$= min\{\mathcal{F}^{-1}(\mathcal{B})(\ddot{\mathfrak{m}}_1), \mathcal{F}^{-1}(\mathcal{B})(\ddot{\mathfrak{m}}_2)\}$$

Therefor, $\mathcal{F}^{-1}(\mathcal{B}) \in BN(\ddot{\mathfrak{G}}_1)$.

Corollary 3.17. Let $\mathcal{F}$ be a homomorphism mapping from classical groups $\ddot{\mathfrak{G}}_1$ to classical groups $\ddot{\mathfrak{G}}_2$ and $\mathcal{A}$ be a BN-subgroup of $\ddot{\mathfrak{G}}_1$, then $\mathcal{F}^{-1}(\mathcal{F}(\mathcal{A}\ \ )) = \mathcal{A}$.

Corollary 3.18. Let $\mathcal{F}$ be a homomorphism mapping from classical groups $\ddot{\mathfrak{G}}_1$ to classical groups $\ddot{\mathfrak{G}}_2$ and $\mathcal{A}$ be a BN-subgroup of $\ddot{\mathfrak{G}}_1$, then $\mathcal{F}(\mathcal{A}\ \ ) = \mathcal{A}$ if and only if $\mathcal{F}^{-1}(\mathcal{A}\ \ ) = \mathcal{A}$.

4. Bipolar neutrosophic normal subgroup (BNNSG)

Definition 4. 1. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic set on classical group $\ddot{\mathfrak{G}}$. Then $\mathcal{A}$ is named bipolar neutrosophic normal subgroup (BNNSG) of $\ddot{\mathfrak{G}}$ if the following condition satisfies $\mathcal{A}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \geq \mathcal{A}(\ddot{\mathfrak{n}})$ for all $\ddot{\mathfrak{m}}, \ddot{\mathfrak{n}} \in \ddot{\mathfrak{G}}$. That means

$\Pi^{+}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{\mathfrak{n}})$, $\Xi^{+}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \geq \Xi^{+}_{\mathcal{A}}(\ddot{\mathfrak{n}})$ and $\Sigma^{+}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \leq \Sigma^{+}_{\mathcal{A}}(\ddot{\mathfrak{n}})$.

$\Pi^{-}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \leq \Pi^{-}_{\mathcal{A}}(\ddot{\mathfrak{n}})$, $\Xi^{-}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \leq \Xi^{-}_{\mathcal{A}}(\ddot{\mathfrak{n}})$ and $\Sigma^{-}_{\mathcal{A}}(\ddot{\mathfrak{m}} * \ddot{\mathfrak{n}} * \ddot{\mathfrak{m}}^{-1}) \geq \Sigma^{-}_{\mathcal{A}}(\ddot{\mathfrak{n}})$.

Here, the collection of all the bipolar neutrosophic normal subgroups of classical group $\ddot{\mathfrak{G}}$ is indicated by BNNS$(\ddot{\mathfrak{G}})$.

Example 4.2. Assume that the classical group $(\ddot{\mathfrak{G}} = \{1, -1, i, -i\}, .)$ Where . is natural multiplication. Then the bipolar neutrosophic set $\mathcal{A}$ on $\ddot{\mathfrak{G}}$ given as following:

$$\mathcal{A} = \left\{\frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{1}, \frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{-1},\right.$$
$$\left.\frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{i}, \frac{\langle 0.3,0.4,0.5,-0.4,-0.6,-0.7\rangle}{-i}\right\}$$

Based on definition 4.1, it is clear that the bipolar neutrosophic set $\mathcal{A}$ is a bipolar neutrosophic normal subgroup of $\ddot{\mathfrak{G}}$.

Theorem 4.3. Let $\ddot{\mathfrak{G}}$ be a traditional group and $\mathcal{A}, \mathcal{B} \in$ BNNS$(\ddot{\mathfrak{G}})$then $\mathcal{A} \cap \mathcal{B} \in$ BNNS$(\ddot{\mathfrak{G}})$ also.

Proof. In this proof we will only deal with the positive memberships side and the rest of the sides the proof is similar based on above definitions. Therefor,

Since $\mathcal{A},\mathcal{B} \in \mathrm{BNNS}(\ddot{\mathfrak{G}})$,then

$\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{n})$ and $\Pi^{+}_{\mathcal{B}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq \Pi^{+}_{\mathcal{B}}(\ddot{n})$

Now based on the definition of $\cap$ then,

$\Pi^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) = min\{\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}), \Pi^{+}_{\mathcal{B}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1})\}$

$\geq min\{\Pi^{+}_{\mathcal{A}}(\ddot{n}), \Pi^{+}_{\mathcal{B}}(\ddot{n})\}$=$\Pi^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})$.

