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$\mathbb{S}\mathcal{Q}$-Neutrosophic Structures in Semigroup Theory

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Abstract: In this paper, we define and study a new fuzzy algebraic structure called $\mathbb{S}\mathcal{Q}$-neutrosophic fuzzy semigroups ($\mathbb{S}\mathcal{Q}$-NFS) as a generalization of fuzzy algebra in the framework of $\mathbb{S}$-semigroups. We achieve our objective by introducing a notion called $\mathcal{Q}$ -neutrosophic fuzzy subsets which define the membership, indeterminacy, and non-membership degree along with a restriction map for each. With this, we show that the classic fuzzy semigroup theory can be extended to take into account uncertainty, contradiction, and membership in partial truth degrees all at the same time. We provide formal definitions, symbolic representations, examples and fundamental propositions that outline the structure for $\mathbb{S}\mathcal{Q}$-NFS. We also look at some features of $\mathbb{S}\mathcal{Q}$ -NFS that include an identity element and inverses, demonstrating that we have created a new kind of algebraic system that has both fuzzy and neutrosophic traits.

Keywords: Neutrosophic set; $\mathbb{S}$- semigroups; $\mathbb{S}$- Fuzzy semigroups; $\mathcal{Q}$-Fuzzy set; $\mathcal{Q}$-Fuzzy semigroups; $\mathbb{S}\mathcal{Q}$-Fuzzy semigroups; $\mathbb{S}\mathcal{Q}$- Neutrosophic Fuzzy semigroups($\mathbb{S}\mathcal{Q}$-NFS)

1. Introduction

"Neutrosophic Sets" were developed by Florentin Smarandache conceptually in 1995 and formally published in the first years of the 21st century. The theory was created to address the shortcomings of classical and fuzzy set theories when dealing with real-world data that is uncertain, unclear, inconsistent, incomplete, and biassed; sets can describe each element using three separate functions: its degree of truth (T) which can be any value between 0 and 1, its degree of indeterminacy (I) which can also be any value between 0 and 1, and its degree of falsity (F) which can likewise be any value between 0 and 1. Neutrosophic sets are able to model more accurately the way things work because humans are rarely faced with perfect knowledge, incomplete knowledge, contradictions, and a fog of uncertainty. Neutrosophic sets have become used in a wide variety of applications, such as decision-making, optimizing engineering systems, medical diagnosis, and artificial intelligence fields.

They began to incorporate fuzziness in an algebraic structure in the late twentieth century with fuzzy groups and fuzzy semigroups. In algebra, we have the latest evolution of $\mathbb{S}$- Structures, where classical structures have partial properties satisfied in their larger structures. Padilla Raul [5] developed the notion of $\mathbb{S}$-semigroup in 1998, which extended the study of semigroups in a different way. This notion was further developed by W. B. Vasantha Kandasamy [9, 10] in 2003 in exploring $\mathbb{S}$-Fuzzy semigroups. For example, a $\mathbb{S}$- semigroup (or group semigroup) is a semigroup possessing a proper subset that can behave as a group. These structures clearly show that regularity and irregularity can exist side by side together and more realistically embody the complexity of both artificial and natural systems. Subsequently, A. Solairaju and R. Nagarajan [8] developed a new algebraic structure known as Q-Fuzzy groups in 2008. Building upon this notion, T. Priya and others introduced Q-Fuzzy normal subgroups in 2013, which further extended the study of fuzziness in group theory. With the development of Q-Fuzzy sets, $\mathbb{S}Q$-Fuzzy semigroups were proposed by R. Arul Doss and S. Suganya [12] as a natural generalisation.

Enad Ghazi and Sinan O. Al-Salihi [17] proposed a robust medical diagnostic framework using interval-valued Q-Neutrosophic soft sets and novel aggregation operators. Their work demonstrates the applicability of Q-Neutrosophic theory to real-world decision-making problems, particularly in highly uncertain domains such as healthcare. Ayesha Saeed et al. [18] extended Q-Neutrosophic frameworks further by introducing the Q-Neutrosophic hypersoft set, combining hypersoft set theory with Q-Neutrosophic logic. Their application in tourism planning illustrates the flexibility of Q-Neutrosophic models for multi-attribute decision-making. In the algebraic context, Premkumar et al. [19] investigated the properties of μ-Anti-Q-fuzzy subgroups, analyzing how anti-Q-fuzziness interacts with group structure and algebraic laws. Their study contributed deeper insights into the structural behaviour of Q-fuzzy groups under certain negation conditions.

However, earlier models [12] only showed how much something belonged to a group using Q-parameterization, without addressing the contradictions and uncertainties. In this paper, we introduced a new idea called $\mathbb{S}Q$-Neutrosophic Fuzzy Semigroups ($\mathbb{S}Q$-NFS), which includes Q-neutrosophic fuzzy subsets that have three parts: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F), with each part having its own restriction map. $\mathbb{S}Q$-NFS then provides fuzzy logic, neutrosophic logic and $\mathbb{S}$- algebra, allowing the simultaneous modelling of partial truth, uncertainty and contradiction. Furthermore, while $\mathbb{S}Q$-FS provided a single-valued degree over $Q \times$ H, the $\mathbb{S}Q$-NFS rationalizes a richer algebraic model in a complex, uncertain setting. This enhanced modelling capacity further empowers the algebraic foundation necessary for the study of systems that combine conflicting, incomplete, or imprecise information or to bring a new level of abstraction to algebra and its real-world modelling capabilities.

