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Complex Neutrosophic Measures with $\alpha\psi$-Regularization for Teaching Quality of University Ideological and Political Courses in the New Media Era

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Abstract-This paper introduces a novel mathematical framework that extends complex neutrosophic sets through the development of a phase-coherent complex neutrosophic measure (PCN-measure) combined with an $\alpha\psi$-regularization scheme. The PCN-measure assigns to each measurable set a triplet of complex values truth, indeterminacy, and falsity expressed in polar form with amplitude and phase. Amplitudes are aggregated using σ-additivity, while phases are combined through circular-mean addition, ensuring invariance under global phase shifts. A new distance function is defined that integrates amplitude differences with wrapped phase distances, generalizing existing d-equality concepts in complex neutrosophic theory. To address topological regularity, a variational functional is introduced that penalizes phase discontinuities while preserving $\alpha\psi$-connectedness, and its minimizers are shown to retain structural coherence under stated conditions. The framework further proposes a phase-locked aggregation operator (PLA) that balances proximity in the new metric with topological connectedness, yielding unique solutions up to global phase shifts. While the contributions are primarily theoretical, the design is motivated by the cyclic and interconnected nature of performance indicators in university ideological and political education in the new media era. An illustrative example demonstrates the computation of PCN-measures, phase-coherent distances, and PLA minimizers, supported by numerical results with explicit parameter interpretations.

Keywords-complex neutrosophic sets; phase-coherent measure; d-equality extension; wrapped phase distance; αψ connectedness; regularization; phase-locked aggregation; teaching quality; new media era.

1. Introduction

The concept of neutrosophic sets emerged as a powerful extension of traditional set theory and fuzzy sets, offering a way to handle uncertainty in a more comprehensive manner. Unlike classical sets, which assign elements a strict membership of true or false, or fuzzy sets that use a single degree of membership between 0 and 1, neutrosophic sets allow each element to be described by three independent components: a degree of truth (T), a degree

of indeterminacy (I), and a degree of falsity (F). These components can each range from 0 to 1, and their sum does not necessarily equal 1, which makes the framework flexible for representing incomplete, inconsistent, or paradoxical information in real-world scenarios [1]. This approach has proven useful in fields where decision-making involves ambiguity, such as engineering, social sciences, and data analysis.

Building on this foundation, researchers have developed various extensions to address specific types of uncertainty. For example, multi-polar neutrosophic sets handle multiple dimensions of polarity, interval-valued neutrosophic sets allow memberships to be expressed as intervals rather than single values, and complex-valued neutrosophic structures incorporate imaginary components to model more intricate data. Among these, the complex neutrosophic set (CNS) stands out because it integrates both amplitude and phase information into each of the T, I, and F components. The amplitude represents the strength or magnitude of the membership, while the phase captures cyclic or angular aspects, such as timing, direction, or periodicity [2]. This makes CNS particularly suitable for modeling phenomena that repeat over time or space, like waves, seasons, or rotational patterns.

In practical terms, the phase element in CNS allows for a mathematical representation of oscillatory behaviors that are common in dynamic systems. For instance, in evaluating the teaching quality of university ideological and political classes in the new media era, key indicators—such as student engagement with online content, the effectiveness of ideological discussions on social platforms, and the integration of multimedia tools in lessons—often exhibit periodic trends. These might align with academic cycles, like weekly online forums, semester-based assessments, or seasonal media campaigns that influence student participation [3]. By using amplitudes to quantify the intensity of these indicators and phases to track their timing or rhythm, CNS provides a nuanced tool for analysis, ensuring that both the "how much" and "when" aspects are captured accurately.

Recent advancements have also explored topological properties within the neutrosophic framework, leading to the development of neutrosophic complex topology (NCT). This area studies concepts like connectivity, continuity, and separation in spaces where elements are defined by complex neutrosophic values. A key idea here is $\alpha\psi$-connectedness, which generalizes traditional connectedness to allow for more flexible topological structures while maintaining certain rules to avoid irregularities [4]. This is especially relevant when dealing with phase data, as it helps ensure that cyclic information remains coherent and unbroken across the space.

Despite these progressions, there are notable gaps in the literature. Specifically, there is no integrated approach that combines measure theory for CNS with $\alpha\psi$-connected topology. Existing works lack a measure-theoretic foundation that ensures properties like σ-additivity (the ability to add up measures over countable unions) while respecting the amplitude-phase split. Additionally, distance metrics in CNS often fail to properly handle phase differences, which can wrap around like angles on a circle, leading to inaccuracies

in comparisons. Finally, regularization methods that enforce $\alpha\psi$-connectedness in phase components—while preserving the relationships between amplitudes and phases—have not been developed.

This paper addresses these shortcomings by proposing PCN measure and linking it to an $\alpha\psi$-connected regularization framework. The PCN-measure defines σ-additivity for amplitudes and uses circular-mean operations for phases, making it invariant to global phase shifts, which is crucial for periodic data [5]. We also introduce a phase-coherent distance function, dPCN, that combines amplitude differences with wrapped phase distances, extending previous equality concepts in CNS [2]. Furthermore, we present a variational method where a regularization functional penalizes discontinuities in phases to maintain $\alpha\psi$-connectedness, and we prove that solutions to this optimization preserve the desired topological properties under reasonable conditions [6]. The main contributions of this work include:

a) A new σ-additive measure for CNS that maintains phase-shift invariance.
b) An enhanced metric that integrates amplitude and phase differences effectively.
c) A regularization functional to ensure topological regularity in CNS phases.
d) An optimization-based aggregator that balances distance proximity with connectedness.

While the proposed framework is general and applicable to various domains, we illustrate its utility in the context of assessing teaching quality in university ideological and political classes during the new media era, where indicators are interconnected and cyclical [3].

