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# Exploring Teaching Cognitive Skills Development in University Art Education Using Extended Bipolar Neutrosophic Sets

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**Abstract:** Cognitive skills development, including critical thinking, creativity, and problem-solving, is a fundamental goal of university art education, overwhelmingly influencing academic success and lifelong learning. However, assessment of these multifaceted cognitive attributes leftovers challenging due to the inherent uncertainty in educational environments. In this study, we present our attempt to address the above challenge through developing a novel framework employing Bipolar Neutrosophic Sets (BNS), which simultaneously capture bipolar indeterminacy within the analysis of Cognitive Skills Development. Central to our framework is the design of a Component-wise Hybrid Harmonic-Geometric Aggregation Operator (CHHGAO), which integrates the strengths of harmonic and geometric means to aggregate BNS information expressively. Moreover, we propose a new distancing measure tailored for BNS that enhances algorithmic decision-making with accurate quantification of the closeness of alternatives to ideal solutions in the evaluation space. Furthermore, we demonstrate the practical applicability of our approach through a detailed case study involving reflecting cognitive skill development dimensions and resource considerations. Quantitative results obtained not only improve the handling of ambiguous and conflicting expert evaluations but also yield insightful decisions informing curriculum design and pedagogical strategies.

**Keywords**: Bipolar neutrosophic sets (BNSs), Uncertainty, Neutrosophic Theory, Teaching Quality, University Art Education.

## 1. Introduction

University art education plays a pivotal role in fostering a broad spectrum of cognitive skills, including creativity, critical thinking, problem-solving, spatial reasoning, and memory [1], [2]. These skills are essential not only for artistic expression but also for enhancing academic performance and lifelong learning capabilities [3]. Theoretical frameworks like Howard Gardner's theory of multiple intelligences highlighted how arts education nurtures various forms of intelligence— in range from spatial to interpersonal—that contribute to cognitive growth [4]. Empirical studies further demonstrated that engaging in artistic activities can activate some brain regions responsible for decision-making, planning, and creative thinking, thereby improving cognitive functions crucial for academic and personal success [5].

Despite the recognized benefits, traditional assessment methods in university art education often struggle to capture the full complexity of cognitive skill development [6]. Old evaluation approaches tend to be subjective, relying heavily on qualitative judgments that can vary widely among educators [7]. Moreover, these methods frequently face uncertainty and ambiguity when assessing abstract cognitive attributes like creativity as well as critical thinking. For example, two evaluators might assign different scores to the same artwork based on personal biases or differing interpretations of creativity, leading to inconsistent and sometimes unfair assessments.

To overcome these limitations, Neutrosophic Sets [8], [9], and in particular Bipolar Neutrosophic Sets (BNS), have presented a powerful tool for handling uncertainty, indeterminacy, and dual perspectives in complex decision-making problems. BNS extended the fuzzy and intuitionistic fuzzy sets [10] by adding bipolar degrees of truth, indeterminacy, and falsity distinctly. The primary objective of this study is to develop and apply a BNS-based framework for evaluating cognitive skills development in university art education. To this end, this article contributes to the body of neutrosophic knowledge as follows. First, we formalize and apply the BNS principles for assessing the cognitive skill development within University art education. Second, in our framework, we design a Component-wise Hybrid Harmonic-Geometric Aggregation Operator (CHHGAO) to integrate the strengths of harmonic and geometric means to expressively aggregate BNS information. In addition, a new distancing measure is integrated into the BNS method to enhance algorithmic quantification of the closeness of alternatives to ideal solutions in the decision space.

## 2. Preliminaries & Definitions

Definition 1. BNSs [11] define a dual directed degree for each of the three components of the original neutrosophic theory.

$$\overleftrightarrow{B} = \left\{e, \langle \begin{matrix} \Psi^+(e), \phi^+(e), \Delta^+(\Delta^+), \\ \Psi^-(e), \phi^-(e), \Delta^-(\Delta^-), \end{matrix} \rangle \,|\, e \in E\right\}, \quad (1)$$

Whereas

$$\Psi^+(e), \phi^+(e), \Delta^+(e): E \to [0,1]$$

$$\Psi^-(e), \phi^-(e), \Delta^-(e): E \to [-1,0]$$

Definition 2. Given a BNS symbolized as $\mathcal{S}_1 = \left\{e, \langle {\Psi_1}^+(e), {\phi_1}^+(e), {\Delta_1}^+(e), {\Psi_1}^-(e), \phi^-_1(e), \Delta^-_1(e) \rangle \middle| e \in E\right\}$, then the complement of $\mathcal{S}_1$ is defined as:

$$\mathcal{S}_1^c = \begin{Bmatrix} (\{1^+\} - \Psi^+(e)), (\{1^+\} - \phi^+(e)), (\{1^+\} - \Delta^+(e)), \\ (\{1^-\} - \Psi^-(e)), (\{1^-\} - \phi^-(e)), (\{1^-\} - \Delta^-(e)) \end{Bmatrix} \quad (2)$$

Definition 3. Given two BNSs $\mathcal{S}_1 = \{e, \langle \Psi_1{}^{+}(e), \phi_1{}^{+}(e), \Delta_1{}^{+}(e), \Psi_1{}^{-}(e), \phi_1{}^{-}(e), \Delta_1{}^{-}(e) \rangle \mid e \in E\}$, and
$\mathcal{S}_2 = \{e, \langle \Psi_2{}^{+}(e), \phi_2{}^{+}(e), \Delta_2{}^{+}(e), \Psi_2{}^{-}(e), \phi_2{}^{-}(e), \Delta_2{}^{-}(e) \rangle \mid e \in E\}$, The union of them is defined as:

$$\mathcal{S}_1 \cup \mathcal{S}_2 = \left\{ e, \begin{pmatrix} \max\left(\Psi_1^{+}(e), \Psi_2^{+}(e)\right), \frac{\phi_1^{+}(e)+\phi_2^{+}(e)}{2}, \min\left(\Delta_1^{+}(e), \Delta_2^{+}(e)\right), \\ \min\left(\Psi_1^{-}(e), \Psi_2^{-}(e)\right), \frac{\phi_1^{-}(e)+\phi_2^{-}(e)}{2}, \max\left(\Delta_1^{-}(e), \Delta_2^{-}(e)\right) \end{pmatrix} \mid e \in E \right\} \quad (3)$$

Definition 4. Given two BNSs $\mathcal{S}_1 = \{e, \langle \Psi_1{}^{+}(e), \phi_1{}^{+}(e), \Delta_1{}^{+}(e), \Psi_1{}^{-}(e), \phi_1{}^{-}(e), \Delta_1{}^{-}(e) \rangle \mid e \in E\}$, and
$\mathcal{S}_2 = \{e, \langle \Psi_2{}^{+}(e), \phi_2{}^{+}(e), \Delta_2{}^{+}(e), \Psi_2{}^{-}(e), \phi_2{}^{-}(e), \Delta_2{}^{-}(e) \rangle \mid e \in E\}$, The intersection of them is defined as:

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \left\{ e, \begin{pmatrix} \min\left(\Psi_1^{+}(e), \Psi_2^{+}(e)\right), \frac{\phi_1^{+}(e)+\phi_2^{+}(e)}{2}, \max\left(\Delta_1^{+}(e), \Delta_2^{+}(e)\right), \\ \max\left(\Psi_1^{-}(e), \Psi_2^{-}(e)\right), \frac{\phi_1^{-}(e)+\phi_2^{-}(e)}{2}, \min\left(\Delta_1^{-}(e), \Delta_2^{-}(e)\right) \end{pmatrix} \mid e \in E \right\} \quad (4)$$

Definition 5. Given two BNSs $\mathcal{S}_1 = \{e, \langle \Psi_1{}^{+}(e), \phi_1{}^{+}(e), \Delta_1{}^{+}(e), \Psi_1{}^{-}(e), \phi_1{}^{-}(e), \Delta_1{}^{-}(e) \rangle \mid e \in E\}$, and
$\mathcal{S}_2 = \{e, \langle \Psi_2{}^{+}(e), \phi_2{}^{+}(e), \Delta_2{}^{+}(e), \Psi_2{}^{-}(e), \phi_2{}^{-}(e), \Delta_2{}^{-}(e) \rangle \mid e \in E\}$, $\mathcal{S}_1 = \mathcal{S}_2$ if, and only if,

$$\begin{gathered} \Psi_1^{+}(e) = \Psi_2^{+}(e), \Delta_1^{+}(e) = \Delta_2^{+}(e), \phi_1^{+}(e) = \phi_2^{+}(e) \\ \Psi_1^{-}(e) = \Psi_2^{-}(e), \Delta_1^{-}(e) = \Delta_2^{-}(e), \phi_1^{-}(e) = \phi_2^{-}(e) \end{gathered} \quad (5)$$