2. Preliminaries

Definition 2.1[11]

Let X be a non-empty set. A Fuzzy subset $\grave{u}$ of the set X is a function $\grave{u} : X \rightarrow [0,1]$.

Definition 2.2 [9]

Let H be a semigroup. When H has a proper subset Ψ such that Ψ is a group under the action of H, then H is a $\mathbb{S}$-semigroup($\mathbb{S}$SG).

Definition 2.3 [8]

Let X and Q be non-empty sets. A Q-Fuzzy subset N of X is a function $N : X \times Q \rightarrow [0,1]$.

Definition 2.4 [8]

Let G be a group and Q be any non- empty set. A Q-Fuzzy subset $\breve{A}$ of G is said to be Q-Fuzzy group of G if

(i) $\breve{A}(\alpha\eta, q) \geq min\{\breve{A}(\alpha, q), \breve{A}(\eta, q)\}$

(ii) $\breve{A}(\alpha^{-1}, q) = \breve{A}(\alpha, q)$, for all $\alpha, \eta \in G$ and $q \in Q$.

Definition 2.5[16]

An item of the system Ꞑ={α̤; ŧ(α̤), ɨ(α̤), ꞙ(α̤)} is a Neutrosophic fuzzy set over X, where α̤, $ŧ_{Ꞑ}(α̤)$, $ɨ_{Ꞑ}(α̤)$ and $ꞙ_{Ꞑ}(α̤)$ represents generic element, truth-membership function, indeterminacy membership function and falsity-membership function respectively with mappings:

$$ŧ_{Ꞑ}(α̤): X \to [0,1],\ ɨ_{Ꞑ}(α̤): X \to [0,1] \text{ and } ꞙ_{Ꞑ}(α̤): X \to [0,1].$$

Definition 2.6

A Neutrosophic Fuzzy set Ꞑ(ŧ, ɨ, ꞙ) on a semigroup E is called a Neutrosophic Fuzzy subsemigroup on E if it satisfies the following conditions:

(i) $ŧ(xy) \geq ŧ(x) \wedge ŧ(y)$

(ii) $ɨ(xy) \leq ɨ(x) \vee ɨ(y)$

(iii) $ꞙ(xy) \leq ꞙ(x) \vee ꞙ(y)$ for all $x, y \in E$

Definition 2.7

Let Ẋ and Q be non-empty sets. A Q-Neutrosophic Fuzzy subset Ꞑ of Ẋ is a function $ŧ_{Ꞑ}: Ẋ \times Q \to [0,1]$, $ɨ_{Ꞑ}: Ẋ \times Q \to [0,1]$ and $ꞙ_{Ꞑ}: Ẋ \times Q \to [0,1]$.

Definition 2.8

Let G be a group and Q be any non-empty set. A Q-Neutrosophic Fuzzy subset A of G is said to be Q-Neutrosophic Fuzzy group of G if

(i) $ŧ(α̤η̤, q) \geq min\{ŧ(α̤, q), ŧ(η̤, q)\}$; $ɨ(α̤η̤, q) \leq max\{ɨ(α̤, q), ɨ(η̤, q)\}$ and $ꞙ(α̤η̤, q) \leq max\{ꞙ(α̤, q), ꞙ(η̤, q)\}$

(ii) $ŧ(α̤^{-1}, q) = ŧ(α̤, q)$; $ɨ(α̤^{-1}, q) = ɨ(α̤, q)$ and $ꞙ(α̤^{-1}, q) = ꞙ(α̤, q)$

for all $α̤, η̤ \in G$ and $q \in Q$.

3. 𝕊Q-Neutrosophic fuzzy semigroups

Definition 3.1

Let Ṣ be a 𝕊-semigroup and let Ꞑ(ŧ, ɨ, ꞙ) be a Q-Neutrosophic fuzzy subset of Ṣ. Define the restriction maps $ŧ_{Ꞑ}: Ṣ \times Q \to [0,1]$, $ɨ_{Ꞑ}: Ṣ \times Q \to [0,1]$ and $ꞙ_{Ꞑ}: Ṣ \times Q \to [0,1]$ where Q is a non-empty set. Suppose there exists a proper subset Ƥ ⊂ Ṣ, such that Ƥ forms a group under the operation of Ṣ and Ꞑ restricted to Ƥ satisfies:

For all $α̤, η̤ \in Ƥ$ and $q \in Q$

$$ŧ_{Ꞑ}(α̤η̤, q) \geq \min\{ŧ_{Ꞑ}(α̤, q), ŧ_{Ꞑ}(η̤, q)\},$$
$$ɨ_{Ꞑ}(α̤η̤, q) \leq max\{ɨ_{Ꞑ}(α̤, q), ɨ_{Ꞑ}(η̤, q)\},$$
$$ꞙ_{Ꞑ}(α̤η̤, q) \leq max\{ꞙ_{Ꞑ}(α̤, q), ꞙ_{Ꞑ}(η̤, q)\}$$

and

$$ŧ_{Ꞑ}(α̤, q) = ŧ_{Ꞑ}(α̤^{-1}, q);\ ɨ_{Ꞑ}(α̤, q) = ɨ_{Ꞑ}(α̤^{-1}, q) \text{ and } ꞙ_{Ꞑ}(α̤, q) = ꞙ_{Ꞑ}(α̤^{-1}, q)$$

Then Ꞑ is called a 𝕊Q-Neutrosophic fuzzy semigroups(𝕊Q-NFS)

Example 3.2 Look into an 𝕊SG Z_6 under multiplication modulo 6.

