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On the Construction of the Neutrosophic Itô Integral

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Abstract. This research pioneers a rigorous framework for stochastic calculus in the neutrosophic paradigm, explicitly modeling systems with truth, indeterminacy, and falsehood degrees. We introduce foundational constructs including **Canonical Neutrosophic Brownian Motion**, **Simple Neutrosophic Processes**, and the space $\widetilde{\mathcal{V}}^2[0,T]$. Building upon these, we formally define the **Neutrosophic Itô Integral** for elementary processes and extend it to general integrands in $\widetilde{\mathcal{V}}^2[0,T]$, establishing **Neutrosophic Martingales**, d-**Dimensional Neutrosophic Brownian Motion**, **Matrix-Valued Neutrosophic Integrands**, and the **Multi-Dimensional Neutrosophic Itô Integral**. The theory is further generalized through **Local Neutrosophic Martingales**, **Locally Square-Integrable Neutrosophic Integrands**, and **Neutrosophic Itô Processes**.

Key theorems demonstrate the successful extension of classical stochastic calculus: The **Density of Elementary Neutrosophic Processes** enables integral construction, while the **Neutrosophic Itô Isometry** and its consistent extension ensure well-defined integration. Crucially, we prove **Martingale Characterization** with **Path Regularity**, construct integrals for local martingales via stopping times, and establish the **Itô Formula** for both neutrosophic Brownian motion and general Itô processes.

Significantly, essential structural properties–including isometry, martingale preservation, sample path continuity, localization efficacy, and the Itô formula–remain rigorously valid in this generalized uncertainty framework. This work provides a powerful mathematical toolkit for stochastic systems under three-valued uncertainty, enabling applications across finance, engineering, and decision science.

Keywords: Neutrosophic Stochastic Calculus, Neutrosophic Brownian Motion, Neutrosophic Itô Integral, Neutrosophic Martingales, Neutrosophic Itô Formula, Neutrosophic Probability.

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1. introduction

In the rapidly evolving landscape of applied mathematics and modern probability theory, the imperative for advanced theoretical frameworks addressing the intricate interplay between randomness and epistemic uncertainty in dynamical systems has become paramount [6, 7, 15].

This study advances the proposition that synthesizing Neutrosophic Philosophy [19, 20] with Stochastic Calculus [8, 9] enables transformative modeling of phenomena characterized by ontological indeterminacy and aleatory uncertainty.

Neutrosophic Logic—a revolutionary extension of classical fuzzy logic—has equipped researchers with sophisticated tools for analyzing data exhibiting contradiction, indeterminacy, and paraconsistent properties [19, 20, 22]. Grounded in the Truth-Indeterminacy-Falsity (T-I-F) triad, this framework has redefined analytical approaches across disciplines, from multi-criteria decision-support systems [1] to probabilistic modeling [3, 16, 27]. Recent theoretical advances have formalized neutrosophic integration methodologies [2, 4, 22], probability distributions [3, 16, 21], and stochastic processes [27, 28], with demonstrated applications in physical systems [25, 26] and statistical modeling [16].

Concurrently, Itô Calculus [8] has fundamentally reshaped our comprehension of stochastic processes, providing rigorous mathematical foundations for solving complex problems in mathematical physics [15] and financial engineering [9, 10, 17]. Modern extensions of the Itô integral [5, 11] and stochastic differential equations (SDEs) [7, 12, 14] continue to address emerging challenges in modeling stochastic phenomena, particularly in financial markets [9, 17, 23] and control systems [6]. Its efficacy in analyzing systems where randomness interacts with structural uncertainty remains unparalleled [10, 13, 15].

The integration of these domains presents a unique opportunity to bridge aleatory stochasticity and epistemic uncertainty. While Itô Calculus addresses objective randomness, Neutrosophy provides formal mechanisms for quantifying subjective indeterminacy [2, 4, 22], enabling novel methodologies for phenomena where both factors converge [27, 28]. This synthesis addresses critical gaps identified in contemporary stochastic analysis [5, 11] and neutrosophic measure theory [21, 22].

Our research contributes a unified framework that synergizes the theoretical rigor of Itô Calculus with the expressive power of Neutrosophic Logic. This integration manifests through three interconnected innovations: the development of **Neutrosophic-Itô Integration**, extending foundational stochastic integration concepts [2, 4, 5, 8, 11]; the formalization of **Neutrosophic Brownian Motion**, generalizing classical Wiener processes to incorporate epistemic uncertainty [10, 23, 27]; and the derivation of the **Neutrosophic Stochastic Itô Formula**, building upon fundamental stochastic calculus principles to accommodate truth-indeterminacy-falsehood dynamics [8, 12, 15]. These innovations collectively establish a novel mathematical architecture where stochastic randomness and neutrosophic indeterminacy coexist within a unified analytical framework.

These developments address theoretical limitations in conventional stochastic calculus [5, 7, 14] and neutrosophic probability [16, 21, 26], while providing practical tools for real-world applications requiring simultaneous treatment of randomness and indeterminacy.

The implications extend to financial market analysis under Knightian uncertainty [9, 17], complex ecological system modeling, and decision-making in information-scarce environments [1]. By establishing new research pathways in artificial intelligence and hybrid dynamical systems, this work demonstrates the transformative potential of Neutrosophic-Stochastic integration for advancing modeling capabilities in systems where traditional approaches prove insufficient [3, 13, 25, 28].

This paper begins by laying out the foundations of neutrosophic stochastic calculus, providing the necessary theoretical background for the subsequent developments. Building upon these foundations, the concept of neutrosophic Brownian motion is introduced and examined in detail. The paper then proceeds to the construction of the neutrosophic Itô integral, followed by an exploration of its key properties. Further extensions are discussed, including the generalization of integrators to neutrosophic martingales, multi-dimensional settings, and localization techniques. Finally, the neutrosophic Itô formula is derived, offering a crucial tool for applications in neutrosophic stochastic analysis.

2. Foundations of Neutrosophic Stochastic Calculus

Definition 2.1. [24][Neutrosophic Number] A *neutrosophic number* is an ordered triple

$$\widetilde{x} = (x^T, x^I, x^F) \in \widetilde{\mathbb{N}}, \tag{1}$$

where $\widetilde{\mathbb{N}} := \{(T, I, F) \in [0,1]^3 \mid 0 \leq T + I + F \leq 3\}$ and:

- x^T represents the **truth component**,
- x^I represents the **indeterminacy component**,
- x^F represents the **falsity component**.

Algebraic operations on $\widetilde{\mathbb{N}}$ are defined component-wise. The *Hadamard product* is

$$\widetilde{a} \odot \widetilde{b} := (a^T b^T,\ a^I b^I,\ a^F b^F), \quad \widetilde{a}, \widetilde{b} \in \widetilde{\mathbb{N}}. \tag{2}$$

The Euclidean norm is $\|\widetilde{x}\| := \sqrt{(x^T)^2 + (x^I)^2 + (x^F)^2}$.

This representation allows one to model phenomena where randomness is entangled with epistemic uncertainty, enabling stochastic analysis that is sensitive to all three dimensions.

Definition 2.2. [21, 26][Neutrosophic Probability Space] A *neutrosophic probability space* is a quadruple $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$, where:

- $\widetilde{\Omega}$ is a non-empty sample space.
- $\widetilde{\mathcal{F}}$ is a σ-algebra of *neutrosophic events*.

- $\widetilde{\mathbb{P}} = (\mathbb{P}^T, \mathbb{P}^I, \mathbb{P}^F)$ is a vector-valued measure assigning, respectively, the *truth*, *indeterminacy*, and *falsity* probabilities, where $\mathbb{P}^k : \widetilde{\mathcal{F}} \to [0,1]$ are probability measures for $k \in \{T, I, F\}$.
- $\Theta : \widetilde{\mathcal{F}} \to [0,1]^3$ is a mapping such that for every $A \in \widetilde{\mathcal{F}}$,

$$\Theta(A) = (T_A, I_A, F_A) \in \widetilde{\mathbb{N}},$$

where:

T_A: degree of truth-membership of A,

I_A: degree of indeterminacy-membership of A,

F_A: degree of falsity-membership of A.

