NSS

Neutrosophic Sets and Systems, Vol. 95, 2026

University of New Mexico

Optimal Neutrosophic Difference to Log-Type Estimator for Population Mean: Some Numerical and Simulation Studies

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Abstract. Indeterminacy in the data are commonly observed in various fields like biomedicine, finance, marketing and other sphere of sciences, where classical statistical methods is a challenging task to deal such type of data. This situation may be effecitively handled under neutrosophic framework. In this paper, we proposed a novel optimal neutrosophic difference to log-type estimator for estimation of population mean, utilizing the dual of an auxiliary variable. We derived the expression for bias and mean squared error (MSE) of the proposed estimator up to the first-order of approximation and determined optimal situation for real constants based on minimum MSE values. We conducted a numerical study using two real-life datasets related to temperature and atmospheric conditions and the results are validated through a simulation study consisting two artificially generated datasets. The findings indicates that the proposed neutrosophic estimator exhibited greater efficiency compared to existing estimators when dealing with uncertain, indeterminate, interval and neutrosophic data. These results highlights its potential applicability in analyzing uncertain or interval-valued data across various scientific domains, including environmental monitoring, social sciences and atmospheric sciences.

Keywords: Study variable; Auxiliary variable; Neutrosophic data; Mean squared error; Percent relative efficiency.

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1. Introduction

In sample surveys, accurate estimation of population parameters, such as the population mean $\bar{Y}$ of a study variable Y, is crucial for researchers to draw cost-effective and time-efficient inferences. The sample mean $\bar{y}$ is an unbiased estimator of $\bar{Y}$, but its sampling variance can be large, particularly when the study variable exhibits high variability or sample sizes are limited. This can lead to imprecise estimates of the true population mean. To address this, survey sampling theory often leverages auxiliary variables to enhance estimator precision. An auxiliary variable X, strongly correlated (positively or negatively) with Y, can significantly reduce variance when incorporated into estimation techniques. For instance, ratio estimation is effective for positively correlated variables, while product estimation is suitable for negatively correlated ones, often assuming a linear relationship through the origin, although adaptations exist for cases where the relationship does not pass through the origin. These methods are widely applied in fields such as agriculture (e.g., estimating crop yield using area and production), economics (e.g., correlating income with investment), and healthcare (e.g., linking hospital infrastructure to health indicators). However, traditional estimators may struggle with complex data structures, such as those involving uncertainty or non-linear relationships. To address these challenges, this study develops efficient estimation techniques that leverage auxiliary information to overcome these limitations, offering improved accuracy and robustness for practical applications across diverse domains.

1.1. *Classical Statistics*

In classical sampling theory, ratio, product and regression estimators leverage auxiliary information to enhance the accuracy of population mean estimation. Ratio estimators, effective for positively correlated study and auxiliary variables, are introduced by [3] for estimating the population mean $\bar{Y}$. Product estimators, suitable for negatively correlated variables, are developed by [7] to improve precision in such scenarios. Regression estimators, proposed by [4], model linear relationships between the study variable Y and auxiliary variable X. Subsequent research has extended these methods to address specific challenges. For instance, [2] introduced exponential ratio and product estimators to handle non-linear relationships, while [12] proposed estimators incorporating known population proportions of auxiliary variables. [15] developed a family of estimators using the known median of the study variable under simple random sampling and [14] explored estimators combining traditional and non-traditional auxiliary variables. Recently, [8] proposed generalized exponential-type estimators under post-stratified sampling and [17] suggested an exponential ratio-type estimator in stratified random sampling using a linear cost function to optimize efficiency. [1] recommended estimators that account for extreme values in stratified double-phase sampling, validated through simulations and real-world data.