The rest of the paper is structured as follows. Section 2 reviews the basics of CNS and $\alpha\psi$-connectedness. Section 3 defines the PCN-measure and discusses its properties. Section 4 introduces the phase-coherent distance and its extensions. Section 5 describes the $\alpha\psi$-connected regularization and includes a proof of preservation. Section 6 presents the phase-locked aggregation operator. Section 7 offers a detailed example with numerical results in tables. Section 8 applies the concepts to the teaching quality scenario in the new media era. Section 9 concludes with key insights and future directions.

2. Preliminaries

In this section, we review the foundational concepts of CNS and $\alpha\psi$-connectedness in NCT, which provide the mathematical backbone for the developments in later sections. Only the essential definitions and properties needed for our framework are included.

2.1 Complex Neutrosophic Sets

Let U be a non-empty universe of discourse. A complex neutrosophic set $\tilde{A}$ over U assigns to each element $x \in U$ a triplet:

$$\tilde{A}(x) = (T_A(x), I_A(x), F_A(x)),$$

where:

$T_A(x) \in \mathbb{D} \subset \mathbb{C}$ is the truth-membership value.

$I_A(x) \in \mathbb{D}$ is the indeterminacy-membership value.

$F_A(x) \in \mathbb{D}$ is the falsity-membership value.
Here $\mathbb{D} = \{z \in \mathbb{C} || z | \leq 1\}$ denotes the unit disk in the complex plane.
Each complex value is expressed in polar form:

$$T_A(x) = r_T(x)e^{i\theta_T(x)}, I_A(x) = r_I(x)e^{i\theta_I(x)}, F_A(x) = r_F(x)e^{i\theta_F(x)}$$

where $r.(x) \in [0,1]$ is the amplitude and $\theta.(x) \in [0,2\pi)$ is the phase of the respective membership component [2].

2.2 Interpretation of Amplitude and Phase

In a CNS:

a) The amplitude $r.(x)$ quantifies the magnitude or degree of truth, indeterminacy, or falsity.
b) The phase $\theta.(x)$ encodes additional cyclic, directional, or temporal information about the membership value.

This dual representation makes CNS particularly suitable for modeling periodic or oscillatory uncertainty, where the when or orientation of a membership value is as important as its magnitude. In the teachingquality context, amplitude may represent the strength of a particular performance indicator, while phase may capture its timing within a semester or engagement cycle.

2.3 Basic Set Operations in CNS

Given two CNSs $\tilde{A}$ and $\tilde{B}$ over U, basic operations such as complement, union, and intersection are defined componentwise:

Complement:

$$T_{\bar{A}}(x) = 1 - T_A(x), I_{\bar{A}}(x) = 1 - I_A(x), F_{\bar{A}}(x) = 1 - F_A(x)$$

Union:

$T_{A\cup B}(x) = \max\{T_A(x), T_B(x)\}$ (in amplitude, phase handled separately),

with analogous rules for I and F.

Intersection:

$$T_{A\cap B}(x) = \min\{T_A(x), T_B(x)\},$$

again with analogous handling for I and F[1]. In amplitude-phase form, "max" and "min" apply to amplitudes, while phase combination is defined by circular-mean rules to avoid discontinuities at 2π.

2.4 Neutrosophic Complex Topology

Let X be a non-empty set and $\mathcal{N}_C(X)$ the family of all CNSs over X.
A neutrosophic complex topology (NCT) on X is a collection $\tau \subseteq \mathcal{N}_C(X)$ satisfying:

1. The empty set and full set are in τ.
2. Arbitrary unions of sets in τ are in τ.
3. Finite intersections of sets in τ are in τ.

The pair (X, τ) is then called a neutrosophic complex topological space [2].

2. 5αψ-Connectedness

The concept of αψ-connectedness generalizes classical connectedness to neutrosophic complex topology. A set $A \in \mathcal{N}_C(X)$ is αψ-connected if it cannot be represented as:

$$A = B \cup C$$

where:

$B, C \in \tau$ are disjoint non-empty CNSs.

The membership values of B and C in each component differ by at least a phase gap of α and an amplitude gap of ψ in the relevant metric.

Intuitively, αψ-connectedness prevents "splitting" a set into two disjoint parts that are separated both in magnitude and in phase. This is crucial for our later regularization functional, which will be designed to preserve αψ-connectedness while smoothing phase discontinuities.

2.6 Distance Measures in CNS

Existing CNS literature [1] defines various distance measures between sets $\tilde{A}$ and $\tilde{B}$ based on differences in their T, I, F components, often considering amplitude differences in $[0,1]$ and ignoring phase or treating it separately. Our framework will extend these distances to a phase-coherent form d_{PCN} that integrates amplitude and wrapped phase differences into a single measure while ensuring invariance under global phase shifts. Throughout the paper:

a) r_T, r_I, r_F denote amplitudes of truth, indeterminacy, and falsity.
b) $\theta_T, \theta_I, \theta_F$ denote phases of truth, indeterminacy, and falsity.
c) arg (z) is the phase of complex number z in $[0,2\pi)$.
d) $\mathbb{S}^1$ denotes the unit circle in $\mathbb{C}$, representing the space of phases.

3. Phase-Coherent Complex Neutrosophic Measures

We now introduce the PCN-measure, which extends the classical measure concept to the complex neutrosophic setting, incorporating both amplitude and phase for each membership component.

3.1 Definition of PCN-Measure

Let (X, Σ) be a measurable space, where Σ is a σ-algebra of subsets of X. A Phase-Coherent Complex Neutrosophic Measure is a set function

$$\mu_{CN}: \Sigma \to \mathbb{D}^3$$

such that for each $E \in \Sigma$:

$$\mu_{CN}(E) = (T_E, I_E, F_E)$$

where $T_E, I_E, F_E \in \mathbb{D}$ are expressed in polar form:

$$T_E = r_T(E)e^{i\theta_T(E)}, I_E = r_I(E)e^{i\theta_I(E)}, F_E = r_F(E)e^{i\theta_F(E)}$$

3.2 Axioms

The PCN-measure satisfies the following axioms:

(A1) Null Set:

$$\mu_{CN}(\emptyset) = (0,0,0)$$

(A2) σ-Additivity on Amplitudes:

If $\{E_k\}_{k=1}^{\infty}$ are pairwise disjoint in Σ, then:

$$r_T\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} r_T(E_k)$$

and similarly for r_I and r_F, provided the sums converge to values in [0,1].
(A3) Circular-Mean Additivity on Phases:
For the same family $\{E_k\}$:

$$\theta_T\left(\bigcup_{k=1}^{\infty} E_k\right) = \mathrm{Arg}\left(\sum_{k=1}^{\infty} r_T(E_k)e^{i\theta_T(E_k)}\right)$$

and analogously for θ_I and θ_F.
Here Arg denotes the principal argument in $[0,2\pi)$.
(A4) Global Phase-Shift Invariance:
For any $\phi \in [0,2\pi)$, replacing every phase $\theta.(E)$ by $\theta.(E) + \phi$ leaves all metric properties and aggregation results unchanged.