Definition 2.3. [27][Neutrosophic Random Variables and Its Expectation]: A *neutrosophic random variable* is a measurable function $\widetilde{X} : \widetilde{\Omega} \to \widetilde{\mathbb{N}}$ such that

$$\widetilde{X}(\widetilde{\omega}) = \left(X^T(\omega^T), X^I(\omega^I), X^F(\omega^F)\right) \in \widetilde{\mathbb{N}} \quad \forall \omega^k \in \widetilde{\Omega}. \tag{3}$$

The *neutrosophic expectation operator* is defined as:

$$\widetilde{\mathbb{E}}[\widetilde{X}] := \left(\mathbb{E}[X^T], \mathbb{E}[X^I], \mathbb{E}[X^F]\right), \tag{4}$$

where for each component $k \in \{T, I, F\}$:

$$\mathbb{E}[X^k] = \int_{\widetilde{\Omega}} X^k(\omega^k) \mathrm{d}\mathbb{P}^k(\omega^k). \tag{5}$$

Definition 2.4. [18][Neutrosophic Filtration] Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$ be a neutrosophic probability space. A family $\{\widetilde{\mathcal{F}}_t\}_{t \geq 0}$ is called a *neutrosophically admissible filtration* if:

(1) (*Monotonicity*) $\widetilde{\mathcal{F}}_s \subseteq \widetilde{\mathcal{F}}_t$ for all $0 \leq s < t < \infty$.
(2) (*Right-continuity*) $\widetilde{\mathcal{F}}_t = \bigcap_{u>t} \widetilde{\mathcal{F}}_u$ for all $t \geq 0$.
(3) (*Completeness*) $\widetilde{\mathcal{F}}_0$ contains all $\widetilde{\mathbb{P}}$-null sets.
(4) (*Neutrosophic encoding*) For every $t \geq 0$ and every $A \in \widetilde{\mathcal{F}}_t$.

3. Neutrosophic Brownian Motion

Definition 3.1 (Canonical Neutrosophic Brownian Motion)**.** Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$ be a neutrosophic probability space. A *canonical neutrosophic Brownian motion* is a trivariate stochastic process

$$\widetilde{B}_t := \left(B_t^T,\, B_t^I,\, B_t^F\right), \quad t \geq 0, \tag{6}$$

where each component B_t^T, B_t^I, and B_t^F is a one-dimensional Brownian motion with its own parameters, and the components may be mutually correlated. For each $\widetilde{\omega} \in \widetilde{\Omega}$ and $t \geq 0$,

$$\widetilde{B}_t(\widetilde{\omega}) = \left(T_t(\omega^T),\, I_t(\omega^I),\, F_t(\omega^F)\right) \in \widetilde{\mathbb{N}}, \tag{7}$$

where:

- $T_t(\omega^T)$ — the *truth-membership* value at time t,
- $I_t(\omega^I)$ — the *indeterminacy-membership* value at time t,
- $F_t(\omega^F)$ — the *falsity-membership* value at time t.

The process $\widetilde{B}_t$ satisfies the following properties:

(1) **Origin condition:** $B_0^k = 0$ $\widetilde{\mathbb{P}}$-almost surely for each $k \in \{T, I, F\}$.

(2) **Independent increments:** For all $0 \le s < t$, the increment $\widetilde{B}_t - \widetilde{B}_s$ is independent of the past σ-algebra $\widetilde{\mathcal{F}}_s$ with respect to $\widetilde{\mathbb{P}}$.

(3) **Gaussian increments:** Each increment $\widetilde{B}_t - \widetilde{B}_s$ follows a trivariate (possibly neutrosophic-parameterized) normal distribution:

$$\widetilde{B}_t - \widetilde{B}_s \sim \mathcal{N}(\mathbf{0},\, (t-s)\Sigma)\,, \tag{8}$$

where Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_T^2 & \rho_{TI}\sigma_T\sigma_I & \rho_{TF}\sigma_T\sigma_F \\ \rho_{TI}\sigma_T\sigma_I & \sigma_I^2 & \rho_{IF}\sigma_I\sigma_F \\ \rho_{TF}\sigma_T\sigma_F & \rho_{IF}\sigma_I\sigma_F & \sigma_F^2 \end{pmatrix}, \tag{9}$$

satisfying:

(a) **Non-negative variances:**

$$\sigma_T^2 \ge 0, \quad \sigma_I^2 \ge 0, \quad \sigma_F^2 \ge 0.$$

(b) **Correlation bounds:**

$$\rho_{TI}, \rho_{TF}, \rho_{IF} \in [-1, 1].$$

(c) **Positive semi-definiteness:** Σ is symmetric and all its eigenvalues are non-negative (equivalently, all principal minors ≥ 0). For a correlation matrix, this is equivalent to:

$$1 - \rho_{TI}^2 - \rho_{TF}^2 - \rho_{IF}^2 + 2\rho_{TI}\rho_{TF}\rho_{IF} \ge 0.$$

(4) **Continuity of sample paths:** For $\widetilde{\mathbb{P}}$-almost every $\widetilde{\omega} \in \widetilde{\Omega}$, the trajectory $t \mapsto \widetilde{B}_t(\widetilde{\omega})$ is continuous in t.

In essence, a canonical neutrosophic Brownian motion models the joint evolution of the truth, indeterminacy, and falsity components over time, each evolving as a correlated Gaussian process while jointly forming a valid neutrosophic number.

4. Construction of the Neutrosophic Itô Integral

In this section, we extend the classical Itô integration framework to the *neutrosophic* setting, where uncertainty is modeled not only by randomness but also by explicit representation of indeterminacy and falsity degrees.

4.1. *Elementary Processes*

Definition 4.1 (Simple Neutrosophic Process)**.** Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$ be a neutrosophic probability space. A stochastic process $\widetilde{\phi}_t : [0,T] \times \widetilde{\Omega} \to \mathbb{R}^3$ is called a *simple neutrosophic process* if it can be expressed in the form

$$\widetilde{\phi}_t(\widetilde{\omega}) = \sum_{j=0}^{n-1} \widetilde{\xi}_j(\widetilde{\omega}) \cdot \mathbf{1}_{[t_j,t_{j+1})}(t), \tag{10}$$

where:

- $0 = t_0 < t_1 < \cdots < t_n = T$ is a finite partition of $[0,T]$,
- Each $\widetilde{\xi}_j = (\xi_j^T, \xi_j^I, \xi_j^F)$ is $\widetilde{\mathcal{F}}_{t_j}$-measurable,
- The component-wise integrability holds:

$$\mathbb{E}_T[\|\widetilde{\xi}_j\|^2] < \infty, \quad \mathbb{E}_I[\|\widetilde{\xi}_j\|^2] < \infty, \quad \mathbb{E}_F[\|\widetilde{\xi}_j\|^2] < \infty, \quad \forall j. \tag{11}$$

The set of all such processes is denoted by $\mathcal{S}_{\text{ns}}(0,T)$.

Definition 4.2 (Neutrosophic Itô Integral for Elementary Processes)**.** For $\widetilde{\phi} \in \mathcal{S}_{\text{ns}}(0,T)$, the *neutrosophic Itô integral* with respect to a canonical neutrosophic Brownian motion $\widetilde{B}_t$ is defined as

$$\int_0^T \widetilde{\phi}_t \mathrm{d}\widetilde{B}_t := \sum_{j=0}^{n-1} \widetilde{\xi}_j \odot \big(\widetilde{B}_{t_{j+1}} - \widetilde{B}_{t_j}\big). \tag{12}$$

4.2. *Extension to General Integrands*

Definition 4.3. (Space $\widetilde{\mathcal{V}}^2([0,T])$). The space $\widetilde{\mathcal{V}}^2([0,T])$ consists of all stochastic processes $\{\widetilde{\phi}_t\}_{t\in[0,T]}$ with values in $\mathbb{R}^3$ such that:

(1) $\widetilde{\phi}_t$ is adapted to a neutrosophically admissible filtration $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$.

(2) The component-wise expectations are finite:

$$\mathbb{E}^T\left[\int_0^T \|\widetilde{\phi}_t\|^2 \mathrm{d}t\right] < \infty, \quad \mathbb{E}^I\left[\int_0^T \|\widetilde{\phi}_t\|^2 \mathrm{d}t\right] < \infty, \quad \mathbb{E}^F\left[\int_0^T \|\widetilde{\phi}_t\|^2 \mathrm{d}t\right] < \infty, \tag{13}$$

where $\|\widetilde{\phi}_t\| = \sqrt{(\phi_t^T)^2 + (\phi_t^I)^2 + (\phi_t^F)^2}$ is the Euclidean norm.