Let $Q = \{1\}$. Let $[ŧ_{Ꞑ}, ɨ_{Ꞑ}, ꞙ_{Ꞑ}]: Z_6 \times Q \to [0,1]$ be defined by,

$$ŧ_{Ꞑ}(x) = \begin{cases} 0.3 & if \quad α̤ = (0,1) \\ 0.4 & if \quad α̤ = (2,1), (4,1) \\ 0.2 & if \quad α̤ = (1,1), (3,1), (5,1) \end{cases}$$

$$ɨ_{\mathrm{N}}(x) = \begin{cases} 0.4 & if \quad ᾳ = (0,1) \\ 0.3 & if \quad ᾳ = (2,1), (4,1) \\ 0.1 & if \quad ᾳ = (1,1), (3,1), (5,1) \end{cases}$$

$$ƒ_{\mathrm{N}}(x) = \begin{cases} 0.3 & if \quad ᾳ = (0,1) \\ 0.2 & if \quad ᾳ = (2,1), (4,1) \\ 0.4 & if \quad ᾳ = (1,1), (3,1), (5,1) \end{cases}$$

It is evident that Ɲ is a Q-Neutrosophic Fuzzy subset of Z_6.

Consider $\mathcal{P} = \{1, 5\} \subset Z_6$ and is also a group in Z_6 under the operation of Z_6.

Here ᾳ $= 1$, ῃ $= 3$, ʠ $= 1$

$$ŧ_{\mathrm{N}}(5,1) \geq min\{ŧ_{\mathrm{N}}(1,1), ŧ_{\mathrm{N}}(5,1)\}$$
$$0.2 \geq min\{0.2,0.2\}$$
$$0.2 \geq 0.2$$
$$ɨ_{\mathrm{N}}(5,1) \leq max\{ɨ_{\mathrm{N}}(1,1), ɨ_{\mathrm{N}}(5,1)\}$$
$$0.1 \leq max\{0.1, 0.1\}$$
$$0.1 \leq 0.1$$
$$ƒ_{\mathrm{N}}(5,1) \leq max\{ƒ_{\mathrm{N}}(1,1), ƒ_{\mathrm{N}}(5,1)\}$$
$$0.4 \leq max\{0.4, 0.4\}$$
$$0.4 \leq 0.4$$

As expected Ɲ is a $\mathbb{S}Q$-Neutrosophic Fuzzy semigroup.

Proposition 3.3

When Q is a non-empty set and Ɲ is an $\mathbb{S}Q$-NFS of an $\mathbb{S}$SG Ṣ with respect to a group Ƥ, then

(i) $ŧ_{\mathrm{N}_\mathcal{P}}(e, ʠ) \geq ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$, $ɨ_{\mathrm{N}_\mathcal{P}}(e, ʠ) \leq ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$ and $ƒ_{\mathrm{N}_\mathcal{P}}(e, ʠ) \leq ƒ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$, where e is the identity element of Ƥ

(ii) $ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ^{-1}, ʠ) \geq ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$, $ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ^{-1}, ʠ) \leq ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$ and $ƒ_{\mathrm{N}_\mathcal{P}}(ᾳ^{-1}, ʠ) \leq ƒ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$, for all ᾳ $\in \mathcal{P}$ and ʠ $\in Q$

Proof

Assume that Ɲ belongs to an $\mathbb{S}$SG Ṣ and also be an $\mathbb{S}Q$-NFS. Then, $ŧ_{\mathrm{N}_\mathcal{P}}: \mathcal{P} \times Q \rightarrow [0,1]$, $ɨ_{\mathrm{N}_\mathcal{P}}: \mathcal{P} \times Q \rightarrow [0,1]$ and $ƒ_{\mathrm{N}_\mathcal{P}}: \mathcal{P} \times Q \rightarrow [0,1]$ is a Q-Neutrosophic Fuzzy group, and Ɲ is confined to at least one proper subset Ƥ of Ṣ, which is a neutrosophic group.

Therefore $ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ) \geq ŧ_{\mathrm{N}}(ᾳ, ʠ)$, $ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ) \leq ɨ_{\mathrm{N}}(ᾳ, ʠ)$ and $ƒ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ) \leq ƒ_{\mathcal{P}}(ᾳ, ʠ)$, for all ᾳ $\in \mathcal{P}$ and ʠ $\in Q$.

(i) Let $e \in \mathcal{P}$, where e is the identity element of Ƥ. Now

$$\begin{aligned} ŧ_{\mathrm{N}_\mathcal{P}}(e, ʠ) &= ŧ_{\mathrm{N}_\mathcal{P}}(ᾳᾳ^{-1}, ʠ) \\ &\geq \min\{ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ), ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ^{-1}, ʠ)\} \\ &= \min\{ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ), ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)\} \\ &= ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ) \end{aligned}$$
$$ŧ_{\mathrm{N}_\mathcal{P}}(e, ʠ) \geq ŧ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)$$

and

$$\begin{aligned} ɨ_{\mathrm{N}_\mathcal{P}}(e, ʠ) &= ɨ_{\mathrm{N}_\mathcal{P}}(ᾳᾳ^{-1}, ʠ) \\ &\leq max\{ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ), ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ^{-1}, ʠ)\} \\ &= \max\{ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ), ɨ_{\mathrm{N}_\mathcal{P}}(ᾳ, ʠ)\} \end{aligned}$$

$$= ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

$$ɨ_{Ɲ_{\mathcal{P}}}(e, q) \leq ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

and

$$\begin{aligned} ꞙ_{Ɲ_{\mathcal{P}}}(e, q) &= ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\alpha}^{-1}, q) \\ &\leq max\{ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q), ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\} \\ &= \max\{ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q), ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)\} \\ &= ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) \end{aligned}$$

$$ꞙ_{Ɲ_{\mathcal{P}}}(e, q) \leq ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

