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A novel hybrid distance-based similarity measures for n-valued neutrosophic trapezoidal numbers and its applications

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Abstract: This paper presents an innovative decision-making method that integrates similarity and Hybrid distance measure techniques within the context of N-valued neutrosophic trapezoidal (NVNT) numbers. The theoretical foundations and essential properties of the proposed similarity measure are rigorously established. Building upon this, a method is developed to address multi-criteria decision-making (MCDM) problems utilizing the proposed measure within the NVNT environment. To validate the effectiveness of the approach, a detailed numerical example is provided. Furthermore, a comparative analysis is conducted against existing MCDM methodologies designed for NVNT-numbers. The results reveal that the proposed method demonstrates superior performance and enhanced flexibility in managing uncertainty and imprecision when compared to conventional techniques.

Keywords: *Neutrosophic sets; N-Vaued Neutrosophic Trapezoidal Numbers; Hybrid Distance Measure; Similarity Measure.*

1. Introduction

In recent years, various decision-making frameworks have been proposed to address real-world problems arising in domains such as education, economics, industry, healthcare, engineering, and other fields. Often, decision-related information cannot be precisely characterized due to insufficiency, ambiguity, and inconsistency. Consequently, researchers in modern artificial intelligence have increasingly sought advanced mathematical modeling techniques capable of handling the inherent uncertainty and vagueness in such data.

Zadeh [1] introduced the foundational concept of fuzzy sets, which has since played a pivotal role in numerous scientific disciplines. Building upon this, Atanassov [2] extended the fuzzy set theory to intuitionistic fuzzy sets, where the sum of membership and non-membership degrees lies within the interval [0,1]. To further enhance this framework, Yager [3] introduced the concept of Pythagorean fuzzy sets. Batool et al. [4] developed decision-making approaches utilizing Pythagorean probabilistic hesitant fuzzy sets. Additionally, Shahzaib et al. [5] proposed the spherical fuzzy set, which allows the sum of the squares of the membership, non-membership, and hesitation degrees to remain within [0,1]. In response to this, Ashraf et al. [6] explored aggregation operators for global linguistic fuzzy information, while Ashraf et al. [7,8] and Muhammad et al. [9] introduced applications based on cosine similarity and fuzzy distance measures.

In 1999, Smarandache introduced the concept of neutrosophic sets, a generalization of fuzzy logic that incorporates neutral thinking. Neutrosophic logic characterizes information through three independent components: the degree of truth (T), the degree of indeterminacy (I), and the degree of falsity (F). This framework offers greater flexibility for modeling uncertainty, as it accommodates fully independent, partially dependent, or correlated components. Neutrosophic set theory has since gained considerable attention, with notable developments in extensions [10–13], neutrosophic algebra [14], decision-making [15–18], and applications across various neutrosophic environments [19–22].

To address structures containing indistinguishable repeated elements, Yager [23] proposed multisets, which permit element repetition. This was further generalized to fuzzy multisets by Sebestian and Ramakrishnan [24,25], who also defined fundamental operations and properties. Later, they introduced multiple fuzzy subgroups and their normal variants [26]. Shinoj and John [27,28] extended this work through intuitionistic fuzzy multisets, combining features of both fuzzy multisets and intuitionistic fuzzy sets.

While fuzzy multisets are effective in modeling uncertainty, they may not adequately capture inconsistency. In response, Smarandache [29] and Ye and Ye [30] introduced neutrosophic multisets, where one or more elements may occur multiple times, either with the same or different neutrosophic components. In this framework, an element is characterized by truth, indeterminacy, and falsity membership degrees each belonging to the interval [0,1] and these degrees may vary over time, enhancing the model's capacity to represent dynamic and uncertain environments. To formalize and extend these ideas, Deli et al. [31,32] redefined key operations such as union, intersection, convexity, strong convexity, and distance measures for neutrosophic sets. Chatterjee et al. [33] later introduced distance and similarity measures for single-valued neutrosophic multisets, applying them effectively to medical diagnosis problems.

As interest in fuzzy multisets has grown, numerous researchers—including Sahin et al. [34–36], Syropoulos [37,38], Ulucay et al. [39], Ejegua and Awolola [41], Rajarajeswari and Uma [42], Broumi and Deli [43], Pramanik et al. [44], and Mondal et al. [45] have contributed to its theoretical and applied development. Sahin et al. [46], Fan et al. [47], and Ulucay et al. [48,49] have further expanded these models. In parallel, the integration of similarity and distance measures into decision-making techniques has become increasingly popular, with methods such as TOPSIS (Hwang and Yoon [50]), PROMETHEE (Brans et al. [51]), VIKOR (Opricovic and Tzeng [52]), and ELECTRE (Roy et al. [53]) forming the foundation of many applications.

More recently, Ulucay et al. [39,49] introduced the concepts of trapezoidal fuzzy multi-numbers and intuitionistic trapezoidal fuzzy multi-numbers, while Deli et al. [55] presented N-valued neutrosophic trapezoidal multi-numbers, a specialized neutrosophic multiset over the real numbers. These models were further enriched by defining appropriate arithmetic operations and by studying extensions involving similarity and distance measures. Examples include the Hamming distance (Ulucay et al. [39]), Jaccard similarity and Dice similarity (Sahin et al. [35]), and normalized similarity measures (Ulucay et al. [36]). Notably, these metrics are special cases of the Minkowski distance with specific parameter values. In recent years, studies in the areas mentioned above have been continuing rapidly[56-64].

In fuzzy multi-number environments, elements may possess multiple membership values in the interval [0,1]; this extends to intuitionistic fuzzy multi-numbers, where elements have multi membership and non-membership values. However, these structures may still fall short in capturing the full complexity of uncertain and inconsistent data. To address this gap, the present study proposes a novel MCDM method based on NVNT-numbers, using a generalized similarity measure grounded in the Hybrid Distance Measure.

The key contributions of this paper are as follows:

1. **A comprehensive framework** for modelling uncertainty and imprecision in decision-making scenarios is developed using NVNT-numbers.
2. **A more detailed representation** of complex decision criteria is achieved using NVNT-numbers.
3. **The incorporation of the Hybrid Distance Measure** enhances the flexibility and adaptability of the proposed similarity measure across a wide range of decision-making contexts.

