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An efficient computational technique for differential systems under epistemic uncertainty

Augus Kurian[1], and I R Sumathi [2,*]
[1,2,*]Department of Mathematics, Amrita School of Physical Sciences, Coimbatore,
Amrita Vishwa Vidyapeetham, India; [1] auguskurian@gmail.com
*Correspondence: ir_sumathi@cb.amrita.edu.

Abstract. This paper presents an innovative framework for addressing the solutions of non-homogeneous linear systems of differential equations by employing neutrosophic forcing functions and neutrosophic numbers as initial conditions. In our study, the forcing function is treated as a neutrosophic set of real-valued functions, which is assumed to be hexagonal. This technique differs from existing approaches by representing the solution as a neutrosophic set of real vector functions (a neutrosophic bunch), rather than as a vector of neutrosophic functions. Each component of this solution set satisfies the system to varying degrees of truth, indeterminacy, and falsity, capturing the multi-dimensional uncertainty inherent in complex systems. The solution will be in the form of a neutrosophic set, with (α, β, γ)-cuts representing parallelepipeds that describe the uncertainty boundaries. An important contribution of this study is the comparative analysis with the differential inclusion-based method. Unlike differential inclusions, which often overestimate uncertainties and can lose essential qualitative properties such as periodicity and boundedness, the proposed approach preserves these characteristics and produces narrower, more precise solution sets. The method avoids the divergence of uncertainty over time by fixing admissible functions across the entire interval. This comparative insight shows the effectiveness of the proposed approach in maintaining structural integrity in dynamic systems with inconsistent or incomplete data. A numerical example illustrates its practicality and robustness.

Keywords: Hexagonal Neutrosophic Number, Hexagonal Neutrosophic Bunch, Non-homogeneous system of differential equations, Differential Inclusion.

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1. Introduction

Differential equations are fundamental tools in mathematical modeling that are used in various scientific fields to describe dynamic systems. Considering the complexity and imprecision present in real-world phenomena, it has become essential to include uncertainty in these models. Fuzzy logic [1] was put forward to address uncertainty in real-world problems.

A comprehensive structure for simulating systems with uncertainty in parameters, initial conditions, or forcing function is offered by fuzzy differential equations (FDEs). By tackling solutions to FDEs with fuzzy initial conditions, Ahmad et al. [2] and Buckley et al. [3] established a solid foundation. Shen [4] expanded on this paradigm by taking into account fuzzy numbers that are linearly correlated. Dehghan et al. [5], Mosleh and Otadi [6], and Shams et al. [7] have developed computational techniques. Of these, Najariyan et al. [8] have focused on systems of first-order equations. Aspects of control and stability been investigated in Nguyen and Kreinovich [10] and Abbasi and Jalali [9]. Gasilov and co-authors made a significant contribution by proposing geometric solution techniques [12], methods for non-homogeneous problems [11], and methods for boundary value problems with fuzzy conditions [13–15]. However, Gasilov et al. [16] significantly advanced the field by formulating a non-homogeneous linear system of interval differential equations, where both the initial conditions and the forcing terms are represented as intervals. This formulation facilitated further progress in the study of uncertainty in differential equations. In parallel, various methodologies have been proposed to handle fuzzy forcing functions combined with fuzzy initial conditions. Mondal et al. [17] introduced a geometric approach for solving linear differential equations with real coefficients and fuzzy initial conditions. Their method yields a fuzzy set of vector-valued functions that satisfy the fuzzy linear differential system and geometrically form an n-dimensional parallelepiped.

Intuitionistic fuzzy set theory, introduced by Atanassov [18], is a generalization of classical fuzzy set theory. It enhances the representation of uncertainty by incorporating both the degrees of membership and non-membership, thereby capturing hesitation more effectively. This framework has inspired numerous studies on intuitionistic fuzzy differential equations and their diverse applications. As a further extension of fuzzy and intuitionistic fuzzy logic, neutrosophic logic [19, 20] provides a comprehensive structure for modeling indeterminacy, uncertainty, and inconsistency. Analytical and numerical techniques for solving neutrosophic differential equations have been explored in detail in [21, 22].

The neutrosophic differential equation encompasses the neutrosophic number [23–26], that includes the truth, indeterministic, and falsity functions. Applications of differential equations in the neutrosophic environment have been discussed in [27–31]. Motivated by various aspects of neutrosophic numbers in differential equations, we have developed the solution of system of differential equation in neutrosophic environment.

Motivation:

Many real-world problems involve uncertainty or imprecision, which cannot be represented using classical Differential equations. Neutrosophic differential equations provide a powerful framework to model such uncertainty in dynamic systems. Only a limited number of research articles have addressed the System of Neutrosophic differential equations [32]. This lack of extensive literature motivated the present study, which aims to explore and develop effective solution methods for such equations. Existing approaches to neutrosophic differential equations, such as those based on Hukuhara derivatives and generalized differentiability, face significant limitations. They often require restrictive assumptions, with ambiguity in uncertainty propagation, and may not guarantee unique or stable solutions. Moreover, differential inclusion-based methods, although widely used, tend to overestimate uncertainty, leading to excessively conservative solution sets and a loss of qualitative system properties such as periodicity and stability.

The main objective is to use a geometric technique to solve a neutrosophic linear system of differential equations with crisp real coefficients and an initial condition represented by a vector of hexagonal neutrosophic numbers. We interpret a vector of hexagonal neutrosophic numbers as a rectangular prism in n-dimensional space, resulting in an n-dimensional parallelepiped as a neutrosophic set of real vector functions. Each member of the solution set provides the system with a certain possibility of truth, indeterminacy, and falsity membership. That is a system of differential equations that involves a bunch of hexagonal Neutrosophic functions, and their solution is a neutrosophic vector function, i.e., a function that returns a vector of hexagonal neutrosophic functions.

In addition, we have incorporated a detailed comparison of the proposed solution method with the differential inclusion [33] approach. This comparison highlights the advantages of our method in preserving important qualitative properties of the system, such as periodicity, boundedness, and the realistic representation of uncertainties over time. While the differential inclusion method allows the solution set to expand as time progresses, often leading to divergence and loss of structural features, our method maintains consistency by fixing admissible functions throughout the interval. As a result, the proposed approach captures the inherent dynamics of the system more accurately and avoids the magnification of uncertainty, which is a common limitation of differential inclusion techniques.

Contributions:

The core contributions of this paper are summarized as follows:

- We present a solution framework for non-homogeneous linear systems of neutrosophic differential equations using the hexagonal neutrosophic bunch structure.
- The method provides a compact representation of uncertainty, maintaining tighter bounds in the solution space and avoiding the overestimation of uncertainties.

- A detailed comparative study with the differential inclusion approach demonstrates the efficiency of the proposed method in preserving qualitative system properties.
- Numerical examples are provided to illustrate the applicability and effectiveness of the framework in solving problems with complex and dynamic uncertainty.

The subsequent sections of this paper are structured as follows: In Section 2, the preliminaries relevant to our study are discussed. Section 3 delineates the mathematical formulation of the proposed model and defines neutrosophic functions which highlights significant contributions. Section 4 presents the systematic approach for solving a non-homogeneous system of differential equations. Section 5 illustrates our methodology with a numerical example. Finally, Section 6 provides a comparison between the proposed solution method and the differential inclusion-based approach followed by the conclusion 7 and the future work.

2. Preliminary Concepts

Definition 2.1. [23] The generalized asymmetric non-linear hexagonal neutrosophic number (GANHNN) is expressed as

$$\eta_{\text{GANHNN}} = \left\{ \begin{array}{l} T\left(d_1, d_2, d_3, d_4, d_5, d_6; \theta, \kappa; \zeta\right)_{(u_1,u_2,u_3,u_4)}, \\ I\left(e_1, e_2, e_3, e_4, e_5, e_6; \theta_1, \kappa_1; \sigma\right)_{(v_1,v_2,v_3,v_4)}, \\ F\left(f_1, f_2, f_3, f_4, f_5, f_6; \theta_2, \kappa_2; \psi\right)_{(w_1,w_2,w_3,w_4)} \end{array} \right\}.$$

where the corresponding membership function is,

$$T_{\eta_{\text{GANHNN}}}(x) = \begin{cases} \theta\left(\dfrac{x-d_1}{d_2-d_1}\right)^{u_1} & \text{, if } d_1 \le x < d_2 \\ \theta + (\zeta-\theta)\left(\dfrac{x-d_2}{d_3-d_2}\right)^{u_2} & \text{, if } d_2 \le x < d_3 \\ \zeta & \text{, if } d_3 \le x < d_4 \\ \kappa + (\zeta-\kappa)\left(\dfrac{x-d_5}{d_4-d_5}\right)^{u_3} & \text{, if } d_4 \le x < d_5 \\ \kappa\left(\dfrac{x-d_6}{d_5-d_6}\right)^{u_4} & \text{, if } d_5 \le x \le d_6 \\ 0 & \text{, otherwise.} \end{cases}$$

the indeterminacy function,

$$I_{\eta_{\text{GANHNN}}}(x)=\begin{cases}1-\theta_1\left(\dfrac{x-e_1}{e_2-e_1}\right)^{v_1} & \text{, if } e_1\le x<e_2\\ 1-\theta_1+(\theta_1-\sigma)\left(\dfrac{x-e_2}{e_3-e_2}\right)^{v_2} & \text{, if } e_2\le x<e_3\\ 1-\sigma & \text{, if } e_3\le x<e_4\\ 1-\kappa_1+(\kappa_1-\sigma)\left(\dfrac{x-e_5}{e_4-e_5}\right)^{v_3} & \text{, if } e_4\le x<e_5\\ 1-\kappa_1\left(\dfrac{x-e_6}{e_5-e_6}\right)^{v_4} & \text{, if } e_5\le x\le e_6\\ 1 & \text{, otherwise.}\end{cases}$$

and the non-membership function,

$$F_{\eta_{\text{GANHNN}}}(x)=\begin{cases}1-\theta_2\left(\dfrac{x-f_1}{f_2-f_1}\right)^{w_1} & \text{, if } f_1\le x<f_2\\ 1-\theta_2+(\theta_2-\psi)\left(\dfrac{x-f_2}{f_3-f_2}\right)^{w_2} & \text{, if } f_2\le x<f_3\\ 1-\psi & \text{, if } f_3\le x<f_4\\ 1-\kappa_2+(\kappa_2-\psi)\left(\dfrac{x-f_5}{f_4-f_5}\right)^{w_3} & \text{, if } f_4\le x<f_5\\ 1-\kappa_2\left(\dfrac{x-f_6}{f_5-f_6}\right)^{w_4} & \text{, if } f_5\le x\le f_6\\ 1 & \text{, otherwise.}\end{cases}$$

where $d_1<d_2<d_3<d_4<d_5<d_6$, $e_1<e_2<e_3<e_4<e_5<e_6$, and $f_1<f_2<f_3<f_4<f_5<f_6$, and all d_i, e_i, f_i $(i=1,\ldots,6)$ are real constants, and $0<\theta,\kappa<\zeta$, $1-\sigma<\theta_1,\kappa_1<1$, $1-\psi<\theta_2,\kappa_2<1$, with $\zeta,\sigma,\psi\in[0,1]$.