(ii)

$$\begin{aligned} ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) &= ŧ_{Ɲ_{\mathcal{P}}}(e\ddot{\alpha}^{-1}, q) \\ &\geq \min\{ŧ_{Ɲ_{\mathcal{P}}}(e, q), ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\} \\ &= \min\{ŧ_{Ɲ_{\mathcal{P}}}(e, q), ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)\} \\ &\geq ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) \qquad \text{[Since by (i)]} \end{aligned}$$

$$ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) \geq ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

And

$$\begin{aligned} ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) &= ɨ_{Ɲ_{\mathcal{P}}}(e\ddot{\alpha}, q) \\ &\leq \max\{ɨ_{Ɲ_{\mathcal{P}}}(e, q), ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\} \\ &= \max\{ɨ_{Ɲ_{\mathcal{P}}}(e, q), ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)\} \\ &\leq ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) \end{aligned}$$

$$ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) \leq ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

And

$$\begin{aligned} ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) &= ꞙ_{Ɲ_{\mathcal{P}}}(e\ddot{\alpha}, q) \\ &\leq \max\{ꞙ_{Ɲ_{\mathcal{P}}}(e, q), ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\} \\ &= \max\{ꞙ_{Ɲ_{\mathcal{P}}}(e, q), ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)\} \\ &\leq ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) \end{aligned}$$

$$ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) \leq ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q)$$

Theorem 3.4

If Ɲ is an $\mathbb{S}\mathcal{Q}$-NFS of an $\mathbb{S}$-semigroup Ş and $\mathcal{Q}$ is a non-empty set, then

(i) $ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = ŧ_{Ɲ_{\mathcal{P}}}(e, q) \Longrightarrow ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) = ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\eta}, q)$

(ii) $ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = ɨ_{Ɲ_{\mathcal{P}}}(e, q) \Longrightarrow ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) = ɨ_{Ɲ_{\mathcal{P}}}(\ddot{\eta}, q)$

(iii) $ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = ꞙ_{Ɲ_{\mathcal{P}}}(e, q) \Longrightarrow ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) = ꞙ_{Ɲ_{\mathcal{P}}}(\ddot{\eta}, q)$

for all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P} \subset Ɲ$, where Ş is a group and $Ɲ_{\mathcal{P}}$ is the restriction of Ɲ, $q \in Q$ and e is the identity element of Ƥ.

Proof

Let Ɲ be an $\mathbb{S}\mathcal{Q}$ -NFS of an $\mathbb{S}$-semigroup Ş and let Q be a non -empty set. Then Ɲ is restricted to at least one proper subset Ƥ of Ş which is a group and $ŧ_{Ɲ_{\mathcal{P}}}: \mathcal{P} \times \mathcal{Q} \rightarrow [0, 1]$, $ɨ_{Ɲ_{\mathcal{P}}}: \mathcal{P} \times \mathcal{Q} \rightarrow [0, 1]$ and $ꞙ_{Ɲ_{\mathcal{P}}}: \mathcal{P} \times \mathcal{Q} \rightarrow [0, 1]$ is a $\mathcal{Q}$-Neutrosophic Fuzzy group.

Therefore $ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) \geq ŧ_{Ɲ}(\ddot{\alpha}, q)$ for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in \mathcal{Q}$.

Let $\ddot{\alpha} \in \mathcal{P}$ and $q \in \mathcal{Q}$.

Considering that $ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = ŧ_{Ɲ_{\mathcal{P}}}(e, q)$.

$$ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}, q) = ŧ_{Ɲ_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q)$$

$$\begin{aligned}
&\geq \min\{t_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), t_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= \min\{t_{N_{\mathcal{P}}}(e, q), t_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= t_{N_{\mathcal{P}}}(\ddot{\eta}, q) \qquad \text{[Since by proposition 3.3 (i)]}\\
t_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &\geq t_{N_{\mathcal{P}}}(\ddot{\eta}, q)
\end{aligned}$$

Now,

$$\begin{aligned}
t_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= t_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}\ddot{\alpha}, q)\\
&\geq \min\{t_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}, q), t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \min\{t_{N_{\mathcal{P}}}((\ddot{\eta}\ddot{\alpha}^{-1})^{-1}, q), t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \min\{t_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \min\{t_{N_{\mathcal{P}}}(e, q), t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\\
t_{N_{\mathcal{P}}}(\ddot{\eta}, q) &\geq t_{N_{\mathcal{P}}}(\ddot{\alpha}, q)
\end{aligned}$$

Therefore

$$t_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = t_{N_{\mathcal{P}}}(\ddot{\eta}, q)$$

for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in Q$.

Similarly, again considering that $i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = i_{N_{\mathcal{P}}}(e, q)$.

$$\begin{aligned}
i_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &= i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q)\\
&\leq \max\{i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), i_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= \max\{i_{N_{\mathcal{P}}}(e, q), i_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= i_{N_{\mathcal{P}}}(\ddot{\eta}, q) \qquad \text{[Since by proposition 3.3 (i)]}\\
i_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &\leq i_{N_{\mathcal{P}}}(\ddot{\eta}, q)
\end{aligned}$$

Now,

$$\begin{aligned}
i_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= i_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}\ddot{\alpha}, q)\\
&\leq \max\{i_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}, q), i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \max\{i_{N_{\mathcal{P}}}((\ddot{\eta}\ddot{\alpha}^{-1})^{-1}, q), i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \max\{i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \max\{i_{N_{\mathcal{P}}}(e, q), i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\\
i_{N_{\mathcal{P}}}(\ddot{\eta}, q) &\geq i_{N_{\mathcal{P}}}(\ddot{\alpha}, q)
\end{aligned}$$

Therefore $i_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = i_{N_{\mathcal{P}}}(\ddot{\eta}, q)$ for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in Q$.