3.3 Consequences of the Axioms

From (A2) and (A3), the polar decomposition is internally consistent:

a) Amplitudes obey classical σ-additivity.
b) Phases are aggregated in a way that respects the geometry of the unit circle, avoiding discontinuities at $0/2\pi$.

From (A4), the measure captures relative phase structure, not absolute phase - an important property in cyclic phenomena such as weekly class engagement patterns in the new media era.

3.4 Basic Properties

Proposition 3.1 (Monotonicity in Amplitude).
If $E, F \in \Sigma$ and $E \subseteq F$, then

$$r.(E) \leq r.(F), \cdot \in \{T, I, F\}.$$

Proof.
Since $F = E \cup (F \setminus E)$ with disjoint union, σ-additivity (A2) implies:

$$r.(F) = r.(E) + r.(F \setminus E) \geq r.(E)$$

Proposition 3.2 (Finite Additivity in Amplitude and Phase).
If $E, F \in \Sigma$ are disjoint:

$$T_{E\cup F} = (r_T(E) + r_T(F))e^{i\theta_T(E\cup F)},$$

with

$$\theta_T(E \cup F) = \mathrm{Arg}\left(r_T(E)e^{i\theta_T(E)} + r_T(F)e^{i\theta_T(F)}\right),$$

and analogously for I and F.
Proposition 3.3 (Absolute Boundedness).
For any $E \in \Sigma$:

$$0 \leq r.(E) \leq 1, \cdot \in \{T, I, F\}$$

This follows directly from the fact that $T_E, I_E, F_E \in \mathbb{D}$.

3.5 Example

Let $X = \{x_1, x_2, x_3\}, \Sigma = 2^X$. Define:

$$\mu_{CN}(\{x_1\}) = \left(0.4e^{i\pi/6}, 0.3e^{i\pi/3}, 0.2e^{i\pi/2}\right),$$
$$\mu_{CN}(\{x_2\}) = \left(0.5e^{i\pi/4}, 0.2e^{i\pi/3}, 0.1e^{i\pi/2}\right),$$
$$\mu_{CN}(\{x_3\}) = \left(0.3e^{i\pi/3}, 0.4e^{i\pi/6}, 0.3e^{i\pi/2}\right).$$

For $E = \{x_1, x_2\}$, amplitudes are:

$$r_T(E) = 0.4 + 0.5 = 0.9$$

and

$$\theta_T(E) = \mathrm{Arg}\left(0.4e^{i\pi/6} + 0.5e^{i\pi/4}\right) \approx 0.576\mathrm{rad}$$

This computation respects σ-additivity of amplitudes and circular-mean aggregation of phases.

4. Phase-Coherent Distance and d-Equality Extension

We now extend the existing CNS distance concepts [2] to a phase-coherent metric, d_{PCN}, that simultaneously accounts for amplitude differences and wrapped phase differences, while remaining invariant under global phase shifts. This section also defines the corresponding d-equality relation, generalizing the CNS d-equality in [2] to our phase-sensitive setting.

4.1 Definition of Wrapped Phase Distance

Let $\Delta_\theta(\alpha, \beta)$ denote the wrapped phase distance on the unit circle $\mathbb{S}^1$:

$$\Delta_\theta(\alpha, \beta) = \min_{k\in\mathbb{Z}} |\alpha - \beta + 2\pi k|, \alpha, \beta \in [0, 2\pi)$$

It satisfies:

1. Symmetry: $\Delta_\theta(\alpha, \beta) = \Delta_\theta(\beta, \alpha)$.
2. Range: $\Delta_\theta(\alpha, \beta) \in [0, \pi]$.
3. Triangle Inequality: Follows from the geodesic structure of $\mathbb{S}^1$.

4.2 Definition of d_{PCN}

Let $\tilde{A}, \tilde{B}$ be two CNSs over the same finite universe $U = \{x_1, \dots, x_n\}$.

Write

$$T_A(x_j) = r_T^A(j)e^{i\theta_T^A(j)}, T_B(x_j) = r_T^B(j)e^{i\theta_T^B(j)}$$

and analogously for I and F.

We define:

$$d_{PCN}(\tilde{A}, \tilde{B}) = \left[\frac{1}{3n}\sum_{\cdot\in\{T,I,F\}}\sum_{j=1}^{n}\left(\left|r_\cdot^A(j) - r_\cdot^B(j)\right|^2 + w_\cdot^2\Delta_\theta\left(\theta_\cdot^A(j), \theta_\cdot^B(j)\right)^2\right)\right]^{1/2},$$

where $w_\cdot > 0$ is a phase weight for component $\cdot$.

4.3 Properties of d_{PCN}

Proposition 4.1 (Non-Negativity and Identity of Indiscernibles).

For all $\tilde{A}, \tilde{B}$:

$$d_{PCN}(\tilde{A}, \tilde{B}) \geq 0, d_{PCN}(\tilde{A}, \tilde{B}) = 0 \Leftrightarrow \tilde{A} = \tilde{B}.$$

Proof.

Non-negativity follows from the fact that each squared term is non-negative.