Definition 2.2. [23] The (α,β,γ)-cut of GANHNN is defined as,

$$\eta_{(\alpha,\beta,\gamma)}=\left\{x\in X\,|\,T_{\eta_{\text{GANHNN}}}(x)\ge\alpha,\ I_{\eta_{\text{GANHNN}}}(x)\le\beta,\ F_{\eta_{\text{GANHNN}}}(x)\le\gamma\right\}.$$

α-Cut (T_α)

Case 1: $\theta_0\le\kappa_0$.

$$T_\alpha=\begin{cases}\left[d_1+\left(\dfrac{\alpha}{\theta_0}\right)^{\frac{1}{u_1}}(d_2-d_1),\ d_6+\left(\dfrac{\alpha}{\kappa_0}\right)^{\frac{1}{u_4}}(d_5-d_6)\right] & ,0\le\alpha<\theta_0\\ \left[d_2+\left(\dfrac{\alpha-\theta_0}{\zeta-\theta_0}\right)^{\frac{1}{u_2}}(d_3-d_2),\ d_6+\left(\dfrac{\alpha}{\kappa_0}\right)^{\frac{1}{u_4}}(d_5-d_6)\right] & ,\theta_0\le\alpha<\kappa_0\\ \left[d_2+\left(\dfrac{\alpha-\theta_0}{\zeta-\theta_0}\right)^{\frac{1}{u_2}}(d_3-d_2),\ d_5+\left(\dfrac{\alpha-\kappa_0}{\zeta-\kappa_0}\right)^{\frac{1}{u_3}}(d_4-d_5)\right] & ,\kappa_0\le\alpha<\zeta\\ [d_3,d_4] & ,\alpha=\zeta.\end{cases}$$

Case 2: $\kappa_0 \leq \theta_0$.

$$T_\alpha = \begin{cases} \left[d_1 + \left(\dfrac{\alpha}{\theta_0} \right)^{\frac{1}{u_1}} (d_2 - d_1),\, d_6 + \left(\dfrac{\alpha}{\kappa_0} \right)^{\frac{1}{u_4}} (d_5 - d_6) \right] & , 0 \leq \alpha < \kappa_0 \\ \left[d_1 + \left(\dfrac{\alpha}{\theta_0} \right)^{\frac{1}{u_1}} (d_2 - d_1),\, d_5 + \left(\dfrac{\alpha - \kappa_0}{\zeta - \kappa_0} \right)^{\frac{1}{u_3}} (d_4 - d_5) \right] & , \kappa_0 \leq \alpha < \theta_0 \\ \left[d_2 + \left(\dfrac{\alpha - \theta_0}{\zeta - \theta_0} \right)^{\frac{1}{u_2}} (d_3 - d_2),\, d_5 + \left(\dfrac{\alpha - \kappa_0}{\zeta - \kappa_0} \right)^{\frac{1}{u_3}} (d_4 - d_5) \right] & , \theta_0 \leq \alpha < \zeta \\ [d_3, d_4] & , \alpha = \zeta. \end{cases}$$

β-Cut (I_β)

Case 1: $\theta_1 \leq \kappa_1$.

$$I_\beta = \begin{cases} [e_3, e_4] & , \beta = 1 - \sigma \\ \left[e_2 + \left(\dfrac{1 - \beta - \theta_1}{\sigma - \theta_1} \right)^{\frac{1}{v_2}} (e_3 - e_2),\, e_5 + \left(\dfrac{1 - \beta - \kappa_1}{\sigma - \kappa_1} \right)^{\frac{1}{v_3}} (e_4 - e_5) \right] & , 1 - \sigma < \beta \leq 1 - \kappa_1 \\ \left[e_2 + \left(\dfrac{1 - \beta - \theta_1}{\sigma - \theta_1} \right)^{\frac{1}{v_2}} (e_3 - e_2),\, e_6 + \left(\dfrac{1 - \beta}{\kappa_1} \right)^{\frac{1}{v_4}} (e_5 - e_6) \right] & , 1 - \kappa_1 < \beta \leq 1 - \theta_1 \\ \left[e_1 + \left(\dfrac{1 - \beta}{\theta_1} \right)^{\frac{1}{v_1}} (e_2 - e_1),\, e_6 + \left(\dfrac{1 - \beta}{\kappa_1} \right)^{\frac{1}{v_4}} (e_5 - e_6) \right] & , 1 - \theta_1 < \beta \leq 1. \end{cases}$$

Case 2: $\kappa_1 \leq \theta_1$.

$$I_\beta = \begin{cases} [e_3, e_4] & , \beta = 1 - \sigma \\ \left[e_2 + \left(\dfrac{1 - \beta - \theta_1}{\sigma - \theta_1} \right)^{\frac{1}{v_2}} (e_3 - e_2),\, e_5 + \left(\dfrac{1 - \beta - \kappa_1}{\sigma - \kappa_1} \right)^{\frac{1}{v_3}} (e_4 - e_5) \right] & , 1 - \sigma < \beta \leq 1 - \theta_1 \\ \left[e_1 + \left(\dfrac{1 - \beta}{\theta_1} \right)^{\frac{1}{v_1}} (e_2 - e_1),\, e_5 + \left(\dfrac{1 - \beta - \kappa_1}{\sigma - \kappa_1} \right)^{\frac{1}{v_3}} (e_4 - e_5) \right] & , 1 - \theta_1 < \beta \leq 1 - \kappa_1 \\ \left[e_1 + \left(\dfrac{1 - \beta}{\theta_1} \right)^{\frac{1}{v_1}} (e_2 - e_1),\, e_6 + \left(\dfrac{1 - \beta}{\kappa_1} \right)^{\frac{1}{v_4}} (e_5 - e_6) \right] & , 1 - \kappa_1 < \beta \leq 1. \end{cases}$$

γ-Cut (F_γ)

Case 1: $\theta_2 \leq \kappa_2$.

$$F_\gamma = \begin{cases} [f_3, f_4] & , \gamma = 1 - \psi \\ \left[f_2 + \left(\dfrac{1 - \gamma - \theta_2}{\psi - \theta_2} \right)^{\frac{1}{w_2}} (f_3 - f_2),\, f_5 + \left(\dfrac{1 - \gamma - \kappa_2}{\psi - \kappa_2} \right)^{\frac{1}{w_3}} (f_4 - f_5) \right] & , 1 - \psi < \gamma \leq 1 - \kappa_2 \\ \left[f_2 + \left(\dfrac{1 - \gamma - \theta_2}{\psi - \theta_2} \right)^{\frac{1}{w_2}} (f_3 - f_2),\, f_6 + \left(\dfrac{1 - \gamma}{\kappa_2} \right)^{\frac{1}{w_4}} (f_5 - f_6) \right] & , 1 - \kappa_2 < \gamma \leq 1 - \theta_2 \\ \left[f_1 + \left(\dfrac{1 - \gamma}{\theta_2} \right)^{\frac{1}{w_1}} (f_2 - f_1),\, f_6 + \left(\dfrac{1 - \gamma}{\kappa_2} \right)^{\frac{1}{w_4}} (f_5 - f_6) \right] & , 1 - \theta_2 < \gamma \leq 1. \end{cases}$$

Case 2: $\kappa_2 \leq \theta_2$.

$$F_\gamma = \begin{cases} [f_3, f_4] & ,\gamma = 1-\psi \\ \left[f_2 + \left(\dfrac{1-\gamma-\theta_2}{\psi-\theta_2}\right)^{\frac{1}{w_2}}(f_3 - f_2),\ f_5 + \left(\dfrac{1-\gamma-\kappa_2}{\psi-\kappa_2}\right)^{\frac{1}{w_3}}(f_4 - f_5)\right] & ,1-\psi < \gamma \leq 1-\theta_2 \\ \left[f_1 + \left(\dfrac{1-\gamma}{\theta_2}\right)^{\frac{1}{w_1}}(f_2 - f_1),\ f_5 + \left(\dfrac{1-\gamma-\kappa_2}{\psi-\kappa_2}\right)^{\frac{1}{w_3}}(f_4 - f_5)\right] & ,1-\theta_2 < \gamma \leq 1-\kappa_2 \\ \left[f_1 + \left(\dfrac{1-\gamma}{\theta_2}\right)^{\frac{1}{w_1}}(f_2 - f_1),\ f_6 + \left(\dfrac{1-\gamma}{\kappa_2}\right)^{\frac{1}{w_4}}(f_5 - f_6)\right] & ,1-\kappa_2 < \gamma \leq 1. \end{cases}$$

Definition 2.3. [26] Let $B = (T_B, I_B, F_B)$ be a neutrosophic set of a classical vector space V over the field $\mathbb{F}$. For any $x, y \in V$ and $\alpha, \beta \in \mathbb{F}$, if it satisfy

$$T_B(\alpha x + \beta y) \geq \min\{T_B(x), T_B(y)\}, I_B(\alpha x + \beta y) \leq \max\{I_B(x), I_B(y)\}$$

and $F_B(\alpha x + \beta y) \leq \max\{F_B(x), F_B(y)\}$ then B is called a neutrosophic vector space V.

3. A novel approach to neutrosophic functions

In this section, we consider a neutrosophic function to be a neutrosophic set (bunch) of real functions, each of which possesses specific degrees of truth, indeterminacy, and falsity membership. If $\tilde{A}(t)$ of such a neutrosophic function $\tilde{A}$ at time t implies a neutrosophic set comprising the outputs of real-valued functions evaluated at t.

Definition 3.1. Let U denote the set of all continuous functions defined over an interval $\mathbb{I}$. Suppose that the functions $T_a(\cdot), \ldots, T_f(\cdot), \quad I_a(\cdot), \ldots, I_f(\cdot), \quad F_a(\cdot), \ldots, F_f(\cdot) \in U$. A neutrosophic subset $\tilde{A} \subseteq U$ is defined as:

$$\tilde{A} = \left\{(f, T(f), I(f), F(f)) \mid f \in U\right\},$$

where $T(f)$, $I(f)$, and $F(f)$ indicate the degrees of truth, indeterminacy, and falsity associated with the function f.

$$T_{\tilde{A}}(y(\cdot)) = \begin{cases} \alpha, & y = T_a + \left(\frac{\alpha}{\theta}\right)^{\frac{1}{u_1}}(T_b - T_a), \quad 0 < \alpha \leq \theta, \\ \alpha, & y = T_f + \left(\frac{\alpha}{\kappa}\right)^{\frac{1}{u_4}}(T_e - T_f), \quad 0 < \alpha \leq \kappa, \\ \alpha, & y = T_b + \left(\frac{\alpha-\theta}{\zeta-\theta}\right)^{\frac{1}{u_2}}(T_c - T_b), \quad \theta < \alpha \leq \zeta, \\ \alpha, & y = T_e + \left(\frac{\alpha-\kappa}{\zeta-\kappa}\right)^{\frac{1}{u_3}}(T_d - T_e), \quad \kappa < \alpha \leq \zeta, \\ 0, & \text{otherwise}, \end{cases}$$

Indeterminacy Function,

$$I_{\tilde{A}}(y(\cdot)) = \begin{cases} \beta, & y = I_b + \left(\frac{1-\beta-\theta_1}{\sigma-\theta_1}\right)^{\frac{1}{v_2}} (I_c - I_b), \quad 1-\sigma < \beta \leq 1-\theta_1, \\ \beta, & y = I_e + \left(\frac{1-\beta-\kappa_1}{\sigma-\kappa_1}\right)^{\frac{1}{v_3}} (I_d - I_e), \quad 1-\sigma < \beta \leq 1-\kappa_1, \\ \beta, & y = I_f + \left(\frac{1-\beta}{\kappa_1}\right)^{\frac{1}{v_4}} (I_e - I_f), \quad 1-\kappa_1 < \beta \leq 1, \\ \beta, & y = I_a + \left(\frac{1-\beta}{\theta_1}\right)^{\frac{1}{v_1}} (I_b - I_a), \quad 1-\theta_1 < \beta \leq 1, \\ 1, & \text{otherwise,} \end{cases}$$

and non-membership function

$$F_{\tilde{A}}(y(\cdot)) = \begin{cases} \gamma, & y = F_b + \left(\frac{1-\gamma-\theta_2}{\psi-\theta_2}\right)^{\frac{1}{w_2}} (F_c - F_b), \quad 1-\psi < \gamma \leq 1-\theta_2, \\ \gamma, & y = F_e + \left(\frac{1-\gamma-\kappa_2}{\psi-\kappa_2}\right)^{\frac{1}{w_3}} (F_d - F_e), \quad 1-\psi < \gamma \leq 1-\kappa_2, \\ \gamma, & y = F_f + \left(\frac{1-\gamma}{\kappa_2}\right)^{\frac{1}{w_4}} (F_e - F_f), \quad 1-\kappa_2 < \gamma \leq 1, \\ \gamma, & y = F_a + \left(\frac{1-\gamma}{\theta_2}\right)^{\frac{1}{w_1}} (F_b - F_a), \quad 1-\theta_2 < \gamma \leq 1, \\ 1, & \text{else,} \end{cases}$$

is known as a Hexagonal neutrosophic function and is represented by:

$$\tilde{A} = \{(T_a, T_c, T_e, T_f, T_d, T_b), (I_a, I_c, I_e, I_f, I_d, I_b), (F_a, F_c, F_e, F_f, F_d, F_b)\}.$$

Based on this definition, a hexagonal neutrosophic function represents a neutrosophic set of real functions.

