NSS

Neutrosophic Sets and Systems, Vol. 93, 2025

University of New Mexico

A Neutrosophic Functional-Analytic Model for Urban Sculpture Design Quality Evaluation

Bingbing Diao*

Film Academy of Performing Arts, Sichuan Film and Television University, Chengdu, 610036, Sichuan, China

*Corresponding author, E-mail: dbb20140824@163.com

Abstract

Urban sculpture design must reconcile aesthetics, structural integrity, site context, public engagement, safety, and sustainability under evaluator disagreement and incomplete information. Classical and fuzzy methods compress ambiguity into single-valued scores, masking how much of a judgment is uncertain rather than negative. We propose a neutrosophic functional-analytic framework that models each criterion-level judgment as a triplet $q=(T,I,F)$capturing truth, indeterminacy, and falsity. On top of this informational primitive, we construct a neutrosophic inner-product module and an operator-based evaluation calculus: (i) a neutrosophic inner product returning triplets; (ii) a scalarization that yields well-defined seminorms/norms; (iii) bounded linear evaluation functionals for criteria; and (iv) a Quality Operator producing an institutional Neutrosophic Design Index (NDI). We prove consistency properties (monotonicity, boundedness, order-invariance) and deliver a worked numerical study evaluating three sculpture alternatives on six criteria. Results show the framework separates ambiguity from negativity, enabling targeted design refinement,e.g., decreasing I through clearer briefs or mock-ups, without conflating it with F. The approach is static, mathematically defined, computationally transparent, and directly applicable to urban design juries and public art programs.

Keywords: Neutrosophy; Urban sculpture; Design evaluation; Functional analysis; Hilbert/Banach modules; Indeterminacy; Operator methods; Multi-criteria assessment.

1. Introduction

Urban sculptures shape public spaces by blending aesthetic appeal, structural integrity, and contextual harmony while engaging communities and ensuring safety and sustainability. Evaluating their design quality is complex, as it involves reconciling diverse criteria under conditions of evaluator disagreement, incomplete information, and inherent ambiguity. Traditional evaluation methods, such as weighted scoring or fuzzy logic, often reduce judgments to single-valued scores, conflating uncertainty with

negative assessments [1,2]. This oversimplification obscures the distinction between genuine deficiencies and indeterminacy, limiting the ability to target design improvements effectively.
To address this, we propose a novel neutrosophic functional-analytic framework for urban sculpture design evaluation. Neutrosophy, introduced by Smarandache [1], represents judgments as triplets capturing truth (T), indeterminacy (I), and falsity (F), offering a richer model for handling ambiguity. Building on this, we develop a mathematical structure inspired by functional analysis [3,4], including a neutrosophic inner-product module, scalarized norms, bounded linear evaluation functionals, and a Quality Operator yielding a NDI. This framework preserves the nuanced interplay of T, I, and F, enabling transparent and auditable rankings that distinguish uncertainty from negativity.

Our approach is static, computationally explicit, and designed for practical use by urban design juries and public art programs. Through a numerical study evaluating three sculpture designs across six criteria, we demonstrate how the model supports precise decision-making by identifying whether high indeterminacy or low truth drives a design's ranking. This guides targeted refinements, such as clarifying site narratives or enhancing public previews, before costly redesigns. The framework's consistency is ensured by proven properties like monotonicity, boundedness, and stability [5], making it a robust tool for multi-criteria assessment in urban design.

1.1. Preliminaries and Notation

We work on a finite criteria set $\mathcal{C} = \{1, \dots, \mathrm{m}\}$. A neutrosophic number is a triplet

$$\mathrm{q} = (\mathrm{T}, \mathrm{I}, \mathrm{F}) \in [0,1]^3, 0 \leq \mathrm{T} + \mathrm{I} + \mathrm{F} \leq 3$$

interpreted as degrees of truth, indeterminacy, and falsity. For urban sculpture design s, the criterion-level judgment is $\mathrm{q_k(s)} = (\mathrm{T_k}, \mathrm{I_k}, \mathrm{F_k})$ $[1-2]$.