Theorem 4.4 (Density of Elementary Neutrosophic Processes)**.** *Let $\mathcal{S}_{\text{ns}}(0,T)$ denote the set of* elementary neutrosophic processes *on $[0,T]$, consisting of finite linear combinations of indicator functions on stochastic intervals with neutrosophic coefficients. Then, $\mathcal{S}_{\text{ns}}(0,T)$ is* *dense in the Hilbert space* $\widetilde{\mathcal{V}}^2[0,T]$ *of square-integrable neutrosophic processes. Formally, for any* $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2[0,T]$*, there exists a sequence* $\{\widetilde{\phi}^{(n)}\} \subset \mathcal{S}_{\mathrm{ns}}(0,T)$ *such that*

$$\lim_{n\to\infty} \mathbb{E}^k \left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \,\mathrm{d}t\right] = (0,0,0), \tag{14}$$

where the limit is taken componentwise in the neutrosophic sense.

Proof. Let $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2[0,T]$. By definition, we have

$$\widetilde{\mathbb{E}}\left[\int_0^T \|\widetilde{\phi}_t\|^2 \mathrm{d}t\right] < \infty, \tag{15}$$

which means $\widetilde{\phi} \in L^2(\widetilde{\Omega} \times [0,T], \mathrm{d}\widetilde{\mathbb{P}} \times \mathrm{d}t)$.

We now construct a unifying finite positive measure $\widetilde{\mathbb{Q}}$ on $\widetilde{\Omega}$ by

$$\mathrm{d}\widetilde{\mathbb{Q}} = \frac{1}{3}\left(\mathrm{d}\mathbb{P}^T + \mathrm{d}\mathbb{P}^I + \mathrm{d}\mathbb{P}^F\right). \tag{16}$$

Since $\widetilde{\phi}$ belongs to each L^2 space, it follows that

$$\int_{\widetilde{\Omega}} \int_0^T \|\widetilde{\phi}_t(\widetilde{\omega})\|^2 \mathrm{d}t \,\mathrm{d}\widetilde{\mathbb{Q}}(\widetilde{\omega}) < \infty, \tag{17}$$

and thus $\widetilde{\phi} \in L^2(\widetilde{\Omega} \times [0,T], \mathrm{d}\widetilde{\mathbb{Q}} \times \mathrm{d}t)$.

From the classical theory of L^2 spaces, elementary processes are dense under product measures. Therefore, there exists $\{\widetilde{\phi}^{(n)}\} \subset \mathcal{S}_{\mathrm{ns}}(0,T)$ such that

$$\lim_{n\to\infty} \int_{\widetilde{\Omega}} \int_0^T \|\widetilde{\phi}_t(\widetilde{\omega}) - \widetilde{\phi}_t^{(n)}(\widetilde{\omega})\|^2 \mathrm{d}t \,\mathrm{d}\widetilde{\mathbb{Q}}(\widetilde{\omega}) = 0. \tag{18}$$

Moreover, the Radon–Nikodym theorem yields $\mathrm{d}\widetilde{\mathbb{P}} \ll \mathrm{d}\widetilde{\mathbb{Q}}$ with $\mathrm{d}\widetilde{\mathbb{P}} \leq 3\,\mathrm{d}\widetilde{\mathbb{Q}}$. Consequently,

$$\int_{\widetilde{\Omega}} \int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \mathrm{d}t \,\mathrm{d}\widetilde{\mathbb{P}} \leq 3 \int_{\widetilde{\Omega}} \int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \mathrm{d}t \,\mathrm{d}\widetilde{\mathbb{Q}}. \tag{19}$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} \widetilde{\mathbb{E}}\left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \mathrm{d}t\right] = 0, \tag{20}$$

for $k \in \{T, I, F\}$. Combining these limits componentwise proves that

$$\lim_{n\to\infty} \mathbb{E}^k \left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \mathrm{d}t\right] = (0,0,0), \tag{21}$$

completing the proof. □

Theorem 4.5 (Neutrosophic Itô Isometry for Elementary Processes)**.** *Let* $\widetilde{\phi} \in \mathcal{S}_{\mathrm{ns}}(0,T)$ *be an elementary neutrosophic process. Then:*

$$\widetilde{\mathbb{E}}\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] = \widetilde{\mathbb{E}}\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right], \tag{22}$$

where $\widetilde{B}_t$ is a canonical neutrosophic Brownian motion and $\widetilde{\mathbb{E}}$ denotes the component-wise neutrosophic expectation.

Proof. Let $\widetilde{\phi} \in \mathcal{S}_{\mathrm{ns}}(0,T)$ be represented as

$$\widetilde{\phi}_t(\widetilde{\omega}) = \sum_{i=0}^{m-1} \widetilde{\xi}_i(\widetilde{\omega})\, \mathbf{1}_{(t_i,t_{i+1}]}(t), \quad 0 = t_0 < t_1 < \cdots < t_m = T, \tag{23}$$

where each $\widetilde{\xi}_i$ is $\widetilde{\mathcal{F}}_{t_i}$-measurable and bounded. The stochastic integral is given by:

$$\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t = \sum_{i=0}^{m-1} \widetilde{\xi}_i\big(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}\big). \tag{24}$$

Its squared norm expands as:

$$\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2 = \sum_{i=0}^{m-1} \widetilde{\xi}_i^2(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})^2 + 2 \sum_{0\le i<j\le m-1} \widetilde{\xi}_i\widetilde{\xi}_j(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})(\widetilde{B}_{t_{j+1}} - \widetilde{B}_{t_j}). \tag{25}$$

For each neutrosophic component $k \in \{T, I, F\}$, the expectation satisfies:

$$\mathbb{E}^k\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] = \widetilde{\mathbb{E}}\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] = \sum_{i=0}^{m-1} \widetilde{\mathbb{E}}\left[\widetilde{\xi}_i^2(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})^2\right], \tag{26}$$

since the cross terms vanish by independence of increments and the property

$$\widetilde{\mathbb{E}}\left[\widetilde{B}_{t_{j+1}} - \widetilde{B}_{t_j}\right] = 0. \tag{27}$$

Using the variance property of neutrosophic Brownian motion:

$$\widetilde{\mathbb{E}}\left[(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})^2\right] = t_{i+1} - t_i, \tag{28}$$

we obtain:

$$\widetilde{\mathbb{E}}\left[\widetilde{\xi}_i^2(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i})^2\right] = \widetilde{\mathbb{E}}\left[\widetilde{\xi}_i^2\right](t_{i+1} - t_i). \tag{29}$$

Thus:

$$\widetilde{\mathbb{E}}\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] = \sum_{i=0}^{m-1} \widetilde{\mathbb{E}}\left[\|\widetilde{\xi}_i\|^2\right](t_{i+1} - t_i). \tag{30}$$

Meanwhile:

$$\widetilde{\mathbb{E}}\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right] = \sum_{i=0}^{m-1} \mathbb{E}^k\left[\|\widetilde{\xi}_i\|^2\right](t_{i+1} - t_i). \tag{31}$$

Since this holds for all $k \in \{T, I, F\}$, the vectorial equality follows:

$$\widetilde{\mathbb{E}}\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] = \widetilde{\mathbb{E}}\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right]. \tag{32}$$

□

4.3. *General Neutrosophic Itô Integral*

Definition 4.6 (General Neutrosophic Itô Integral)**.** For $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2[0,T]$, we define

$$\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t := \operatorname*{plim}_{n\to\infty} \int_0^T \widetilde{\phi}_t^{(n)} \,\mathrm{d}\widetilde{B}_t, \tag{33}$$

where $\{\widetilde{\phi}^{(n)}\} \subset \mathcal{S}_{\mathrm{ns}}(0,T)$ approximates $\widetilde{\phi}$ in $L^2(\widetilde{\mathbb{P}})$.

Theorem 4.7 (Consistency and Isometry of the Neutrosophic Itô Integral)**.** *Let* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$ *be a neutrosophic probability space and let* $\widetilde{B}_t = (B_t^T, B_t^I, B_t^F)$ *be a canonical neutrosophic Brownian motion adapted to* $\widetilde{\mathcal{F}}$ *with independent components. For any* $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2([0,T])$*, the neutrosophic Itô integral* $\int_0^T \widetilde{\phi}_t \mathrm{d}\widetilde{B}_t$ *is well-defined and satisfies:*

(1) ***Independence of the approximating sequence****: The integral is independent of the choice of approximating sequence in* $\mathcal{S}_{\mathrm{ns}}(0,T)$.

(2) ***Neutrosophic Itô isometry****:*

$$\begin{aligned}
\mathbb{E}^T\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] &= \mathbb{E}^T\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right] \\
\mathbb{E}^I\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] &= \mathbb{E}^I\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right] \\
\mathbb{E}^F\left[\left\|\int_0^T \widetilde{\phi}_t \,\mathrm{d}\widetilde{B}_t\right\|^2\right] &= \mathbb{E}^F\left[\int_0^T \|\widetilde{\phi}_t\|^2 \,\mathrm{d}t\right]
\end{aligned} \tag{34}$$

Proof. The proof extends classical Itô integral arguments to each neutrosophic component $k \in \{T, I, F\}$.