We interpret a neutrosophic function as a neutrosophic collection of real functions, rather than the conventional neutrosophic-valued function. To clarify the distinction between these two types of neutrosophic functions, we provide explanations with illustrative examples.

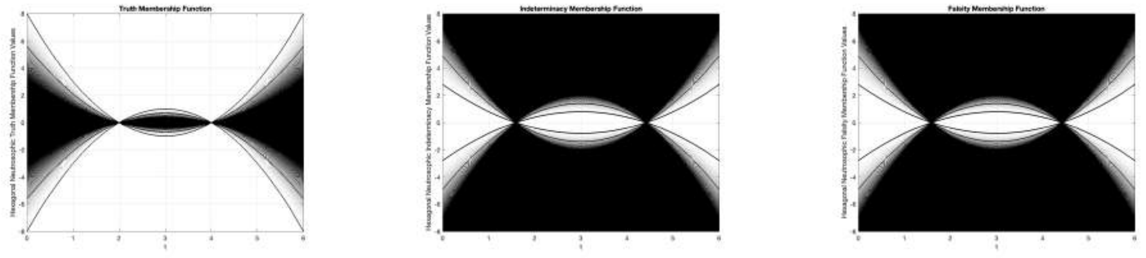
Example 3.2. Let $\tilde{A} : [0, 6] \to \mathbb{H}_{\mathbb{R}}$ be a function that maps any continous function defined over the interval $[0, 6]$ into the set of hexagonal neutrosophic numbers, denoted by $\mathbb{H}_{\mathbb{R}}$. Let the function $\tilde{A}$ is defined as follows:

$$\tilde{A}(t) = \{(T_a, T_c, T_e, T_f, T_d, T_b), (I_a, I_c, I_e, I_f, I_d, I_b), (F_a, F_c, F_e, F_f, F_d, F_b)\}$$

where,

$$T_a = -\left|t^2-6t+8\right|, T_b = \left|t^2-6t+8\right|, T_c = -\left|0.7\left(t^2-6t+8\right)\right|, T_d = \left|0.7\left(t^2-6t+8\right)\right|,$$
$$T_e = -\left|0.4\left(t^2-6t+8\right)\right|, T_f = \left|0.4\left(t^2-6t+8\right)\right|, I_a = -\left|1-(t^2-6t+8)\right|, I_b = \left|1-(t^2-6t+8)\right|,$$
$$I_c = -\left|0.7(1-\left(t^2-6t+8\right))\right|, I_d = \left|0.7(1-\left(t^2-6t+8\right))\right|, I_e = -\left|0.4(1-\left(t^2-6t+8\right))\right|,$$
$$I_f = \left|(0.4(1-\left(t^2-6t+8\right))\right| .F_a = -\left|1-(t^2-6t+8)\right|, F_b = \left|1-(t^2-6t+8)\right|,$$
$$F_c = -\left|0.7(1-\left(t^2-6t+8\right))\right|, F_d = \left|0.7(1-\left(t^2-6t+8\right))\right|, F_e = -\left|0.4(1-\left(t^2-6t+8\right))\right|,$$
$$F_f = \left|0.4(1-\left(t^2-6t+8\right))\right|.$$

At $t = 0, 0.1, 0.2, 0.3 \ldots, 5.9, 6$ (with a step size of $\Delta t = 0.1$), the values of $\tilde{A}$ are
$\langle(-8, -5.6, -3.2, 3.2, 5.6, 8), (-6.41, -4.487, -2.564, 2.564, 4.487, 6.41),$
$(-6.41, -4.487, -2.564, 2.564, 4.487, 6.41)\rangle,$
$\langle(-7.41, -5.187, -2.964, 2.964, 5.187, 7.41), (-5.84, -4.088, -2.336, 2.336, 4.088, 5.84),$
$(-5.84, -4.088, -2.336, 2.336, 4.088, 5.84)\rangle,$
$\langle(-6.84, -4.788, -2.736, 2.736, 4.788, 6.84), (-5.29, -3.703, -2.116, 2.116, 3.703, 5.29),$
$(-5.29, -3.703, -2.116, 2.116, 3.703, 5.29)\rangle, \ldots,$
$\langle(-7.41, -5.187, -2.964, 2.964, 5.187, 7.41), (-6.41, -4.487, -2.564, 2.564, 4.487, 6.41),$
$(-6.41, -4.487, -2.564, 2.564, 4.487, 6.41)\rangle,$
$\langle(-8, -5.6, -3.2, 3.2, 5.6, 8), (-7, -4.9, -2.8, 2.8, 4.9, 7), (-7, -4.9, -2.8, 2.8, 4.9, 7)\rangle$
respectively. The values of truth, indeterminacy and falsity membership grades are depicted in Figure 1. The figure employs a grayscale representation, where black corresponds to a truth membership degree of 1 in figure 1a, and white indicates a truth membership of degree 0. Then in figure 1b and 1c where black corresponds to a indeterminacy and falsity membership degree is 0, and white indicates indeterminacy and falsity membership degree is 1. As t varies continuously, the graph visualizes the behavior of the neutrosophic function in terms of truth (T), indeterminacy (I), and falsity (F) membership degrees.



(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 1. Hexagonal Neutrosophic function

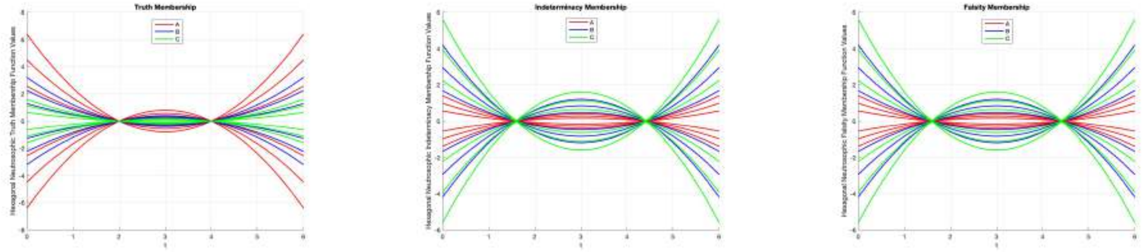
In the following example, a neutrosophic function is represented by a set of real-valued functions, each linked to specific degrees of truth-membership, indeterminacy, and falsity.

Example 3.3. Let U denote the universal set comprising all continuous functions defined on the interval $[0,6]$, i.e., $U = C[0,6]$. A neutrosophic subset $\tilde{A}$ is defined within U as follows:

$\tilde{A}(t) = \{(T_{a\alpha}, T_{c\alpha}, T_{e\alpha}, T_{f\alpha}, T_{d\alpha}, T_{b\alpha}), (I_{a\beta}, I_{c\beta}, I_{e\beta}, I_{f\beta}, I_{d\beta}, I_{b\beta}), (F_{a\gamma}, F_{c\gamma}, F_{e\gamma}, F_{f\gamma}, F_{d\gamma}, F_{b\gamma})\}$ where,

$$T_{a\alpha} = -(1-\alpha)\left|t^2-6t+8\right|, T_{b\alpha} = (1-\alpha)\left|t^2-6t+8\right|, T_{c\alpha} = -(1-\alpha)(0.7\left|t^2-6t+8\right|),$$
$$T_{d\alpha} = (1-\alpha)(0.7\left|t^2-6t+8\right|), T_{e\alpha} = -(1-\alpha)(0.4\left|t^2-6t+8\right|), T_{f\alpha} = (1-\alpha)(0.4\left|t^2-6t+8\right|),$$
$$I_{a\beta} = -\beta\left|1-(t^2-6t+8)\right|, I_{b\beta} = \beta\left|1-(t^2-6t+8)\right|, I_{c\beta} = -0.7\beta\left|1-(t^2-6t+8)\right|,$$
$$I_{d\beta} = 0.7\beta\left|1-(t^2-6t+8)\right|, I_{e\beta} = -0.4\beta\left|1-(t^2-6t+8)\right|, I_{f\beta} = 0.4\beta\left|1-(t^2-6t+8)\right|.$$
$$F_{a\gamma} = -\gamma\left|1-(t^2-6t+8)\right|, F_{b\gamma} = \gamma\left|1-(t^2-6t+8)\right|, F_{c\gamma} = -0.7\gamma\left|1-(t^2-6t+8)\right|,$$
$$F_{d\gamma} = 0.7\gamma\left|1-(t^2-6t+8)\right|, F_{e\gamma} = -0.4\gamma\left|1-(t^2-6t+8)\right|, F_{f\gamma} = 0.4\gamma\left|1-(t^2-6t+8)\right|.$$

where α represents the membership degree, such that: $T_{\tilde{A}}(T_{i\alpha}(\cdot)) = \alpha, \quad T_{\tilde{A}}(T_{j\alpha}(\cdot)) = \alpha$.



(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 2. Hexagonal Neutrosophic Bunch where $\alpha, \beta, \gamma \in \{0.2, 0.6, 0.8\}$

where $i, j \in \{a,b,c,d,e,f\}$, β represents the indeterminacy membership degree, such that: $I_{\tilde{A}}(I_{i\beta}(\cdot)) = \beta, \quad I_{\tilde{A}}(I_{j\beta}(\cdot)) = \beta$. where $i, j \in \{a,b,c,d,e,f\}$, and γ represents the non-membership degree, such that: $F_{\tilde{A}}(F_{i\gamma}(\cdot)) = \gamma, \quad F_{\tilde{A}}(F_{j\gamma}(\cdot)) = \gamma$. where $i, j \in \{a,b,c,d,e,f\}$.

Remark 3.4. Let the hexagonal neutrosophic number be represented by

$$\tilde{A}(t) = \left\{ (T_i)_{i=1}^{6}, \ (I_i)_{i=1}^{6}, \ (F_i)_{i=1}^{6} \right\},$$

where T_i, I_i, and F_i $(i = 1, \ldots, 6)$ denote the respective truth, indeterminacy, and falsity membership components.

The function $\tilde{A}$ can be decomposed into a crisp part and an uncertainty structure as

$$\tilde{A} = \langle T_{\text{cr}}, I_{\text{cr}}, F_{\text{cr}} \rangle + \left\langle (T_i)_{i=1}^{6}, (I_i)_{i=1}^{6}, (F_i)_{i=1}^{6} \right\rangle,$$

where the crisp components are given by

$$T_{\text{cr}} = \frac{T_3 + T_4}{2}, \quad I_{\text{cr}} = \frac{I_3 + I_4}{2}, \quad F_{\text{cr}} = \frac{F_3 + F_4}{2}.$$

Furthermore, the intervals (T_3, T_4), (I_3, I_4), and (F_3, F_4) are centered at zero and characterize the symmetric uncertainty inherent in the system.

Definition 3.5. Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be a 2×2 real matrix, and let the vectors

$$\vec{A} = \langle (AT_i, AI_i, AF_i) \rangle_{i=a}^{f}, \quad \vec{B} = \langle (BT_i, BI_i, BF_i) \rangle_{i=a}^{f}$$

represent hexagonal neutrosophic numbers. Then the product

$$B \cdot \begin{bmatrix} \vec{A} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} \langle (b_{11}AT_i + b_{12}BT_i,\ AI_i + b_{12}BI_i,\ AF_i + b_{12}BF_i) \rangle_{i=a}^{f} \\ \langle (b_{21}AT_i + b_{22}BT_i,\ b_{21}AI_i + b_{22}BI_i,\ b_{21}AF_i + b_{22}BF_i) \rangle_{i=a}^{f} \end{bmatrix}$$

defines the matrix multiplication of a real matrix with hexagonal neutrosophic Number.