Algebra on triplets

For $\mathrm{q} = (\mathrm{T}, \mathrm{I}, \mathrm{F}), \mathrm{q}' = (\mathrm{T}', \mathrm{I}', \mathrm{F}')$:

a) Componentwise partial order: $\mathrm{q} \preceq \mathrm{q}' \Leftrightarrow \mathrm{T} \leq \mathrm{T}', \mathrm{I} \geq \mathrm{I}', \mathrm{F} \geq \mathrm{F}'$ (more desirable means larger T and smaller I, F).

b) Convex combination: for $\lambda \in [0,1]$,

$$\lambda\mathrm{q} + (1-\lambda)\mathrm{q}' = (\lambda\mathrm{T} + (1-\lambda)\mathrm{T}', \lambda\mathrm{I} + (1-\lambda)\mathrm{I}', \lambda\mathrm{F} + (1-\lambda)\mathrm{F}').$$

Scalarization and admissible penalty

To compare triplets numerically we define, for $\lambda \in [0,1]$, the admissible scalarization

$$\mathrm{sc_\lambda(q)} := \mathrm{T} - \mathrm{F} - \lambda\mathrm{I} \in [-1,1]$$

It preserves design preference: if $q \preceq q'$ then $sc_\lambda(q) \le sc_\lambda(q')$.

Weights and design vectors

Let $w = (w_1, \ldots, w_m)$ be positive weights with $\sum_k w_k = 1$. A design s is represented by the vector of triplets

$$q(s) = (q_1(s), \ldots, q_m(s)) \in ([0,1]^3)^m$$

2. Neutrosophic Functional-Analytic Layer

We construct a neutrosophic inner-product module $(\mathcal{H}_N, \langle\langle\cdot,\cdot\rangle\rangle)$ over $\mathbb{R}$, where inner products return triplets [3].

Inner product returning triplets

For $q, r \in \mathcal{H}_N = ([0,1]^3)^m$, define

$$\langle\langle q, r\rangle\rangle = \left(\sum_{k=1}^{m} w_k T_k R_k, \sum_{k=1}^{m} w_k \Phi_I(q_k, r_k), \sum_{k=1}^{m} w_k \Phi_F(q_k, r_k)\right),$$

where $q_k = (T_k, I_k, F_k), r_k = (R_k, J_k, G_k)$, and coupling functions

$$\Phi_I(q_k, r_k) := T_k J_k + I_k R_k + I_k J_k, \Phi_F(q_k, r_k) := T_k G_k + F_k R_k + F_k G_k$$

Properties:

a) Symmetry: $\langle\langle q, r\rangle\rangle = \langle\langle r, q\rangle\rangle$.
b) Positive T-component: $\sum w_k T_k^2 \ge 0$ with equality only if all $T_k = 0$.
c) Monotonicity: If $q \preceq r$ componentwise, then each component of $\langle\langle q, q\rangle\rangle$ does not exceed that of $\langle\langle r, r\rangle\rangle$ in desirability sense (higher T, lower I, F).

Scalarized seminorm and norm

Using sc_λ we induce the scalar inner product

$$\langle q, r\rangle_\lambda := sc_\lambda(\langle\langle q, r\rangle\rangle),$$

and a seminorm $\|q\|_\lambda := \sqrt{\langle q, q\rangle_\lambda}$. For design evaluation we restrict to sets where $\|\cdot\|_\lambda$ is positive definite (e.g., exclude trivial all-zero T with maximal I, F), giving a normed space ($\mathcal{H}_N, \|\cdot\|_\lambda$) [4-5].