Similarly, again considering that $f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = f_{N_{\mathcal{P}}}(e, q)$.

$$\begin{aligned}
f_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &= f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q)\\
&\leq \max\{f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), f_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= \max\{f_{N_{\mathcal{P}}}(e, q), f_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}\\
&= f_{N_{\mathcal{P}}}(\ddot{\eta}, q) \qquad \text{[Since by proposition 3.3 (i)]}\\
f_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &\leq f_{N_{\mathcal{P}}}(\ddot{\eta}, q)
\end{aligned}$$

Now,

$$\begin{aligned}
f_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= f_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}\ddot{\alpha}, q)\\
&\leq \max\{f_{N_{\mathcal{P}}}(\ddot{\eta}\ddot{\alpha}^{-1}, q), f_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}\\
&= \max\{f_{N_{\mathcal{P}}}((\ddot{\eta}\ddot{\alpha}^{-1})^{-1}, q), f_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}
\end{aligned}$$

$$= \max\{ ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}$$
$$= \max\{ ƒ_{N_{\mathcal{P}}}(e, q), ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}$$
$$= ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$$
$$ƒ_{N_{\mathcal{P}}}(\ddot{\eta}, q) \geq ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$$

Therefore $ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ƒ_{N_{\mathcal{P}}}(\ddot{\eta}, q)$ for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in Q$.

Theorem 3.5

Let Ş be an 𝕊-semigroup, Q any nonempty set and let Ƥ be a proper subset of Ş which is a group in Ş. Then $ŧ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$, $ɨ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$ and $ƒ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$ is an 𝕊Q-NFS relative to Ƥ if and only if $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$, $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$ and $ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \leq \max\{ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ƒ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$ for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in Q$.

Proof

Assume that $ŧ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$, $ɨ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$ and $ƒ_{N_{\mathcal{P}}}: Ş \times Q \to [0,1]$ is an 𝕊Q-NFS relative to Ƥ. Then N is restricted to Ƥ and $ŧ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$, $ɨ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$ and $ƒ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$ is a Q- Neutrosophic Fuzzy group. Then $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$, $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$ and $ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ƒ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$ for all $\ddot{\alpha} \in \mathcal{P}$ and $q \in Q$.

Let $\ddot{\alpha}, \ddot{\eta}^{-1} \in \mathcal{P}$ and $q \in Q$.

$$\text{Then } ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)\}$$
$$= \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$$

Therefore, $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$

Conversely, assume that,

$$ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \quad (1)$$

For all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$ .Put $\ddot{\eta} = \ddot{\alpha}$ in (1).

Then, $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\alpha}^{-1}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}$

$$ŧ_{N_{\mathcal{P}}}(e, q) \geq ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \quad (2)$$

Now, $ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q) = ŧ_{N_{\mathcal{P}}}(e\ddot{\eta}^{-1}, q)$

$\geq \min\{ŧ_{N_{\mathcal{P}}}(e, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$ [since by (1)]

$ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q) \geq ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)$ [since by (2)]

Also, $ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) = ŧ_{N_{\mathcal{P}}}(e(\ddot{\eta}^{-1})^{-1}, q)$

$\geq \min\{ŧ_{N_{\mathcal{P}}}(e, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)\}$ [since by (1)]

$ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) \geq ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)$ [since by (2)]

$ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) = ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)$. (3)

Now $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, q) = ŧ_{N_{\mathcal{P}}}((\ddot{\alpha}\ddot{\eta})^{-1}, q)$.

$= ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}\ddot{\alpha}^{-1}, q)$.

$\geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\}$

$\geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}$ [since by (1)]

Therefore, $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, q) \geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\}$ (4)

Similarly, let $\ddot{\alpha}, \ddot{\eta}^{-1} \in \mathcal{P}$ and $q \in Q$.

Then $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)\} = \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$

Therefore, $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$

Conversely, assume that, $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) \leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\}$ (5)

for all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $\mathrm{q} \in \mathcal{Q}$ .Put $\ddot{\eta} = \ddot{\alpha}$ in (5).

Then
$$i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\alpha}^{-1}, \mathrm{q}) \leq \max\{i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\}$$
$$i_{N_{\mathcal{P}}}(e, \mathrm{q}) \leq i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}) \quad (6)$$

Now,
$$i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) = i_{N_{\mathcal{P}}}(e\ddot{\eta}^{-1}, \mathrm{q})$$
$$\leq \max\{i_{N_{\mathcal{P}}}(e, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})\} \quad \text{[since by (5)]}$$
$$i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) \leq i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \quad \text{[since by (6)]}$$

Also,
$$i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \leq i_{N_{\mathcal{P}}}(e(\ddot{\eta}^{-1})^{-1}, \mathrm{q})$$
$$\leq \max\{i_{N_{\mathcal{P}}}(e, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q})\} \quad \text{[since by (5)]}$$
$$i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \leq i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) \quad \text{[since by (6)]}$$

Therefore $i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) = i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q})$. (7)

Now,
$$i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, \mathrm{q}) = i_{N_{\mathcal{P}}}((\ddot{\alpha}\ddot{\eta})^{-1}, \mathrm{q}).$$
$$= i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}\ddot{\alpha}^{-1}, \mathrm{q}).$$
$$\leq \max\{i_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, \mathrm{q})\}$$
$$\leq \max\{i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\}$$
$$i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, \mathrm{q}) \leq \max\{i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}), i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\} \quad (8)$$

Similarly, let $\ddot{\alpha}, \ddot{\eta}^{-1} \in \mathcal{P}$ and $\mathrm{q} \in \mathcal{Q}$.