4. The Method of Solving a Non-homogeneous system of DE with initial values in the Neutrosophic Context

We demonstrate how to find a solution for Non-homogeneous system of DE with initial values as a neutrosophic set.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \tag{1}$$

with the initial condition as,

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix} \tag{2}$$

where $b_{11}, b_{12}, b_{21}, b_{22}$ are real numbers; $\tilde{A}$, $\tilde{B}$ are given hexagonal neutrosophic functions and $\tilde{C}$, $\tilde{D}$ are given hexagonal neutrosophic numbers.

The neutrosophic initial value problem (NIVP), given by equations (1) and (2), can be expressed in matrix form as:

$$\mathbf{x}' = B\mathbf{x} + \tilde{\mathbf{E}}, \tag{3}$$

$$\mathbf{x}(0) = \tilde{\mathbf{f}}, \tag{4}$$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

The problem (equation (1-2))(or equivalently equation(3-4)) has neutrosophic functions as inputs. Therefore, the solution is also a neutrosophic function. The characteristics of neutrosophic functions and numbers depends on the forcing terms, the initial conditions, and the arithmetic operations of neutrosophic numbers defined in [23].

The following approaches can be employed to interpret these components: (1) forcing functions can be expressed as neutrosophic-valued functions or as a neutrosophic set (bunch) of real functions; (2) the neutrosophic derivative may follow the Hukuhara approach in [21]. For example, when the forcing functions in equations (1) and (2) are interpreted as neutrosophic-valued functions, and both differentiation and arithmetic operations are carried out according to neutrosophic rules [21,23], then the resulting solutions $x(t)$ and $y(t)$ are neutrosophic-valued functions.

The NIVP is interpreted as a family of crisp initial value problems (IVPs), constructed by selecting individual elements from each of the four neutrosophic sets: $l(\cdot) \in \tilde{A}$, $m(\cdot) \in \tilde{B}$, $o \in \tilde{C}$, and $p \in \tilde{D}$. Consequently, the NIVP defined in (1)–(2) is reformulated as a set of classical IVPs:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} l(.) \\ m(.) \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} n \\ o \end{bmatrix}, \tag{6}$$

where $l(\cdot) \in \tilde{A}$, $m(\cdot) \in \tilde{B}$, $n \in \tilde{C}$, and $o \in \tilde{D}$.

The solution of 5-6 is a real vector function, say $(x_{lmno}(\cdot), y_{lmno}(\cdot))$.

Using definition 3.1 $T_1 = T_{\tilde{A}}(l(\cdot))$, $T_2 = T_{\tilde{B}}(m(\cdot))$, $T_3 = T_{\tilde{C}}(n)$, $T_4 = T_{\tilde{D}}(o), I_1 = I_{\tilde{A}}(l(\cdot))$, $I_2 = I_{\tilde{B}}(m(\cdot))$, $I_3 = I_{\tilde{C}}(n)$, $I_4 = I_{\tilde{D}}(o), F_1 = F_{\tilde{A}}(l(\cdot))$, $F_2 = F_{\tilde{B}}(m(\cdot))$, $F_3 = F_{\tilde{C}}(n)$, and $F_4 = T_{\tilde{D}}(o)$. Define:

$$T = \min\{T_1, T_2, T_3, T_4\}, I = \max\{I_1, I_2, I_3, I_4\} \text{ and } F = \max\{F_1, F_2, F_3, F_4\}. \tag{7}$$

To the vector function $(x_{lmno}(\cdot), y_{lmno}(\cdot))$, we assign the membership, Indeterminacy and falsity degrees as T, I and F respectively. The set comprising all these vector-valued functions is called the neutrosophic solution and is represented by $\tilde{X}$.

Remark 4.1. It is important to emphasize that equation (1) serves as a formal representation of a differential equation system involving neutrosophic uncertainties. A more precise formulation of the problem is as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}. \tag{8}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \in \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix}. \tag{9}$$

Definition 4.2. Let U represent the universal set composed of vector-valued functions that are continuous over the interval $[0, \infty)$ and differentiable on $(0, \infty)$. The neutrosophic set $\tilde{X}$ defined on U is assigned to a membership function given as follows:

$$T_{\tilde{X}}(x(\cdot), y(\cdot)) = \min\{T_{\tilde{C}}(x(0)), T_{\tilde{D}}(y(0)), T_{\tilde{A}}(x' - b_{11}x - b_{12}y), T_{\tilde{B}}(y' - b_{21}x - b_{22}y)\}, \tag{10}$$

Indeterminacy function

$$I_{\tilde{X}}(x(\cdot), y(\cdot)) = \max\{I_{\tilde{C}}(x(0)), I_{\tilde{D}}(y(0)), I_{\tilde{A}}(x' - b_{11}x - b_{12}y), I_{\tilde{B}}(y' - b_{21}x - b_{22}y)\}, \tag{11}$$

and Falsity function

$$F_{\tilde{X}}(x(\cdot), y(\cdot)) = \max\{F_{\tilde{C}}(x(0)), F_{\tilde{D}}(y(0)), F_{\tilde{A}}(x' - b_{11}x - b_{12}y), F_{\tilde{B}}(y' - b_{21}x - b_{22}y)\} \tag{12}$$

is referred to as the solution of the NIVP described by equations (1) and (2).

$\tilde{X}(t)$ represents the neutrosophic solution at the time t. Based on Definition 4.2, $\tilde{X}$ consists of neutrosophic-valued vector functions. At any specific time t, each function yields a vector, corresponding to a point in the coordinate space. Therefore, $\tilde{X}(t)$ can be understood as a neutrosophic set of these points—a neutrosophic region—where every point carries the associated degrees of truth, indeterminacy, and falsity from its respective vector function.

If multiple vector functions map to the same point, the point is assigned the **maximum** truth-membership degree among them and the **minimum** for indeterminacy and falsity degrees.

Formally, for a point (ξ, ζ), we define:

$$T_{\tilde{X}}(t)(\xi, \zeta) = \alpha, \quad \text{if} \quad \exists (x(\cdot), y(\cdot)) : \big(T_{\tilde{X}}(x(\cdot), y(\cdot)) = \alpha \wedge (x(t), y(t)) = (\xi, \zeta)\big) \tag{13}$$

$$\forall (p(\cdot), q(\cdot)) : \big(T_{\tilde{X}}(p(\cdot), q(\cdot)) > \alpha \rightarrow (p(t), q(t)) \neq (\xi, \zeta)\big) \tag{14}$$

Similarly, the indeterminacy-membership is defined as:

$$I_{\tilde{X}}(t)(\xi, \zeta) = \beta, \quad \text{if} \quad \exists (x(\cdot), y(\cdot)) : \big(I_{\tilde{X}}(x(\cdot), y(\cdot)) = \beta \wedge (x(t), y(t)) = (\xi, \zeta)\big) \tag{15}$$

$$\forall (p(\cdot), q(\cdot)) : \big(I_{\tilde{X}}(p(\cdot), q(\cdot)) < \beta \rightarrow (p(t), q(t)) \neq (\xi, \zeta)\big) \tag{16}$$

The falsity-membership is similarly given by:

$$F_{\tilde{X}}(t)(\xi, \zeta) = \gamma, \quad \text{if} \quad \exists (x(\cdot), y(\cdot)) : \big(F_{\tilde{X}}(x(\cdot), y(\cdot)) = \gamma \wedge (x(t), y(t)) = (\xi, \zeta)\big) \tag{17}$$

$$\forall (p(\cdot), q(\cdot)) : \big(F_{\tilde{X}}(p(\cdot), q(\cdot)) < \gamma \rightarrow (p(t), q(t)) \neq (\xi, \zeta)\big) \tag{18}$$

Here, the symbols "$\wedge$" and "$\rightarrow$" denote logical conjunction and implication, respectively. Here, we have applied the standard t-norm for truth-membership and t-conorm operations for indeterminacy and falsity memberships:

$$T(x, y) = \min\{T(x), T(y)\}, \quad I(x, y) = \max\{I(x), I(y)\}, \quad F(x, y) = \max\{F(x), F(y)\}. \tag{19}$$

Key Properties

The following properties of linear transformations and neutrosophic vectors should hold for the solution:

(1) A linear transformation maps the zero vector to itself.

(2) A linear transformation preserves parallelism, meaning it maps one pair of parallel straight lines to another. As a result, a parallelepiped is transformed into another parallelepiped.

(3) The neutrosophic vector $\tilde{b}$ defines a neutrosophic region in $\mathbb{R}^n$, where each (α, β, γ)-cut corresponds to a nested parallelepiped.

(4) According to the existence and uniqueness theorem, solutions starting from distinct initial points remain distinct for all t.

(5) The general solution of $X'(t) = AX(t)$ is given by $X(t) = e^{At}v$ where $v \in \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix}$.

As a result, the initial points create a parallelepiped, and the solution at each moment in time also takes the form of a parallelepiped

4.1. *Limitations*

While the proposed method offers an effective framework for solving non-homogeneous linear systems of differential equations in the neutrosophic context, it is subject to certain limitations.

(i) The approach assumes that the truth, indeterminacy, and falsity membership functions are well-defined and exhibit smooth or piecewise-smooth behavior.

(ii)The method is not yet generalized to handle the truth or indeterminacy and falsity member functions fails to be convexity or concavity respectively, which could arise in practical applications.

(iii) When the dimension of the system increases, the computational effort required is more to solve the system.

4.2. *Algorithm*

Step 1 : The system described in equations (1)–(2) can be reformulated in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} A_{\text{cr}} \\ B_{\text{cr}} \end{bmatrix} + \begin{bmatrix} A \\ B \end{bmatrix},$$

subject to the initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} C_{\text{cr}} \\ D_{\text{cr}} \end{bmatrix} + \begin{bmatrix} C \\ D \end{bmatrix}.$$

Here, the terms are defined from remark 3.4 as follows:

- $\tilde{A} = \langle AT_{\text{cr}}, AI_{\text{cr}}, AF_{\text{cr}}\rangle + \left\langle (AT_i, AI_i, AF_i)_{i=a}^{f} \right\rangle$, where the uncertainty components (AT_c, AT_d), (AI_c, AI_d), and (AF_c, AF_d) are intervals symmetric about zero.
- $\tilde{B} = \langle BT_{\text{cr}}, BI_{\text{cr}}, BF_{\text{cr}}\rangle + \left\langle (BT_i, BI_i, BF_i)_{i=a}^{f} \right\rangle$, with the same structure: intervals (BT_c, BT_d), (BI_c, BI_d), and (BF_c, BF_d) centered at zero.
- $\tilde{C} = \langle CT_{\text{cr}}, CI_{\text{cr}}, CF_{\text{cr}}\rangle + \left\langle (CT_i, CI_i, CF_i)_{i=a}^{f} \right\rangle$, where the intervals (CT_c, CT_d), (CI_c, CI_d), and (CF_c, CF_d) are likewise centered at zero.
- $\tilde{D} = \langle DT_{\text{cr}}, DI_{\text{cr}}, DF_{\text{cr}}\rangle + \left\langle (DT_i, DI_i, DF_i)_{i=a}^{f} \right\rangle$, again featuring symmetric intervals (DT_c, DT_d), (DI_c, DI_d), and (DF_c, DF_d).

Considering the linearity of the system, the complete solution can be decomposed into the superposition of three independent subproblems. This decomposition not only clarifies the contributions of the crisp, interval, and neutrosophic uncertainties separately but also significantly simplifies the computational procedure for obtaining the overall solution.