Step 1: Independence of the approximating sequence. Let $\{\widetilde{\phi}^{(n)}\}$ and $\{\widetilde{\psi}^{(n)}\}$ be sequences in $\mathcal{S}_{\mathrm{ns}}(0,T)$ satisfying:

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}\|^2 \mathrm{d}t\right] = 0 \quad \forall k \in \{T, I, F\}, \tag{35}$$

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\psi}_t^{(n)}\|^2 \mathrm{d}t\right] = 0 \quad \forall k \in \{T, I, F\}. \tag{36}$$

By the triangle inequality in $L^2(\mathbb{P}^k)$ for each component:

$$\begin{aligned}
\left(\mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t^{(n)} - \widetilde{\psi}_t^{(n)}\|^2 \,\mathrm{d}t\right]\right)^{1/2} \leq\ & \underbrace{\left(\mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t^{(n)} - \widetilde{\phi}_t\|^2 \,\mathrm{d}t\right]\right)^{1/2}}_{\to 0} \\
& + \underbrace{\left(\mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t - \widetilde{\psi}_t^{(n)}\|^2 \,\mathrm{d}t\right]\right)^{1/2}}_{\to 0}.
\end{aligned} \tag{37}$$

The Itô isometry for elementary processes under $\mathbb{P}^k$ gives:

$$\mathbb{E}^k\left[\left|\int_0^T (\widetilde{\phi}_t^{(n)} - \widetilde{\psi}_t^{(n)})\mathrm{d}B_t^k\right|^2\right] = \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t^{(n)} - \widetilde{\psi}_t^{(n)}\|^2\mathrm{d}t\right] \to 0. \tag{38}$$

Thus, the integral limits coincide for all approximating sequences in each component.

Step 2: Neutrosophic Itô isometry. Let $\{\widetilde{\phi}^{(n)}\} \subset \mathcal{S}_{\mathrm{ns}}(0,T)$ approximate $\widetilde{\phi}$. For elementary processes, componentwise Itô isometry holds:

$$\mathbb{E}^k\left[\left(\int_0^T \widetilde{\phi}_t^{(n)}\mathrm{d}B_t^k\right)^2\right] = \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t^{(n)}\|^2\mathrm{d}t\right] \quad \forall k. \tag{39}$$

By $L^2(\mathbb{P}^k)$-continuity of the integral and dominated convergence:

$$\lim_{n\to\infty} \mathbb{E}^k\left[\left|\int_0^T \widetilde{\phi}_t\mathrm{d}B_t^k - \int_0^T \widetilde{\phi}_t^{(n)}\mathrm{d}B_t^k\right|^2\right] = 0, \tag{40}$$

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T \left|\|\widetilde{\phi}_t\|^2 - \|\widetilde{\phi}_t^{(n)}\|^2\right| \mathrm{d}t\right] = 0. \tag{41}$$

Taking limits componentwise:

$$\begin{aligned}\mathbb{E}^k\left[\left(\int_0^T \widetilde{\phi}_t\mathrm{d}B_t^k\right)^2\right] &= \lim_{n\to\infty} \mathbb{E}^k\left[\left(\int_0^T \widetilde{\phi}_t^{(n)}\mathrm{d}B_t^k\right)^2\right] \\ &= \lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t^{(n)}\|^2\mathrm{d}t\right] \\ &= \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t\|^2\mathrm{d}t\right]. \end{aligned} \tag{42}$$

For the vector integral, by independence of $\{B_t^T, B_t^I, B_t^F\}$:

$$\mathbb{E}^k\left[\left\|\int_0^T \widetilde{\phi}_t\mathrm{d}\widetilde{B}_t\right\|^2\right] = \sum_{j\in\{T,I,F\}} \mathbb{E}^k\left[\left(\int_0^T \widetilde{\phi}_t\mathrm{d}B_t^j\right)^2\right] = \mathbb{E}^k\left[\int_0^T \|\widetilde{\phi}_t\|^2\mathrm{d}t\right], \tag{43}$$

where cross-terms vanish since $\mathbb{E}^k[\mathrm{d}B_t^j\mathrm{d}B_t^m] = \delta_{jm}\mathrm{d}t$. This establishes the isometry. □

5. Properties of the Neutrosophic Itô Integral

Theorem 5.1 (Neutrosophic Martingale Characterization). *Let $\{\widetilde{B}_t\}_{t\geq 0}$ be a canonical neutrosophic Brownian motion on a neutrosophic probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$, where $\widetilde{\mathbb{P}} = (\mathbb{P}^T, \mathbb{P}^I, \mathbb{P}^F)$ is a vector-valued measure. Let $\widetilde{\mathbb{E}}[\tilde{\cdot} \mid \widetilde{\mathcal{F}}_t]$ denote the* neutrosophic conditional expectation*, defined component-wise as:*

$$\widetilde{\mathbb{E}}[\widetilde{X} \mid \widetilde{\mathcal{F}}_t] = \left(\mathbb{E}[X^T \mid \mathcal{F}_t^T],\, \mathbb{E}[X^I \mid \mathcal{F}_t^I],\, \mathbb{E}[X^F \mid \mathcal{F}_t^F]\right). \tag{44}$$

If $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2([0,T])$ is a predictable process such that:

$$\mathbb{E}^k\left[\int_0^T |\widetilde{\phi}_t|^2\mathrm{d}t\right] < \infty, \quad \forall k \in \{T, I, F\}, \tag{45}$$

then the stochastic integral

$$\widetilde{M}_t := \int_0^t \widetilde{\phi}_s \mathrm{d}\widetilde{B}_s \tag{46}$$

is a neutrosophic martingale*, i.e.,*

$$\widetilde{\mathbb{E}}\big[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\big] = \widetilde{M}_s, \quad \mathbb{P}^k\text{-}a.s., \quad 0 \le s < t \le T, \quad \forall k \in \{T, I, F\}. \tag{47}$$

Proof. The proof is established component-wise for each measure $\mathbb{P}^k$ ($k \in \{T, I, F\}$) in two steps.

Step 1: Elementary integrands. Let $\widetilde{\phi} \in \mathcal{S}_{\text{ns}}(0,T)$ be given by

$$\widetilde{\phi}_u = \sum_{i=0}^{m-1} \widetilde{\xi}_i \mathbf{1}_{(t_i, t_{i+1}]}(u), \quad 0 = t_0 < t_1 < \cdots < t_m = T, \tag{48}$$

where each $\widetilde{\xi}_i$ is bounded and $\widetilde{\mathcal{F}}_{t_i}$-measurable. For $0 \le s < t \le T$ with $s = t_j$, $t = t_\ell$, the integral is

$$\widetilde{M}_t = \sum_{i=0}^{\ell-1} \widetilde{\xi}_i \big(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}\big). \tag{49}$$

For each $k \in \{T, I, F\}$, condition on $\widetilde{\mathcal{F}}_s$ and use the independence of Brownian increments under $\mathbb{P}^k$:

$$\mathbb{E}^k\big[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\big] = \sum_{i=0}^{j-1} \widetilde{\xi}_i \big(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}\big) + \sum_{i=j}^{\ell-1} \mathbb{E}^k\big[\widetilde{\xi}_i \big(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}\big) \mid \widetilde{\mathcal{F}}_s\big] = \widetilde{M}_s, \tag{50}$$

since $\mathbb{E}^k\big[\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i} \mid \widetilde{\mathcal{F}}_s\big] = 0$ $\mathbb{P}_k$-a.s. for $i \ge j$. Thus:

$$\widetilde{\mathbb{E}}\big[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\big] = \big(\mathbb{E}[M_t^T \mid \mathcal{F}_s^T], \mathbb{E}[M_t^I \mid \mathcal{F}_s^I], \mathbb{E}[M_t^F \mid \mathcal{F}_s^F]\big) = (\widetilde{M}_s, \widetilde{M}_s, \widetilde{M}_s) = \widetilde{M}_s. \tag{51}$$

Step 2: General integrands. For $\widetilde{\phi} \in \widetilde{\mathcal{V}}^2([0,T])$, choose $\{\widetilde{\phi}^{(n)}\} \subset \mathcal{S}_{\text{ns}}(0,T)$ such that for each $k \in \{T, I, F\}$:

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T \big|\widetilde{\phi}_u - \widetilde{\phi}_u^{(n)}\big|^2 \mathrm{d}u\right] = 0. \tag{52}$$