To demonstrate the practical utility of the proposed methodology, the remainder of this paper is structured as follows:

- **Section 2** presents foundational concepts, including definitions related to fuzzy sets, intuitionistic fuzzy multisets, neutrosophic multisets, and related structures.
- **Section 3** introduces the new similarity measure based on NVNT-numbers and the Minkowski distance, along with a study of its fundamental properties.
- **Section 4** describes an MCDM model incorporating normalized NVNT-numbers.

- **Section 5** applies the proposed similarity measure to solve a representative problem.
- **Section 6** provides a comparative analysis between the proposed approach and existing MCDM methods.
- **Section 7** concludes the paper and outlines directions for future research.

2. Preliminaries

Definition 1 [65] Let $\mathcal{U}$ be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{< u, (\mu_{\mathcal{A}}(u), v_{\mathcal{A}}(u), w_{\mathcal{A}}(u)) >: u \in \mathcal{U}\}$$

where, $\mu_{\mathcal{A}}(u)$, $v_{\mathcal{A}}(u)$ and $w_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$\mu_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \qquad v_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \qquad w_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq \mu_{\mathcal{A}}(u) + v_{\mathcal{A}}(u) + w_{\mathcal{A}}(u) \leq 3^{+}$.

Definition 2 [66] Assume that E is the universe. Then, a single valued neutrosophic set (N-set) A in E defined as

$$A = \{< \mathrm{x}, \mathrm{T_A(x)}, \mathrm{I_A(x)}, \mathrm{F_A(x)} >: x \in E\} \tag{1}$$

where $\mathrm{T_A(x)}, \mathrm{I_A(x)}, \mathrm{F_A(x)} \in [0,1]$ for each point x in such that $0 \leq \mathrm{T_A(x)} + \mathrm{I_A(x)} + \mathrm{F_A(x)} \leq 3$.

Definition 3 [67] Let E be a universe. $\mathrm{A_1}$ neutrosophic multi-set set $\mathrm{A_1}$ on E can be defined as follows:

$$\mathrm{A_1} = \{< x, (\mathrm{T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x)}), (\mathrm{I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^P_{A_1}(x)}), (\mathrm{F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x)}) >: \mathrm{x} \in \mathrm{E}\},$$

where

$$\mathrm{T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x), I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^p_{A_1}(x), F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x)}: \mathrm{E} \to [0,1]$$

such that $0 \leq \sup \mathrm{T^i_{A_1}}(\mathrm{x}) + \sup \mathrm{I^i_{A_1}}(\mathrm{x}) + \sup \mathrm{F^i_{A_1}}(\mathrm{x}) \leq 3$ (i=1,2,…,P) for any $\mathrm{x} \in \mathrm{E}$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Definition 4 [55] Let $\eta^i_{A_1}, \vartheta^i_{A_1}, \theta^i_{A_1} \in [0,1]$ $(i \in \{1,2,\dots,p\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then, an N-valued neutrosophic trapezoidal number (NVNT-number)

$\tilde{a} = \langle [a, b, c, d]; (\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle$ is a neutrosophic multi-set on the real number set $\mathbb{R}$, whose truth-membership functions, indeterminacy-membership functions and falsity-membership functions are defined as, respectively.

$$\mathrm{T}^i_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}\eta^i_{\tilde{\alpha}}, & a \leq x < b \\ \eta^i_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{(d-x)}{(d-c)}\eta^i_{\tilde{\alpha}}, & c < x \leq d \\ 0, & \text{otherwise}, \end{cases} \quad , \quad \mathrm{I}^i_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{(b-x) + \vartheta^i_{\tilde{\alpha}}(x-a)}{(b-a)}, & a \leq x < b \\ \vartheta^i_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{(x-c) + \vartheta^i_{\tilde{\alpha}}(d-x)}{(d-c)}, & c < x \leq d \\ 1, & \text{otherwise}, \end{cases}$$

and

$$\mathrm{F}^i_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{(b-x) + \theta^i_{A}(x-a)}{(b-a)}, & a \leq x < b \\ \theta^i_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{(x-c) + \theta^i_{A}(d-x)}{(d-c)}, & c < x \leq d \\ 1, & \text{otherwise}, \end{cases}$$

Note that the set of all NVNT-numbers on $\mathbb{R}$ will be denoted by Λ.

Definition 5 [55] Let $\mathrm{A_1} = \langle (a_1, b_1, c_1, d_1); (\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle \in \Lambda$. If $\mathrm{A_1}$ is not normalized NVTN-number $(a_1, b_1, c_1, d_1 \notin [0,1])$, the normalized NVNT-number of $\mathrm{A_1}$, denoted by $\overline{\mathrm{A_1}}$ is given by;

$$\overline{\mathrm{A}}_1 = \langle\left[\frac{a_1}{a_1+b_1+c_1+d_1},\frac{b_1}{a_1+b_1+c_1+d_1},\frac{c_1}{a_1+b_1+c_1+d_1},\frac{d_1}{a_1+b_1+c_1+d_1}\right]; \left(\eta^1_{\mathrm{A}_1},\eta^2_{\mathrm{A}_1},\ldots,\eta^{\mathrm{P}}_{\mathrm{A}_1}\right),\left(\vartheta^1_{\mathrm{A}_1},\vartheta^2_{\mathrm{A}_1},\ldots,\vartheta^{\mathrm{P}}_{\mathrm{A}_1}\right),\left(\theta^1_{\mathrm{A}_1},\theta^2_{\mathrm{A}_1},\ldots,\theta^{\mathrm{P}}_{\mathrm{A}_1}\right)\rangle. \tag{2}$$