Step 2 : The first subproblem is associated with the corresponding crisp non-homogeneous initial value problem:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} A_{cr} \\ B_{cr} \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} C_{cr} \\ D_{cr} \end{bmatrix}.$$

This subproblem admits a unique solution, given by

$$x_{cr}(t) = \begin{bmatrix} (Tx_{cr}(t), Ix_{cr}(t), Fx_{cr}(t)) \\ (Ty_{cr}(t), Iy_{cr}(t), Fy_{cr}(t)) \end{bmatrix},$$

which can be obtained through either analytical techniques or numerical computation.

Step 3 : The next subproblem involves solving the neutrosophic homogeneous problem:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}.$$

At each time t, the solution takes the shape of a parallelogram in the truth, indeterminacy and falsity part.

$$x_{homo}(t) = e^{At} \begin{bmatrix} C \\ D \end{bmatrix} = \left\{ \mathbf{v} \mid \mathbf{v} = e^{At}\mathbf{u},\, \mathbf{u} = \begin{bmatrix} C \\ D \end{bmatrix},\, C \in \tilde{C},\, D \in \tilde{D} \right\},$$

where $A = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ is the system matrix, and the membership degree is:
$T(\mathbf{v}) = \min\left\{ T_{\tilde{C}}(c), T_{\tilde{D}}(d) \right\}$, the indeterminacy degree is: $I(\mathbf{v}) = \max\left\{ I_{\tilde{C}}(c), I_{\tilde{D}}(d) \right\}$, and the non-membership degree is: $F(\mathbf{v}) = \max\left\{ F_{\tilde{C}}(c), F_{\tilde{D}}(d) \right\}$.

Remark 4.3. The Hexagonal neutrosophic zero $\tilde{0}$ is defined as

$$\tilde{0} = \{\langle 0,0,0,0,0,0\rangle,\ \langle 0,0,0,0,0,0\rangle,\ \langle 0,0,0,0,0,0\rangle\}.$$

where T($\tilde{0}$)=0, I($\tilde{0}$)=1 and F($\tilde{0}$)=1.

Step 4 : The final subproblem addresses the neutrosophic non-homogeneous system, formulated as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \langle (AT_i, AI_i, AF_i)_{i=a}^{f} \rangle \\ \langle (BT_i, BI_i, BF_i)_{i=a}^{f} \rangle \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix}. \tag{20}$$

As noted earlier, the neutrosophic system described by equation 20 is treated as a family of corresponding crisp systems. In this formulation, the forcing terms are represented by the neutrosophic bunch

$$\langle (AT_i, AI_i, AF_i)_{i=a}^{f} \rangle \quad \text{and} \quad \langle (BT_i, BI_i, BF_i)_{i=a}^{f} \rangle,$$

which are considered as linear combinations of the form

$$\langle \xi\, AT_i + \zeta\, BT_j, \quad \xi\, AI_i + \zeta\, BI_j, \quad \xi\, AF_i + \zeta\, BF_j \rangle\,,$$

where $\xi, \zeta \in [0,1]$ and $i, j \in \{a, b, c, d, e, f\}$.

Consequently, the neutrosophic system reduces to solving three families of crisp initial value problems. Each subproblem can be expressed individually as:

(1) For the truth membership forcing term:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \xi\, AT_i \\ \zeta\, BT_j \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix}. \tag{21}$$

(2) For the indeterminacy membership forcing term:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \xi\, AI_i \\ \zeta\, BI_j \end{bmatrix}, \quad \begin{bmatrix} x0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix}. \tag{22}$$

(3) For the falsity membership forcing term:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \xi\, AF_i \\ \zeta\, BF_j \end{bmatrix}, \quad \begin{bmatrix} \xi(0) \\ \zeta(0) \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix}. \tag{23}$$

Thus, the neutrosophic non-homogeneous problem can be resolved by analyzing the ensemble of these crisp systems, corresponding to different combinations of the neutrosophic components. For each combination of the solution for equation 21, 22 and 23 $x(.)$ and $y(.)$, the corresponding membership, indeterminacy, and non-membership degrees are defined as:

$$T_{\tilde{X}} = \min\{1-\xi, 1-\zeta\},$$

$$I_{\tilde{X}} = \max\{\xi, \zeta\},$$

$$F_{\tilde{X}} = \max\{\xi, \zeta\}.$$

The collection of solutions to all systems 21, 22 and 23 forms the neutrosophic solution for 20. To describe this neutrosophic solution efficiently, we employ the following approach.
Each of the systems 21, 22 and 23 can be expressed in the form

$$\mathbf{x}' = B\ \mathbf{x}\ + \mathbf{T}, \quad \mathbf{x}(0) = \mathbf{0} \tag{24}$$

$$\mathbf{x}' = B\ \mathbf{x}\ + \mathbf{I}, \quad \mathbf{x}(0) = \mathbf{0} \tag{25}$$

$$\mathbf{x}' = B\ \mathbf{x}\ + \mathbf{F}, \quad \mathbf{x}(0) = \mathbf{0} \tag{26}$$

The solutions for equations 24, 25 and 26 corresponding to six specific cases of the forcing term T, I, F, associated with the components $AT_i, AI_i, AF_i, BT_i, BI_i, BF_i$ for $i \in \{a, b, c, d, e, f\}$, are solved.

- For $T = \begin{bmatrix} AT_i \\ 0 \end{bmatrix}$, the solution is: $\tilde{T}_i(t) = \begin{bmatrix} x\tilde{T}_i(t) \\ y\tilde{T}_i(t) \end{bmatrix}$.
- For $I = \begin{bmatrix} AI_i \\ 0 \end{bmatrix}$, the solution is: $\tilde{I}_i(t) = \begin{bmatrix} x\tilde{I}_i(t) \\ y\tilde{I}_i(t) \end{bmatrix}$.
- For $F = \begin{bmatrix} AF_i \\ 0 \end{bmatrix}$, the solution is: $\tilde{F}_i(t) = \begin{bmatrix} x\tilde{F}_i(t) \\ y\tilde{F}_i(t) \end{bmatrix}$.
- For $T = \begin{bmatrix} 0 \\ BT_i \end{bmatrix}$, the solution is: $\hat{T}_i(t) = \begin{bmatrix} x\hat{T}_i(t) \\ y\hat{T}_i(t) \end{bmatrix}$.
- For $I = \begin{bmatrix} 0 \\ BI_i \end{bmatrix}$, the solution is: $\hat{I}_i(t) = \begin{bmatrix} x\hat{I}_i(t) \\ y\hat{I}_i(t) \end{bmatrix}$.
- For $F = \begin{bmatrix} 0 \\ BF_i \end{bmatrix}$, the solution is: $\hat{F}_i(t) = \begin{bmatrix} x\hat{F}_i(t) \\ y\hat{F}_i(t) \end{bmatrix}$. where i=a,b,...,f

The solution of equation 21, 22 and 23 (where $\mathbf{T}= \xi\tilde{T}_i + \zeta\hat{T}_i$, $\mathbf{I}= \xi\tilde{I}_i + \zeta\hat{I}_i$ and $\mathbf{F}= \xi\tilde{F}_i + \zeta\hat{F}_i$) is $\mathbf{x}=\xi\Phi + \zeta\Lambda$ (where $\Phi \in \{\tilde{T}_i(t), \tilde{I}_i(t), \tilde{F}_i(t)\},\ \Lambda \in \{\hat{T}_i(t), \hat{I}_i(t), \hat{F}_i(t)\}$).

Theorem 4.4. *Consider the NIVP defined by equation 20, where the forcing terms A and B are given as hexagonal neutrosophic functions:*

$$A = \left\langle (AT_i)_{i=a}^{f}, (AI_i)_{i=a}^{f}, (AF_i)_{i=a}^{f} \right\rangle, \quad B = \left\langle (BT_i)_{i=a}^{f}, (BI_i)_{i=a}^{f}, (BF_i)_{i=a}^{f} \right\rangle.$$

We define the corresponding solution components as follows:

$$\tilde{T}_i(t) = \begin{bmatrix} x\tilde{T}_i(t) \\ y\tilde{T}_i(t) \end{bmatrix}, \quad \tilde{I}_i(t) = \begin{bmatrix} x\tilde{I}_i(t) \\ y\tilde{I}_i(t) \end{bmatrix}, \quad \tilde{F}_i(t) = \begin{bmatrix} x\tilde{F}_i(t) \\ y\tilde{F}_i(t) \end{bmatrix},$$

$$\hat{T}_i(t) = \begin{bmatrix} x\hat{T}_i(t) \\ y\hat{T}_i(t) \end{bmatrix}, \quad \hat{I}_i(t) = \begin{bmatrix} x\hat{I}_i(t) \\ y\hat{I}_i(t) \end{bmatrix}, \quad \hat{F}_i(t) = \begin{bmatrix} x\hat{F}_i(t) \\ y\hat{F}_i(t) \end{bmatrix},$$

where each pair $(\tilde{T}_i, \tilde{T}_i)$, $(\tilde{I}_i, \tilde{I}_i)$, and $(\tilde{F}_i, \tilde{F}_i)$ corresponds to the particular forcing terms

$$AT_i(t) = \begin{bmatrix} AT_i(t) \\ 0 \end{bmatrix}, \quad AI_i(t) = \begin{bmatrix} AI_i(t) \\ 0 \end{bmatrix}, \quad AF_i(t) = \begin{bmatrix} AF_i(t) \\ 0 \end{bmatrix},$$

and

$$BT_i(t) = \begin{bmatrix} 0 \\ BT_i(t) \end{bmatrix}, \quad BI_i(t) = \begin{bmatrix} 0 \\ BI_i(t) \end{bmatrix}, \quad BF_i(t) = \begin{bmatrix} 0 \\ BF_i(t) \end{bmatrix},$$

respectively.

It follows that there exists a unique neutrosophic solution $\tilde{X}(t)$ associated with this problem, which can be interpreted as a neutrosophic ensemble of vector functions. The complete solution is then expressed through a membership function that aggregates the contributions from the truth, indeterminacy, and falsity components.

The membership function $T_{\tilde{X}}(x(\cdot))$ is defined as:

$$T_{\tilde{X}}(x(\cdot)) = \begin{cases} \min\{1-\xi, 1-\zeta\}, & \textit{if } \exists i,j \in \{a,b,c,d,e,f\},\ \exists \xi,\zeta \in [0,1] : \boldsymbol{T} = \xi\tilde{T}_i + \zeta\hat{T}_j, \\ 0, & \textit{otherwise.} \end{cases}$$

The indeterminacy function $I_{\tilde{X}}(x(\cdot))$ is defined as:

$$I_{\tilde{X}}(x(\cdot)) = \begin{cases} \max\{\xi, \zeta\}, & \textit{if } \exists i,j \in \{a,b,c,d,e,f\}, \exists \xi,\zeta \in [0,1] : \boldsymbol{I} = \xi\tilde{I}_i + \zeta\hat{I}_j, \\ 0, & \textit{otherwise.} \end{cases}$$

The non-membership function $F_{\tilde{X}}(x(\cdot))$ is defined as:

$$F_{\tilde{X}}(x(\cdot)) = \begin{cases} \max\{\xi, \zeta\}, & \textit{if } \exists i,j \in \{a,b,c,d,e,f\}, \exists \xi,\zeta \in [0,1] : \boldsymbol{F} = \xi\tilde{F}_i + \zeta\hat{F}_j, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The proof follows from the decomposition of the neutrosophic initial value problem (20) into three families of crisp systems for the truth-membership, indeterminacy, and non-membership components. By solving these systems for all possible combinations of parameters $\xi, \zeta \in [0,1]$ and indices $i, j \in \{a,b,c,d,e,f\}$, we construct the neutrosophic solution $\tilde{X}(t)$.

The bunch of all such vector functions $\mathbf{x}$ defines the neutrosophic solution x_{nonhomo} of 20, characterized by its corresponding membership, indeterminacy, and falsity functions as specified below,

$$x_{\text{nonhomo}}\big(\xi\Phi + \zeta\Lambda\big) = \left\langle T_{\tilde{X}}(x(t)), I_{\tilde{X}}(x(t)), F_{\tilde{X}}(x(t)) \right\rangle,$$

where $x_{\text{nonhomo}}(t)$ denotes the nonhomogeneous part of the solution.