By a similar step result, we get

$\Xi^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq$=$\Xi^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})$ and $\Sigma^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq$=$\Sigma^{+}_{\mathcal{A}\cap\mathcal{B}}(\ddot{n})$.

Therefore, the intersection of two BNNSGs is also a BNNSG.

Proposition 4.4. Assume that $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ is a bipolar neutrosophic set on classical group $\ddot{\mathfrak{G}}$. Then the following point are fulfilled.

i. $\mathcal{A} \in BNNS(\mathfrak{G})$.

ii. $\mathcal{A}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq \mathcal{A}(\ddot{n})$ for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$.

iii. $\mathcal{A}(\ddot{m} * \ddot{n}) = \mathcal{A}(\ddot{n} * \ddot{m})$ for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$.

Proof. In this proof we will only deal with the positive memberships side and the rest of the sides the proof is similar based on above definitions. Therefor,

(i) $\Rightarrow$ (ii): Let $\mathcal{A}$ is a bipolar neutrosophic normal subgroup of classical group $\ddot{\mathfrak{G}}$ and for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$. Then by definition

$\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{n})$, , $\Xi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \geq \Xi^{+}_{\mathcal{A}}(\ddot{n})$ and $\Sigma^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \leq \Sigma^{+}_{\mathcal{A}}(\ddot{n})$.

Thus, taking advantage of $\ddot{\mathfrak{G}}$, i.e., the arbitrary property of $\ddot{m}$, the following is obtained for the truth-membership of $\mathcal{A}$.

$\Pi^{+}_{\mathcal{A}}(\ddot{m}^{-1} * \ddot{n} * m) = \Pi^{+}_{\mathcal{A}}(\ddot{m}^{-1} * \ddot{n} * (\ddot{m}^{-1})^{-1}) \geq \Pi^{+}_{\mathcal{A}}(\ddot{n})$

Therefor, $\Pi^{+}_{\mathcal{A}}(\ddot{n}) = \Pi^{+}_{\mathcal{A}}(\ddot{m}^{-1} * (\ddot{m} * \ddot{n} * \ddot{m}^{-1}) * \ddot{m}) \leq \Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) \leq \Pi^{+}_{\mathcal{A}}(\ddot{n})$,

i.e., $\Pi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) = \Pi^{+}_{\mathcal{A}}(\ddot{n})$.

And with steps similar to the rest of the borders,

$\Xi^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) = \Xi^{+}_{\mathcal{A}}(\ddot{n})$ and and $\Sigma^{+}_{\mathcal{A}}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) = \Sigma^{+}_{\mathcal{A}}(\ddot{n})$.

Now, (ii) $\rightarrow$(iii) It can be accessed directly through substitution $\ddot{n}$ for $\ddot{n} * \ddot{m}$ in proof steps of (ii) .

Now, (iii) $\rightarrow$(i) regarding $\mathcal{A}(\ddot{m} * \ddot{n}) = \mathcal{A}(\ddot{n} * \ddot{m})$ for all $\ddot{m}, \ddot{n} \in \ddot{\mathfrak{G}}$. Then $\mathcal{A}(\ddot{m} * \ddot{n} * \ddot{m}^{-1}) = \mathcal{A}(\ddot{m} * \ddot{m}^{-1} * \ddot{n}) = \mathcal{A}(\ddot{n}) \geq \mathcal{A}(\ddot{n})$.

Hence it satisfied.

Proposition 4.5. Let $\ddot{\mathfrak{G}}$ be a traditional group and $\mathcal{A}$ be a bipolar neutrosophic normal subgroup of $\ddot{\mathfrak{G}}$ if and only if for all $\beta \in [-1,1]$, β −levelset of $\mathcal{A}$ $\Pi^{+}_{(\mathcal{A})_\beta}$,$\Pi^{-}_{(\mathcal{A})_\beta}$, $\Xi^{+}_{(\mathcal{A})_\beta}$, $\Xi^{-}_{(\mathcal{A})_\beta}$, $\Sigma^{+}_{(\mathcal{A})_\beta}$, $\Sigma^{-}_{(\mathcal{A})_\beta}$ are classical subgroups of traditional group $\ddot{\mathfrak{G}}$.

Proof. Clear and similar to the proof steps mentioned in above Proposition.