Definition 6 [55] Let $\overline{\mathcal{A}} = \langle[a_1,b_1,c_1,d_1];(\eta^1_{\mathcal{A}},\eta^2_{\mathcal{A}},\ldots,\eta^{\mathrm{P}}_{\mathcal{A}}),(\vartheta^1_{\mathcal{A}},\vartheta^2_{\mathcal{A}},\ldots,\vartheta^{\mathrm{P}}_{\mathcal{A}}),(\theta^1_{\mathcal{A}},\theta^2_{\mathcal{A}},\ldots,\theta^{\mathrm{P}}_{\mathcal{A}})\rangle$ and $\overline{\mathcal{B}} = \langle[a_2,b_2,c_2,d_2];(\eta^1_{\mathcal{B}},\eta^2_{\mathcal{B}},\ldots,\eta^{\mathrm{P}}_{\mathcal{B}}),(\vartheta^1_{\mathcal{B}},\vartheta^2_{\mathcal{B}},\ldots,\vartheta^{\mathrm{P}}_{\mathcal{B}}),(\theta^1_{\mathcal{B}},\theta^2_{\mathcal{B}},\ldots,\theta^{\mathrm{P}}_{\mathcal{B}})\rangle$ be two normalized NVNT-numbers and $r > 0$. Then, the generalized distance measures between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, denoted by $d_r(\overline{\mathcal{A}},\overline{\mathcal{B}})$, is given below;

$$\begin{aligned} d_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \; & \frac{1}{16p}.\Bigg(\sum_{i=1}^{p}\Big[\big(\left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})a_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})a_2\right|\big)^r + \\ & \big(\left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})b_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})b_2\right|\big)^r + \\ & \big(\left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})c_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})c_2\right|\big)^r + \\ & \big(\left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})d_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})d_2\right|\big)^r\Bigg)^{\frac{1}{r}} \end{aligned} \tag{3}$$

3. Hybrid Distance-Based Similarity Measures for NVNT-numbers

In this study, Hamming and Euclidean distances in Definition 6, which are widely used in practice, are considered as basic distance measures corresponding to the cases r = 1 and r = 2, respectively, and are used in the proposed novel similarity measures.

Definition 7 Let $\overline{\mathcal{A}} = \langle[a_1,b_1,c_1,d_1];(\eta^1_{\mathcal{A}},\eta^2_{\mathcal{A}},\ldots,\eta^{\mathrm{P}}_{\mathcal{A}}),(\vartheta^1_{\mathcal{A}},\vartheta^2_{\mathcal{A}},\ldots,\vartheta^{\mathrm{P}}_{\mathcal{A}}),(\theta^1_{\mathcal{A}},\theta^2_{\mathcal{A}},\ldots,\theta^{\mathrm{P}}_{\mathcal{A}})\rangle$ and $\overline{\mathcal{B}} = \langle[a_2,b_2,c_2,d_2];(\eta^1_{\mathcal{B}},\eta^2_{\mathcal{B}},\ldots,\eta^{\mathrm{P}}_{\mathcal{B}}),(\vartheta^1_{\mathcal{B}},\vartheta^2_{\mathcal{B}},\ldots,\vartheta^{\mathrm{P}}_{\mathcal{B}}),(\theta^1_{\mathcal{B}},\theta^2_{\mathcal{B}},\ldots,\theta^{\mathrm{P}}_{\mathcal{B}})\rangle$ be two NVNT-numbers. Then, hybrid distance measure between NVNT-numbers $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, denoted

$$HybD_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \lambda\big(d_r(\overline{\mathcal{A}},\overline{\mathcal{B}})\big) + (1-\lambda)\big(d_{r+1}(\overline{\mathcal{A}},\overline{\mathcal{B}})\big) \text{ (For r=1)}$$

$$\begin{aligned} HybD_1(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \; & \lambda\Bigg(\frac{1}{16p}.\sum_{i=1}^{p}\Big[\left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})a_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})a_2\right| + \\ & \left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})b_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})b_2\right| + \\ & \left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})c_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})c_2\right| + \\ & \left|(1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})d_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})d_2\right|\Big) + \\ & (1-\lambda).\Bigg(\frac{1}{16p}.\sum_{i=1}^{p}\Big[\big((1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})a_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})a_2\big)^2 + \\ & \big((1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})b_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})b_2\big)^2 + \\ & \big((1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})c_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})c_2\big)^2 + \\ & \big((1+\eta^i_{\mathcal{A}}-\vartheta^i_{\mathcal{A}}-\theta^i_{\mathcal{A}})d_1-(1+\eta^i_{\mathcal{B}}-\vartheta^i_{\mathcal{B}}-\theta^i_{\mathcal{B}})d_2\big)^2\Big]^{\frac{1}{2}}\Bigg) \end{aligned} \tag{4}$$

Similarity and distance (or dissimilarity) are complementary: as one increases, the other decreases. The distance measure and similarity measure are therefore dual concepts. Thus,

$$s_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) = 1 - HybD_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) \tag{5}$$

and reversely. The properties of distance measures correspond complementarily to those of similarity measures.

Theorem 1 The similarity measure $s_r(\mathcal{A},\mathcal{B})$ for satisfies the following properties.

$s_r\_1)\ 0 \leq s_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) \leq 1,$

$s_r\_2)\ \overline{\mathcal{A}} = \overline{\mathcal{B}} \Rightarrow s_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) = 1,$

$s_r\_3)\ s_r(\overline{\mathcal{A}},\overline{\mathcal{B}}) = s_r(\overline{\mathcal{B}},\overline{\mathcal{A}}),$

$s_r\_4)\ If\ \overline{\mathcal{A}} \subseteq \overline{\mathcal{B}} \subseteq \overline{\mathcal{C}}, then\ s_r(\overline{\mathcal{A}},\overline{\mathcal{C}}) \leq s_r(\overline{\mathcal{A}},\overline{\mathcal{B}})\ and\ s_r(\overline{\mathcal{A}},\overline{\mathcal{C}}) \leq s_r(\mathcal{B},\overline{\mathcal{C}}).$

Proof: It is straightforward to verify that $s_r(\overline{\mathcal{A}},\overline{\mathcal{B}})$ satisfies properties $s_r\_1 - s_r\_4$.