By the Itô isometry under each $\mathbb{P}^k$:

$$\mathbb{E}^k\big[\big|\widetilde{M}_t - \widetilde{M}_t^{(n)}\big|^2\big] = \mathbb{E}^k\left[\int_0^t \big|\widetilde{\phi}_u - \widetilde{\phi}_u^{(n)}\big|^2 \mathrm{d}u\right] \xrightarrow{n\to\infty} 0, \quad \forall k \in \{T, I, F\}. \tag{53}$$

Since $L^2(\mathbb{P}^k)$-limits preserve the martingale property for each component, and $\widetilde{M}_t^{(n)}$ is a $\mathbb{P}^k$-martingale for all n (by Step 1), we have for $0 \le s < t \le T$:

$$\mathbb{E}^k\big[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\big] = \lim_{n\to\infty} \mathbb{E}^k\big[\widetilde{M}_t^{(n)} \mid \widetilde{\mathcal{F}}_s\big] = \lim_{n\to\infty} \widetilde{M}_s^{(n)} = \widetilde{M}_s \quad \mathbb{P}^k\text{-a.s.} \tag{54}$$

Thus, $\widetilde{\mathbb{E}}\big[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\big] = \widetilde{M}_s$ holds for all $k \in \{T, I, F\}$. □

Theorem 5.2 (Path Regularity). *Let $\{\widetilde{B}_t\}_{t\geq 0}$ be a canonical neutrosophic Brownian motion on a neutrosophic probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$, where $\widetilde{\mathbb{P}} = (\mathbb{P}^T, \mathbb{P}^I, \mathbb{P}^F)$. Let $\widetilde{\phi}_t$ be a predictable process with respect to the neutrosophic filtration $\{\widetilde{\mathcal{F}}_t\}$ such that:*

$$\mathbb{E}^k\left[\int_0^t |\widetilde{\phi}_s|^2 \mathrm{d}s\right] < \infty \quad \forall t > 0, \quad \forall k \in \{T, I, F\}. \tag{55}$$

Then the stochastic integral

$$\widetilde{M}_t := \int_0^t \widetilde{\phi}_s \mathrm{d}\widetilde{B}_s \tag{56}$$

admits a modification $\widetilde{M}_t$ that is $\mathbb{P}^k$-almost surely continuous for each $k \in \{T, I, F\}$.

Proof. The existence of a continuous modification is established component-wise for each measure $\widetilde{\mathbb{P}}$. For each $k \in \{T, I, F\}$:

(1) **Square-Integrability:** The condition

$$\mathbb{E}^k\left[\int_0^t |\widetilde{\phi}_s|^2 \mathrm{d}s\right] < \infty \quad \forall t > 0 \tag{57}$$

ensures the stochastic integral $\widetilde{M}_t = \int_0^t \widetilde{\phi}_s \mathrm{d}\widetilde{B}_s$ is well-defined and a square-integrable martingale under $\mathbb{P}^k$ (by the Martingale Characterization Theorem).

(2) **Approximation:** Choose a sequence $\{\widetilde{\phi}^{(n)}\}_{n\in\mathbb{N}}$ of simple processes such that:

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T |\widetilde{\phi}_s^{(n)} - \widetilde{\phi}_s|^2 \mathrm{d}s\right] = 0 \quad \forall T > 0. \tag{58}$$

This is possible by density of elementary processes in $L^2(\mathbb{P}^k)$.

(3) **Continuity of Approximants:** The integrals

$$\widetilde{M}_t^{(n)} = \int_0^t \widetilde{\phi}_s^{(n)} \mathrm{d}\widetilde{B}_s \tag{59}$$

are continuous under $\mathbb{P}^k$ by construction (as finite sums of scaled Brownian increments).

(4) **Uniform Convergence:** By Itô's isometry under $\mathbb{P}^k$:

$$\mathbb{E}^k\left[\sup_{0\leq s\leq t} |\widetilde{M}_s^{(n)} - \widetilde{M}_s|^2\right] \leq 4\,\mathbb{E}^k\left[\int_0^t |\widetilde{\phi}_s^{(n)} - \widetilde{\phi}_s|^2 \mathrm{d}s\right] \xrightarrow{n\to\infty} 0. \tag{60}$$

Thus, $\widetilde{M}^{(n)} \to \widetilde{M}$ in $L^2(\mathbb{P}^k)$-norm uniformly on $[0, t]$, implying uniform convergence on compacts in probability under $\mathbb{P}^k$.

(5) **Continuous Modification:** By the standard martingale regularization theorem (applied under each $\mathbb{P}^k$), there exists a $\mathbb{P}^k$-null set N^k and a process $\widetilde{M}_t^{(k)}$ such that: (i) $\widetilde{M}_t^{(k)} = \widetilde{M}_t$ $\mathbb{P}^k$-a.s. for all t (ii) $t \mapsto \widetilde{M}_t^{(k)}$ is continuous for all $\widetilde{\omega} \notin N^k$.

To construct a single modification $\widetilde{M}_t$ continuous $\mathbb{P}^k$-a.s. for all k, define $N = N^T \cup N^I \cup N^F$ and:

$$\widetilde{M}_t(\widetilde{\omega}) = \begin{cases} M_t(\widetilde{\omega}) & \widetilde{\omega} \notin N \\ 0 & \widetilde{\omega} \in N \end{cases}. \tag{61}$$

For each $k \in \{T, I, F\}$, $\mathbb{P}^k(N) = 0$ and $\widetilde{M}_t$ $\mathbb{P}^k$-a.s., with paths $t \mapsto \widetilde{M}_t(\widetilde{\omega})$ continuous for all $\omega \in \widetilde{\Omega}$. Thus, $\widetilde{M}_t$ is the required modification. □

Remark. The continuity of $\widetilde{M}_t$ can be established component-wise via the Kolmogorov continuity criterion. For each $k \in \{T, I, F\}$, consider the process under $\mathbb{P}^k$. For $0 \leq s < t \leq T$, the Burkholder-Davis-Gundy inequality (under $\mathbb{P}^k$) yields for $p \geq 2$:

$$\mathbb{E}^k\left[|\widetilde{M}_t - \widetilde{M}_s|^p\right] \leq C_p^{(k)} \mathbb{E}^k\left[\left(\int_s^t |\widetilde{\phi}_u|^2 \mathrm{d}u\right)^{p/2}\right], \tag{62}$$

where $C_p^{(k)} > 0$ is a constant depending on p and the measure $\mathbb{P}^k$. Under the square-integrability condition:

$$\mathbb{E}^k\left[\int_0^T |\widetilde{\phi}_u|^2 \mathrm{d}u\right] < \infty, \tag{63}$$

we have:

$$\mathbb{E}^k\left[|\widetilde{M}_t - \widetilde{M}_s|^p\right] \leq C_p^{(k)} \left(\mathbb{E}^k\left[\int_s^t |\widetilde{\phi}_u|^2 \mathrm{d}u\right]\right)^{p/2} |t-s|^{p/2} \leq K^k |t-s|^{p/2}, \tag{64}$$

for some constant $K_k > 0$. Choosing $p > 2$ satisfies the condition $p/2 > 1$ required by Kolmogorov's continuity criterion under $\mathbb{P}_k$. Thus, for each k, there exists a continuous modification under $\mathbb{P}_k$. The universal modification $\widetilde{M}_t$ constructed in the proof is continuous $\mathbb{P}^k$-a.s. for all $k \in \{T, I, F\}$.

Theorem 5.3 (Linearity of the Neutrosophic Itô Integral)**.** *Let $\{\widetilde{B}_t\}_{t\geq 0}$ be a canonical Neutrosophic Brownian motion on a neutrosophic probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \Theta)$ with $\widetilde{\mathbb{P}} = (\mathbb{P}^T, \mathbb{P}^I, \mathbb{P}^F)$. Let $\widetilde{\phi}, \widetilde{\psi} \in \widetilde{\mathcal{V}}^2([0, T])$ be predictable processes such that for all $k \in \{T, I, F\}$:*

$$\mathbb{E}^k\left[\int_0^T |\widetilde{\phi}_t|^2 \mathrm{d}t\right] < \infty, \quad \mathbb{E}^k\left[\int_0^T |\widetilde{\psi}_t|^2 \mathrm{d}t\right] < \infty. \tag{65}$$

For any deterministic scalars $\alpha, \beta \in \mathbb{R}$, the following equality holds $\mathbb{P}^k$-almost surely for each $k \in \{T, I, F\}$:

$$\int_0^T (\alpha\widetilde{\phi}_t + \beta\widetilde{\psi}_t)\mathrm{d}\widetilde{B}_t = \alpha \int_0^T \widetilde{\phi}_t \mathrm{d}\widetilde{B}_t + \beta \int_0^T \widetilde{\psi}_t \mathrm{d}\widetilde{B}_t. \tag{66}$$

Proof. We establish component-wise linearity for each $k \in \{T, I, F\}$.