Definition 6. Given two BNSs $\mathcal{S}_1 = \{e, \langle \Psi_1{}^{+}(e), \phi_1{}^{+}(e), \Delta_1{}^{+}(e), \Psi_1{}^{-}(e), \phi_1{}^{-}(e), \Delta_1{}^{-}(e) \rangle \mid e \in E\}$, and
$\mathcal{S}_2 = \{e, \langle \Psi_2{}^{+}(e), \phi_2{}^{+}(e), \Delta_2{}^{+}(e), \Psi_2{}^{-}(e), \phi_2{}^{-}(e), \Delta_2{}^{-}(e) \rangle \mid e \in E\}$, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if, and only if,

$$\begin{gathered} \Psi^{+}(e) < \Psi^{+}(e), \Delta^{+}(e) < \Delta^{+}(e), \phi^{+}(e) \geq \phi^{+}(e) \\ \Psi^{+}(e) < \Psi^{+}(e), \Delta^{+}(e) < \Delta^{+}(e), \phi^{+}(e) \geq \phi^{+}(e) \end{gathered} \quad (6)$$

$$\lambda \mathcal{S}_1 = \left\{ e, \begin{pmatrix} \left(1 - \left(1 - \Psi_1^{+}(e)\right)^{\lambda}\right), \phi_1^{+}(e)^{\lambda}, \Delta_1^{+}(e), \\ -\left(-\Psi_1^{-}(e)\right)^{\lambda}, -\left(-\phi_1^{-}(e)\right)^{\lambda}, -\left(1 - \left(1 - \Delta_1^{-}(e)\right)\right)^{\lambda} \end{pmatrix} \mid e \in E \right\} \quad (7)$$

$$\mathcal{S}_1^{\lambda} = \left\{ e, \begin{pmatrix} \left(\Psi_1^+(e)\right)^{\lambda}, 1-\left(1-\phi_1^+(e)\right)^{\lambda}, 1-\left(1-\Delta_1^+(e)\right)^{\lambda}, \\ -\left(1-\left(1-\left(-\Psi_1^-(e)\right)\right)\right)^{\lambda}, -\left(-\phi_1^-(e)\right)^{\lambda}, -\left(-\Delta_1^-(e)\right)^{\lambda} \end{pmatrix} \mid e \in E \right\} \tag{8}$$

$$\lambda\mathcal{S}_1 = \left\{ e, \begin{pmatrix} \left(1-\left(1-\Psi_1^+(e)\right)^{\lambda}\right), \phi_1^+(e)^{\lambda}, \Delta_1^+(e), \\ -\left(-\Psi_1^-(e)\right)^{\lambda}, -\left(-\phi_1^-(e)\right)^{\lambda}, -\left(1-\left(1-\Delta_1^-(e)\right)\right)^{\lambda} \end{pmatrix} \mid e \in E \right\} \tag{9}$$

$$\mathcal{S}_1 + \mathcal{S}_2 = \left\{ e, \begin{pmatrix} \left(\Psi_1^+(e)+\Psi_2^+(e)-\Psi_1^+(e)\cdot\Psi_2^+(e)\right), \phi_1^+(e)\cdot\phi_2^+(e), \\ \Delta_1^+(e)\cdot\Delta_2^+(e), \left(-\Psi_1^-(e)-\Psi_2^-(e)\right), \\ -\left(-\phi_1^-(e)-\phi_2^-(e)-\phi_1^-(e)\cdot\phi_2^-(e)\right), \\ -\left(-\Delta_1^-(e)-\Delta_2^-(e)-\Delta_1^-(e)\cdot\phi_2^-(e)\right) \end{pmatrix} \mid e \in E \right\} \tag{10}$$

$$\mathcal{S}_1 \cdot \mathcal{S}_2 = \left\{ e, \begin{pmatrix} \left(\Psi_1^+(e)\cdot\Psi_2^+(e)\right), \phi_1^+(e)+\phi_2^+(e)-\phi_1^+(e)\cdot\phi_2^+(e), \\ \Delta_1^+(e)+\Delta_2^+(e)-\Delta_1^+(e)\cdot\Delta_2^+(e), \\ \left(-\Psi_1^-(e)-\Psi_2^-(e)-\Psi_1^-(e)\cdot\Psi_2^-(e)\right), \\ \left(-\phi_1^-(e)\cdot\phi_2^-(e)\right), \left(-\Delta_1^-(e)\cdot\phi_2^-(e)\right) \end{pmatrix} \mid e \in E \right\} \tag{11}$$

Definition 7. Given an BNS $\mathcal{S}_1 = \{e, \langle {\Psi_1}^+(e), {\phi_1}^+(e), {\Delta_1}^+(e), {\Psi_1}^-(e), {\phi_1}^-(e), {\Delta_1}^-(e)\rangle \mid e \in E\}$,The de-neutrosophication of BNS refers to the action of converting the BNS into a crisp number. This was achieved using the following scoring function [12], [13]:

$$S(\mathcal{S}_1) = \frac{\left(\Psi_1^+(e)+1-\phi_1^+(e)+1-\Delta_1^+(e)+1+\Psi_1^-(e)-\phi_1^-(e)-\Delta_1^-(e)\right)}{6}. \tag{12}$$

However, this definition can be followed in many ways.

Definition 8.Let $W = (w_1, w_2, \ldots, w_n)$ be a weight vector such that $w_j \in W$ and $\sum_{j=1}^{n} w_j = 1$, BNS Weighted Average (BNSWA) Operator $A_w$ aggregates the bipolar neutrosophic numbers $\tilde{s}_j$Into a single BNN defined as:

(13)

$$A_w(s_1, \tilde{s}_2, \dots, \tilde{s}_n) = \sum_{j=1}^{n} w_j \tilde{s}_j$$

$$= \left\langle \begin{array}{c} \left(1 - \prod_{j=1}^{n}\left(1 - \Psi_j^{+}(e)\right)^{w_j}\right), \prod_{j=1}^{n}\left(\phi_j^{+}(e)\right)^{w_j}, \prod_{j=1}^{n}\left(\Delta_j^{+}(e)\right)^{w_j}, \\ -\prod_{j=1}^{n}\left(-\Psi_j^{-}(e)\right)^{w_j}, -\left(1 - \prod_{j=1}^{n}\left(1 - \left(-\phi_j^{-}(e)\right)\right)^{w_j}\right), \\ -\left(1 - \prod_{j=1}^{n}\left(1 - \left(-\Delta_j^{-}(e)\right)\right)^{w_j}\right) \end{array} \right\rangle$$

3. Materials and Methods

A leading university's Art Department seeks to evaluate six undergraduate art courses for their effectiveness in fostering cognitive skills development. The department aims to select the most effective course for a new curriculum initiative, using a structured neutrosophic approach. The assessment incorporates both benefit and cost criteria to ensure a balanced, real-world evaluation. The six alternatives represent distinct art courses, namely A1 (Studio Painting), A2 (Digital Media Arts), A3 (Sculpture & Installation), A4 (Art History & Criticism), A5 (Printmaking), and A6 (Mixed Media Workshop). Five criteria are used to evaluate each alternative, based on literature and real-world assessment practices in university art education. are three *benefit* criteria: C1 (Enhancement of Critical Thinking), C2 (Creativity Stimulation), and C3(Problem-Solving Skill Development), and two are *cost* criteria including C4 (Resource Intensity), and C5 (Time Commitment). In Table 1, we present the linguistic terms used for each assessment criterion in the case study, along with their corresponding BNS values.