Example 1 Let $\overline{\mathcal{A}} = \langle[0.2,0.3,0.5,0.8]; (0.1,0.3,0.6,0.8), (0.2,0.5,0.4,0.3), (0.5,0.6,0.3,0.1)\rangle$ and $\overline{\mathcal{B}} = \langle[0.1,0.4,0.6,0.7]; (0.4,0.5,0.7,0.8), (0.1,0.2,0.3,0.5), (0.2,0.3,0.4, 0.5)\rangle$ be two normalized NVNT-numbers then, similarity measure $s_1(\overline{\mathcal{A}},\overline{\mathcal{B}})$ between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are found below;

$$
\begin{aligned}
\mathrm{d}_1\ (\overline{\mathcal{A}},\overline{\mathcal{B}}) = \ & \frac{1}{64}.[|(1+0.1-0.2-0.5)0.2-(1+0.4-0.1-0.2)0.1| + |(1+0.3-0.5-0.6)0.2 \\
& -(1+0.5-0.2-0.3)0.1| + |(1+0.6-0.4-0.3)0.2-(1+0.7-0.3-0.4)0.1| \\
& |(1+0.8-0.3-0.1)0.2-(1+0.8-0.5-0.5)0.1| + |(1+0.1-0.2-0.5)0.3 \\
& -(1+0.4-0.1-0.2)0.4| + |(1+0.3-0.5-0.6)0.3-(1+0.5-0.2-0.3)0.4| + \\
& |(1+0.6-0.4-0.3)0.3-(1+0.7-0.3-0.4)0.4| + \left|\left((1+0.8-0.3-0.1)0.3-\right.\right. \\
& (1+0.8-0.5-0.5)0.4| + |(1+0.1-0.2-0.5)0.5-(1+0.4-0.1-0.2)0.6| + \\
& |(1+0.3-0.5-0.6)0.5-(1+0.5-0.2-0.3)0.6| + |(1+0.6-0.4-0.3)0.5- \\
& (1+0.7-0.3-0.4)0.6| + |(1+0.8-0.3-0.1)0.5-(1+0.8-0.5-0.5)0.6| + \\
& |(1+0.1-0.2-0.5)0.8-(1+0.4-0.1-0.2)0.7| + |(1+0.3-0.5-0.6)0.8- \\
& (1+0.5-0.2-0.3)0.7| + |(1+0.6-0.4-0.3)0.8-(1+0.7-0.3-0.4)0.7| \\
& |(1+0.8-0.3-0.1)0.8-(1+0.8-0.5-0.5)0.7|] \\
= \ & 0.024
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}_2\ (\overline{\mathcal{A}},\overline{\mathcal{B}}) = \ & \frac{1}{64}.\Big[\big((1+0.1-0.2-0.5)0.2-(1+0.4-0.1-0.2)0.1\big)^2 + \big((1+0.3-0.5-0.6)0.2 \\
& -(1+0.5-0.2-0.3)0.1)^2 + \big((1+0.6-0.4-0.3)0.2-(1+0.7-0.3-0.4)0.1\big)^2 \\
& \big((1+0.8-0.3-0.1)0.2-(1+0.8-0.5-0.5)0.1\big)^2 + \big((1+0.1-0.2-0.5)0.3 \\
& -(1+0.4-0.1-0.2)0.4)^2 + \big((1+0.3-0.5-0.6)0.3-(1+0.5-0.2-0.3)0.4\big)^2 +
\end{aligned}
$$

$$\big((1+0.6-0.4-0.3)0.3-(1+0.7-0.3-0.4)0.4\big)^2+\big((1+0.8-0.3-0.1)0.3-$$

$$(1+0.8-0.5-0.5)0.4)^2+\big((1+0.1-0.2-0.5)0.5-(1+0.4-0.1-0.2)0.6\big)^2+$$

$$\big((1+0.3-0.5-0.6)0.5-(1+0.5-0.2-0.3)0.6\big)^2+\big((1+0.6-0.4-0.3)0.5-$$

$$-(1+0.7-0.3-0.4)0.6)^2+\big((1+0.8-0.3-0.1)0.5-(1+0.8-0.5-0.5)0.6\big)^2+$$

$$\big((1+0.1-0.2-0.5)0.8-(1+0.4-0.1-0.2)0.7\big)^2+\big((1+0.3-0.5-0.6)0.8-$$

$$-(1+0.5-0.2-0.3)0.7)^2+\big((1+0.6-0.4-0.3)0.8-(1+0.7-0.3-0.4)0.7\big)^2$$

$$\big((1+0.8-0.3-0.1)0.8-(1+0.8-0.5-0.5)0.7\big)^2\Big]^{1/2}$$

$$= 0.019$$

$$HybD_1(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \lambda(0.24)+(1-\lambda)(0.019) \quad (\text{for } \lambda=0.3)$$

$$= 0.0853$$

$$s_1(\overline{\mathcal{A}},\overline{\mathcal{B}}) = 1-Hybd(\overline{\mathcal{A}},\overline{\mathcal{B}})$$

$$= 1-0.0853$$

$$= 0.915$$

4. Decision-Making Algorithm

In this section, we present a MCDM method based on normalized NVNT-numbers.

Assume that $K=\{K_1,K_2,\dots,K_m\}$ be the set of alternatives and $G=\{g_1,g_2,\dots,g_n\}$ be the set of criterias. In Deli et al. [55], the normalized NVNT-numbers decision matrix is given as;

$$(K_{kj})_{mxn} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ K_{m1} & K_{m2} & \cdots & K_{mn} \end{pmatrix}$$

such that

$K_{kj}=\langle[a_{kj},b_{kj},c_{kj},d_{kj}],\left(\eta^1_{kj},\eta^2_{kj},\eta^3_{kj},\dots,\eta^p_{kj}\right),\left(\vartheta^1_{kj},\vartheta^2_{kj},\vartheta^3_{kj},\dots,\vartheta^p_{kj}\right),\left(\theta^1_{kj},\theta^2_{kj},\theta^3_{kj},\dots,\theta^p_{kj}\right)\rangle$, (k=1, 2, …, m) and (j=1,2, …, n).

It is carried out the following algorithm to get best choice:

Algorithm:

Step 1: Create an evaluation matrix $(K_{kj})_{mxn}$, (k=1,2, …, m; j=1, 2, …, n)

Step 2: Find the new similarity measure $s_r(K_{kj},\mathcal{P}^+)$ between $\mathcal{P}^+$ ($or\ \mathcal{P}^-$) the positive (or negative) ideal and

$$s_r(K_{kj},\mathcal{P}^+) = 1-HybD_r(K_{kj},\mathcal{P}^+)$$

where for all $1<k<m$ and $1<j<n$.