Step 1: Simple processes. Let $\widetilde{\phi}_t = \sum_{i=0}^{n-1} a_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$ and $\widetilde{\psi}_t = \sum_{i=0}^{n-1} b_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$ where $0 = t_0 < \cdots < t_n = T$, with each a_i, b_i bounded and $\widetilde{\mathcal{F}}_{t_i}$-measurable. The linear combination is:

$$\alpha\widetilde{\phi}_t + \beta\widetilde{\psi}_t = \sum_{i=0}^{n-1} (\alpha a_i + \beta b_i)\mathbf{1}_{(t_i, t_{i+1}]}(t). \tag{67}$$

By definition of the Itô integral for simple processes under each $\mathbb{P}^k$:

$$\begin{aligned}\int_0^T (\alpha\widetilde{\phi}_t + \beta\widetilde{\psi}_t)\mathrm{d}\widetilde{B}_t &= \sum_{i=0}^{n-1}(\alpha a_i + \beta b_i)(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}) \\ &= \alpha\sum_{i=0}^{n-1} a_i(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}) + \beta\sum_{i=0}^{n-1} b_i(\widetilde{B}_{t_{i+1}} - \widetilde{B}_{t_i}) \\ &= \alpha\int_0^T \widetilde{\phi}_t\mathrm{d}\widetilde{B}_t + \beta\int_0^T \widetilde{\psi}_t\mathrm{d}\widetilde{B}_t.\end{aligned} \tag{68}$$

Step 2: General processes. For each $k \in \{T, I, F\}$, choose sequences of simple processes $\{\widetilde{\phi}^{(n)}\}, \{\widetilde{\psi}^{(n)}\}$ such that:

$$\lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T |\widetilde{\phi}_t - \widetilde{\phi}_t^{(n)}|^2\mathrm{d}t\right] = 0, \quad \lim_{n\to\infty} \mathbb{E}^k\left[\int_0^T |\widetilde{\psi}_t - \widetilde{\psi}_t^{(n)}|^2\mathrm{d}t\right] = 0. \tag{69}$$

By Step 1:

$$\int_0^T (\alpha\widetilde{\phi}_t^{(n)} + \beta\widetilde{\psi}_t^{(n)})\mathrm{d}\widetilde{B}_t = \alpha\int_0^T \widetilde{\phi}_t^{(n)}\mathrm{d}\widetilde{B}_t + \beta\int_0^T \widetilde{\psi}_t^{(n)}\mathrm{d}\widetilde{B}_t. \tag{70}$$

Applying the Itô isometry under each $\mathbb{P}^k$:

$$\mathbb{E}^k\left[\left|\int_0^T (\alpha\widetilde{\phi}_t + \beta\widetilde{\psi}_t)\mathrm{d}\widetilde{B}_t - \int_0^T (\alpha\widetilde{\phi}_t^{(n)} + \beta\widetilde{\psi}_t^{(n)})\mathrm{d}\widetilde{B}_t\right|^2\right] \xrightarrow{n\to\infty} 0, \tag{71}$$

and similarly for the right-hand side. Thus, the equality holds in $L^2(\mathbb{P}^k)$ for each $k \in \{T, I, F\}$.

□

6. Extensions

6.1. *Integrators as Neutrosophic Martingales*

Definition 6.1 (Neutrosophic Martingale)**.** An $\widetilde{\mathcal{F}}_t$-adapted process $\widetilde{M}_t = (M_t^T, M_t^I, M_t^F)$ is a *neutrosophic square-integrable martingale* if:

(1) $\mathbb{E}^k\left[(M_t^k)^2\right] < \infty$ for all $t \geq 0$ and $k \in \{T, I, F\}$.
(2) $\mathbb{E}^k\left[\widetilde{M}_t \mid \widetilde{\mathcal{F}}_s\right] = \widetilde{M}_s$ for $0 \leq s \leq t$ and $k \in \{T, I, F\}$.

Theorem 6.2 (Martingale Integral Construction)**.** *Let $\widetilde{M}_t$ be a continuous neutrosophic square-integrable martingale with component-wise predictable quadratic variations*

$$\langle M^k\rangle_t = \int_0^t (\sigma_s^k)^2\mathrm{d}s, \quad k \in \{T, I, F\}. \tag{72}$$

For any $\widetilde{\phi}_s = (\phi_s^T, \phi_s^I, \phi_s^F) \in \mathcal{V}^2([0, T])$ where each $\phi^k \in L^2(\mathrm{d}\mathbb{P}_k \otimes \mathrm{d}\langle M^k\rangle_s)$, the stochastic integral

$$\int_0^T \widetilde{\phi}_s \odot \mathrm{d}\widetilde{M}_s := \left(\int_0^T \phi_s^T\mathrm{d}M_s^T, \int_0^T \phi_s^I\mathrm{d}M_s^I, \int_0^T \phi_s^F\mathrm{d}M_s^F\right), \tag{73}$$

exists and satisfies the neutrosophic Itô isometry component-wise:

$$\mathbb{E}^k\left[\left(\int_0^T \phi_s^k \mathrm{d}M_s^k\right)^2\right] = \mathbb{E}^k\left[\int_0^T (\phi_s^k)^2 \mathrm{d}\langle M^k\rangle_s\right], \quad k \in \{T, I, F\}. \tag{74}$$

Proof. We proceed in three steps for each component $k \in \{T, I, F\}$.

Step 1: Simple processes. For $\phi_s^k = \sum_{i=0}^{n-1} a_i^k \mathbf{1}_{(t_i,t_{i+1}]}(s)$ with a_i^k bounded $\mathcal{F}_{t_i}$-measurable:

$$\int_0^T \phi_s^k \mathrm{d}M_s^k := \sum_{i=0}^{n-1} a_i^k (M_{t_{i+1}}^k - M_{t_i}^k). \tag{75}$$

Cross-terms vanish by the martingale property, and

$$\mathbb{E}^k\left[(M_{t_{i+1}}^k - M_{t_i}^k)^2 \mid \mathcal{F}_{t_i}^k\right] = \langle M^k\rangle_{t_{i+1}} - \langle M^k\rangle_{t_i}. \tag{76}$$

Summation yields the isometry for simple processes.

Step 2: Bounded integrands. For bounded ϕ^k, choose simple $\phi^{k,(n)}$ with

$$\mathbb{E}^k\left[\int_0^T \left(\phi_s^k - \phi_s^{k,(n)}\right)^2 \mathrm{d}\langle M^k\rangle_s\right] \to 0. \tag{77}$$

Integrals form an $L^2(\mathbb{P}^k)$-Cauchy sequence; extend by continuity.

Step 3: General integrands. Truncate via $\phi_s^{k,(n)} := \phi_s^k \mathbf{1}_{\{|\phi_s^k| \leq n\}}$. Apply Step 2 and pass to the limit. □

6.2. *Multi-Dimensional Extension*

Definition 6.3 (d-Dimensional Neutrosophic Brownian Motion)**.** A *d-dimensional neutrosophic Brownian motion* is a vector

$$\widetilde{B}_t = \left(\widetilde{B}_t^{(1)}, \ldots, \widetilde{B}_t^{(d)}\right), \tag{78}$$

where each $\widetilde{B}_t^{(j)} = (B_t^{T,(j)}, B_t^{I,(j)}, B_t^{F,(j)})$ is a one-dimensional neutrosophic Brownian motion with component-wise variations:

$$\langle B_\cdot^{k_1}, B_\cdot^{k_2}\rangle_t = \int_0^t \rho_{ij}^{k_1 k_2} \mathrm{d}s, \quad k_1, k_2 \in \{T, I, F\}. \tag{79}$$

The cross-variation between dimensions i, j is characterized by a neutrosophic correlation matrix $\widetilde{\rho} = [\widetilde{\rho}_{ij}]$ where $\widetilde{\rho}_{ij} = (\rho_{ij}^T, \rho_{ij}^I, \rho_{ij}^F) \in \widetilde{\mathbb{N}}$.

Definition 6.4 (Matrix-Valued Neutrosophic Integrand)**.** Let $\widetilde{\Phi}_t = [\widetilde{\phi}_{ij}(t)]_{m\times d}$ where $\widetilde{\phi}_{ij}(t) = (\phi_{ij}^T(t), \phi_{ij}^I(t), \phi_{ij}^F(t))$. The process is *admissible* if:

(1) Each $\phi_{ij}^k \in L^2(\mathrm{d}\mathbb{P}^k \otimes \mathrm{d}t)$ for $k \in \{T, I, F\}$.