If $d_{PCN}(\tilde{A}, \tilde{B}) = 0$, then every term in the sum is zero, so $r_\cdot^A(j) = r_\cdot^B(j)$ and $\Delta_\theta\left(\theta_\cdot^A(j), \theta_\cdot^B(j)\right) = 0$ for all $\cdot, j$. The latter implies $\theta_\cdot^A(j) = \theta_\cdot^B(j) \mathrm{mod} 2\pi$, hence equality in polar form. The converse is immediate.

Proposition 4.2 (Symmetry).
$d_{PCN}(\tilde{A}, \tilde{B}) = d_{PCN}(\tilde{B}, \tilde{A})$.
Proof.
Absolute value and Δ_θ are symmetric in their arguments; the sums are identical upon swapping A and B.

Proposition 4.3 (Triangle Inequality).
For any $\tilde{A}, \tilde{B}, \tilde{C}$:

$$d_{PCN}(\tilde{A}, \tilde{C}) \leq d_{PCN}(\tilde{A}, \tilde{B}) + d_{PCN}(\tilde{B}, \tilde{C})$$

Proof.
Fix a component $\cdot$ and index j.
By the standard triangle inequality in $\mathbb{R}$:

$$\left|r_{\cdot}^{A}(j) - r_{\cdot}^{C}(j)\right| \leq \left|r_{\cdot}^{A}(j) - r_{\cdot}^{B}(j)\right| + \left|r_{\cdot}^{B}(j) - r_{\cdot}^{C}(j)\right|.$$

Similarly, by the geodesic triangle inequality on $\mathbb{S}^1$:

$$\Delta_\theta\left(\theta_{\cdot}^{A}(j), \theta_{\cdot}^{C}(j)\right) \leq \Delta_\theta\left(\theta_{\cdot}^{A}(j), \theta_{\cdot}^{B}(j)\right) + \Delta_\theta\left(\theta_{\cdot}^{B}(j), \theta_{\cdot}^{C}(j)\right).$$

Squaring and summing preserves subadditivity up to a constant (by Minkowski's inequality), and averaging over $\cdot, j$ yields the stated triangle inequality.

4.4 Global Phase-Shift Invariance
Proposition 4.4.
For any $\phi \in [0,2\pi)$, define $\tilde{A}^{\phi}$ by replacing each phase $\theta_{\cdot}^{A}(j)$ with $\theta_{\cdot}^{A}(j) + \phi$. Then:

$$d_{PCN}\left(\tilde{A}^{\phi}, \tilde{B}^{\phi}\right) = d_{PCN}(\tilde{A}, \tilde{B})$$

Proof.
Since $\Delta_\theta(\alpha + \phi, \beta + \phi) = \Delta_\theta(\alpha, \beta)$, the phase terms are unaffected by uniform shifts; amplitudes are unchanged. Hence the formula for d_{PCN} is invariant.

4.5 d-Equality Extension
We define a d-equality relation $\equiv_{d_{PCN}}$ on CNSs as:

$$\tilde{A} \equiv_{d_{PCN}} \tilde{B} \iff d_{PCN}(\tilde{A}, \tilde{B}) \leq \varepsilon,$$

for some pre-specified tolerance $\varepsilon \geq 0$.
When $\varepsilon = 0, \equiv_{d_{PCN}}$ reduces to exact equality of amplitudes and phases modulo 2π.
For $\varepsilon > 0$, it defines an approximate equality tolerant to small amplitude or phase differences - crucial when working with real-world periodic signals where perfect equality is unattainable.

4.6 Example
Let $U = \{x_1, x_2\}$, phase weight w. $= 1$ for all components.
Suppose:

$$T_A(x_1) = 0.8e^{i\pi/6}, \quad T_B(x_1) = 0.7e^{i\pi/4}$$
$$T_A(x_2) = 0.6e^{i\pi/3}, \quad T_B(x_2) = 0.6e^{i\pi/2}$$

Then for x_1 :

$$|0.8 - 0.7|^2 = 0.01, \Delta_\theta(\pi/6, \pi/4) = \pi/12 \approx 0.2618, (\Delta_\theta)^2 \approx 0.0686.$$

For x_2, amplitude term $= 0$, phase term:

$$\Delta_\theta(\pi/3, \pi/2) = \pi/6 \approx 0.5236, (\Delta_\theta)^2 \approx 0.2742.$$

Summing over both points and all components (for illustration, using only T here) and normalizing gives a concrete d_{PCN} value.

4.7 Role in Our Framework

The metric d_{PCN} :

a) Provides the proximity measure in the Phase-Locked Aggregation Operator.
b) Ensures that both amplitude and phase differences are treated consistently.
c) Guarantees invariance under uniform phase shifts, making it robust to global timing offsets in cyclic signals, such as those found in teaching-quality cycles in the new media era.

αψ-Connected Regularization Functional

We now define a regularization functional $\mathcal{R}_{\alpha\psi}$ for CNS data, designed to enforce αψ-connectedness in both amplitude and phase. The goal is to smooth phase discontinuities and amplitude gaps while ensuring the resulting CNS remains topologically coherent in the sense of Sec 2.5.

5.1 Motivation

In practical datasets, e.g., periodic teaching-quality metrics in the New Media Era - performance indicators often exhibit small phase jumps or amplitude inconsistencies between consecutive elements. Unregularized, such jumps may violate αψ - connectedness, splitting the CNS into disconnected clusters. The functional $\mathcal{R}_{\alpha\psi}$ penalizes such violations.

5. 2αψ-Separation Metric

Given two CNS elements p, q (which may represent points, time steps, or features), define the αψ separation:

$$\delta_{\alpha\psi}(p,q) = \max_{\cdot\in\{T,I,F\}} \left[\frac{|r.(p) - r.(q)|}{\psi}, \frac{\Delta_\theta(\theta.(p), \theta.(q))}{\alpha}\right]$$

If $\delta_{\alpha\psi}(p,q) > 1$, the pair is considered disconnected for our purposes.

5.3 Definition of the Functional

Let $U = \{x_1, \dots, x_n\}$ and $\mathcal{N}_C(U)$ be the set of CNSs on U.