$$\mathcal{P}^+ = \left\langle\left[a^+_{kj},b^+_{kj},c^+_{kj},d^+_{kj}\right];\left(\eta^{1\,+}_{kj},\eta^{2\,+}_{kj},\eta^{3\,+}_{kj},\dots,\eta^{p\,+}_{kj}\right),\left(\vartheta^{1\,+}_{kj},\vartheta^{2\,+}_{kj},\vartheta^{3\,+}_{kj},\dots,\vartheta^{p\,+}_{kj}\right),\left(\theta^{1\,+}_{kj},\theta^{2\,+}_{kj},\theta^{3\,+}_{kj},\dots,\theta^{p\,+}_{kj}\right)\right\rangle$$

$$= \left\langle\left[\max_k\{a_{kj}\},\max_k\{b_{kj}\},\max_k\{c_{kj}\},\max_k\{d_{kj}\}\right];\left(\max_k\{\eta^1_{kj}\},\max_k\{\eta^2_{kj}\},\max_k\{\eta^3_{kj}\},\dots,\max_k\left\{\eta^p_{kj}\right\}\right)\right.$$

$$\left.\left(\min_k\{\vartheta^1_{kj}\},\min_k\{\vartheta^2_{kj}\},\min_k\{\vartheta^3_{kj}\},\dots,\min_k\left\{\vartheta^p_{kj}\right\}\right),\left(\min_k\{\theta^1_{kj}\},\min_k\{\theta^2_{kj}\},\min_k\{\theta^3_{kj}\},\dots,\min_k\left\{\theta^p_{kj}\right\}\right)\right\rangle$$

$$\mathcal{P}^{-} = \left\langle \left[a_{kj}^{-}, b_{kj}^{-}, c_{kj}^{-}, d_{kj}^{-}\right]; \left(\eta_{kj}^{1\,-}, \eta_{kj}^{2\,-}, \eta_{kj}^{3\,-}, \dots, \eta_{kj}^{p\,-}\right), \left(\vartheta_{kj}^{1\,-}, \vartheta_{kj}^{2\,-}, \vartheta_{kj}^{3\,-}, \dots, \vartheta_{kj}^{p\,-}\right), \left(\theta_{kj}^{1\,-}, \theta_{kj}^{2\,-}, \theta_{kj}^{3\,-}, \dots, \theta_{kj}^{p\,-}\right) \right\rangle$$

$$= \left\langle \left[\max_k\{a_{kj}\}, \max_k\{b_{kj}\}, \max_k\{c_{kj}\}, \max_k\{d_{kj}\}\right]; \left(\max_k\{\eta_{kj}^1\}, \max_k\{\eta_{kj}^2\}, \max_k\{\eta_{kj}^3\}, \dots, \max_k\{\eta_{kj}^p\}\right)\right.$$
$$\left. \left(\min_k\{\vartheta_{kj}^1\}, \min_k\{\vartheta_{kj}^2\}, \min_k\{\vartheta_{kj}^3\}, \dots, \min_k\{\vartheta_{kj}^p\}\right), \left(\min_k\{\theta_{kj}^1\}, \min_k\{\theta_{kj}^2\}, \min_k\{\theta_{kj}^3\}, \dots, \min_k\{\theta_{kj}^p\}\right) \right\rangle.$$

Step 3: Order all similarity values $s_r(\mathrm{K}_{kj}, \mathcal{P}^{+})$ from highest to lowest.

Step 4: Determine the top choice.

In Figure 1, you can see the flowchart of the MCDM method for normalized NVNT-numbers.

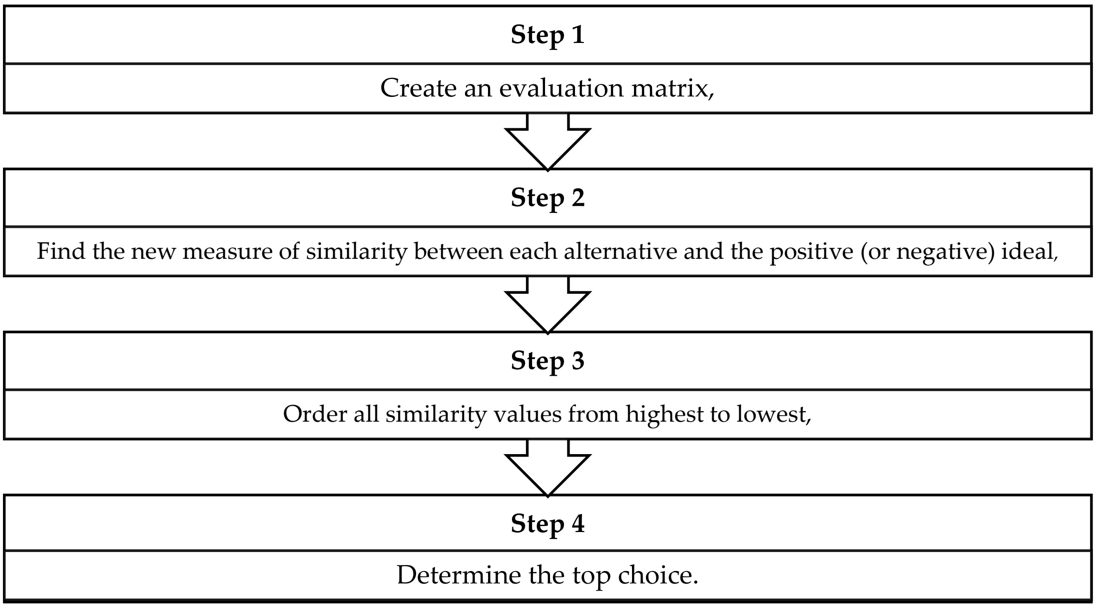


Figure 1. Flowchart for the normalized NVNT-numbers MCDM method.

5. Application of Decision-Making Algorithm

Suppose a manufacturing company wants to select the most suitable supplier for a critical raw material. The company has identified five potential suppliers and will evaluate them based on three key criteria. These criteria reflect the company's priorities in ensuring quality, timely delivery, and cost efficiency. The set of alternative suppliers is: $\mathrm{K} = \{\mathrm{K}_1, \mathrm{K}_2, \dots, \mathrm{K}_5\}$. The evaluation is based on the following criteria: $\mathrm{G} = \{\mathrm{g}_1 = \text{Product Quality}, \mathrm{g}_2 = \text{Delivery Time}, \mathrm{g}_3 = \text{Cost}\}$.

Each supplier will be assessed using a multi-criteria decision-making algorithm that incorporates normalized NVNT-numbers to ensure consistency and comparability among different evaluation metrics.

The objective is to rank the alternatives K_k for all k=1, 2, …, 5 and ultimately identify the best supplier for long-term collaboration.