(2) The integrability holds component-wise:

$$\mathbb{E}^k\left[\int_0^T \sum_{i=1}^m \sum_{j=1}^d |\phi_{ij}^k(t)|^2 \mathrm{d}t\right] < \infty, \quad k \in \{T, I, F\}. \tag{80}$$

Definition 6.5 (Multi-Dimensional Neutrosophic Itô Integral)**.** For admissible $\widetilde{\Phi}_t$ and $\widetilde{B}_t$, the integral is defined component-wise as

$$\left(\int_0^T \widetilde{\Phi}_t \odot \mathrm{d}\widetilde{B}_t\right)_i := \left(\sum_{j=1}^d \int_0^T \phi_{ij}^T \mathrm{d}B_t^{T,(j)}, \sum_{j=1}^d \int_0^T \phi_{ij}^I \mathrm{d}B_t^{I,(j)}, \sum_{j=1}^d \int_0^T \phi_{ij}^F \mathrm{d}B_t^{F,(j)}\right), \tag{81}$$

for $i = 1, \ldots, m$.

6.3. *Localization*

Definition 6.6 (Local Neutrosophic Martingale)**.** An $\widetilde{\mathcal{F}}_t$-adapted process $\widetilde{M}_t = (M_t^T, M_t^I, M_t^F)$ is a *local neutrosophic martingale* if there exist stopping times $\tau_n \uparrow \infty$ such that for each n and $k \in \{T, I, F\}$:

(1) $\mathbb{E}^k[|M_{t\wedge\tau_n}^k|] < \infty$.
(2) $\mathbb{E}^k[M_{t\wedge\tau_n}^k \mid \mathcal{F}_s^k] = M_{s\wedge\tau_n}^k$ for $0 \leq s \leq t$.

Definition 6.7 (Locally Square-Integrable Integrands)**.** $\widetilde{\phi}_t = (\phi_t^T, \phi_t^I, \phi_t^F) \in \widetilde{\mathcal{V}}_{\text{loc}}^2$ if there exist stopping times $\tau_n \uparrow \infty$ such that for each $k \in \{T, I, F\}$:

$$\mathbb{E}^k\left[\int_0^{T\wedge\tau_n} |\phi_t^k|^2 \mathrm{d}t\right] < \infty. \tag{82}$$

Theorem 6.8 (Localized Neutrosophic Stochastic Integral)**.** *Let $\widetilde{M}_t$ be a continuous local neutrosophic martingale and $\widetilde{\phi} \in \widetilde{\mathcal{V}}_{\text{loc}}^2$. Then there exists a unique continuous local neutrosophic martingale*

$$\int_0^t \widetilde{\phi}_s \odot \mathrm{d}\widetilde{M}_s := \left(\int_0^t \phi_s^T \mathrm{d}M_s^T, \int_0^t \phi_s^I \mathrm{d}M_s^I, \int_0^t \phi_s^F \mathrm{d}M_s^F\right), \tag{83}$$

such that for any localizing sequence $\{\tau_n\}$:

$$\left(\int_0^t \widetilde{\phi}_s \odot \mathrm{d}\widetilde{M}_s\right)_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} \widetilde{\phi}_s \odot \mathrm{d}\widetilde{M}_s^{\tau_n}. \tag{84}$$

Proof. For each $k \in \{T, I, F\}$, apply the classical localization procedure to the pairs (ϕ_t^k, M_t^k) using stopping times τ_n^k that localize both. Define $\tau_n = \tau_n^T \wedge \tau_n^I \wedge \tau_n^F \uparrow \infty$. The integral for each component is constructed as:

$$\int_0^t \phi_s^k \mathrm{d}M_s^k := \lim_{n\to\infty} \int_0^{t\wedge\tau_n} \phi_s^k \mathbf{1}_{\{s\leq\tau_n\}} \mathrm{d}M_s^k, \tag{85}$$

where the right-hand side integrals are well-defined by the global case. The triple satisfies the required properties component-wise. □

7. Neutrosophic Itô Formula

This structural decomposition embodies the neutrosophic framework established by Smarandache, wherein uncertainty is formally characterized through the explicit and independent modeling of three constituent dimensions: truth, indeterminacy, and falsity. Algebraic operations on neutrosophic numbers are executed componentwise. Within stochastic analysis, each component may undergo distinct dynamical evolution while rigorously preserving the integrated uncertainty structure.

Theorem 7.1 (Itô Formula for Neutrosophic Brownian Motion)**.** *Let* $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ *be* C^1 *in* t *and* C^2 *in* $\widetilde{x} = (x^T, x^I, x^F)$*. Consider a* canonical neutrosophic Brownian motion

$$\widetilde{B}_t = (B_t^T, B_t^I, B_t^F),$$

with quadratic variations and covariations:

$$\mathrm{d}\langle B^k, B^l\rangle_t = \rho^{kl}\mathrm{d}t, \quad k,l \in \{T,I,F\}.$$

Then, for the process $f(\widetilde{B}_t, t)$*:*

$$f(\widetilde{B}_t,t) = f(\widetilde{B}_0,0) + \int_0^t \frac{\partial f}{\partial s}\mathrm{d}s + \sum_{k\in\{T,I,F\}} \int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}B_s^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle B^k, B^l\rangle_s, \tag{86}$$

where all integrals are defined under $\mathbb{P}^k$*.*

Proof. Fix $k \in \{T,I,F\}$. Under the measure $\mathbb{P}^k$, the canonical neutrosophic Brownian motion $\widetilde{B}_t = (B_t^T, B_t^I, B_t^F)$ is a vector of correlated Brownian motions with quadratic variations and covariations given by:

$$\mathrm{d}\langle B^k, B^l\rangle_t = \rho^{kl}\mathrm{d}t, \quad \text{where} \quad \rho^{kl} \in \mathbb{R}. \tag{87}$$

Specifically, $\rho^{kk} = 1$ for each k, and $\rho^{kl} = \rho^{lk}$ for $k \neq l$.

Since f is C^1 in t and C^2 in $\widetilde{x}$, we apply the multi-dimensional Itô formula to the vector process $\widetilde{B}_t$ under $\mathbb{P}^k$. Consider the Taylor expansion of f:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{k\in\{T,I,F\}} \frac{\partial f}{\partial x^k}\mathrm{d}B_t^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}B_t^k\mathrm{d}B_t^l + R, \tag{88}$$

where R contains higher-order terms. By the properties of Brownian motion:

- $\mathrm{d}B_t^k\mathrm{d}B_t^l = \mathrm{d}\langle B^k, B^l\rangle_t + o(\mathrm{d}t) = \rho^{kl}\mathrm{d}t + o(\mathrm{d}t)$
- Terms of order $(\mathrm{d}t)^2$, $\mathrm{d}t \cdot \mathrm{d}B_t^k$, and higher vanish in the limit

Thus, we obtain:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{k\in\{T,I,F\}} \frac{\partial f}{\partial x^k}\mathrm{d}B_t^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle B^k, B^l\rangle_t. \tag{89}$$

Integrating (100) from 0 to t yields:

$$\int_0^t \mathrm{d}f = f(\widetilde{B}_t, t) - f(\widetilde{B}_0, 0) = \int_0^t \frac{\partial f}{\partial s}\mathrm{d}s + \sum_{k\in\{T,I,F\}} \int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}B_s^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle B^k, B^l\rangle_s. \tag{90}$$

Rearranging terms gives the stated formula.

The integrals are well-defined under $\mathbb{P}^k$ because:

(1) The partial derivatives $\frac{\partial f}{\partial x^k}$ are continuous (hence bounded on compacts) and adapted, so the Itô integrals $\int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}B_s^k$ exist as continuous local martingales under $\mathbb{P}^k$.