Table 1. Linguistic and mathematical mapping for use in Bipolar Neutrosophic.

| Criterion | Linguistic Term | BNS Value |
|---|---|---|
| Enhancement of Critical Thinking | Low | (0.2, 0.3, 0.5), (0.6, 0.3, 0.1) |
| | Medium | (0.5, 0.3, 0.2), (0.3, 0.4, 0.3) |
| | High | (0.8, 0.1, 0.1), (0.1, 0.2, 0.7) |
| Creativity Stimulation | Low | (0.2, 0.3, 0.5), (0.6, 0.3, 0.1) |
| | Medium | (0.5, 0.3, 0.2), (0.3, 0.4, 0.3) |
| | High | (0.8, 0.1, 0.1), (0.1, 0.2, 0.7) |

| | | |
|---|---|---|
| Problem-Solving Skill Development | Low | (0.2, 0.3, 0.5), (0.6, 0.3, 0.1) |
| | Medium | (0.5, 0.3, 0.2), (0.3, 0.4, 0.3) |
| | High | (0.8, 0.1, 0.1), (0.1, 0.2, 0.7) |
| Resource Intensity | Low | (0.8, 0.1, 0.1), (0.1, 0.2, 0.7) |
| | Medium | (0.5, 0.3, 0.2), (0.3, 0.4, 0.3) |
| | High | (0.2, 0.3, 0.5), (0.6, 0.3, 0.1) |
| Time Commitment | Low | (0.8, 0.1, 0.1), (0.1, 0.2, 0.7) |
| | Medium | (0.5, 0.3, 0.2), (0.3, 0.4, 0.3) |
| | High | (0.2, 0.3, 0.5), (0.6, 0.3, 0.1) |

Team of faculty and external assessors rate each alternative against the criteria using a standardized scale, which enables transparent comparison and supports advanced tools based on BNSs. In Table 2-4, we present three BNS decision matrices, which represent valuations from three different assessors for the six art courses against five criteria, using BNS values based on the linguistic definitions provided earlier.

Table 2. BNS-Decision Matrix from Expert 1.

| Alternative | C1: Critical Thinking | C2: Creativity | C3: Problem-Solving | C4: Resource Intensity | C5: Time Commitment |
|---|---|---|---|---|---|
| A1 | (0.42,0.48,0.49),(-0.58,-0.24,-0.26) | (0.23,0.45,0.44),(-0.67,-0.21,-0.59) | (0.7,0.18,0.27),(-0.25,-0.28,-0.41) | (0.46,0.22,0.44),(-0.21,-0.27,-0.35) | (0.47,0.41,0.28),(-0.51,-0.35,-0.22) |
| A2 | (0.56,0.17,0.23),(-0.86,-0.44,-0.52) | (0.38,0.14,0.47),(-0.45,-0.23,-0.4) | (0.22,0.46,0.3),(-0.63,-0.28,-0.41) | (0.53,0.17,0.59),(-0.72,-0.43,-0.56) | (0.56,0.47,0.24),(-0.26,-0.21,-0.33) |
| A3 | (0.43,0.21,0.53),(-0.39,-0.27,-0.42) | (0.28,0.42,0.23),(-0.89,-0.39,-0.28) | (0.2,0.43,0.48),(-0.68,-0.39,-0.23) | (0.42,0.15,0.55),(-0.6,-0.28,-0.23) | (0.39,0.23,0.49),(-0.61,-0.42,-0.39) |
| A4 | (0.27,0.39,0.5),(-0.55,-0.39,-0.4) | (0.51,0.27,0.21),(-0.19,-0.21,-0.45) | (0.39,0.3,0.56),(-0.3,-0.3,-0.5) | (0.34,0.13,0.32),(-0.23,-0.43,-0.52) | (0.58,0.45,0.52),(-0.25,-0.42,-0.42) |
| A5 | (0.68,0.46,0.33),(-0.19,-0.26,-0.37) | (0.69,0.44,0.2),(-0.51,-0.3,-0.29) | (0.27,0.24,0.58),(-0.36,-0.33,-0.48) | (0.42,0.49,0.58),(-0.3,-0.32,-0.32) | (0.37,0.11,0.44),(-0.5,-0.21,-0.31) |

| | | | | | |
|---|---|---|---|---|---|
| A6 | (0.74,0.2,0.26),(-0.49,-0.45,-0.3) | (0.6,0.4,0.3),(-0.68,-0.29,-0.45) | (0.58,0.31,0.24),(-0.77,-0.28,-0.27) | (0.22,0.34,0.47),(-0.11,-0.33,-0.29) | (0.59,0.17,0.48),(-0.41,-0.43,-0.26) |

Table 3. BNS-Decision Matrix from Expert 2.

| Alternative | C1: Critical Thinking | C2: Creativity | C3: Problem-Solving | C4: Resource Intensity | C5: Time Commitment |
|---|---|---|---|---|---|
| A1 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A2 | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) |
| A3 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A4 | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) |
| A5 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A6 | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) |

Table 4. BNS-Decision Matrix from Expert 3.

| Alternative | C1: Critical Thinking | C2: Creativity | C3: Problem-Solving | C4: Resource Intensity | C5: Time Commitment |
|---|---|---|---|---|---|
| A1 | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A2 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.8,0.1,0.1),(0.1,0.2,0.7) |
| A3 | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) |
| A4 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A5 | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.2,0.3,0.5),(0.6,0.3,0.1) | (0.5,0.3,0.2),(0.3,0.4,0.3) |
| A6 | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.8,0.1,0.1),(0.1,0.2,0.7) | (0.5,0.3,0.2),(0.3,0.4,0.3) | (0.2,0.3,0.5),(0.6,0.3,0.1) |

## 4. Component-wise Hybrid Harmonic-Geometric Aggregation

In the following, we provide our proposed approach BNSs for making appropriate decisions about the quality of teaching of the English language and related literature.

Definition 8.Let $W=(w_1,w_2,\dots,w_n)$ be a weight vector such that $w_j\in W$ and $\sum_{j=1}^{n} w_j=1$, Component-wise Hybrid Harmonic-Geometric Aggregation Operator ($CHHGAO$) Operator $A_w$ aggregates the bipolar neutrosophic numbers $\tilde{s}_j$into a single BNN defined as:

$$CHHGAO_W(\tilde{a}_1,\dots,\tilde{a}_n)=\sum_{j=1}^{n} w_j\tilde{a}_j$$

$$=\langle \left(\prod_{j=1}^{n}\left(\Psi_j^{+}\right)^{w_j}\right),\left(\frac{\sum_{j=1}^{n} w_j}{\sum_{j=1}^{n}\frac{w_j}{\left|\varphi_j^{+}\right|+\epsilon}}\right),\left(\prod_{j=1}^{n}\left(\Delta_j^{+}\right)^{w_j}\right), -\prod_{j=1}^{n}\left(-\Psi_j^{-}(e)\right)^{w_j},\left(-\frac{\sum_{j=1}^{n} w_j}{\sum_{j=1}^{n}\frac{w_j}{\left|\varphi_j^{-}\right|+\epsilon}}\right), \left(-\prod_{i=1}^{n}\left(-\Delta_j^{-}\right)^{w_j}\right)\rangle \quad (14)$$

where $\epsilon>0$ is a small positive constant.
Then the operator satisfies Idempotency property stating that If $\tilde{a}_1=\tilde{a}_2=\cdots=\tilde{a}_n=\tilde{a}$, then

$$CHHGAO_W(\tilde{a},\dots,\tilde{a})=\tilde{a}.$$

<u>Proof</u>

Assume $\tilde{a}_1=\tilde{a}_2=\cdots=\tilde{a}_n=\tilde{a}=\langle\Psi^{+},\varphi^{+},\Delta^{+},\Psi^{-},\varphi^{-},\Delta^{-}\rangle$, then

$$\Psi^{+}=\prod_{j=1}^{n}\left(\Psi_j^{+}\right)^{w_j}=\prod_{j=1}^{n}\left(\Psi^{+}\right)^{w_j}=\left(\Psi^{+}\right)^{\sum_{j=1}^{n} w_j}=\left(\Psi^{+}\right)^{1}=\Psi^{+}$$

$$\varphi^{+}=\frac{\sum_{j=1}^{n} w_j}{\sum_{j=1}^{n}\frac{w_j}{\varphi_j^{+}+\epsilon}}=\frac{1}{\frac{1}{\varphi^{+}+\epsilon}}=\varphi^{+}+\epsilon\approx\varphi^{+}$$

$$\Delta^{+}=\prod_{j=1}^{n}\left(\Delta_j^{+}\right)^{w_j}=\left(\Delta^{+}\right)^{\sum w_j}=\Delta^{+}.$$

$$\Psi^{-} = -\prod_{j=1}^{n} \left(-\Psi_j^{-}\right)^{w_j} = -\prod_{j=1}^{n} (-\Psi^{-})^{w_j} = -(-\Psi^{-})^{\sum w_j} = -(-\Psi^{-})^{1} = \Psi^{-}$$

$$\varphi^{-} = -\frac{\sum w_j}{\sum \frac{w_j}{|\varphi_j^{-}| + \epsilon}} = -\frac{1}{\frac{1}{|\varphi^{-}| + \epsilon}} = -(|\varphi^{-}| + \epsilon) \approx \varphi^{-}$$

noting $\varphi^{-} \le 0$ and $|\varphi^{-}| = -\varphi^{-}$.

$$\Delta^{-} = -\prod_{j=1}^{n} \left(-\Delta_j^{-}\right)^{w_j} = -(-\Delta^{-})^{\sum w_j} = \Delta^{-}$$

The definition of our aggregator satisfies the Boundedness property for each component. $\min_j x_j \le$ aggregated $\le \max_j x_j$, where $x_j$ Denotes the respective membership degree inputs.