Algorithm:

Step 1: The evaluation matrix$(\mathrm{K}_{kj})_{5x3}$ is given by an expert as;

$$(\mathrm{K}_{kj})_{5x3} = \begin{array}{c} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{array} \begin{pmatrix} \langle[0.13,0.14,0.15,0.16]; (0.3,0.5,0.7,0.8), (0.1,0.3,0.2,0.3), (0.6,0.3,0.5,0.6)\rangle \\ \langle[0.21,0.22,0.27,0.33]; (0.4,0.2,0.3,0.5), (0.2,0.5,0.7,0.6), (0.7,0.5,0.6,0.8)\rangle \\ \langle[0.42,0.53,0.62,0.73]; (0.7,0.6,0.4,0.8), (0.5,0.5,0.5,0.6), (0.4,0.3,0.4,0.5)\rangle \\ \langle[0.02,0.23,0.46,0.81]; (0.8,0.7,0.5,0.6), (0.4,0.3,0.7,0.5), (0.1,0.5,0.7,0.7)\rangle \\ \langle[0.13,0.35,0.41,0.58]; (0.7,0.8,0.9,0.9), (0.2,0.7,0.5,0.6), (0.1,0.7,0.8,0.4)\rangle \end{pmatrix}$$

$$\begin{array}{l}
\langle[0.28,0.32,0.38,0.43];(0.5,0.3,0.4,0.6),(0.2,0.1,0.5,0.4),(0.4,0.6,0.5,0.7)\rangle \\
\langle[0.11,0.15,0.18,0.23];(0.3,0.7,0.9,0.9),(0.1,0.2,0.3,0.7),(0.3,0.4,0.7,0.5)\rangle \\
\langle[0.35,0.66,0.72,0.75];(0.6,0.8,0.9,0.8),(0.2,0.5,0.4,0.3),(0.2,0.3,0.6,0.6)\rangle \\
\langle[0.08,0.15,0.27,0.37];(0.3,0.9,0.8,0.4),(0.8,0.7,0.6,0.5),(0.1,0.1,0.4,0.3)\rangle \\
\langle[0.09,0.13,0.19,0.69];(0.2,0.5,0.7,0.9),(0.1,0.3,0.5,0.4),(0.6,0.7,0.8,0.8)\rangle
\end{array}$$

$$\left.\begin{array}{l}
\langle[0.12,0.28,0.60,0.65];(0.2,0.7,0.8,0.9),(0.4,0.3,0.8,0.5),(0.2,0.5,0.6,0.4)\rangle \\
\langle[0.22,0.48,0.43,0.73];(0.3,0.8,0.9,0.7),(0.5,0.7,0.7,0.6),(0.1,0.4,0.8,0.6)\rangle \\
\langle[0.14,0.33,0.43,0.83];(0.1,0.6,0.9,0.5),(0.8,0.5,0.6,0.7),(0.1,0.3,0.5,0.8)\rangle \\
\langle[0.63,0.75,0.84,0.93];(0.4,0.5,0.7,0.6),(0.2,0.6,0.8,0.3),(0.3,0.6,0.7,0.2)\rangle \\
\langle[0.21,0.23,0.69,0.76];(0.5,0.7,0.8,0.3),(0.1,0.2,0.6,0.4),(0.4,0.2,0.9,0.7)\rangle
\end{array}\right)$$

Step 2: The value of $\mathrm{K_k} = s_r(\mathrm{K_{kj}}, \mathcal{P}^+)$, computed according to given formula

Table 1. Analysing how the proposed method performs compared to other methods

Proposed Methods	Value of $s_r(\mathrm{K_{kj}}, \mathcal{P}^+)$	Ranking
$s_1(\mathrm{K_{kj}}, \mathcal{P}^+)$	$\mathrm{K_1} = s_1(\mathrm{K_{1j}}, \mathcal{P}^+) = 0.9525$ $\mathrm{K_2} = s_1(\mathrm{K_{2j}}, \mathcal{P}^+) = 0.9505$ $\mathrm{K_3} = s_1(\mathrm{K_{3j}}, \mathcal{P}^+) = 0.9613$ $\mathrm{K_4} = s_1(\mathrm{K_{4j}}, \mathcal{P}^+) = 0.9542$ $\mathrm{K_5} = s_1(\mathrm{K_{5j}}, \mathcal{P}^+) = 0.9529$	$\mathrm{K_3} > \mathrm{K_4} > \mathrm{K_5} > \mathrm{K_1} > \mathrm{K_2}$
$s_2(\mathrm{K_{kj}}, \mathcal{P}^+)$	$\mathrm{K_1} = s_2(\mathrm{K_{1j}}, \mathcal{P}^+) = 0.9767$ $\mathrm{K_2} = s_2(\mathrm{K_{2j}}, \mathcal{P}^+) = 0.9757$ $\mathrm{K_3} = s_2(\mathrm{K_{3j}}, \mathcal{P}^+) = 0.9765$ $\mathrm{K_4} = s_2(\mathrm{K_{4j}}, \mathcal{P}^+) = 0.9745$ $\mathrm{K_5} = s_2(\mathrm{K_{5j}}, \mathcal{P}^+) = 0.9763$	$\mathrm{K_1} > \mathrm{K_3} > \mathrm{K_5} > \mathrm{K_2} > \mathrm{K_4}$
$s_{27}(\mathrm{K_{kj}}, \mathcal{P}^+)$	$\mathrm{K_1} = s_{27}(\mathrm{K_{1j}}, \mathcal{P}^+) = 0.8761$ $\mathrm{K_2} = s_{27}(\mathrm{K_{2j}}, \mathcal{P}^+) = 0.875815$ $\mathrm{K_3} = s_{27}(\mathrm{K_{3j}}, \mathcal{P}^+) = 0.8739$ $\mathrm{K_4} = s_{27}(\mathrm{K_{4j}}, \mathcal{P}^+) = 0.875819$ $\mathrm{K_5} = s_{27}(\mathrm{K_{5j}}, \mathcal{P}^+) = 0.8754$	$\mathrm{K_1} > \mathrm{K_4} > \mathrm{K_2} > \mathrm{K_5} > \mathrm{K_3}$
$s_{79}(\mathrm{K_{kj}}, \mathcal{P}^+)$	$\mathrm{K_1} = s_{79}(\mathrm{K_{1j}}, \mathcal{P}^+) = 0.6286$ $\mathrm{K_2} = s_{79}(\mathrm{K_{2j}}, \mathcal{P}^+) = 0.6283$ $\mathrm{K_3} = s_{79}(\mathrm{K_{3j}}, \mathcal{P}^+) = 0.6264$ $\mathrm{K_4} = s_{79}(\mathrm{K_{4j}}, \mathcal{P}^+) = 0.6284$ $\mathrm{K_5} = s_{79}(\mathrm{K_{5j}}, \mathcal{P}^+) = 0.6279$	$\mathrm{K_1} > \mathrm{K_4} > \mathrm{K_2} > \mathrm{K_5} > \mathrm{K_3}$