For $\tilde{A} \in \mathcal{N}_C(U)$, define:

$$\mathcal{R}_{\alpha\psi}(\tilde{A}) = \frac{1}{n-1}\sum_{j=1}^{n-1} \sum_{\cdot\in\{T,I,F\}} \left[\frac{\left|r.\left(x_{j+1}\right) - r.\left(x_j\right)\right|^2}{\psi^2} + \frac{\Delta_\theta\left(\theta.\left(x_{j+1}\right), \theta.\left(x_j\right)\right)^2}{\alpha^2}\right]$$

Interpretation:

a) The amplitude term penalizes jumps greater than ψ.
b) The phase term penalizes jumps greater than α.
c) The sum is normalized by $n - 1$ for comparability across datasets.

5.4 Minimization Problem

Given an initial CNS $\tilde{A}_0$, we consider:

$$\tilde{A}^* = \arg \min_{\tilde{A} \in \mathcal{N}_C(U)} \left[E(\tilde{A}) + \lambda \mathcal{R}_{\alpha\psi}(\tilde{A}) \right]$$

where:

$E(\tilde{A})$ is a fidelity term (e.g., $d^2_{PCN}(\tilde{A}, \tilde{A}_0)$).

$\lambda > 0$ controls the regularization strength.

5.5 Preservation Theorem

Theorem 5.1.

If $\mathcal{R}_{\alpha\psi}(\tilde{A}^*) \leq 1$, then $\tilde{A}^*$ is αψ-connected.

Proof.

For any consecutive x_j, x_{j+1}, the squared amplitude and phase differences normalized by ψ and α sum to at most 1 .

Thus:

$$\frac{|r_\cdot(x_{j+1}) - r_\cdot(x_j)|}{\psi} \leq 1, \frac{\Delta_\theta\left(\theta_\cdot(x_{j+1}), \theta_\cdot(x_j)\right)}{\alpha} \leq 1$$

for all $\cdot$.

By definition of αψ-connectedness (Section 2.5), no adjacent pair is disconnected, and transitivity over the finite chain $x_1, \ldots, x_n$ yields global αψ-connectedness.

5.6 Existence of Minimizers

Proposition 5.2.

If U is finite and $E(\tilde{A})$ is continuous in amplitudes and phases, then a minimizer $\tilde{A}^*$ exists.

Proof.

The feasible set $\mathcal{N}_C(U)$ is compact in the product topology $[0,1]^{3n} \times (\mathbb{S}^1)^{3n}$.

The objective function $E(\tilde{A}) + \lambda \mathcal{R}_{\alpha\psi}(\tilde{A})$ is continuous.

By the Weierstrass theorem, a minimum is attained.

5.7 Example

Let $U = \{t_1, t_2, t_3\}$ represent three consecutive weeks of a course.

Truth amplitudes: $r_T = (0.8, 0.75, 0.7)$.

Truth phases: $\theta_T = (0, \pi/8, \pi/4)$.

Let $\psi = 0.1, \alpha = \pi/6$.

For week $1 \to 2$:

$$\frac{|0.75 - 0.8|^2}{0.1^2} = 0.25, \frac{\Delta_\theta(0, \pi/8)^2}{(\pi/6)^2} \approx 0.5625.$$

$$\text{Sum } = 0.8125.$$

For week $2 \to 3$:

$$\frac{|0.7 - 0.75|^2}{0.1^2} = 0.25, \frac{\Delta_\theta(\pi/8, \pi/4)^2}{(\pi/6)^2} \approx 0.0625$$

Sum $= 0.3125$.

Thus:

$$\mathcal{R}_{\alpha\psi}(\tilde{A}) \approx \frac{0.8125 + 0.3125}{2} = 0.5625 \leq 1,$$

hence αψ-connectedness is preserved.

6. Phase-Locked Aggregation Operator

We now introduce an optimization-based aggregator that fuses several complex neutrosophic datasets while locking phases and enforcing $\alpha\psi$-regularity.

6.1 Problem formulation

Let $U = \{x_1, \dots, x_n\}$ and suppose we have m CNS $\tilde{A}^{(s)}$ on U (e.g., multiple assessments or sources), $s = 1, \dots, m$. For each component $\cdot \in \{T, I, F\}$ and each x_j, write

$$\tilde{A}_{\cdot}^{(s)}(x_j) = r_{\cdot}^{(s)}(j)e^{i\theta_{\cdot}^{(s)}(j)}, r_{\cdot}^{(s)}(j) \in [0,1], \theta_{\cdot}^{(s)}(j) \in [0,2\pi).$$

Let nonnegative source weights $w_s > 0$ (normalized or not), and phase component weights $w_{\cdot} > 0$.

We seek a CNS $\tilde{Z}$ that minimizes the phase-locked functional

$$\mathcal{J}(\tilde{Z}) = \frac{1}{3n} \sum_{\cdot \in \{T,I,F\}} \sum_{j=1}^{n} \sum_{s=1}^{m} w_s \left(\left| r_{\cdot}(j) - r_{\cdot}^{(s)}(j) \right|^2 + w_{\cdot}^2 \Delta_\theta \left(\theta_{\cdot}(j), \theta_{\cdot}^{(s)}(j) \right)^2 \right) + \gamma \mathcal{R}_{\alpha\psi}(\tilde{Z})$$

where Δ_θ is the wrapped phase distance (Sec. 4.1), and $\mathcal{R}_{\alpha\psi}$ is the αψ-regularizer from Sec. 5.3 with parameter $\gamma \geq 0$. The Phase-Locked Aggregation operator (PLA) is

$$\mathrm{PLA}\left(\tilde{A}^{(1)}, \dots, \tilde{A}^{(m)}\right) = \arg\min_{\tilde{Z} \in \mathcal{N}_C(U)} \mathcal{J}(\tilde{Z}).$$

6.2 Existence, decomposition, and invariances

Theorem 6.1 (Existence & decomposition).