The collection of all such individual solutions constitutes the neutrosophic solution $\tilde{X}$, described through the membership functions:

$$T_{\tilde{X}}(x(t)) = \min\{1-\xi, 1-\zeta\},$$

the indeterminacy function:

$$I_{\tilde{X}}(x(t)) = \max\{\xi, \zeta\},$$

and the non-membership function:

$$F_{\tilde{X}}(x(t)) = \max\{\xi, \zeta\}.$$

Next, we analyze $x_{\text{nonhomo}}(t)$, which represents the solution value at a particular time t. The corresponding $(0,1,1)$-cut of the solution is expressed as:

$$x_{\text{nonhomo},(0,1,1)}(t) = \bigcup \text{Parallelogram}\{\Phi, \Lambda\}, \tag{21}$$

where

$$\text{Parallelogram}\{p, q\} = \{\xi p + \zeta q \mid 0 \le \xi \le 1, 0 \le \zeta \le 1\}$$

denotes the set spanned by vectors p and q.

The membership, indeterminacy, and non-membership functions are then obtained by associating the degrees $\min\{1-\xi, 1-\zeta\}$ and $\max\{\xi, \zeta\}$ with each combination, as specified in the theorem. Uniqueness and existence of the solution follow from the linearity of the system and the properties of the neutrosophic set construction. By the linearity of the problem, the (α, β, γ)-cut can be similarly determined using scaling similarity:

$$x_{\text{nonhomo},(\alpha,\beta,\gamma)}(t) = (1-\alpha)x_{\text{nonhomo},0}(t) \cup \beta x_{\text{nonhomo},1}(t) \cup \gamma x_{\text{nonhomo},1}(t).$$

Therefore, the neutrosophic solution of the differential equation takes the form:

$$x(t) = x_{cr}(t) + x_{homo}(t) + x_{\text{nonhomo}}(t). \tag{27}$$

□

5. **Numerical Example**

An example is given to illustrate the suggested approach.

Example 5.1. We now proceed to analyze the set of differential equations outlined below:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \tag{28}$$

with initial conditions:

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix} = \begin{bmatrix} \langle (13,14,15,16,17,18),(13,14,15,16,17,18),(13,14,15,16,17,18) \rangle \\ \langle (3,4,5,6,7,8),(3,4,5,6,7,8),(3,4,5,6,7,8) \rangle \end{bmatrix}. \tag{29}$$

Here, the forcing functions and initial conditions are defined as:

$$\tilde{A} = A_{\text{cr}} + \tilde{a}, \quad A_{\text{cr}} = \left\langle 4t^2 - 14t - 24, 4t^2 - 14t - 24, 4t^2 - 14t - 24 \right\rangle, \tilde{a} = \langle (-3e^{-t}, -2e^{-t}, -e^{-t}, e^{-t}, 2e^{-t}, 3e^{-t}),$$

$$(-3e^{-t}, -2.5e^{-t}, -1.5e^{-t}, 1.5e^{-t}, 2.5e^{-t}, 3e^{-t}), (-3e^{-t}, -1.5e^{-t}, -.5e^{-t}, .5e^{-t}, 1.5e^{-t}, 3e^{-t}) \rangle,$$

$$\tilde{B} = B_{\text{cr}} + \tilde{b}, \quad B_{\text{cr}} = \left\langle 9t^2 - 9t - 39, 9t^2 - 9t - 39, 9t^2 - 9t - 39 \right\rangle, \tilde{b} = \langle (-3e^{-2t}, -2e^{-2t}, -e^{-2t}, 1, 2, 3)$$

$$(-3e^{-2t}, -2.5e^{-2t}, -1.5e^{-2t}, 1.5, 2.5, 3), (-3e^{-2t}, -1.5e^{-2t}, -.5e^{-2t}, 0.5, 1.5, 3) \rangle.$$

(1) Solving the crisp problem The crisp problem is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} A_{\text{cr}} \\ B_{\text{cr}} \end{bmatrix},$$

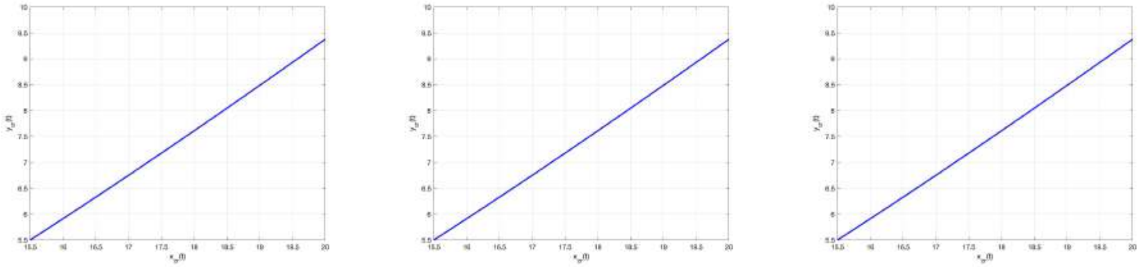
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \langle 15.5, 15.5, 15.5 \rangle \\ \langle 5.5, 5.5, 5.5 \rangle \end{bmatrix}.$$

The crisp solution is:

$$x_{\text{cr}}(t) = \left\langle 5t + 11e^{2t} + 3t^2 + \frac{9}{2}, 5t + 11e^{2t} + 3t^2 + \frac{9}{2}, 5t + 11e^{2t} + 3t^2 + \frac{9}{2} \right\rangle$$

$$y_{\text{cr}}(t) = \left\langle 11e^{2t} + 8t^2 - \frac{11}{2}, 11e^{2t} + 8t^2 - \frac{11}{2}, 11e^{2t} + 8t^2 - \frac{11}{2} \right\rangle.$$

Figure 3 illustrate these results.



(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 3. Crisp Solution

(2) Uncertainty due to initial conditions.

The homogeneous system is: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$

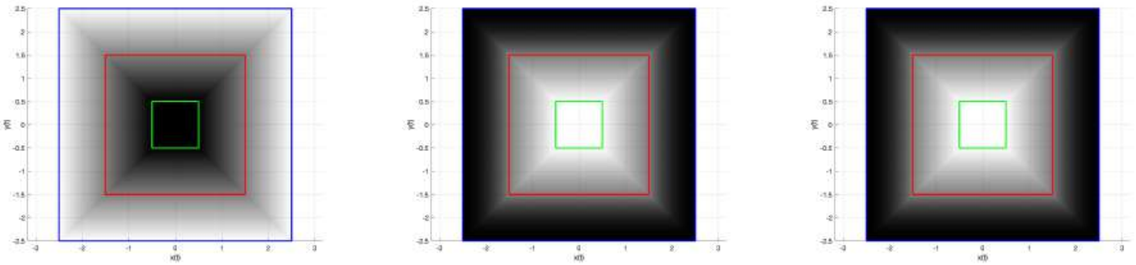
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \langle (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5) \rangle \\ \langle (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5) \rangle \end{bmatrix}.$$

The uncertainty due to initial values is expressed as:

$$x_{homo}(t) = e^{At} \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5e^{2t} - 2e^{-t} & -2e^{2t} + 2e^{-t} \\ 5e^{2t} - 5e^{-t} & 5e^{-t} - 2e^{2t} \end{bmatrix} \times$$

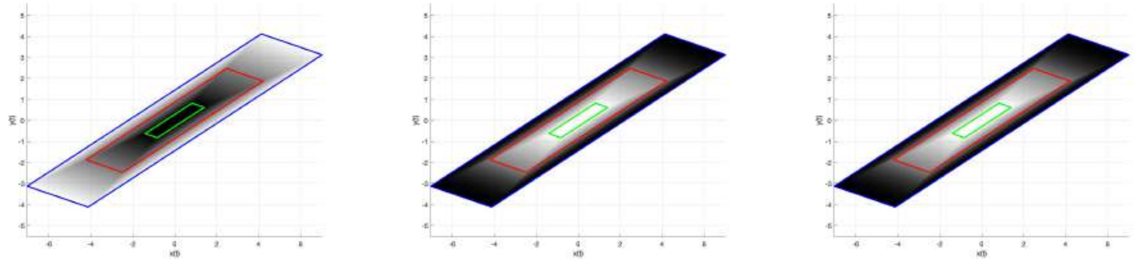
$$\begin{bmatrix} \langle (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5) \rangle \\ \langle (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5), (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5) \rangle \end{bmatrix}$$

Figures 4, 5 and 6 illustrate these results.



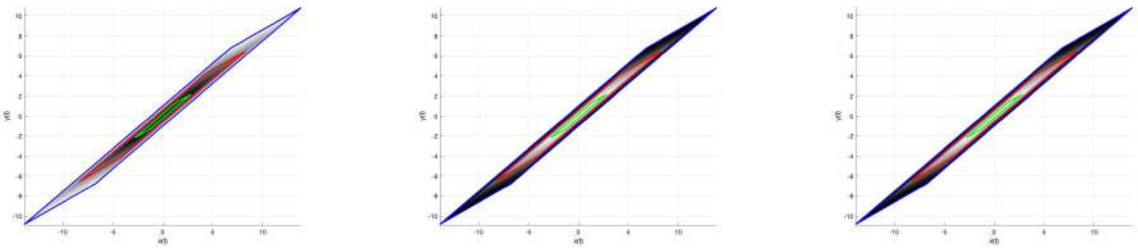
(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 4. Homogeneous Solution (Uncertainty due to initial condition) at $t = 0$



(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 5. Homogeneous Solution (Uncertainty due to initial condition) at $t = 0.25$



(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 6. Homogeneous Solution (Uncertainty due to initial condition) at $t = 0.5$

(3) Uncertainty due to forcing functions. The corresponding non-homogeneous neutrosophic system is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \left[\begin{array}{c} \left\langle \begin{array}{l} (-3e^{-t},\, -2e^{-t},\, -e^{-t},\, e^{-t},\, 2e^{-t},\, 3e^{-t}), \\ (-3e^{-t},\, -2.5e^{-t},\, -1.5e^{-t},\, 1.5e^{-t},\, 2.5e^{-t},\, 3e^{-t}), \\ (-3e^{-t},\, -1.5e^{-t},\, -0.5e^{-t},\, 0.5e^{-t},\, 1.5e^{-t},\, 3e^{-t}) \end{array} \right\rangle \\ \left\langle \begin{array}{l} (-3e^{-2t},\, -2e^{-2t},\, -e^{-2t},\, 1,\, 2,\, 3), \\ (-3e^{-2t},\, -2.5e^{-2t},\, -1.5e^{-2t},\, 1.5,\, 2.5,\, 3), \\ (-3e^{-2t},\, -1.5e^{-2t},\, -0.5e^{-2t},\, 0.5,\, 1.5,\, 3) \end{array} \right\rangle \end{array}\right],$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \end{bmatrix}.$$