Step 3-4: Based on the $\mathrm{K_k} = s_r(\mathrm{K_{kj}}, \mathcal{P}^+)$ values, the alternatives $\mathrm{K_k}$ $(k = 1,2,3,4,5)$ are ranked in Table 1 and Figure 2 as

$$\mathrm{K_3} > \mathrm{K_4} > \mathrm{K_1} > \mathrm{K_2} > \mathrm{K_5}.$$

Hence $\mathrm{K_3}$ emerges as the top choice.

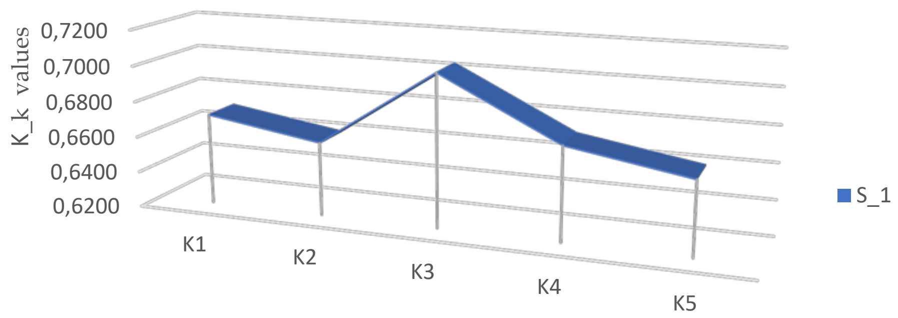


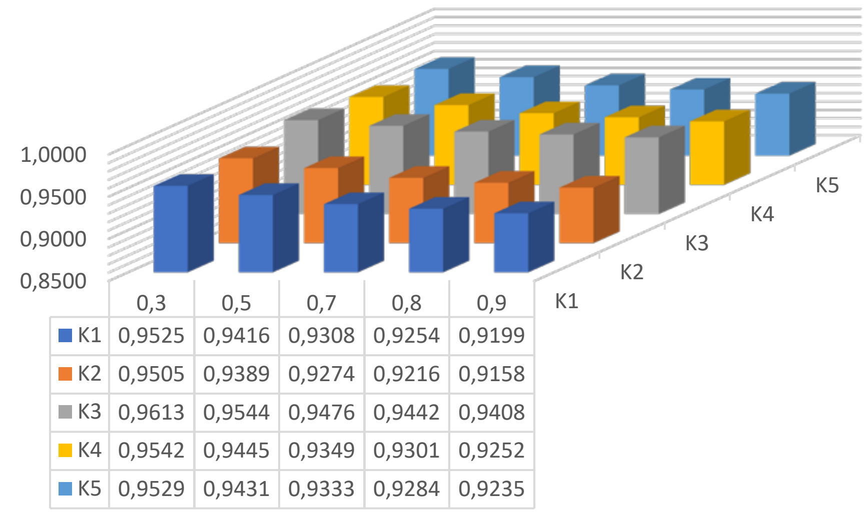
Figure 2: Ranking of K_k according to S_1 values

6. Comparison analysis

A primary contribution of this study is the development of a novel similarity measure tailored for NVNT-numbers. To rigorously evaluate its performance, the proposed measure is benchmarked against several established multi-criteria decision-making (MCDM) methods. Specifically, this section presents a comparative analysis of similarity measurement techniques, including those based on the 1rd-order Hybrid Distance, the 2rd-order Hybrid Distance, the 27rd-order Hybrid Distance, and the 79th-order Hybrid Distance. Furthermore, the proposed approach is compared with existing methods introduced by Ulucay and Okumus [68], Wang [69], Ulucay [70], and Meng et al. [71]. The comparative results are provided in Table 2 and visualized in Figure 3-4.

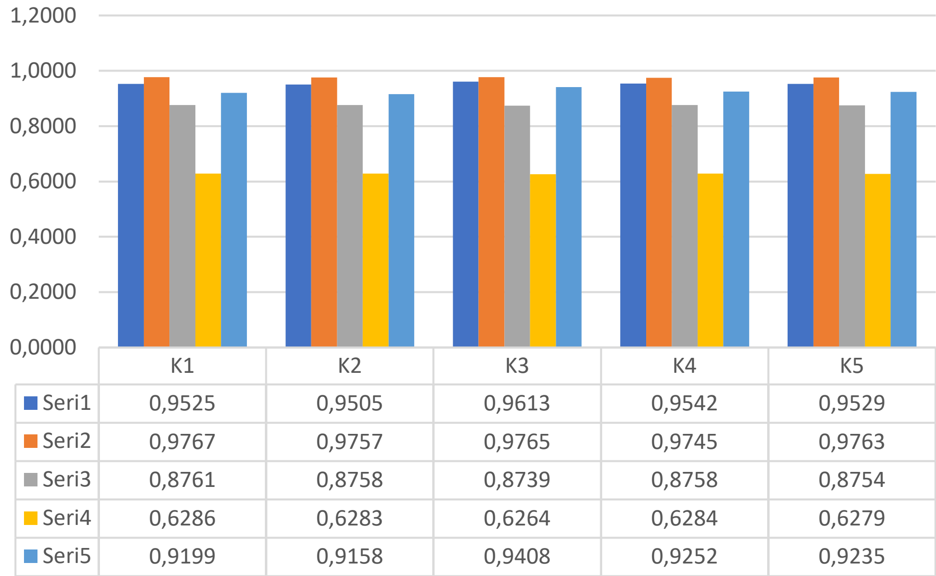
Table 2. Comparing the performance of the proposed method with other existing methods