On finite U, a minimizer $\tilde{Z}^\star$ exists. Moreover, $\mathcal{J}$ splits additively into an amplitude part and a phase part:

$$\mathcal{J}(\tilde{Z}) = \underbrace{\frac{1}{3n} \sum_{\cdot,j} \sum_{s} w_s \left| r_{\cdot}(j) - r_{\cdot}^{(s)}(j) \right|^2}_{\mathcal{J}_{\mathrm{amp}}} + \underbrace{\frac{1}{3n} \sum_{\cdot,j} \sum_{s} w_s w_{\cdot}^2 \Delta_\theta \left(\theta_{\cdot}(j), \theta_{\cdot}^{(s)}(j) \right)^2}_{\mathcal{J}_{\mathrm{phase}}} + \gamma \mathcal{R}_{\alpha\psi}(\tilde{Z}).$$

Proof. Compactness of $[0,1]^{3n} \times (\mathbb{S}^1)^{3n}$ and continuity of all terms (Sec. 5.6) yield existence. Separability follows because amplitudes and phases appear in disjoint summands.

Proposition 6.2 (Global phase-shift invariance).

For any $\phi \in [0,2\pi)$, replacing every $\theta_{\cdot}(j)$ and $\theta_{\cdot}^{(s)}(j)$ by $" + \phi"$ leaves $\mathcal{J}$ unchanged (Sec. 4.4), hence PLA is invariant up to a global phase.

6.3 Closed-form amplitudes

Theorem 6.3 (Amplitude solution).

For each $\cdot, j$,

$$r_{\cdot}^\star(j) = \Pi_{[0,1]} \left(\frac{\sum_{s=1}^{m} w_s r_{\cdot}^{(s)}(j)}{\sum_{s=1}^{m} w_s} \right)$$

where $\Pi_{[0,1]}$ denotes projection onto $[0,1]$.

Proof. $\mathcal{J}_{\mathrm{amp}}$ is strictly convex quadratic in $r_{\cdot}(j)$; its unconstrained minimizer is the

weighted average.
Projecting enforces membership bounds.

6.4 Phases: circular means and regularized locking

Let $\varphi_\cdot(j) = \theta_\cdot(j) \in [0,2\pi)$. For $\gamma = 0$ the phase term decouples across j.

Lemma 6.4 (Unregularized phase, $\gamma = 0$).

If all $\left\{\theta_\cdot^{(s)}(j)\right\}_s$ lie in an open semicircle, then the unique minimizer is the weighted circular mean

$$\varphi_\cdot^\star(j) = \mathrm{Arg}\left(\sum_{s=1}^{m} w_s e^{i\theta_\cdot^{(s)}(j)}\right)$$

and uniqueness holds up to a global phase shift.

Proof. The Fréchet mean on $\mathbb{S}^1$ with squared geodesic loss is unique under the semicircle condition; our objective at fixed j,· is exactly that loss.

For $\gamma > 0, \alpha\psi$-regularization couples $\{\varphi_\cdot(j)\}_j$. Assume phase-consistency: there exists an unwrap (branch) with $|\varphi_\cdot(j+1) - \varphi_\cdot(j)| < \pi$ for all adjacent indices.

Theorem 6.5 (Regularized phase, uniqueness up to global shift).

Under phase-consistency and connectedness of the adjacency graph in $\mathcal{R}_{\alpha\psi}$, the regularized phase problem is strictly convex in the unwrapped angles; it has a unique minimizer modulo a global shift. If one pins a reference (e.g., $\sum_j \varphi_\cdot(j) = 0$), the minimizer is unique.

Proof . Choose a branch where all wrapped differences are represented as real differences; then Δ_θ equals Euclidean distance and $\mathcal{R}_{\alpha\psi}$ becomes a positive-semidefinite quadratic form with Laplacian L. The phase objective is

$$Q(\varphi) = \frac{1}{3n}\sum_j \sum_s w_s w_\cdot^2 \left(\varphi_\cdot(j) - \theta_\cdot^{(s)}(j)\right)^2 + \gamma \varphi_\cdot^\top L \varphi_\cdot,$$

which is strictly convex because $\sum_s w_s > 0$ for each j and $L \succeq 0$ on a connected graph. Nullspace corresponds to uniform shifts; pinning removes it.

6.5 Efficient PLA algorithm

For each component · :

1. Amplitudes: compute $r_\cdot^\star(j)$ via Theorem 6.3.
2. Initialization: set $\varphi_\cdot^{(0)}(j) = \mathrm{Arg}\left(\sum_s w_s e^{i\theta_\cdot^{(s)}(j)}\right)$.
3. Unwrap along $j = 1 \to n$ with jumps $< \pi$.
4. Solve the linear system $(W_\cdot + \tilde{\gamma}L)\varphi_\cdot = b_\cdot$, where $W_\cdot = \mathrm{diag}(\sum_s w_s w_\cdot^2), \tilde{\gamma} = 3n\gamma$ and $b_\cdot = \sum_s w_s w_\cdot^2 \theta_\cdot^{(s)}$ (on the unwrap).
5. Wrap back to $[0,2\pi)$ and assemble $\tilde{Z}^\star$.

Complexity is $O(mn)$ for statistics plus $O(n)$ for solving the banded system (chain L); for general sparse L, near-linear solvers apply.

6.6 PLA example ($\gamma = 0$, equal weights)

Let $U = \{t_1, t_2, t_3\}, m = 2, w_1 = w_2 = 1$. For clarity we use exact fractional angles; the semicircle condition holds at each t_j.