Bounded linear evaluation functionals

For each criterion k, define $E_k: \mathcal{H}_N \to [0,1]^3$ by $E_k(q) = q_k$. The aggregate evaluation functional is

$$E(q) = \sum_{k=1}^{m} w_k q_k = \left(\sum w_k T_k, \sum w_k I_k, \sum w_k F_k\right),$$

which is linear (componentwise) and bounded. Its scalarization gives the score

$$S_\lambda(q) = \sum_{k=1}^{m} w_k sc_\lambda(q_k) = \sum w_k(T_k - F_k - \lambda I_k)$$

3. Urban Sculpture Evaluation Model

We fix six criteria typical for public art commissions: $\mathcal{C} = \{\mathrm{C1}$: Context Fit, C2: Structural Integrity, C3: Aesthetic Coherence, C4: Accessibility & Safety, C5: Sustainability, C6: Public Engagement}. Weights (institutional policy):

$$w = (0.20, 0.20, 0.15, 0.15, 0.15, 0.15), \sum w_k = 1.$$

For any design s, expert panels produce $q_k(s) = (T_k, I_k, F_k)$. The Neutrosophic Design Index (NDI) is defined in two equivalent ways:

1. Operator + scalarization

$$NDI_\lambda(s) = sc_\lambda(E(q(s))) = \sum_{k=1}^{m} w_k(T_k - F_k - \lambda I_k).$$

2. Penalty multiplicative form (used for ranking with a visible I-penalty):

$$NDI_\lambda^\times(s) = \left(\sum w_k T_k - \sum w_k F_k\right)\left(1 - \sum w_k I_k\right).$$

Both are monotone in $T \uparrow, I \downarrow, F \downarrow$. We will report both.

4. Numerical Study

We evaluate three alternatives S_1, S_2, S_3. Panel-produced triplets are rounded to two decimals for auditability. We set $\lambda = 1$ for a conservative indeterminacy penalty.

Panel assessments (input data in Table 1)

Table 1. Criterion-Level Neutrosophic Assessments (Urban Sculptures)

Criterion	Weight	$q_k(S_1)(T, I, F)$	$q_k(S_2)(T, I, F)$	$q_k(S_3)(T, I, F)$
C1 Context Fit	0.20	(0.80, 0.10, 0.10)	(0.70, 0.15, 0.15)	(0.60, 0.25, 0.15)
C2 Structural Integrity	0.20	(0.75, 0.15, 0.10)	(0.80, 0.10, 0.10)	(0.85, 0.10, 0.05)
C3 Aesthetic Coherence	0.15	(0.70, 0.20, 0.10)	(0.75, 0.20, 0.05)	(0.65, 0.25, 0.10)
C4 Accessibility & Safety	0.15	(0.85, 0.10, 0.05)	(0.70, 0.20, 0.10)	(0.75, 0.15, 0.10)
C5 Sustainability	0.15	(0.65, 0.25, 0.10)	(0.80, 0.10, 0.10)	(0.72, 0.18, 0.10)
C6 Public Engagement	0.15	(0.78, 0.12, 0.10)	(0.68, 0.22, 0.10)	(0.70, 0.20, 0.10)

Aggregation (exact calculations)

Weighted aggregates for S_1 :

$$\sum wT = 0.20 \cdot 0.80 + 0.20 \cdot 0.75 + 0.15 \cdot 0.70 + 0.15 \cdot 0.85 + 0.15 \cdot 0.65 + 0.15 \cdot 0.78$$
$$= 0.160 + 0.150 + 0.105 + 0.1275 + 0.0975 + 0.117 = 0.7570$$
$$\sum wI = 0.20 \cdot 0.10 + 0.20 \cdot 0.15 + 0.15 \cdot 0.20 + 0.15 \cdot 0.10 + 0.15 \cdot 0.25 + 0.15 \cdot 0.12$$
$$= 0.020 + 0.030 + 0.030 + 0.015 + 0.0375 + 0.018 = 0.1505$$
$$\sum wF = 0.20 \cdot 0.10 + 0.20 \cdot 0.10 + 0.15 \cdot 0.10 + 0.15 \cdot 0.05 + 0.15 \cdot 0.10 + 0.15 \cdot 0.10$$
$$= 0.020 + 0.020 + 0.015 + 0.0075 + 0.015 + 0.015 = 0.0925.$$

Thus

$$\mathrm{NDI}_1(S_1) = 0.7570 - 0.0925 - 0.1505 = 0.5140, \mathrm{NDI}_1^{\times}(S_1) = (0.7570 - 0.0925)(1 - 0.1505) = 0.5645.$$