The solutions for specific forcing terms are:

$$x_{\tilde{T}a}(t) = x_{\tilde{I}a}(t) = x_{\tilde{F}a}(t) = x = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{3}, y_{\tilde{T}a}(t) = y_{\tilde{I}a}(t) = y_{\tilde{F}a}(t) = \frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{3},$$

$$x_{\tilde{T}b}(t) = \frac{2e^{-t}\left(6t - 5e^{3t} + 5\right)}{9}, y_{\tilde{T}b}(t) = \frac{10e^{-t}\left(3t - e^{3t} + 1\right)}{9}, x_{\tilde{I}b}(t) = \frac{5e^{-t}\left(6t - 5e^{3t} + 5\right)}{18},$$

$$y_{\tilde{I}b}(t) = \frac{25e^{-t}\left(3t - e^{3t} + 1\right)}{18}, x_{\tilde{F}b}(t) = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{6}, y_{\tilde{F}b}(t) = \frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{6},$$

$$x_{\tilde{T}c}(t) = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{9}, y_{\tilde{T}c}(t) = \frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{9}, x_{\tilde{I}c}(t) = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{6},$$

$$y_{\tilde{I}c}(t) = \frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{6}, x_{\tilde{F}c}(t) = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{18}, y_{\tilde{F}c}(t) = \frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{18},$$

$$x_{\tilde{T}d}(t) = -\frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{9}, y_{\tilde{T}d}(t) = -\frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{9}, x_{\tilde{I}d}(t) = -\frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{6},$$

$$y_{\tilde{I}d}(t) = -\frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{6}, x_{\tilde{F}d}(t) = -\frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{18}, y_{\tilde{F}d}(t) = -\frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{18},$$

$$x_{\tilde{T}e}(t) = -\frac{2e^{-t}\left(6t - 5e^{3t} + 5\right)}{9}, y_{\tilde{T}e}(t) = -\frac{10e^{-t}\left(3t - e^{3t} + 1\right)}{9}, x_{\tilde{I}e}(t) = -\frac{5e^{-t}\left(6t - 5e^{3t} + 5\right)}{18},$$

$$y_{\tilde{I}e}(t) = -\frac{25e^{-t}\left(3t - e^{3t} + 1\right)}{18}, x_{\tilde{F}e}(t) = -\frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{6}, y_{\tilde{F}e}(t) = -\frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{6},$$

$$x_{\tilde{T}f}(t) = x_{\tilde{I}f}(t) = x_{\tilde{F}f}(t) = -\frac{e^{-t}\left(6t - 5e^{3t} + 5\right)}{3}, y_{\tilde{T}f}(t) = y_{\tilde{I}f}(t) = y_{\tilde{F}f}(t) = -\frac{5e^{-t}\left(3t - e^{3t} + 1\right)}{3},$$

$$x_{\hat{Ta}}(t)=x_{\hat{Ia}}(t)=x_{\hat{Fa}}(t)=\frac{3e^{-2t}}{2}-2e^{-t}+\frac{e^{2t}}{2},y_{\hat{Ta}}(t)=y_{\hat{Ia}}(t)=y_{\hat{Fa}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{2},$$

$$x_{\hat{Tb}}(t)=e^{-2t}-\frac{4e^{-t}}{3}+\frac{e^{2t}}{3},y_{\hat{Tb}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{3},x_{\hat{Ib}}(t)=\frac{5e^{-2t}}{4}-\frac{5e^{-t}}{3}+\frac{5e^{2t}}{12},$$

$$y_{\hat{Ib}}(t)=\frac{5e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{12}x_{\hat{Fb}}(t)=\frac{3e^{-2t}}{4}-e^{-t}+\frac{e^{2t}}{4},y_{\hat{Fb}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{4},$$

$$x_{\hat{Tc}}(t)=\frac{e^{-2t}}{2}-\frac{2e^{-t}}{3}+\frac{e^{2t}}{6},y_{\hat{Tc}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{6},x_{\hat{Ic}}(t)=\frac{3e^{-2t}}{4}-e^{-t}+\frac{e^{2t}}{4},$$

$$y_{\hat{Ic}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{4},x_{\hat{Fc}}(t)=\frac{e^{-2t}}{4}-\frac{e^{-t}}{3}+\frac{e^{2t}}{12},y_{\hat{Fc}}(t)=\frac{e^{-2t}\left(e^{4t}-10e^{t}+9\right)}{12},$$

$$x_{\hat{Td}}(t)=1-\frac{e^{2t}}{3}-\frac{2e^{-t}}{3},y_{\hat{Td}}(t)=2-\frac{e^{2t}}{3}-\frac{5e^{-t}}{3},x_{\hat{Id}}(t)=\frac{3}{2}-\frac{e^{2t}}{2}-e^{-t},$$

$$y_{\hat{Id}}(t)=3-\frac{e^{2t}}{2}-\frac{5e^{-t}}{2},x_{\hat{Fd}}(t)=\frac{1}{2}-\frac{e^{2t}}{6}-\frac{e^{-t}}{3},y_{\hat{Fd}}(t)=1-\frac{e^{2t}}{6}-\frac{5e^{-t}}{6},$$

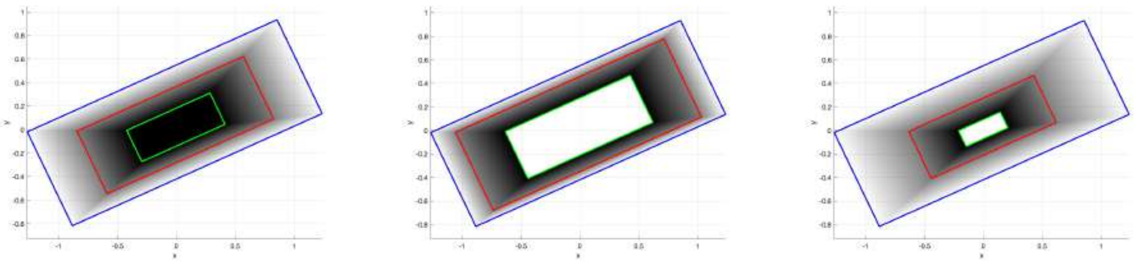
$$x_{\hat{Te}}(t)=2-\frac{2e^{2t}}{3}-\frac{4e^{-t}}{3},y_{\hat{Te}}(t)=4-\frac{2e^{2t}}{3}-\frac{10e^{-t}}{3},x_{\hat{Ie}}(t)=\frac{5}{2}-\frac{5e^{2t}}{6}-\frac{5e^{-t}}{3},$$

$$y_{\hat{Ie}}(t)=5-\frac{5e^{2t}}{6}-\frac{25e^{-t}}{6},x_{\hat{Fe}}(t)=\frac{3}{2}-\frac{e^{2t}}{2}-e^{-t},y_{\hat{Fe}}(t)=3-\frac{e^{2t}}{2}-\frac{5e^{-t}}{2},$$

$$x_{\hat{Tf}}(t)=x_{\mathrm{if}}(t)=x_{\hat{Ff}}(t)=3-e^{2t}-2e^{-t},y_{\hat{Tf}}(t)=y_{\mathrm{if}}(t)=y_{\hat{Ff}}(t)=6-e^{2t}-5e^{-t}.$$

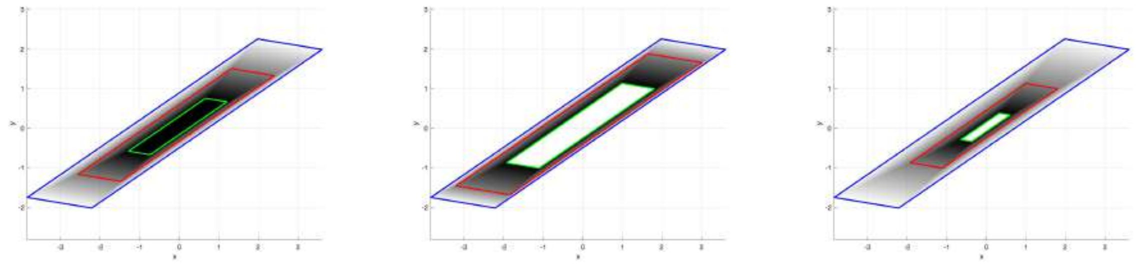
Figures 7, and 8 illustrate these results.

The Total Uncertainty of equation 28 from step 2 and 3 is given by figure 9 and 10.



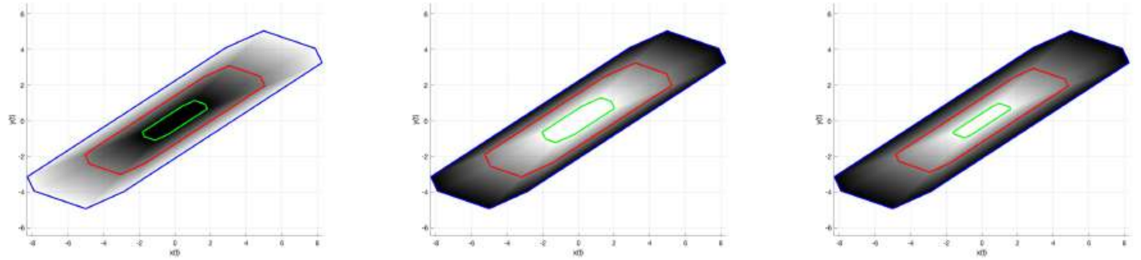
(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 7. Non-Homogeneous Solution (Forcing term) at $t = 0.25$



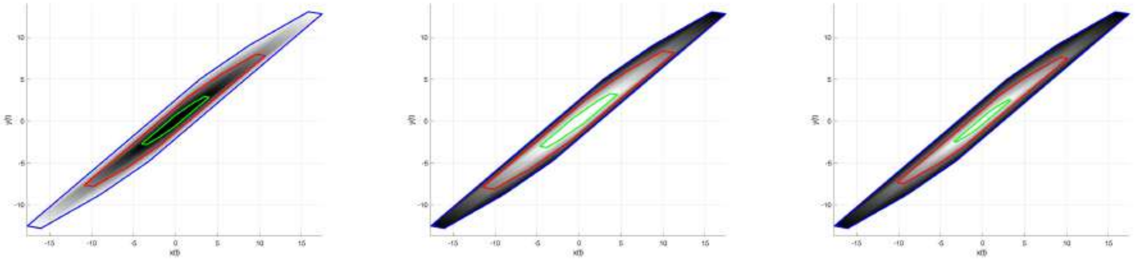
(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 8. Non-Homogeneous Solution (Forcing term) at $t = 0.5$



(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 9. Total Uncertainty at $t = 0.25$



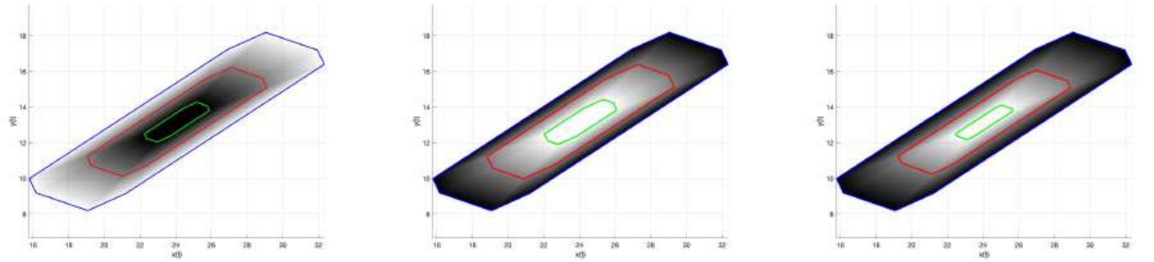
(A) Membership function (B) Indeterminacy function (C) Non-membership function

Figure 10. Total Uncertainty at $t = 0.5$

The General solution of equation 28 with initial condition 29 is given by figure 11, 12 and 13 using 27.

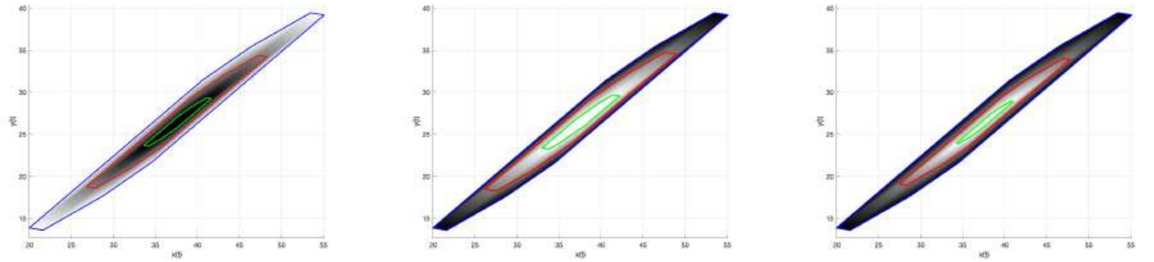
6. Comparison with the Differential Inclusion-Based Approach

In this section, we contrast the Neutrosophic solution obtained through our method with the one derived using the classical differential inclusion (DI) framework. The DI approach addresses neutrosophic differential equations by interpreting the neutrosophic right-hand side as a family of time-dependent intervals and solving the resulting differential inclusion accordingly.