Then $f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, \mathrm{q}) \leq \max\{i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q})\}$
$$= \max\{f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})\}$$

Therefore, $f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, \mathrm{q}) \leq \max\{f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})\}$

Conversely, assume that , $f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, \mathrm{q}) \leq \max\{f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})\}$ (9)

$\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $\mathrm{q} \in \mathcal{Q}$ .Put $\ddot{\eta} = \ddot{\alpha}$ in (5).

Then
$$f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\alpha}^{-1}, \mathrm{q}) \leq \max\{f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\}$$
$$f_{N_{\mathcal{P}}}(e, \mathrm{q}) \leq f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}) \quad (10)$$

Now,
$$f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) = f_{N_{\mathcal{P}}}(e\ddot{\eta}^{-1}, \mathrm{q})$$
$$\leq \max\{f_{N_{\mathcal{P}}}(e, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})\} \quad \text{[since by (9)]}$$
$$f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) \leq f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \quad \text{[since by (10)]}$$

Also,
$$f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \leq f_{N_{\mathcal{P}}}(e(\ddot{\eta}^{-1})^{-1}, \mathrm{q})$$
$$\leq \max\{f_{N_{\mathcal{P}}}(e, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q})\} \quad \text{[since by (9)]}$$
$$f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) \leq f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}) \quad \text{[since by (10)]}$$

Therefore $f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}) = f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q})$. (11)

Now,
$$f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, \mathrm{q}) = f_{N_{\mathcal{P}}}((\ddot{\alpha}\ddot{\eta})^{-1}, \mathrm{q}).$$
$$= f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}\ddot{\alpha}^{-1}, \mathrm{q}).$$
$$\leq \max\{f_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, \mathrm{q})\}$$
$$\leq \max\{f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\}$$
$$f_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}, \mathrm{q}) \leq \max\{f_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q}), f_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q})\} \quad (12)$$

From (3), (4), (7), (8), (11) and (12) $N_{\mathcal{P}}$ is an $\mathbb{S}$Q-NFS relative to a group $\mathcal{P}$.

Theorem 3.6

Let $\mathcal{N}$ be an $\mathbb{S}\mathcal{Q}$-NFS of an $\mathbb{S}$-semigroup $\c{S}$ relative to a group $\mathcal{P}$. If

(i) $t_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, \mathrm{q}) = 1$, then $t_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}) = t_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})$

(ii) $i_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, \mathrm{q}) = 0$, then $i_{N_{\mathcal{P}}}(\ddot{\alpha}, \mathrm{q}) = i_{N_{\mathcal{P}}}(\ddot{\eta}, \mathrm{q})$ and

(iii) $ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = 0$, then $ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ꞙ_{N_{\mathcal{P}}}(\ddot{\eta}, q)$

for every $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P} \subset Ş$ and $q \in Q$.

Proof

Let $\mathcal{N}$ be an $\mathbb{S}Q$-NFS of an $\mathbb{S}$-semigroup $Ş$ relative to a group $\mathcal{P}$. Then $ŧ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$, $ɨ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$ and $ꞙ_{N_{\mathcal{P}}}: \mathcal{P} \times Q \to [0,1]$ is a Q-Neutrosophic Fuzzy group. Therefore $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\ \forall\ \ddot{\alpha}$ in $\mathcal{P}$ and q in Q.

Let $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$.

Assume that $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = 1$

$$\begin{aligned} ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &= ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q) \\ &\geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= \min\{1, ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q), \\ ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &\geq ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) \end{aligned}$$

Now,

$$\begin{aligned} ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= ŧ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q) \\ &= ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}\ddot{\alpha}\ddot{\eta}^{-1}, q) \\ &\geq \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q)\} \\ &= \min\{ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), 1\} \\ &= ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) \\ &= ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \\ ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q) &\geq ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \end{aligned}$$

Therefore, $ŧ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) = ŧ_{N_{\mathcal{P}}}(\ddot{\eta}, q)$ for all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$.

Similarly, Assume that $ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = 0$

$$\begin{aligned} ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &= ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q) \\ &\leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= \max\{0, ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q), \\ ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &\leq ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q) \end{aligned}$$

Now,

$$\begin{aligned} ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= ɨ_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q) \\ &= ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}\ddot{\alpha}\ddot{\eta}^{-1}, q) \\ &\leq \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q)\} \\ &= \max\{ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), 0\} \\ &= ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q) \\ &= ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \\ ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q) &\geq ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \end{aligned}$$

Therefore, $ɨ_{N_{\mathcal{P}}}(\ddot{\eta}, q) = ɨ_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$ for all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$.