Definition 4. 6. Let $(\ddot{\mathfrak{G}},*)$ be a classical group and $\mathcal{A}$ be a bipolar neutrosophic set on classical group $\ddot{\mathfrak{G}}$. The the normalizer of a BN-subgroup and $\mathcal{A}$ classical group $\ddot{\mathfrak{G}}$ is given as following:

$$N(\mathcal{A}) = \{\ddot{\text{n}} \in \ddot{\mathfrak{G}}, \mathcal{A}(\ddot{\text{n}}^{-1} * \ddot{\text{m}} * \ddot{\text{n}}) \geq min\{\mathcal{A}(\ddot{\text{m}}), \mathcal{A}(\ddot{\text{n}}^{-1} * \ddot{\text{m}} * \ddot{\text{n}})\} \leq \mathcal{A}(\ddot{\text{m}}), \forall \ddot{\text{m}} \in \ddot{\mathfrak{G}}\}$$

Theorem 4.7. Let $\ddot{\mathfrak{G}}$ be a traditional group and $\mathcal{A} \in \text{BNSG}(\ddot{\mathfrak{G}})$, then

i. $N(\mathcal{A})$ is subgroup of $\ddot{\mathfrak{G}}$.

ii. $\mathcal{A}$ BNNSG of $\ddot{\mathfrak{G}}$ if and only if $N(\mathcal{A}) = \ddot{\mathfrak{G}}$.

Proof (i). In this proof we will only deal with the positive and negative true memberships side and the rest of the sides the proof is similar based on above definitions. Therefor,

Suppose that $\ddot{\text{n}}_1, \ddot{\text{n}}_2 \in N(\mathcal{A})$ and to prove that $\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1} \in N(\mathcal{A})$.

Therefor for any $\ddot{\text{m}} \in \ddot{\mathfrak{G}}$ we have,

$$\Xi_{\mathcal{A}}^{+}\left(\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)^{-1}\ddot{\text{m}}\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)^{-1}\right) = \Xi_{\mathcal{A}}^{+}\left(\left(\ddot{\text{n}}_2\ddot{\text{n}}_1^{-1}\right)\ddot{\text{m}}\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)\right)$$

$$\Xi_{\mathcal{A}}^{+}\left(\ddot{\text{n}}_2\left(\ddot{\text{n}}_1^{-1}\ddot{\text{m}}\ddot{\text{n}}_1\right)\ddot{\text{n}}_2^{-1}\right) \geq \Xi_{\mathcal{A}}^{+}\left(\ddot{\text{n}}_1^{-1}\ddot{\text{m}}\ddot{\text{n}}_1\right) \geq \Xi_{\mathcal{A}}^{+}(\ddot{\text{m}})$$

and,

$$\Xi_{\mathcal{A}}^{-}\left(\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)^{-1}\ddot{\text{m}}\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)^{-1}\right) = \Xi_{\mathcal{A}}^{-}\left(\left(\ddot{\text{n}}_2\ddot{\text{n}}_1^{-1}\right)\ddot{\text{m}}\left(\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1}\right)\right)$$

$$\Xi_{\mathcal{A}}^{-}\left(\ddot{\text{n}}_2\left(\ddot{\text{n}}_1^{-1}\ddot{\text{m}}\ddot{\text{n}}_1\right)\ddot{\text{n}}_2^{-1}\right) \leq \Xi_{\mathcal{A}}^{-}\left(\ddot{\text{n}}_1^{-1}\ddot{\text{m}}\ddot{\text{n}}_1\right) \leq \Xi_{\mathcal{A}}^{-}(\ddot{\text{m}})$$

Thus, we got

$\ddot{\text{n}}_1\ddot{\text{n}}_2^{-1} \in N(\mathcal{A}) \Rightarrow (\mathcal{A})$ is subgroup of $\ddot{\mathfrak{G}}$.

Proof (ii). Direct depending on the definition of BNNSG of $\ddot{\mathfrak{G}}$ mention above.

5. Conclusion

In continuation of the recent research work presented in the fuzzy algebraic environment, and to explains the apparatus of operation of the bipolar system in the fuzzy algebraic environment. In this studied, we applied this idea in an algebraic environment when we initiated the novel concept of bipolar single value neutrosophic subgroups and proved that every bipolar single value neutrosophic subgroup generates two bipolar single value neutrosophic subgroups. we explained the level set, support, kernel for bipolar single value neutrosophic set, bipolar single value neutrosophic characteristic function, and bipolar single value neutrosophic point. Then, we illuminated the bipolar single value neutrosophic subgroup, bipolar single value neutrosophic normal subgroup, bipolar single value neutrosophic conjugate, normalizer for bipolar single value neutrosophic subgroup, bipolar single value neutrosophic abelian subgroup, and bipolar single value neutrosophic factor group. Furthermore, we presented the linked theorems and examples and prove these theorems. Finally, we discussed the image and pre-image of bipolar single-value neutrosophic subgroups under homomorphism and proved the related theorems. However, the result presented in this work has some limitations as well, and the investigated work cannot be employed for the examination of molecule structures. BN subgroups can be generalised to bipolar complex neutrosophic subgroups, bipolar complex picture fuzzy subgroups, bipolar complex neutrosophic soft groups, etc. In the future, we aim to expand this research to bipolar bipolar complex neutrosophic soft sets [31,32], and complex bipolar complex neutrosophic N-soft sets [33-35]. We hope that these concepts will form the basis for innovative research on subgroups.

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