<u>Proof</u>

We prove boundedness for each component separately. For any set of positive numbers $x_j \in [a, b]$ with $0 < a \le b$, the weighted geometric mean satisfies:

$$\min_j x_j \le \prod_{j=1}^{n} x_j^{w_j} \le \max_j x_j \tag{15}$$

Since $\Psi_j^{+}, \Delta_j^{+} \in$, the property holds for $\Psi^{+}$and $\Delta^{+}$. For negative components, consider $-\Psi_j^{-}$and $-\Delta_j^{-}$Which are positive numbers (since $\Psi_j^{-}, \Delta_j^{-} \le 0$ ), so the same property holds:

$$\min_j\left(-\Psi_j^{-}\right) \le \prod_{j=1}^{n} \left(-\Psi_j^{-}\right)^{w_j} \le \max_j\left(-\Psi_j^{-}\right) \tag{16}$$

Multiplying by -1 reverses inequalities, but since $\Psi^{-} = -\prod \left(-\Psi_j^{-}\right)^{w_j}$, the aggregated $\Psi^{-}$lies between $\min_j \Psi_j^{-}$and $\max_j \Psi_j^{-}$. Recall the weighted harmonic mean:

$H = \frac{\sum_{j=1}^{n} w_j}{\sum_{j=1}^{n} \frac{w_j}{x_j}}$, where $x_j > 0$. It is well-known that

$$\min_j x_j \le H \le \max_j x_j \tag{17}$$

For $\varphi^{+}$, since $\varphi_j^{+} + \epsilon > 0$, boundedness holds. For $\varphi^{-}$, since $\left|\varphi_j\right| + \epsilon > 0$, boundedness holds similarly.

Our aggregator satisfies the Monotonicity property If $\tilde{a}_j \leq \tilde{a}_j^*$ component-wise for all $j$, then $CHHGAO_W(\tilde{a}_1, \dots, \tilde{a}_n) \leq CHHGAO_W(\tilde{a}_1^*, \dots, \tilde{a}_n^*)$

Proof

Assume for all $j$,

$$\begin{aligned} &\Psi_j^+ \leq \Psi_j^*, \quad \varphi_j^+ \leq \varphi_j^{+*}, \quad \Delta_j^+ \leq \Delta_j^{+*} \\ &\Psi_j^- \leq \Psi_j^{**}, \quad \varphi_j^- \leq \varphi_j^{**}, \quad \Delta_j^- \leq \Delta_j^* \end{aligned} \tag{18}$$

We want to show $CHHGAO_W(\tilde{a}_1, \dots, \tilde{a}_n) \leq CHHGAO_W(\tilde{a}_1^*, \dots, \tilde{a}_n^*)$. For geometric mean components $\Psi^+, \Delta^+$Since the weighted geometric mean is monotonically increasing in each argument,

$$\prod_{j=1}^{n} (\Psi_j^+)^{w_j} \leq \prod_{j=1}^{n} (\Psi_j^{+*})^{w_j}, \prod_{j=1}^{n} (\Delta_j^+)^{w_j} \leq \prod_{j=1}^{n} (\Delta_j^{+*})^{w_j}. \tag{19}$$

For harmonic mean components $\varphi^+$The harmonic mean is monotonically increasing in each argument $x_j$: If $\varphi_j^+ \leq \varphi_j^{+*}$, then

$$\frac{\sum w_j}{\sum \frac{w_j}{\varphi_j^+ + \epsilon}} \leq \frac{\sum w_j}{\sum \frac{w_j}{\varphi_j^{+*} + \epsilon}}. \tag{20}$$

For negative geometric mean components $\Psi^-, \Delta^-$: since $\Psi_j^- \leq \Psi_j^{-*} \leq 0$, then $-\Psi_j^- \geq -\Psi_j^{-*} \geq 0$. Because geometric mean is increasing in positive arguments,

$$\prod_{j=1}^{n} (-\Psi_j^-)^{w_j} \geq \prod_{j=1}^{n} (-\Psi_j^{-*})^{w_j}.$$

Multiplying by $-1$ reverses inequality: (21)

$$\Psi^- = -\prod (-\Psi_j^-)^{w_j} \leq -\prod (-\Psi_j^{-*})^{w_j} = \Psi^{-*}$$

Similarly, for $\Delta^-$. For negative harmonic mean component $\varphi^-$: since $\varphi_j^- \leq \varphi_j^{-*} \leq 0$, $|\varphi_j^-| \geq |\varphi_j^{-*}|$. Because harmonic mean is increasing in positive arguments,

$$\frac{\sum w_j}{\sum \frac{w_j}{|\varphi_j^-| + \epsilon}} \geq \frac{\sum w_j}{\sum \frac{w_j}{|\varphi_j^{-*}| + \epsilon}}.$$

Multiplying by $-1$ reverses inequality:

(22)

$$\varphi^- = -\frac{\sum w_j}{\sum \frac{w_j}{|\varphi_j^-| + \epsilon}} \leq -\frac{\sum w_j}{\sum \frac{w_j}{|\varphi_j^{-*}| + \epsilon}} = \varphi^{-*}.$$

## 5. Decision-making approach

Let there be $N_A$ alternatives evaluated against $N_C$ Criteria, with aggregated BNS values obtained from multiple experts. The following steps outline the decision-making procedure:

Step 1: Obtain the Aggregated Decision Matrix

Compute the aggregated decision matrix $A = \left[\tilde{a}_{ij} = \langle \Psi_{ij}^+, \phi_{ij}^+, \Delta_{ij}^+, \Psi_{ij}^-, \phi_{ij}^-, \Delta_{ij}^- \rangle\right]$ In BNS format, where each element represents the aggregated positive truth, indeterminacy, falsity, and negative truth, indeterminacy, and falsity membership degrees of alternative $i$ concerning criterion $j$.

Step 2: Normalize the Aggregated Decision Matrix

Normalize each element $\tilde{a}_{ij} = \langle \Psi_{ij}^+, \phi_{ij}^+, \Delta_{ij}^+, \Psi_{ij}^-, \phi_{ij}^-, \Delta_{ij}^- \rangle$, of the aggregated matrix to ensure comparability across criteria while preserving the BNS structure. The normalization depends on whether criterion $j$ is a benefit or cost criterion. For benefit criteria, the normalized BNS components are computed as:

$$\begin{cases} \Psi_{ij}^{+(norm)} = \dfrac{\Psi_{ij}^{+} - \min\limits_{i}\Psi_{ij}^{+}}{\max\limits_{i}\Psi_{ij}^{+} - \min\limits_{i}\Psi_{ij}^{+} + \varepsilon} \\ \phi_{ij}^{+(norm)} = \dfrac{\phi_{ij}^{+} - \min\limits_{i}\phi_{ij}^{+}}{\max\limits_{i}\phi_{ij}^{+} - \min\limits_{i}\phi_{ij}^{+} + \varepsilon} \\ \Delta_{ij}^{+(norm)} = \dfrac{\Delta_{ij}^{+} - \min\limits_{i}\Delta_{ij}^{+}}{\max\limits_{i}\Delta_{ij}^{+} - \min\limits_{i}\Delta_{ij}^{+} + \varepsilon} \\ \Psi_{ij}^{-(norm)} = \dfrac{\Psi_{ij}^{-} - \min\limits_{i}\Psi_{ij}^{-}}{\max\limits_{i}\Psi_{ij}^{-} - \min\limits_{i}\Psi_{ij}^{-} + \varepsilon} \\ \phi_{ij}^{-(norm)} = \dfrac{\phi_{ij}^{-} - \min\limits_{i}\phi_{ij}^{-}}{\max\limits_{i}\phi_{ij}^{-} - \min\limits_{i}\phi_{ij}^{-} + \varepsilon} \\ \Delta_{ij}^{-(norm)} = \dfrac{\Delta_{ij}^{-} - \min\limits_{i}\Delta_{ij}^{-}}{\max\limits_{i}\Delta_{ij}^{-} - \min\limits_{i}\Delta_{ij}^{-} + \varepsilon} \end{cases}, \begin{cases} \Psi_{ij}^{+(norm)} = \dfrac{\max\limits_{i}\Psi_{ij}^{+} - \Psi_{ij}^{+}}{\max\limits_{i}\Psi_{ij}^{+} - \min\limits_{i}\Psi_{ij}^{+} + \varepsilon} \\ \phi_{ij}^{+(norm)} = \dfrac{\max\limits_{i}\phi_{ij}^{+} - \phi_{ij}^{+}}{\max\limits_{i}\phi_{ij}^{+} - \min\limits_{i}\phi_{ij}^{+} + \varepsilon} \\ \Delta_{ij}^{+(norm)} = \dfrac{\max\limits_{i}\Delta_{ij}^{+} - \Delta_{ij}^{+}}{\max\limits_{i}\Delta_{ij}^{+} - \min\limits_{i}\Delta_{ij}^{+} + \varepsilon} \\ \Psi_{ij}^{-(norm)} = \dfrac{\max\limits_{i}\Psi_{ij}^{-} - \Psi_{ij}^{-}}{\max\limits_{i}\Psi_{ij}^{-} - \min\limits_{i}\Psi_{ij}^{-} + \varepsilon} \\ \phi_{ij}^{-(norm)} = \dfrac{\max\limits_{i}\phi_{ij}^{-} - \phi_{ij}^{-}}{\max\limits_{i}\phi_{ij}^{-} - \min\limits_{i}\phi_{ij}^{-} + \varepsilon} \\ \Delta_{ij}^{-(norm)} = \dfrac{\max\limits_{i}\Delta_{ij}^{-} - \Delta_{ij}^{-}}{\max\limits_{i}\Delta_{ij}^{-} - \min\limits_{i}\Delta_{ij}^{-} + \varepsilon} \end{cases} \tag{23}$$

However, since the aggregated value within acceptable range according to definition of BNS we skip the normalization step.

Step 3: Construct the Weighted Decision Matrix

Given the weights $w_j$ for each criterion $j = 1, 2, \cdots, N_C$ Compute the weighted BNS values $\tilde{b}_{ij}^{(w_j)}$ as follows:

$$\tilde{b}_{ij}^{(w_j)} = \left\langle \Psi_{ij}^{(w_j)+}, \phi_{ij}^{(w_j)+}, \Delta_{ij}^{(w_j)+}, \Psi_{ij}^{(w_j)-}, \phi_{ij}^{(w_j)-}, \Delta_{ij}^{(w_j)-} \right\rangle \text{ where}$$

$$\begin{cases} \Psi_{ij}^{(w_j)+} = 1 - (1 - \Psi_{ij}^{+})^{w_j} \\ \phi_{ij}^{(w_j)+} = (\phi_{ij}^{+})^{w_j} \\ \Delta_{ij}^{(w_j)+} = (\Delta_{ij}^{+})^{w_j} \\ \Psi_{ij}^{(w_j)-} = -(-\Psi_{ij}^{-})^{w_j} \\ \phi_{ij}^{(w_j)-} = -(-\phi_{ij}^{-})^{w_j} \\ \Delta_{ij}^{(w_j)-} = -\left(1 - \left(1 - (-\Delta_{ij}^{-})\right)^{w_j}\right) \end{cases} \tag{24}$$

Step 4: Determine the Bipolar Neutrosophic Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS)

$$PIS = \left\{\tilde{p}_j = \langle \Psi_j^{(w_j)+}, \phi_j^{(w_j)+}, \Delta_j^{(w_j)+}, \Psi_j^{(w_j)-}, \phi_j^{(w_j)-}, \Delta_j^{(w_j)-} \rangle \mid j = 1,2,\dots,N_C\right\}$$
$$NIS = \left\{\tilde{n}_j = \langle \Psi_j^{(w_j)+}, \phi_j^{(w_j)+}, \Delta_j^{(w_j)+}, \Psi_j^{(w_j)-}, \phi_j^{(w_j)-}, \Delta_j^{(w_j)-} \rangle \mid j = 1,2,\dots,N_C\right\} \quad (25)$$

The components of $PIS$ and $NIS$ depend on whether the criterion $j$ is a benefit or cost criterion:

- For benefit criteria ( $j = 1, 2, \cdots, N_C^b$):

$$\begin{cases} \Psi_j^{(w_j)+} = \max_i \Psi_{ij}^{(w_j)+}, \phi_j^{(w_j)+} = \min_i \phi_{ij}^{(w_j)+}, \Delta_j^{(w_j)+} = \min_i \Delta_{ij}^{(w_j)+} \\ \Psi_j^{(w_j)-} = \min_i \Psi_{ij}^{(w_j)-}, \phi_j^{(w_j)-} = \max_i \phi_{ij}^{(w_j)-}, \Delta_j^{(w_j)-} = \max_i \Delta_{ij}^{(w_j)-} \end{cases}$$

$$\begin{cases} \Psi_j^{(w_j)+} \in P, \phi_j^{(w_j)+} \in P, \Delta_j^{(w_j)+} \in P \\ \Psi_j^{(w_j)-} \in P, \phi_j^{(w_j)-} \in P, \Delta_j^{(w_j)-} \in P \end{cases} \begin{cases} \Psi_j^{(w_j)+} \in N, \phi_j^{(w_j)+} \in N, \Delta_j^{(w_j)+} \in N \\ \Psi_j^{(w_j)-} \in N, \phi_j^{(w_j)-} \in N, \Delta_j^{(w_j)-} \in N \end{cases} \quad (26)$$

where the $N$ components are the opposite extrema (min or max swapped accordingly).

- For cost criteria ( $j = N_C^b + 1, \cdots, N_C$):

$$\begin{cases} \Psi_j^{(w_j)+} = \min_i \Psi_{ij}^{(w_j)+}, \phi_j^{(w_j)+} = \max_i \phi_{ij}^{(w_j)+}, \Delta_j^{(w_j)+} = \max_i \Delta_{ij}^{(w_j)+} \\ \Psi_j^{(w_j)-} = \max_i \Psi_{ij}^{(w_j)-}, \phi_j^{(w_j)-} = \min_i \phi_{ij}^{(w_j)-}, \Delta_j^{(w_j)-} = \min_i \Delta_{ij}^{(w_j)-} \end{cases} \quad (27)$$

Step 5: Compute the Distance of Each Alternative from PIS and NIS

For each alternative $S_i$, compute the distance to the PIS and NIS using the Minkowski distance of order $p$:

$$D_{\min}(S_i,P)=\left(\frac{1}{6N_C}\sum_{j=1}^{N_C}\left(\left|\Psi_{ij}^{(w_j)+}-\Psi_{j}^{(w_j)+}\right|^p+\left|\phi_{ij}^{(w_j)+}-\phi_{j}^{(w_j)+}\right|^p+\left|\Delta_{ij}^{(w_j)+}-\Delta_{j}^{(w_j)+}\right|^p+\left|\Psi_{ij}^{(w_j)-}-\Psi_{j}^{(w_j)-}\right|^p+\left|\phi_{ij}^{(w_j)-}-\phi_{j}^{(w_j)-}\right|^p+\left|\Delta_{ij}^{(w_j)-}-\Delta_{j}^{(w_j)-}\right|^p\right)\right)^{\frac{1}{p}} \quad (28)$$

$$D_{\min}(S_i,N)=\left(\frac{1}{6N_C}\sum_{j=1}^{N_C}\left(\left|\Psi_{ij}^{(w_j)+}-\Psi_{j}^{(w_j)+}\right|^p+\left|\phi_{ij}^{(w_j)+}-\phi_{j}^{(w_j)+}\right|^p+\left|\Delta_{ij}^{(w_j)+}-\Delta_{j}^{(w_j)+}\right|^p+\left|\Psi_{ij}^{(w_j)-}-\Psi_{j}^{(w_j)-}\right|^p+\left|\phi_{ij}^{(w_j)-}-\phi_{j}^{(w_j)-}\right|^p+\left|\Delta_{ij}^{(w_j)-}-\Delta_{j}^{(w_j)-}\right|^p\right)\right)^{\frac{1}{p}} \quad (29)$$

Step 6: Calculate the Closeness Coefficient (Level of Nearness) $\mu_i$

For each alternative $ S_i $, compute the closeness coefficient with respect to the positive ideal solution:

$$\mu_i(S_i)=\frac{D_{\min}(S_i,N)}{\max\limits_{1\le k\le N_A} D_{\min}(S_k,N)+D_{\min}(S_i,P)} \text{ for } i=1,2,\dots,N_A \quad (30)$$

Step 7: Compute the Inferior Ratio $\vartheta(i)$ and Rank Alternatives

Calculate the inferior ratio for each alternative:

$$\vartheta(i) = \frac{\mu_i(S_i)}{\min_{1\le k\le N_A} \mu_k(S_k)} \tag{31}$$

where $\vartheta(i) \in (0, \infty)$.