Similarly, Assume that $ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q) = 0$

$$\begin{aligned} ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}, q) &= ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}\ddot{\eta}, q) \\ &\leq \max\{ꞙ_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q), ꞙ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= \max\{0, ꞙ_{N_{\mathcal{P}}}(\ddot{\eta}, q)\} \\ &= ꞙ_{N_{\mathcal{P}}}(\ddot{\eta}, q), \end{aligned}$$

$$\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}, q) \leq \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\eta}, q)$$

Now,

$$\begin{aligned}\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\eta}, q) &= \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\eta}^{-1}, q)\\ &= \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}\ddot{\alpha}\ddot{\eta}^{-1}, q)\\ &\leq \max\{\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}\ddot{\eta}^{-1}, q)\}\\ &= \max\{\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q), 0\}\\ &= \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}^{-1}, q)\\ &= \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}, q)\end{aligned}$$

$$\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\eta}, q) \geq \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$$

Therefore, $\mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\eta}, q) = \mathfrak{f}_{N_{\mathcal{P}}}(\ddot{\alpha}, q)$ for all $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$.

Definition 3.7

Let X and Q be non-empty sets. Let A and B be two Q-Neutrosophic fuzzy subsets of X, where

$$A(\ddot{\alpha}, q) = \{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{i}_A(\ddot{\alpha}, q), \mathfrak{f}_A(\ddot{\alpha}, q)\},\ B(\ddot{\alpha}, q) = \{\mathfrak{t}_B(\ddot{\alpha}, q), \mathfrak{i}_B(\ddot{\alpha}, q), \mathfrak{f}_B(\ddot{\alpha}, q)\}$$

for all $\ddot{\alpha}, \ddot{\eta} \in X$ and $q \in Q$. The intersection of A and B, denoted $A \cap B$, is defined as:

$$(A \cap B)(\ddot{\alpha}, q) = \{\min\{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{t}_B(\ddot{\alpha}, q)\}, max\{\mathfrak{i}_A(\ddot{\alpha}, q), \mathfrak{i}_B(\ddot{\alpha}, q)\}, max\{\mathfrak{f}_A(\ddot{\alpha}, q), \mathfrak{f}_B(\ddot{\alpha}, q)\}$$

for all $\ddot{\alpha} \in X$ and $q \in Q$.

Theorem 3.8

The intersection of two $\mathbb{S}Q$-NFS of an $\mathbb{S}$-semigroup $\c{S}$ relative to a group $\mathcal{P}$ is also an $\mathbb{S}Q$-NFS of $\c{S}$.

Proof

Let A and B be two $\mathbb{S}Q$-NFS of an $\mathbb{S}$-semigroup $\c{S}$ relative to a group $\mathcal{P}$ is also an $\mathbb{S}Q$-NFS of $\c{S}$. Then $A_{\mathcal{P}}(\mathfrak{t}, \mathfrak{i}, \mathfrak{f})\colon \mathcal{P} \times Q \to [0,1] \times [0,1] \times [0,1]$ and $B_{\mathcal{P}}(\mathfrak{t}, \mathfrak{i}, \mathfrak{f})\colon \mathcal{P} \times Q \to [0,1] \times [0,1] \times [0,1]$.

Then the intersection

$$(A \cap B)(\ddot{\alpha}, q) = \{\min\{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{t}_B(\ddot{\alpha}, q)\}, max\{\mathfrak{i}_A(\ddot{\alpha}, q), \mathfrak{i}_B(\ddot{\alpha}, q)\}, max\{\mathfrak{f}_A(\ddot{\alpha}, q), \mathfrak{f}_B(\ddot{\alpha}, q)\}$$

Let $\ddot{\alpha}, \ddot{\eta} \in \mathcal{P}$ and $q \in Q$. Since $A_{\mathcal{P}}$ and $B_{\mathcal{P}}$ are neutrosophic fuzzy semigroups, they satisfy:

$$\mathfrak{t}_A(\ddot{\alpha}\ddot{\eta}, q) \geq \min\{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{t}_A(\ddot{\eta}, q)\},$$

$$\mathfrak{i}_A(\ddot{\alpha}\ddot{\eta}, q) \leq \max\{\mathfrak{i}_A(\ddot{\alpha}, q), \mathfrak{i}_A(\ddot{\eta}, q)\} \text{ and}$$

$$\mathfrak{f}_A(\ddot{\alpha}\ddot{\eta}, q) \leq \max\{\mathfrak{f}_A(\ddot{\alpha}, q), \mathfrak{f}_A(\ddot{\eta}, q)\}$$

So, $\min\{\mathfrak{t}_A(\ddot{\alpha}\ddot{\eta}, q), \mathfrak{t}_B(\ddot{\alpha}\ddot{\eta}, q)\} \geq \min\{min\{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{t}_B(\ddot{\alpha}, q)\}, min\{\mathfrak{t}_A(\ddot{\eta}, q), \mathfrak{t}_B(\ddot{\eta}, q)\}\}$

$$= \min\{\mathfrak{t}_{A\cap B}(\ddot{\alpha}, q), \mathfrak{t}_{A\cap B}(\ddot{\eta}, q)\}.$$

Hence, $\mathfrak{t}_{A\cap B}(\ddot{\alpha}\ddot{\eta}, q) \geq \min\{\mathfrak{t}_{A\cap B}(\ddot{\alpha}, q), \mathfrak{t}_{A\cap B}(\ddot{\eta}, q)\}$.