(2) The Lebesgue-Stieltjes integrals $\int_0^t \frac{\partial f}{\partial s}\mathrm{d}s$ and $\int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle B^k, B^l\rangle_s$ are well-defined because:
   - $\frac{\partial f}{\partial s}$ and $\frac{\partial^2 f}{\partial x^k \partial x^l}$ are continuous (hence bounded on $[0,T]\times K$ for compact $K \subset \mathbb{R}^3$)
   - $\langle B^k, B^l\rangle_t = \rho^{kl} t$ has finite variation

Therefore, the equality holds $\mathbb{P}^k$-almost surely for each $k \in \{T, I, F\}$. □

Definition 7.2 (Neutrosophic Itô Process)**.** A *neutrosophic Itô process* $\widetilde{X}_t = (X_t^T, X_t^I, X_t^F)$ satisfies component-wise SDEs:

$$\mathrm{d}\widetilde{X}_t = \mathrm{d}X_t^k = \mu_t^k \mathrm{d}t + \sum_{k,l\in\{T,I,F\}} \sigma_t^{kl}\mathrm{d}B_t^k, \tag{91}$$

where:

- $\mu_t^k = (\mu_t^T, \mu_t^I, \mu_t^F)$ is adapted with $\int_0^T |\mu_t^k|\mathrm{d}t < \infty$ $\mathbb{P}$-a.s.
- $\sigma_t^{kl} = [\sigma_t^{kl}]_{k,l\in\{T,I,F\}}$ is a 3×3 adapted matrix with $\int_0^T \sum_l |\sigma_t^{kl}|^2 \mathrm{d}t < \infty$ $\mathbb{P}$-a.s.
- $B_t^k = (B_t^T, B_t^I, B_t^F)$ is a canonical neutrosophic Brownian motion.

Theorem 7.3 (Itô Formula for Neutrosophic Itô Processes)**.** *Let $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ be C^1 in t and C^2 in $\widetilde{x}$, and $\widetilde{X}_t = (X_t^T, X_t^I, X_t^F)$ a neutrosophic Itô process with dynamics:*

$$\mathrm{d}X_t^k = \mu_t^k \mathrm{d}t + \sum_{p\in\{T,I,F\}} \sigma_t^{kp}\mathrm{d}B_t^p, \quad k \in \{T, I, F\}, \tag{92}$$

where $\widetilde{B}_t = (B_t^T, B_t^I, B_t^F)$ is a canonical neutrosophic Brownian motion with $\mathrm{d}\langle B^p, B^q\rangle_t = \rho^{pq}\mathrm{d}t$, and σ_t^{kp} are predictable with $\int_0^t (\sigma_s^{kp})^2 \mathrm{d}s < \infty$ a.s. Then:

$$\begin{aligned} f(\widetilde{X}_t, t) = f(\widetilde{X}_0, 0) &+ \int_0^t \frac{\partial f}{\partial s}\mathrm{d}s + \sum_{k\in\{T,I,F\}} \int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}X_s^k \\ &+ \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle X^k, X^l\rangle_s, \end{aligned} \tag{93}$$

where the cross-variation is:

$$\mathrm{d}\langle X^k, X^l\rangle_t = \left(\sum_{p,q\in\{T,I,F\}} \sigma_t^{kp}\sigma_t^{lq}\rho^{pq}\right)\mathrm{d}t, \tag{94}$$

and all integrals are defined under a common neutrosophic probability measure $\mathbb{P}^k$.

Proof. Fix $k \in \{T, I, F\}$ and work under $\mathbb{P}^k$. The neutrosophic Itô process $\widetilde{X}_t = (X_t^T, X_t^I, X_t^F)$ has component-wise dynamics:

$$\mathrm{d}X_t^k = \mu_t^k \mathrm{d}t + \mathrm{d}M_t^k, \quad \mathrm{d}M_t^k := \sum_{p\in\{T,I,F\}} \sigma_t^{kp} \mathrm{d}B_t^p. \tag{95}$$

The martingale parts M_t^k have cross-variation:

$$\mathrm{d}\langle M^k, M^l\rangle_t = \sum_{p,q} \sigma_t^{kp}\sigma_t^{lq} \mathrm{d}\langle B^p, B^q\rangle_t = \left(\sum_{p,q} \sigma_t^{kp}\sigma_t^{lq}\rho^{pq}\right) \mathrm{d}t, \tag{96}$$

since $\mathrm{d}\langle B^p, B^q\rangle_t = \rho^{pq}\mathrm{d}t$. As the finite-variation terms $\mu_t^k\mathrm{d}t$ do not contribute to quadratic variation:

$$\mathrm{d}\langle X^k, X^l\rangle_t = \mathrm{d}\langle M^k, M^l\rangle_t = \left(\sum_{p,q} \sigma_t^{kp}\sigma_t^{lq}\rho^{pq}\right) \mathrm{d}t. \tag{97}$$

Apply the multi-dimensional Itô formula to $f(\widetilde{X}_t, t)$. For $\widetilde{X}_t = X_t^k = (X_t^T, X_t^I, X_t^F)$, Taylor's expansion gives:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{k\in\{T,I,F\}} \frac{\partial f}{\partial x^k}\mathrm{d}X_t^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}X_t^k \mathrm{d}X_t^l + R, \tag{98}$$

where R contains higher-order terms. By Itô's lemma rules:

$$\begin{aligned} \mathrm{d}X_t^k \mathrm{d}X_t^l &= \mathrm{d}\langle X^k, X^l\rangle_t + o(\mathrm{d}t), \\ \mathrm{d}X_t^k \mathrm{d}t &= o(\mathrm{d}t), \quad (\mathrm{d}t)^2 = o(\mathrm{d}t). \end{aligned} \tag{99}$$

Substituting the cross-variation yields:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{k\in\{T,I,F\}} \frac{\partial f}{\partial x^k}\mathrm{d}X_t^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle X^k, X^l\rangle_t. \tag{100}$$

Integrate (100) from 0 to t:

$$f(\widetilde{X}_t, t) - f(\widetilde{X}_0, 0) = \int_0^t \frac{\partial f}{\partial s}\mathrm{d}s + \sum_{k\in\{T,I,F\}} \int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}X_s^k + \frac{1}{2}\sum_{k,l\in\{T,I,F\}} \int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l}\mathrm{d}\langle X^k, X^l\rangle_s. \tag{101}$$

The integrals are well-defined under $\mathbb{P}^k$ because:

(1) **Itô integrals**: For each m, $\int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}X_s^k$ exists since:

$$\int_0^t \frac{\partial f}{\partial x^k}\mathrm{d}M_s^k = \sum_p \int_0^t \frac{\partial f}{\partial x^k}\sigma_s^{kp}\mathrm{d}B_s^p, \tag{102}$$

is a sum of Itô integrals with $\mathbb{E}^k\left[\int_0^t \left|\frac{\partial f}{\partial x^k}\sigma_s^{kp}\right|^2 \mathrm{d}s\right] < \infty$ by the C^2 condition and adaptedness.

(2) **Lebesgue integrals**: The terms $\int_0^t \frac{\partial f}{\partial s} \mathrm{d}s$ and $\int_0^t \frac{\partial^2 f}{\partial x^k \partial x^l} \mathrm{d}\langle X^k, X^l\rangle_s$ are Riemann-Stieltjes integrals with:

$$\int_0^t g_s \mathrm{d}\langle X^k, X^l\rangle_s = \int_0^t g_s \left(\sum_{p,q\in\{T,I,F\}} \sigma_s^{kp} \sigma_s^{lq} \rho^{pq} \right) \mathrm{d}s, \tag{103}$$

which converge absolutely as g_s is continuous and the volatility is integrable.

Thus, the equality holds $\mathbb{P}_k$-almost surely for each $k \in \{T, I, F\}$. □

8. Conclusions

This research has established the theoretical foundations of stochastic calculus within the neutrosophic paradigm, creating a rigorous mathematical framework for systems governed by three-valued uncertainty. Our work introduces essential constructs including **Canonical Neutrosophic Brownian Motion**, the space $\widetilde{\mathcal{V}}^2[0, T]$, and **Neutrosophic Itô Processes**, while demonstrating the consistent extension of classical stochastic integration to this novel setting. Crucially, we have proven that fundamental properties—isometry preservation, martingale characterization with path regularity, effective localization techniques, and the validity of Itô's formula—remain intact when operating under truth-indeterminacy-falsehood dynamics. These theoretical advances provide a powerful toolkit for modeling complex stochastic systems where traditional probability measures cannot adequately capture pervasive indeterminacy.

Looking forward, this foundation enables multiple research trajectories that naturally extend our work. Immediate extensions will focus on developing neutrosophic stochastic differential equations with applications to financial modeling under epistemic uncertainty. Subsequent efforts should address neutrosophic jump processes incorporating indeterminate amplitudes and arrival intensities, while connections to rough path theory may yield relaxed regularity conditions. Important theoretical challenges include developing dependence structures for multivariate neutrosophic processes with interacting truth-indeterminacy relationships. Practical implementation will require creating computational methods for simulating neutrosophic processes and approximating integrals. Finally, exploring connections between neutrosophic uncertainty and quantum superposition states may open new interdisciplinary research avenues at the physics-mathematics interface. Collectively, these directions promise to advance neutrosophic stochastic calculus from theoretical foundation to applicable methodology while preserving the essential mathematical structures established herein.

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