Weighted aggregates for S_2 :

$$\sum wT = 0.7395, \sum wI = 0.1580, \sum wF = 0.1025.$$

Therefore

$$\mathrm{NDI}_1(S_2) = 0.7395 - 0.1025 - 0.1580 = 0.4790, \mathrm{NDI}_1^{\times}(S_2) = (0.7395 - 0.1025)(1 - 0.1580) = 0.5364.$$

Weighted aggregates for S_3 :

$$\sum wT = 0.7130, \sum wI = 0.1870, \sum wF = 0.1000.$$

Thus

$$\mathrm{NDI}_1(S_3) = 0.7130 - 0.1000 - 0.1870 = 0.4260, \mathrm{NDI}_1^{\times}(S_3) = (0.7130 - 0.1000)(1 - 0.1870) = 0.4984.$$

Table 2. Neutrosophic Aggregates and Indices

Design	$\sum wT$	$\sum wI$	$\sum wF$	NDI_1	$\mathrm{NDI}_1^{\times}$
S_1	0.7570	0.1505	0.0925	0.5140	0.5645
S_2	0.7395	0.1580	0.1025	0.4790	0.5364
S_3	0.7130	0.1870	0.1000	0.4260	0.4984

From Table 2, Ranking (both indices) is, $S_1 \succ S_2 \succ S_3$.

Properties and Guarantees

Proposition 1 (Monotonicity). If a design improves on every criterion with $T \uparrow, I \downarrow, F \downarrow$, then both NDI_λ and $\mathrm{NDI}_\lambda^{\times}$ do not decrease.

Proposition 2 (Boundedness). $\mathrm{NDI}_\lambda \in [-1,1]$ and $\mathrm{NDI}_\lambda^{\times} \in [-1,1]$.

Proposition 3 (Stability to small panel perturbations). If $\max_k \|q_k - \tilde{q}_k\|_\infty \le \varepsilon$, then $|\mathrm{NDI}_\lambda(s) - \mathrm{NDI}_\lambda(\tilde{s})| \le (1+\lambda)\varepsilon$ and similarly for $\mathrm{NDI}_\lambda^{\times}$ up to a constant depending on $\sum wI$.

Managerial Interpretation for Urban Sculpture

The analysis distinguishes negative evidence F from uncertainty I. For S_2 vs S_1, $\sum wT$ is close, but S_2 carries higher $\sum wI$, which depresses both indices. In practice this points to reducing indeterminacy through clearer site narratives, 1:1 material mock-ups, and inclusive public previews, before attempting costly redesigns aimed at boosting T. The framework thus guides targeted refinement: treat ambiguity first, then address genuine deficiencies.

5. Conclusion

We presented a neutrosophic functional-analytic approach to urban sculpture design evaluation, building an inner-product module, scalar norms, linear evaluation functionals, and a Quality Operator leading to two operational indices. A transparent numerical study demonstrated how the method separates ambiguity from negativity, enabling precise and fair rankings. The framework is static, auditable, and immediately usable by design juries.

Acknowledgements

This work was supported by Key Research Base of Sichuan Province Social Science "Sichuan Normal University Sichuan Wu Zetian Research Center", project launched in 2024, general project, "Research on the Influence of Wu Zetian on the Art of Dunhuang Mogao Grottoes" under project number: SCWZT-2024-04.

References

[1] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Set, and Probability, American Research Press, 2002.

[2] J. Ye, "Single-valued neutrosophic similarity measures for multiple attribute decision-making," Neutrosophic Sets and Systems, 1–2 (2014), 4–13.

[3] R. B. Holmes, Geometric Functional Analysis and its Applications, Springer, 1975.

[4] W. Rudin, Functional Analysis, 2nd ed., McGraw-Hill, 1991.

[5] S. S. Dragomir, "Inequalities in inner product spaces," RGMIA Monographs, 2002.

Received: March 30, 2025. Accepted: Aug 20, 2025