Finally, rank the alternatives in ascending order of $\vartheta(i)$. The alternative with the lowest $\vartheta(i)$ value is considered the best choice.

## 6. Results and Analysis

This section presents the outcomes of applying the proposed BNS framework to evaluate cognitive skills development across various university art education courses. In Table 5, we present the aggregated BNS decision matrix, synthesizing expert evaluations of six university art education alternatives across five assessment criteria.

Table 5. Aggregated BNS Decision Matrix for University Art Education Alternatives

| Criteria / Alternative | C1 | C2 | C3 | C4 | C5 |
|---|---|---|---|---|---|
| $S_1$ | (0.42,0.26,0.47), (-0.63,-0.29,-0.35) | (0.50,0.40,0.37), (-0.31,-0.22,-0.41) | (0.51,0.27,0.28), (-0.32,-0.35,-0.36) | (0.61,0.25,0.37), (-0.29,-0.28,-0.48) | (0.55,0.20,0.28), (-0.42,-0.29,-0.31) |
| $S_2$ | (0.62,0.25,0.33), (-0.56,-0.36,-0.44) | (0.48,0.23,0.48), (-0.63,-0.26,-0.46) | (0.31,0.35,0.31), (-0.54,-0.33,-0.31) | (0.66,0.19,0.46), (-0.68,-0.40,-0.46) | (0.47,0.36,0.29), (-0.23,-0.29,-0.36) |
| $S_3$ | (0.60,0.33,0.41), (-0.43,-0.22,-0.46) | (0.39,0.45,0.36), (-0.50,-0.29,-0.37) | (0.43,0.22,0.31), (-0.63,-0.32,-0.39) | (0.52,0.14,0.38), (-0.78,-0.27,-0.32) | (0.58,0.34,0.52), (-0.66,-0.32,-0.31) |
| $S_4$ | (0.50,0.37,0.47), (-0.72,-0.30,-0.40) | (0.41,0.30,0.28), (-0.30,-0.26,-0.41) | (0.48,0.22,0.52), (-0.25,-0.29,-0.49) | (0.35,0.24,0.24), (-0.30,-0.31,-0.46) | (0.44,0.40,0.39), (-0.33,-0.40,-0.34) |
| $S_5$ | (0.45,0.37,0.37), (-0.41,-0.32,-0.50) | (0.43,0.26,0.39), (-0.51,-0.31,-0.31) | (0.28,0.25,0.49), (-0.45,-0.33,-0.31) | (0.46,0.33,0.49), (-0.47,-0.32,-0.38) | (0.56,0.17,0.47), (-0.61,-0.29,-0.36) |
| $S_6$ | (0.56,0.27,0.32), (-0.42,-0.25,-0.24) | (0.57,0.40,0.41), (-0.30,-0.32,-0.39) | (0.52,0.33,0.36), (-0.60,-0.31,-0.26) | (0.28,0.37,0.46), (-0.43,-0.37,-0.32) | (0.46,0.28,0.38), (-0.41,-0.38,-0.30) |

Table 6 displays the normalized BNS decision matrix, where each aggregated value from Table 5 is scaled to a uniform range while preserving its bipolar nature. Normalization adjusts for varying scales and measurement units across criteria, facilitating fair cross-criteria comparisons.

Table 6. Normalized BNS Decision Matrix

| Criteria / Alternative | C1 | C2 | C3 | C4 | C5 |
|---|---|---|---|---|---|
| $S_1$ | (0.00,0.0833,1.00),(-0.7097,-0.50,-0.4231) | (0.6111,0.7727,0.4500),(-0.0303,-0.00,-0.6667) | (0.9583,0.3846,0.00),(-0.1842,-1.00,-0.4348) | (0.1316,0.5217,0.48),(-1.00,-0.9231,-0.00) | (0.2143,0.8696,1.00),(-0.5581,-1.00,-0.8333) |
| $S_2$ | (1.00,0.00,0.0667),(-0.4839,-1.0,-0.7692) | (0.50,0.00,1.00),(-1.0000,-0.4000,-1.00) | (0.1250,1.00,0.1250),(-0.7632,-0.6667,-0.2174) | (0.00,0.7826,0.12),(-0.2041,-0.0000,-0.1250) | (0.7857,0.1739,0.9583),(-1.0,-1.00,-0.00) |
| $S_3$ | (0.90,0.6667,0.6000),(-0.0645,-0.00,-0.8462) | (0.0000,1.0000,0.4000),(-0.6061,-0.7000,-0.4000) | (0.6250,0.00,0.1250),(-1.0000,-0.50,-0.5652) | (0.3684,1.0000,0.4400),(-0.00,-1.00,-1.00) | (0.00,0.2609,0.0000),(-0.00,-0.7273,-0.8333) |
| $S_4$ | (0.40,1.0000,1.00),(-1.00,-0.5714,-0.6154) | (0.1111,0.3182,0.0000),(-0.0000,-0.40,-0.6667) | (0.8333,0.00,1.00),(-0.00,-0.00,-1.00) | (0.8158,0.5652,1.00),(-0.9796,-0.6923,-0.1250) | (1.00,0.00,0.5417),(-0.7674,-0.00,-0.3333) |
| $S_5$ | (0.1500,1.0000,0.3333),(-0.00,-0.7143,-1.00) | (0.2222,0.1364,0.5500),(-0.6364,-0.90,-0.00) | (0.00,0.2308,0.8750),(-0.5263,-0.6667,-0.2174) | (0.5263,0.1739,0.0000),(-0.6327,-0.6154,-0.6250) | (0.1429,1.0000,0.2083),(-0.1163,-1.0000,-0.0000) |
| $S_6$ | (0.70,0.1667,0.00),(-0.0323,-0.2143,-0.00) | (1.0000,0.7727,0.6500),(-0.00,-1.00,-0.5333) | (1.00,0.8461,0.3333),(-0.9211,-0.3333,-0.00) | (1.00,0.00,0.12),(-0.7143,-0.2308,-1.00) | (0.8571,0.5217,0.5833),(-0.5814,-0.1818,-1.0000) |

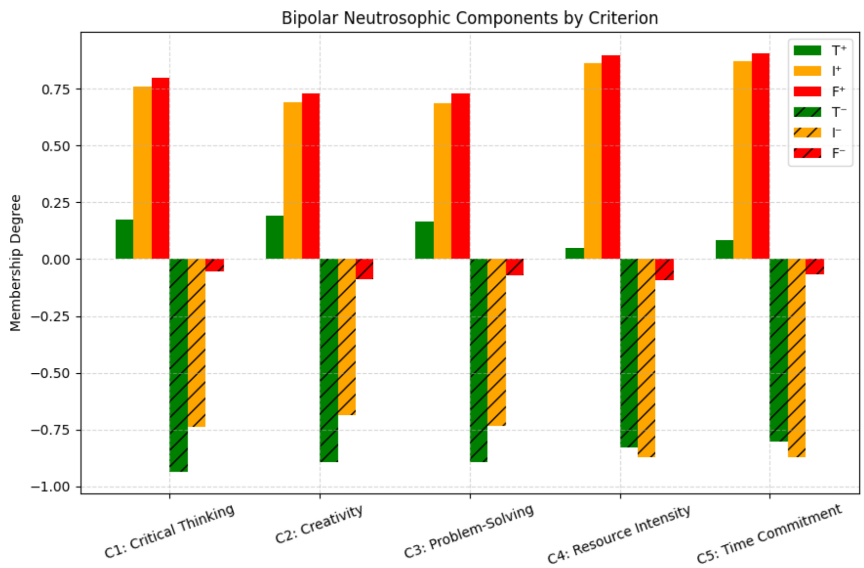
In Table 7, we show the weighted BNS decision matrix obtained by applying criterion weights to the normalized values from Table 5 using an established power-based aggregation formula.