Proposed Methods	Value of $s_r(\mathrm{K}_{\mathrm{kj}}, \mathcal{P}^+)$	Ranking
$s_1(\mathrm{K}_{\mathrm{kj}}, \mathcal{P}^+)$	$\mathrm{K}_1 = s_1(\mathrm{K}_{1\mathrm{j}}, \mathcal{P}^+) = 0.9525$ $\mathrm{K}_2 = s_1(\mathrm{K}_{2\mathrm{j}}, \mathcal{P}^+) = 0.9505$ $\mathrm{K}_3 = s_1(\mathrm{K}_{3\mathrm{j}}, \mathcal{P}^+) = 0.9613$ $\mathrm{K}_4 = s_1(\mathrm{K}_{4\mathrm{j}}, \mathcal{P}^+) = 0.9542$ $\mathrm{K}_5 = s_1(\mathrm{K}_{5\mathrm{j}}, \mathcal{P}^+) = 0.9529$	$\mathrm{K}_3 > \mathrm{K}_4 > \mathrm{K}_5 > \mathrm{K}_1 > \mathrm{K}_2$
$s_2(\mathrm{K}_{\mathrm{kj}}, \mathcal{P}^+)$	$\mathrm{K}_1 = s_2(\mathrm{K}_{1\mathrm{j}}, \mathcal{P}^+) = 0.9767$ $\mathrm{K}_2 = s_2(\mathrm{K}_{2\mathrm{j}}, \mathcal{P}^+) = 0.9757$ $\mathrm{K}_3 = s_2(\mathrm{K}_{3\mathrm{j}}, \mathcal{P}^+) = 0.9765$ $\mathrm{K}_4 = s_2(\mathrm{K}_{4\mathrm{j}}, \mathcal{P}^+) = 0.9745$ $\mathrm{K}_5 = s_2(\mathrm{K}_{5\mathrm{j}}, \mathcal{P}^+) = 0.9763$	$\mathrm{K}_1 > \mathrm{K}_3 > \mathrm{K}_5 > \mathrm{K}_2 > \mathrm{K}_4$
$s_{27}(\mathrm{K}_{\mathrm{kj}}, \mathcal{P}^+)$	$\mathrm{K}_1 = s_{27}(\mathrm{K}_{1\mathrm{j}}, \mathcal{P}^+) = 0.8761$ $\mathrm{K}_2 = s_{27}(\mathrm{K}_{2\mathrm{j}}, \mathcal{P}^+) = 0.875815$ $\mathrm{K}_3 = s_{27}(\mathrm{K}_{3\mathrm{j}}, \mathcal{P}^+) = 0.8739$ $\mathrm{K}_4 = s_{27}(\mathrm{K}_{4\mathrm{j}}, \mathcal{P}^+) = 0.875819$ $\mathrm{K}_5 = s_{27}(\mathrm{K}_{5\mathrm{j}}, \mathcal{P}^+) = 0.8754$	$\mathrm{K}_1 > \mathrm{K}_4 > \mathrm{K}_2 > \mathrm{K}_5 > \mathrm{K}_3$
$s_{79}(\mathrm{K}_{\mathrm{kj}}, \mathcal{P}^+)$	$\mathrm{K}_1 = s_{79}(\mathrm{K}_{1\mathrm{j}}, \mathcal{P}^+) = 0.6286$ $\mathrm{K}_2 = s_{79}(\mathrm{K}_{2\mathrm{j}}, \mathcal{P}^+) = 0.6283$ $\mathrm{K}_3 = s_{79}(\mathrm{K}_{3\mathrm{j}}, \mathcal{P}^+) = 0.6264$ $\mathrm{K}_4 = s_{79}(\mathrm{K}_{4\mathrm{j}}, \mathcal{P}^+) = 0.6284$ $\mathrm{K}_5 = s_{79}(\mathrm{K}_{5\mathrm{j}}, \mathcal{P}^+) = 0.6279$	$\mathrm{K}_1 > \mathrm{K}_4 > \mathrm{K}_2 > \mathrm{K}_5 > \mathrm{K}_3$
Ulucay and Okumus [59]		$\mathrm{K}_2 > \mathrm{K}_3 > \mathrm{K}_4 > \mathrm{K}_5 > \mathrm{K}_1$
Wang [60]		$\mathrm{K}_2 > \mathrm{K}_3 > \mathrm{K}_4 > \mathrm{K}_1 > \mathrm{K}_5$
Ulucay [61]		$\mathrm{K}_3 > \mathrm{K}_4 > \mathrm{K}_2 > \mathrm{K}_5 > \mathrm{K}_1$
Meng et al. [62]		$\mathrm{K}_2 > \mathrm{K}_4 > \mathrm{K}_3 > \mathrm{K}_5 > \mathrm{K}_1$



	0,3	0,5	0,7	0,8	0,9
K1	0,9525	0,9416	0,9308	0,9254	0,9199
K2	0,9505	0,9389	0,9274	0,9216	0,9158
K3	0,9613	0,9544	0,9476	0,9442	0,9408
K4	0,9542	0,9445	0,9349	0,9301	0,9252
K5	0,9529	0,9431	0,9333	0,9284	0,9235

Figure 3: Comparison based on λ values



	K1	K2	K3	K4	K5
Seri1	0,9525	0,9505	0,9613	0,9542	0,9529
Seri2	0,9767	0,9757	0,9765	0,9745	0,9763
Seri3	0,8761	0,8758	0,8739	0,8758	0,8754
Seri4	0,6286	0,6283	0,6264	0,6284	0,6279
Seri5	0,9199	0,9158	0,9408	0,9252	0,9235

Figure 4: Comparison based on similarity measures values

7. Conclusions

This paper presents a novel similarity measure based on NVNT-numbers, developed using the Hybrid Distance Measure. We examine the fundamental properties of the proposed measure and introduce a method for addressing multi-criteria decision-making (MCDM) problems involving NVNT-numbers. The method enables a comprehensive evaluation of complex decision scenarios, incorporates multiple perspectives, supports validation through various techniques, and delivers reliable and robust outcomes. To demonstrate its effectiveness, a numerical example is provided. Furthermore, a comparative analysis is carried out against existing methods used for solving MCDM problems with NVNT-numbers. Our findings underscore the flexibility, practicality, and applicability of the proposed approach in complex decision-making environments. Future research will focus on extending the methodology to incorporate techniques such as VIKOR, TOPSIS, and ELECTRE I, II, and III.

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