Truth (T) inputs

$$r_T^{(1)} = (0.70, 0.75, 0.72), \quad \theta_T^{(1)} = \left(0, \frac{\pi}{8}, \frac{\pi}{6}\right)$$

$$r_T^{(2)} = (0.65, 0.74, 0.77), \quad \theta_T^{(2)} = \left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{5}\right)$$

Indeterminacy (I) inputs

$$r_I^{(1)} = (0.20, 0.22, 0.21), \quad \theta_I^{(1)} = \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}\right),$$

$$r_I^{(2)} = (0.18, 0.24, 0.20), \quad \theta_I^{(2)} = \left(\frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{4}\right).$$

Falsity (F) inputs

$$r_F^{(1)} = (0.10, 0.12, 0.11), \quad \theta_F^{(1)} = \left(\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}\right),$$

$$r_F^{(2)} = (0.12, 0.11, 0.09), \quad \theta_F^{(2)} = \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right).$$

PLA amplitudes (Theorem 6.3): simple averages

$$r_T^\star = (0.675, 0.745, 0.745), r_I^\star = (0.19, 0.23, 0.205), r_F^\star = (0.11, 0.115, 0.10)$$

PLA phases ($\gamma = 0$, Lemma 6.4): equal-weight circular midpoints

$$\theta_T^\star = \left(\frac{\pi}{24}, \frac{7\pi}{48}, \frac{11\pi}{60}\right),$$

$$\theta_I^\star = \left(\frac{9\pi}{40}, \frac{7\pi}{24}, \frac{5\pi}{24}\right),$$

$$\theta_F^\star = \left(\frac{7\pi}{24}, \frac{5\pi}{12}, \frac{5\pi}{12}\right).$$

These results are summarized in Table 1 and are directly cited here.

Table 1. PLA outputs for the fully calculated example ($\gamma = 0$, equal weights)

t_j	$r_T^\star$	$\theta_T^\star$	$r_I^\star$	$\theta_I^\star$	$r_F^\star$	$\theta_F^\star$
t_1	0.675	$\frac{\pi}{24}$	0.190	$\frac{9\pi}{40}$	0.110	$\frac{7\pi}{24}$
t_2	0.745	$\frac{7\pi}{48}$	0.230	$\frac{7\pi}{24}$	0.115	$\frac{5\pi}{12}$
t_3	0.745	$\frac{11\pi}{60}$	0.205	$\frac{5\pi}{24}$	0.100	$\frac{5\pi}{12}$

Table 1 lists the phase-locked aggregation outputs obtained by PLA using the exact closed-form formulas (Theorem 6.3 and Lemma 6.4). Amplitudes are weighted averages; phases are equalweight circular means, demonstrating phase coherence and boundedness within $[0, 2\pi)$.

When $\gamma > 0$, the PLA phases are obtained by solving the banded linear system in Sec. 6.5 on an unwrap consistent with α (Sec. 5). This yields smoother phases across t_1, t_2, t_3 and, by Theorem 6.5, a unique solution up to global shift.

PLA produces a single, coherent CNS profile from multiple sources (e.g., platform logs and classroom observations), preserving both amplitude (strength) and phase (timing

pattern). This respects the periodicity and connectivity principles emphasized by CNS and neutrosophic complex topology [2], without turning the paper into an application study.

7. End-to-End Example

This section completes the theory with explicit numbers, using the PLA outputs from Table 1. We (*i*) compute a normalized PCN-measure for unions of atoms, (*ii*) evaluate the phase-coherent distance d_{PCN} before/after aggregation, and (*iii*) verify αψ-connectedness via the regularizer. Throughout we set the phase weights $w_T = w_I = w_F = 1$ and measure angles in radians.

7.1 Data used (from Table 1)

For $U = \{t_1, t_2, t_3\}$, PLA returned:

Truth: $r_T^\star = (0.675, 0.745, 0.745), \theta_T^\star = \left(\frac{\pi}{24}, \frac{7\pi}{48}, \frac{11\pi}{60}\right)$.

Indeterminacy: $r_I^\star = (0.190, 0.230, 0.205), \theta_I^\star = \left(\frac{9\pi}{40}, \frac{7\pi}{24}, \frac{5\pi}{24}\right)$.

Falsity: $r_F^\star = (0.110, 0.115, 0.100), \theta_F^\star = \left(\frac{7\pi}{24}, \frac{5\pi}{12}, \frac{5\pi}{12}\right)$.

7.2 PCN-measure on unions (σ-additivity on amplitudes + circular mean on phases)

To define a PCN-measure μ_{CN} compatible with Section 3, we normalize amplitudes so that the total amplitude over U is 1 per component:

$$c_T = \frac{1}{0.675 + 0.745 + 0.745} = \frac{1}{2.165} \approx 0.461893, c_I = \frac{1}{0.625} = 1.600000, c_F = \frac{1}{0.325} \approx 3.076923.$$

For a set $E \subseteq U$, amplitudes add:

$$r_T(E) = c_T \sum_{t_j \in E} r_T^\star(t_j), r_I(E) = c_I \sum r_I^\star, r_F(E) = c_F \sum r_F^\star,$$

while phases are the argument of the vector sum:

$$\theta_T(E) = \mathrm{Arg}\left(\sum_{t_j \in E} c_T r_T^\star(t_j) e^{i\theta_T^\star(t_j)}\right), \text{etc.}$$

Example $1 - E = \{t_1, t_2\}$.

$r_T(E) = 0.461893(0.675 + 0.745) = 0.655889$

$\theta_T(E) = \mathrm{Arg}\left(0.461893 \cdot 0.675 e^{i\pi/24} + 0.461893 \cdot 0.745 e^{i7\pi/48}\right) = 0.302663$

$r_I(E) = 1.6(0.190 + 0.230) = 0.672000, \theta_I(E) = 0.821588$

$r_F(E) = 3.076923(0.110 + 0.115) = 0.692308, \theta_F(E) = 1.117068$

Example $2 - E = U$.

$$r_T(U) = 1.000000, \quad \theta_T(U) = 0.397299$$
$$r_I(U) = 1.000000, \quad \theta_I(U) = 0.766641$$
$$r_F(U) = 1.000000, \quad \theta_F(U) = 1.176823$$

These calculations exhibit σ-additivity on amplitudes and circular-mean aggregation of phases as required by Section 3.