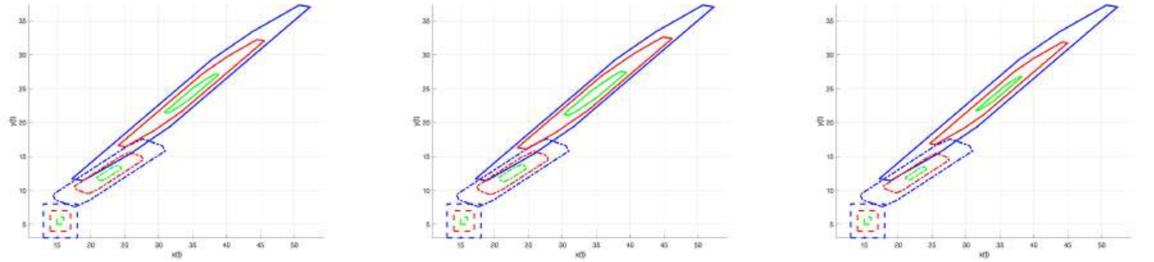
(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 11. Neutrosophic Solution at $t = 0.25$



(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 12. Neutrosophic Solution at $t = 0.5$



(A) Membership function (B) Indeterminacy function (C) Non-membership function

FIGURE 13. Neutrosophic Solution at $t = 0, 0.25, 0.5$

Example 6.1. Consider the following neutrosophic differential problem:

$$y'(s) = \tilde{G}(s), \quad y(0) = \hat{0}$$

where the input $\tilde{G}(s)$ is a Hexagonal Neutrosophic-valued function defined as:

$$\tilde{G}(s) = \left\{ \begin{array}{l} \langle -3\cos s,\ -2\cos s,\ -\cos s,\ \cos s,\ 2\cos s,\ 3\cos s \rangle, \\ \langle -3\sin s,\ -2\sin s,\ -\sin s,\ \sin s,\ 2\sin s,\ 3\sin s \rangle, \\ \langle -\sin s,\ -0.5\sin s,\ -0.2\sin s,\ 0.2\sin s,\ 0.5\sin s, \sin s \rangle \end{array} \right\}$$

and

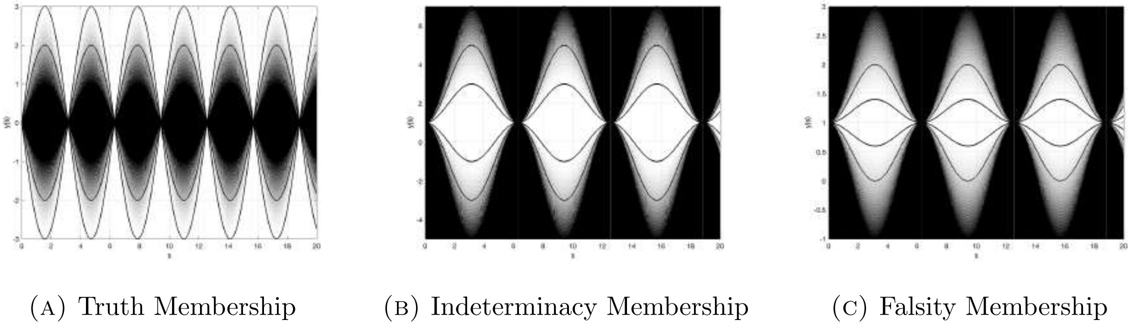
$$\hat{0} = \{\langle 0,0,0,0,0,0\rangle,\ \langle 0,0,0,0,0,0\rangle,\ \langle 0,0,0,0,0,0\rangle\}.$$

Using the Neutrosophic Bunch Interpretation:

Here, we will apply our approch to solve the initial value problem. The solution is given by

$$\tilde{G}(s) = \left\{\begin{array}{l} \langle -3\sin s,\ -2\sin s,\ -\sin s,\ \sin s,\ 2\sin s,\ 3\sin s\rangle, \\ \langle 3\cos s - 2,\ 2\cos s - 1,\ \cos s,\ 2-\cos s,\ 3-2\cos s,\ 4-3\cos s\rangle, \\ \langle \cos s,\ \dfrac{\cos s}{2}+\dfrac{1}{2},\ \dfrac{\cos s}{5}+\dfrac{4}{5},\ \dfrac{6}{5}-\dfrac{\cos s}{5},\ \dfrac{3}{2}-\dfrac{\cos s}{2},\ 2-\cos s\rangle \end{array}\right\}$$

This solution remains hexagonal for each value of s, centered at zero, and exhibits symmetric periodic bounds given in figure 14.



(A) Truth Membership (B) Indeterminacy Membership (C) Falsity Membership

FIGURE 14. Neutrosophic components by Neutrosophic Bunch.

Using the Differential Inclusion Framework:

In contrast, the differential inclusion method views $\tilde{G}(s)$ as an interval-valued function for each s, and the equation becomes:

$$y'(s) = \tilde{G}(s), \quad y(0) = \{\langle 0,0,0,0,0,0\rangle, \langle 1,1,1,1,1,1\rangle, \langle 1,1,1,1,1,1\rangle\}.$$

where

$$\tilde{G}(s) = \left\{\begin{array}{l} \langle -3|\cos s|,\ -2|\cos s|,\ -|\cos s|,\ |\cos s|,\ 2|\cos s|,\ 3|\cos s|\rangle, \\ \langle -3|\sin s|,\ -2|\sin s|,\ -|\sin s|,\ |\sin s|,\ 2|\sin s|,\ 3|\sin s|\rangle, \\ \langle -|\sin s|,\ -0.5|\sin s|,\ -0.2|\sin s|,\ |\sin s|,\ 0.2|\sin s|,\ 0.5|\sin s|\rangle \end{array}\right\}$$

we can arrive at the solution

$$\widehat{y}_1(s) = \begin{cases} 3\sin s, & \text{for } s \in \left[0,\ \dfrac{\pi}{2}\right] \\ 6k + (-1)^k 3\sin s, & \text{for } s \in \left[\dfrac{(2k-1)\pi}{2},\ \dfrac{(2k+1)\pi}{2}\right], \quad k \in \mathbb{N} \end{cases}$$

$$\widehat{y}_2(s) = \begin{cases} 2\sin s, & \text{for } s \in \left[0,\ \dfrac{\pi}{2}\right] \\ 4k + (-1)^k 2\sin s, & \text{for } s \in \left[\dfrac{(2k-1)\pi}{2},\ \dfrac{(2k+1)\pi}{2}\right], \quad k \in \mathbb{N} \end{cases}$$

$$\widehat{y}_3(s) = \begin{cases} \sin s, & \text{for } s \in \left[0,\ \frac{\pi}{2}\right] \\ 2k + (-1)^k \sin s, & \text{for } s \in \left[\frac{(2k-1)\pi}{2},\ \frac{(2k+1)\pi}{2}\right], \quad k \in \mathbb{N} \end{cases}$$

$$\widehat{y}_4(s) = \Big\{ 6k - 2 + (-1)^k 3\cos s, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

$$\widehat{y}_5(s) = \Big\{ 4k - 1 + (-1)^k 2\cos s, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

$$\widehat{y}_6(s) = \Big\{ 2k + (-1)^k \cos s, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

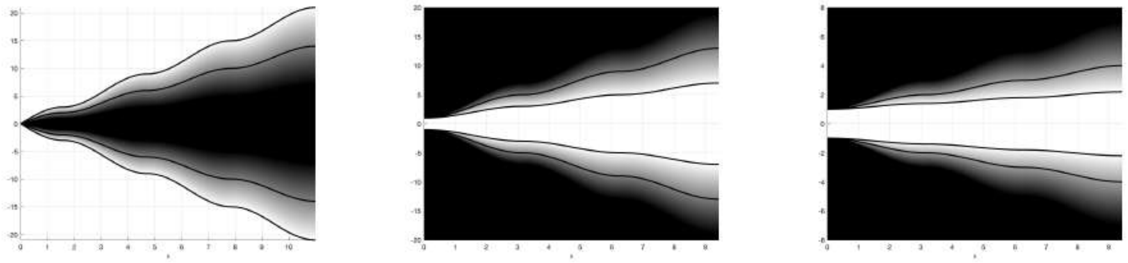
$$\widehat{y}_7(s) = \Big\{ 2k + (-1)^k \cos s, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

$$\widehat{y}_8(s) = \Big\{ \frac{2k + 1 + (-1)^k \cos s}{2}, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

$$\widehat{y}_9(s) = \Big\{ \frac{2k + 4 + (-1)^k \cos s}{5}, \quad \text{for } s \in [(k-1)\pi,\ k\pi], \quad k \in \mathbb{N}$$

Therefore, the neutrosophic-valued solution in this case is:

$$\tilde{y}(s) = \begin{Bmatrix} \langle -\widehat{y}_1(s),\ -\widehat{y}_2(s),\ -\widehat{y}_3(s),\ \widehat{y}_3(s),\ \widehat{y}_2(s),\ \widehat{y}_1(s) \rangle, \\ \langle -\widehat{y}_4(s),\ -\widehat{y}_5(s),\ -\widehat{y}_6(s),\ \widehat{y}_6(s),\ \widehat{y}_5(s),\ \widehat{y}_4(s) \rangle, \\ \langle -\widehat{y}_7(s),\ -\widehat{y}_8(s),\ -\widehat{y}_9(s),\ \widehat{y}_9(s),\ \widehat{y}_8(s),\ \widehat{y}_7(s) \rangle \end{Bmatrix}$$



(A) Truth Membership (B) Indeterminacy Membership (C) Falsity Membership

FIGURE 15. Neutrosophic components by Differential Inclusion.

Unlike the solution obtained via the proposed neutrosophic bunch method, this solution grows over time and lacks the oscillatory behavior of the original system, despite having the same input dynamics as shown in figure 15.

From this comparison, it is evident that the proposed methodology maintains the periodic and bounded nature of the underlying system, accurately reflecting the characteristics of the input function. In contrast, the DI-based method leads to an expanding uncertainty range, which may misrepresent the system's true dynamic behavior.

7. Conclusion

In this paper, neutrosophic hexagonal numbers are used as the initial conditions and forcing functions in linear systems of differential equations and a new approach is proposed to solve this system. The framework that has been proposed enhances the current methods of solving neutrosophic differential equations, making it an efficient tool for managing uncertainty on a broader scale. The investigation provides both theoretical and geometrical interpretations of the solutions by describing the stability of these systems within a neutrosophic framework.

This framework ensures the existence and uniqueness of the solution while offering significant advantages such as a gradual reduction in uncertainty over time and the preservation of periodicity, which are not commonly observed in existing methods. However, this approach currently has limitations. It is not directly applicable to systems involving more complex neutrosophic numbers, particularly those that are non-convex or contain nested uncertainty. Additionally, the method is presently restricted to linear systems and does not yet accommodate nonlinear neutrosophic differential equations.

Future Work

As part of our future work, we aim to extend this framework to linear neutrosophic differential equations using the hexagonal bunch structure and explore generalizations to broader classes of neutrosophic numbers. We can anaylse the approach for addressing systems when the truth, indeterminacy, and falsity membership functions fails to be convex or concave, which would significantly enhance the applicability of the approach to real-world scenarios uncertainty patterns. Another direction involves extending the methodology to non-linear systems of differential equations to determine whether the uncertainty-reducing and periodicity-preserving properties observed in the linear case can be retained. We also intend to investigate computational strategies to improve the scalability of the proposed approach for high-dimensional systems. Additionally, incorporating time-varying uncertainties and dynamic switching of admissible functions could provide a more realistic modeling framework for systems where the nature of uncertainty evolves over time.

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