Table 7. Weighted BNS Decision Matrix Incorporating Criterion Importance

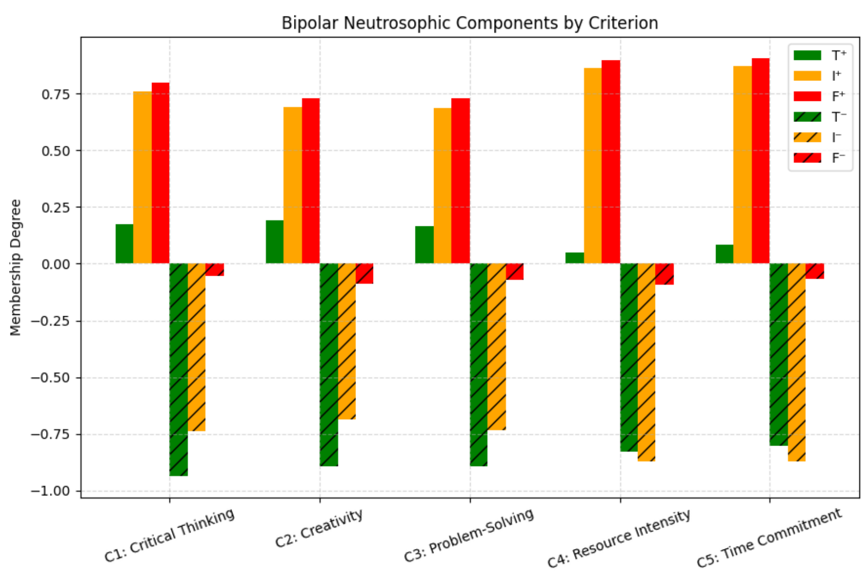
| Criteria / Alternative | C1 | C2 | C3 | C4 | C5 |
|---|---|---|---|---|---|
| $S_1$ | (0.1032,0.7638,0.8598),(-0.9117,-0.7807,-0.0825) | (0.1591,0.7953,0.7799),(-0.7462,-0.6849,-0.1236) | (0.1633,0.7208,0.7274),(-0.7521,-0.7692,-0.1056) | (0.1317,0.8123,0.8615),(-0.8305,-0.8262,-0.0934) | (0.1129,0.7855,0.8262),(-0.8780,-0.8305,-0.0541) |
| $S_2$ | (0.1759,0.7579,0.8011),(-0.8905,-0.8152,-0.1095) | (0.1508,0.6925,0.8324),(-0.8909,-0.7141,-0.1428) | (0.0886,0.7692,0.7462),(-0.8572,-0.7579,-0.0886) | (0.1494,0.7795,0.8900),(-0.9438,-0.8716,-0.0883) | (0.0908,0.8579,0.8305),(-0.8022,-0.8305,-0.0648) |
| $S_3$ | (0.1674,0.8011,0.8367),(-0.8447,-0.7387,-0.1159) | (0.1162,0.8190,0.7746),(-0.8409,-0.7338,-0.1091) | (0.1311,0.6849,0.7462),(-0.8909,-0.7521,-0.1162) | (0.1043,0.7446,0.8649),(-0.9634,-0.8217,-0.0562) | (0.1220,0.8506,0.9066),(-0.9396,-0.8429,-0.0541) |
| $S_4$ | (0.1294,0.8197,0.8598),(-0.9364,-0.7860,-0.0971) | (0.1236,0.7401,0.7274),(-0.7401,-0.7141,-0.1236) | (0.1508,0.6849,0.8492),(-0.7071,-0.7338,-0.1549) | (0.0626,0.8073,0.8073),(-0.8348,-0.8389,-0.0883) | (0.0833,0.8716,0.8683),(-0.8468,-0.8716,-0.0604) |
| $S_5$ | (0.1127,0.8197,0.8197),(-0.8367,-0.7962,-0.1294) | (0.1311,0.7141,0.7903),(-0.8451,-0.7462,-0.0886) | (0.0788,0.7071,0.8367),(-0.8190,-0.7579,-0.0886) | (0.0883,0.8468,0.8985),(-0.8929,-0.8429,-0.0692) | (0.1159,0.7666,0.8929),(-0.9285,-0.8305,-0.0648) |

| | | | | | |
|---|---|---|---|---|---|
| $s_6$ | (0.1514,0.7696,0.7962),(-0.8407,-0.7579,-0.0534) | (0.1902,0.7953,0.8002),(-0.7401,-0.7521,-0.1162) | (0.1676,0.7579,0.7746),(-0.8801,-0.7462,-0.0725) | (0.0481,0.8615,0.8900),(-0.8811,-0.8615,-0.0562) | (0.0883,0.8262,0.8649),(-0.8748,-0.8649,-0.0521) |

Figure 2 illustrates the BNS positive and negative ideal solutions extracted from the weighted decision matrix. Part (a) represents the ideal reference point maximizing positive memberships and minimizing negative components across benefit and cost criteria, respectively. Part (b) depicts the negative ideal solution with opposite extremities.




*A)*




*B)*

*Figure 2. Bipolar Neutrosophic Ideal Solutions: (a) Positive Ideal Solution, (b) Negative Ideal Solution Derived from the Weighted Decision Matrix*

In Figure 3, part (a) illustrates the calculated distances of each alternative to the PIS, representing how close each art course is to the optimal scenario across all criteria. Part (b) presented the distances to the NIS, reflecting how far each alternative is from the least favorable scenario.

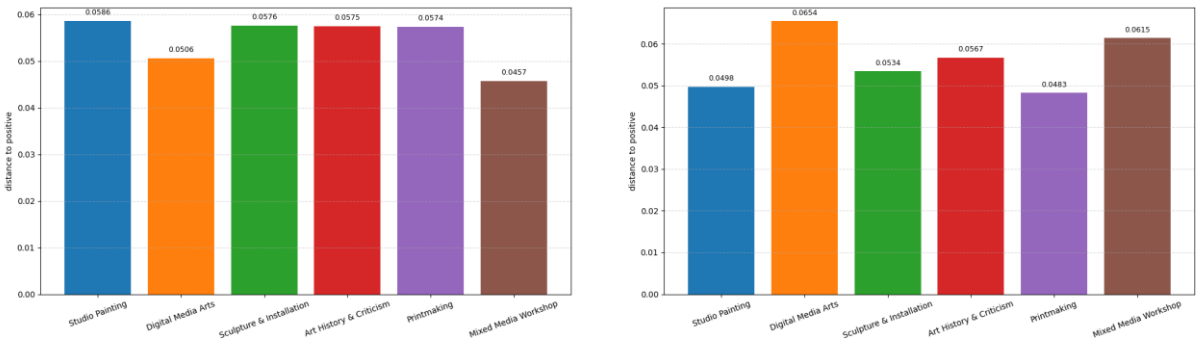



*Figure 3. Distance to Positive Ideal Solution (LEFT:A), and to Negative Ideal Solution (RIGHT:B).*

Figure 4 visualizes the closeness coefficients indicating how close each alternative is to the positive ideal solution while being distant from the negative ideal. These coefficients synthesize distance measures in the BNS space to provide a composite indicator of each alternative's desirability in cognitive skill development. Higher closeness values correspond to better alternatives in the evaluation context.