7.3 Phase-coherent distance before vs. after PLA

Recall (Section 4):

$$d_{PCN}(\tilde{A},\tilde{B}) = \left[\frac{1}{3n}\sum_{\cdot\in\{T,I,F\}}\sum_{j=1}^{n}\left(\left|r_{\cdot}^{A} - r_{\cdot}^{B}\right|^2 + \Delta_\theta\left(\theta_{\cdot}^{A},\theta_{\cdot}^{B}\right)^2\right)\right]^{1/2}$$

Step-by-step illustration (one term).

At t_1 for truth:

$$|0.675 - 0.70|^2 = 0.000625, \Delta_\theta\left(\frac{\pi}{24}, 0\right) = \frac{\pi}{24} = 0.130899, (\Delta_\theta)^2 = 0.017143$$

The other terms are computed analogously and aggregated. Aggregated results (all components, all t_j; $n = 3$):

Distance PLA → Source 1:

$$d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(1)}\right) = 0.1565.$$

Distance PLA → Source 2 :

$$d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(2)}\right) = 0.1565.$$

Source-to-Source distance:

$$d_{PCN}\left(\tilde{A}^{(1)}, \tilde{A}^{(2)}\right) = 0.3131.$$

As expected for equal-weight least-squares on amplitudes and circular means on phases,

$$d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(1)}\right) = d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(2)}\right) = \frac{1}{2} d_{PCN}\left(\tilde{A}^{(1)}, \tilde{A}^{(2)}\right),$$

showing that PLA halves the pairwise distance.

7. 4αψ-regularizer and connectedness check

With $\psi = 0.10$ and $\alpha = \frac{\pi}{5}$ (i.e., $\alpha \approx 0.6283$; cf. Section 5), the total regularizer value for $\tilde{Z}^{\star}$ is

$$\mathcal{R}_{\alpha\psi}\left(\tilde{Z}^{\star}\right) = 0.8596 (\leq 1).$$

By Theorem 5.1, this bound ensures the aggregated CNS is αψ-connected (no disconnected phase or amplitude jumps across the timeline).

Table 2. Distances and regularization summary for the end-to-end example

Quantity	Value
$d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(1)}\right)$	0.1565
$d_{PCN}\left(\tilde{Z}^{\star}, \tilde{A}^{(2)}\right)$	0.1565
$d_{PCN}\left(\tilde{A}^{(1)}, \tilde{A}^{(2)}\right)$	0.3131
$\mathcal{R}_{\alpha\psi}\left(\tilde{Z}^{\star}\right)$ (with $\psi = 0.10, \alpha = \pi/5$)	0.8596

Table 2 compiles the phase-coherent distances and the αψ-regularizer for the PLA solution $\tilde{Z}^{\star}$. PLA halves the source-to-source distance and attains $\mathcal{R}_{\alpha\psi} \leq 1$, which certifies αψ connectedness as per Theorem 5.1.

7.5 Contextual note (new media era teaching-quality)

Interpreting t_1, t_2, t_3 as consecutive instructional periods, the amplitudes capture strength of indicators (e.g., content mastery, engagement), while phases encode their timing within the cycle. PLA yields a single coherent profile that (*i*) reduces disagreement to half the original discrepancy and (*ii*) remains connected under αψ-criteria, reflecting a consistent

pedagogical trajectory rather than disjointed episodes-precisely the behavior expected in a new media era with periodic cycles of interaction.

8. Discussion and Contextualization

The proposed PLA framework, combined with $\alpha\psi$-regularization, introduces a mathematically rigorous method for synthesizing complex neutrosophic datasets while preserving both structural coherence and phase consistency. This approach moves beyond classical aggregation in three key aspects:

1. Dual-domain integration Amplitudes are handled in the Euclidean domain with exact weighted averaging, while phases are aggregated on the unit circle using circular means. This avoids artifacts such as phase wrapping errors, which are critical in cyclic phenomena like academic calendar cycles or instructional engagement rhythms.
2. Topological robustness $\alpha\psi$-regularization enforces smoothness constraints that prevent disconnections in the neutrosophic topology. This ensures that the aggregated profile is not merely a statistical compromise but a connected entity in the $\alpha\psi$-sense, aligning with theoretical advances in neutrosophic topology [2].
3. Practical interpretability The framework produces outputs that can be directly interpreted:
    a) Amplitude represents the strength or magnitude of an indicator (e.g., mastery of ideological concepts, engagement levels).
    b) Phase represents timing or cyclical positioning, e.g., whether engagement peaks early or late in a blended instructional cycle).

By preserving these dimensions, PLA provides richer insights than scalar aggregation methods. In the New Media Era, ideological and political classes in universities face increased temporal variability due to blended formats, asynchronous content delivery, and diversified media channels. PLA offers a mathematical tool to integrate heterogeneous time-stamped assessments such as online quiz performance, discussion forum participation, and in-class feedback into a unified, phase-aware model. This model captures both what is achieved (amplitude) and when it occurs (phase), which is vital for adaptive scheduling and pedagogical interventions.

9. Conclusion

This study presents a rigorous mathematical framework for integrating complex neutrosophic data while preserving both its magnitude and temporal structure. By combining a phase-coherent measure with $\alpha\psi$-regularization, the approach ensures statistical consistency, maintains topological connectivity, and avoids the disruptions that occur in traditional averaging techniques. At the core of the framework is the phase-locked aggregation operator, which unifies heterogeneous data sources into a single coherent representation. This operator captures both amplitude (the strength of indicators) and phase (their timing or cyclical pattern), making it especially well-suited for periodic and interconnected phenomena.

Applied to the context of ideological and political education in the new media era, the framework demonstrates how performance indicators such as student engagement, content mastery, and multimedia integration can be modeled in a phase-aware manner that respects both intensity and timing. The results confirm that the proposed method reduces discrepancies between data sources while preserving $\alpha\psi$-connectedness, leading to more interpretable and coherent profiles of teaching quality.

Beyond education, this framework provides a flexible foundation for analyzing any cyclic or phase-sensitive data, with potential applications in fields such as artificial intelligence, social network analysis, and dynamic system modeling. Future research may focus on extending the framework to large-scale datasets, optimizing computational efficiency, and exploring domain-specific applications.

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