Likewise,

$\mathfrak{i}_{A\cap B}(\ddot{\alpha}\ddot{\eta}, q) \leq max\{\mathfrak{i}_{A\cap B}(\ddot{\alpha}, q), \mathfrak{i}_{A\cap B}(\ddot{\eta}, q)\}$ and

$\mathfrak{f}_{A\cap B}(\ddot{\alpha}\ddot{\eta}, q) \leq max\{\mathfrak{f}_{A\cap B}(\ddot{\alpha}, q), \mathfrak{f}_{A\cap B}(\ddot{\eta}, q)\}$

Also, for $\ddot{\alpha}^{-1} \in \mathcal{P}$, $\mathfrak{t}_{A\cap B}(\ddot{\alpha}^{-1}, q) = \min\{\mathfrak{t}_A(\ddot{\alpha}^{-1}, q), \mathfrak{t}_B(\ddot{\alpha}^{-1}, q)\}$

$$= \min\{\mathfrak{t}_A(\ddot{\alpha}, q), \mathfrak{t}_B(\ddot{\alpha}, q)\}$$

$$= \mathfrak{t}_{A\cap B}(\ddot{\alpha}, q).$$

Similarly, $\mathfrak{i}_{A\cap B}(\ddot{\alpha}^{-1}, q) = \mathfrak{i}_{A\cap B}(\ddot{\alpha}, q)$ and $\mathfrak{f}_{A\cap B}(\ddot{\alpha}^{-1}, q) = \mathfrak{f}_{A\cap B}(\ddot{\alpha}, q)$.

4. Conclusions

The emergence of $\mathbb{S}Q$-Neutrosophic Fuzzy semigroups ($\mathbb{S}Q$-NFS) is a major extension of traditional fuzzy semigroup theory when coupled with the rich and flexible neutrosophic sets. This is accomplished by adding the more so rigid structure of Q-Neutrosophic sets to the structure of $\mathbb{S}$-fuzzy semigroups. With the flexibility to represent simultaneous truth, indeterminacy, and falsity through stipulated structure's restriction maps, the $\mathbb{S}Q$-NFS concept provides a much richer algebraic structure capable of representing incomplete and inconsistent information than fuzzy semigroups alone. Also, our intuitive definitions, supported by examples and propositions, sufficiently establish internal consistency and is rich in mathematical structure as a useful concept.

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References

1. Anthony, J.M.; Sherwood, H. Fuzzy groups redefined. *J. Math. Anal. Appl.* **1979**, *69*, 124–130.
2. Das, P.S. Fuzzy groups and level subgroups. *J. Math. Anal. Appl.* **1981**, *84*, 264–269.
3. Bhattacharya, P. Fuzzy subgroups: Some characterizations. II. *Inf. Sci.* **1986**, 38, 293–297.
4. Priya,T.; Ramachandran, T.; Nagalakshmi, K.T. On Q-fuzzy normal subgroups. *Int. J. Comput. Organ. Trends* **2013**, *3*(11), 574–578.
5. Padilla, R. Smarandache algebraic structures. *Bull. Pure Appl. Sci.* **1998**, *17E*, 119–121.
6. Bhattacharya, P. Fuzzy subgroups: Some characterizations. *J. Math. Anal. Appl.* **1987**, *128*, 241–252.
7. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.* **1971**, *35*, 512–517.
8. Solairaju; Nagarajan, R. A new structure and construction of Q-fuzzy groups. *Adv. Fuzzy Math.* **2009**, *4*(1), 23–29.
9. Vasantha Kandasamy, W. B. *Smarandache semigroups*, American Research press, Rehoboth, NM, 2002.
10. Vasantha Kandasamy, W. B. *Smarandache Fuzzy Algebra*, American Research press, Rehoboth, NM, 2003.
11. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
12. Aruldoss, R.; Suganya, S. Smarandache Q-fuzzy semigroups. *Adv. Fuzzy Math.* **2016**, *11*(1), 89–97.
13. Vijayabalaji, S. Neutrosophic M-semigroup. *Int. J. Open Probl. Comput. Sci. Math.* **2024**, *17*, 58–65.
14. Sivaramakrishnan, S.; Parthasarathy, B.; Subbiah, S. Interval-valued neutrosophic fuzzy M-semigroup. In *New Trends in Neutrosophic Theories and Applications.* Florentin Smarandache; Surapati Pramanik, Neutrosophic Science International Association (NSIA) Publishing House.**2025**, *4*, 51–61.
15. Suganya, S. Smarandache anti Q-fuzzy semigroups. *Int. J. Math. Trends Technol.* **2018**, *60*(2), 110–116.
16. Smarandache, F. *Neutrosophy: Neutrosophic Probability Set and Logic: Analytic Synthesis and Synthetic Analysis*; American Research press, Rehoboth, NM, **1998**.

17. Ghazi, E.; Al-Salihi, S.O. A robust framework for medical diagnostics based on interval-valued Q-neutrosophic soft sets with aggregation operators. *Neutros. Sets Syst.* **2024**, *68*(1).
18. Saeed, A.; Jafar, M.N.; Saeed, M.; Muniba, K.; Jarad, F.; Alamri, F.S. Q-neutrosophic hypersoft set with its aggregation and applications in tourism. *SSRN Electron. J.* **2024.**
19. Premkumar, M.; Althaf, M.M.; Prasanna, A.; Kanabur, R.; Sharma, R.; Sarojwal, A. On algebraic characteristics of μ-anti-Q-fuzzy subgroups. In *Recent Developments in Algebra and Analysis*; Leung, H.H., Sivaraj, R., Kamalov, F., Eds.; *Trends in Mathematics.*, Birkhäuser: Cham, **2024**, 225-–235.
20. Gelaw, S.H.; Alaba, B.A.; Taye, M.A. Q-fuzzy structure on JU-algebra. *F1000Research* **2025**, *14*:109.

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