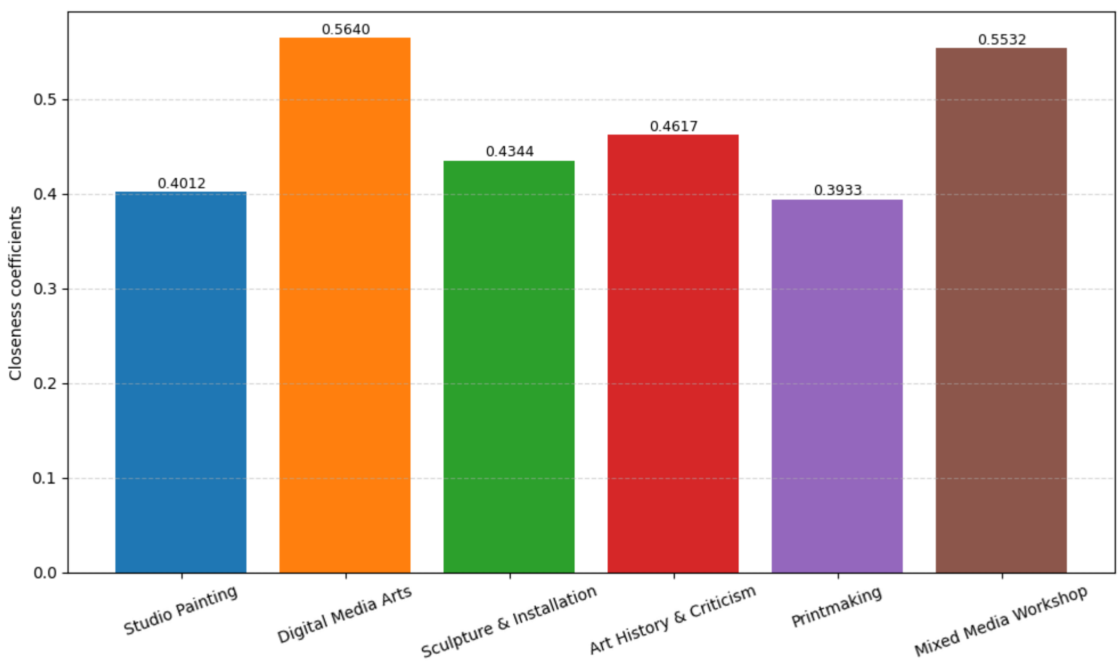



Figure 4. Closeness Coefficients of Alternatives Relative to Bipolar Neutrosophic Ideal Solutions

Figure 5 depicts the inferior ratio values calculated from the closeness coefficients. This metric normalizes the closeness coefficients relative to the most favorable alternative, transforming them into a comparable scale for ranking. Alternatives are ultimately ordered by ascending inferior ratio values, with lower values indicating superior performance in fostering cognitive skills within university art education programs.

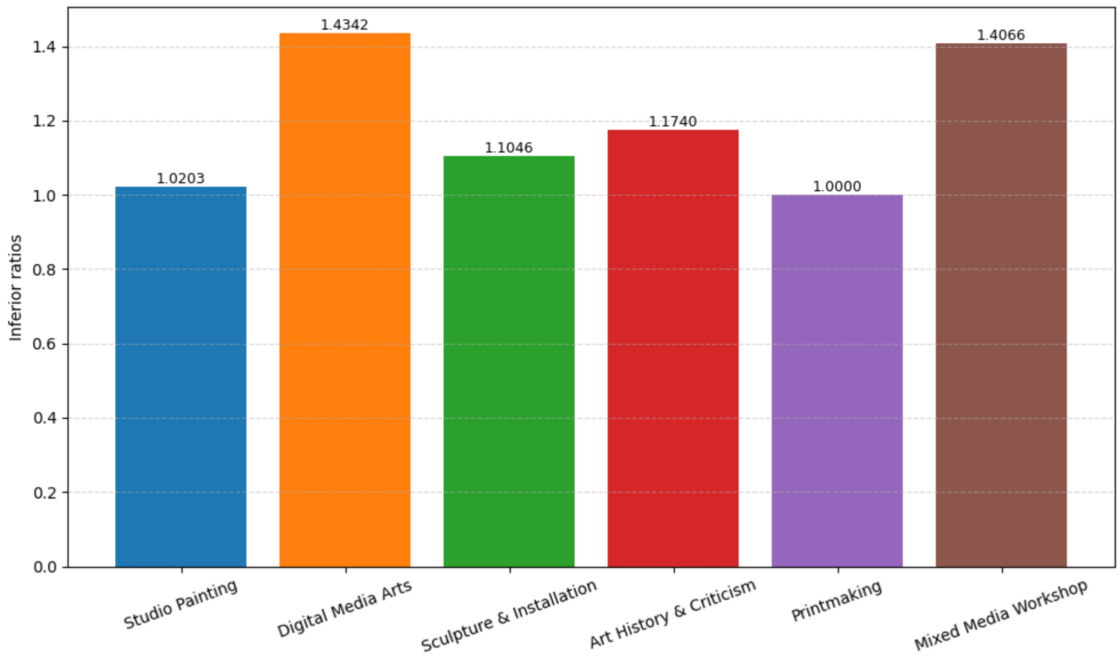



Figure 5. Inferior Ratios of Alternatives for Final Ranking Decision

In Figure 6, we visualize the sensitivity of alternative rankings to changes of weights across six distinct experiments. Each subplot revealed that while some alternatives maintain stable rankings

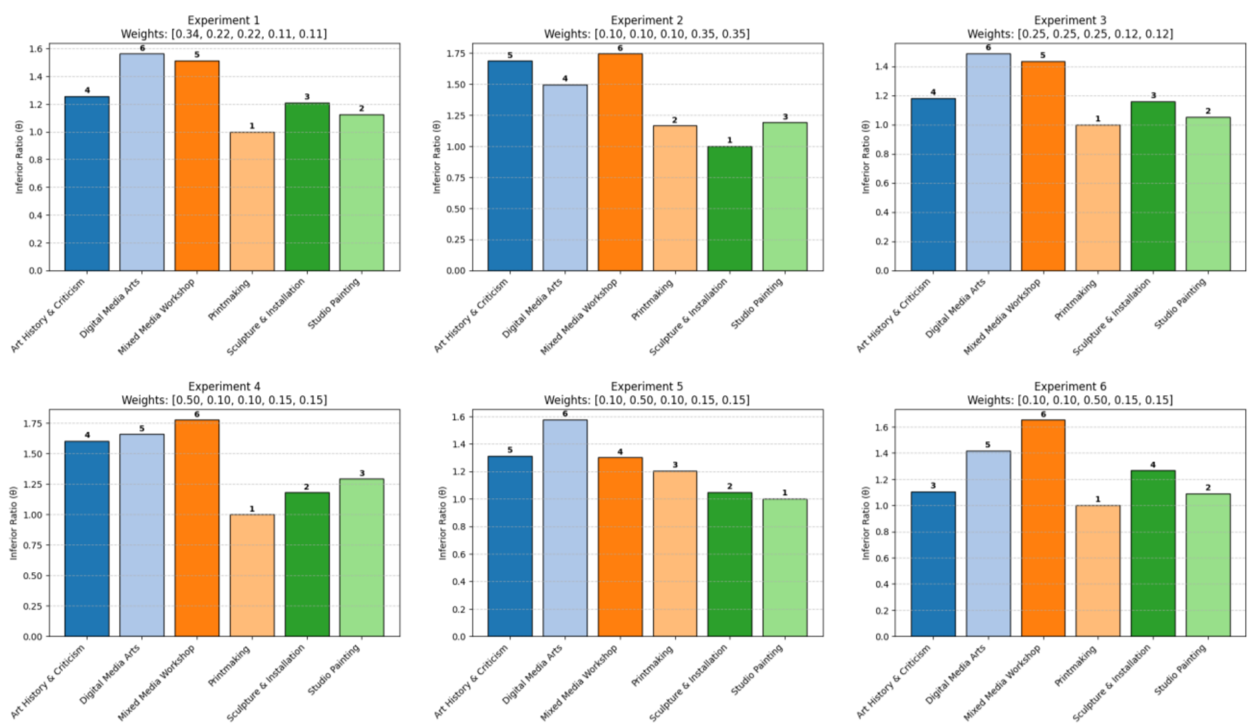



*Figure 6.. Sensitivity analysis of alternative rankings across six weight experiments in university art education assessment*

across different weighting schemes, others show fluctuations that highlight their strengths or weaknesses under erratic evaluation priorities.

## 7. Conclusions

This study successfully demonstrates the significant advantages of employing BNS in assessing complex cognitive skills development within university art education, addressing the inherent subjectivity and uncertainty. Our research has introduced a robust framework that integrates novel CHHGAO that effectively aggregates diverse expert opinions expressed in BNS, capturing both the central tendency and the spread of evaluations while accommodating the dual nature of preferences. Furthermore, we have formulated a new distance measure specifically for BNS, enhancing the precision of algorithmic decision-making by enabling a more accurate quantification of the proximity of alternatives to ideal solutions. The practical utility of our proposed methodology is thoroughly validated through a real-world case study, where the results not only provide a more transparent and equitable assessment of course effectiveness but also highlight how the explicit modeling of indeterminacy and bipolarity.

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