1.2. *Neutrosophic Statistics*

Earlier studies in classical sampling theory have mostly focused on collecting data that is clear, specific and precise. These methods often give a single accurate result. In many real-world situations, however, data are uncertain, unclear, or imprecise—what we call neutrosophic data. Traditional statistical methods cannot handle this kind of data well, so neutrosophic statistics is used instead. Neutrosophic data includes observations that are uncertain and fall within a range of possible values, called interval-valued neutrosophic numbers. This approach helps deal with situations where the exact value of data cannot be determined at the time of collection. Such indeterminate data is common in practice and often more available than clear, precise data, especially when collecting information is expensive or complex.

Addressing this issue, [13] proposed a neutrosophic ratio estimator of the population mean in the presence of an auxiliary variable. Along these lines, the neutrosophic exponential type estimator was developed by [5] for the population mean. [11] proposed a neutrosophic regression cum ratio estimator using population parameters of an auxiliary variable. Some other researchers such as, [10] recommended neutrosophic Horvitz-Thompson estimators for estimation of the population mean in unequal probability sampling with ambiguous data. [9] suggested a robust neutrosophic exponential estimator for population mean estimation under uncertainty in simple random sampling. [6] developed efficient neutrosophic ranked set sampling (NRSS) estimators for accurate population mean estimation under uncertain and imprecise data conditions. They conducted simulation and numerical studies to analyze the effectiveness of the proposed estimators.

The aforementioned researchers have suggested various types of estimators to increase the efficiency for the estimation of population mean. In our knowledge, no studies have ever been addressed using log-type estimator with effective improvement in case of neutrosophic data. Considering this gap, we have proposed an optimal neutrosophic difference to log-type estimator for the estimation of population mean in neutrosophic data, which lays the foundation for establishing more neutrosophic log and difference estimators in sample surveys.

1.3. *Flow Chart*

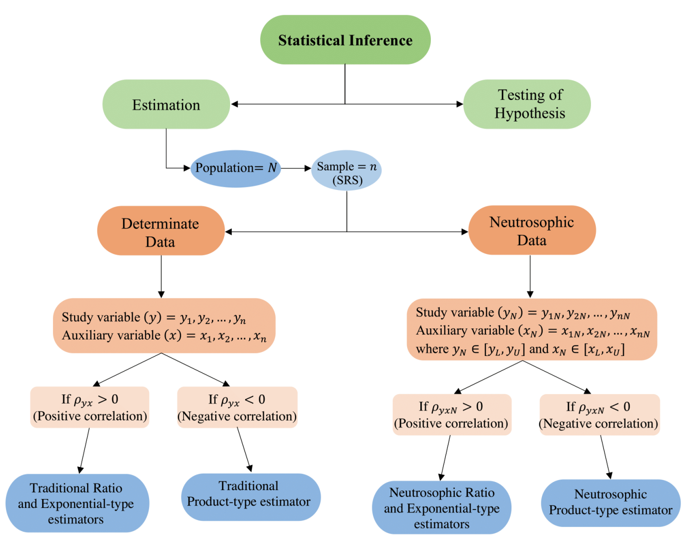

Figure 1: Flow chart of research outline

1.4. *Terminology*

Consider a neutrosophic random sample of size $n_N \in [n_L, n_U]$, which is drawn from a finite population of N units $(\omega_1, \omega_2, \ldots, \omega_N)$. Let y_{Ni} be the i-th sample observation of our neutrosophic data, which is of the form $y_N \in [y_L, y_U]$. Similarly, for the auxiliary variable, let $x_N \in [x_L, x_U]$. Let $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$ be our neutrosophic variable of interest and $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$ be our auxiliary neutrosophic variable, which is correlated with our study variable $\bar{y}_N$. In addition, let $Y_N \in [Y_L, Y_U]$ and $X_N \in [X_L, X_U]$ be the overall mean of the neutrosophic data set. Let $C_{yN} \in [C_{yNL}, C_{yNU}]$ and $C_{xN} \in [C_{xNL}, C_{xNU}]$ be the neutrosophic coefficients of variation for Y_N and X_N, respectively. ρ_{xyN} is the neutrosophic correlation between the neutrosophic variables X_N and Y_N. Furthermore, let $\beta_{2(x)N} \in \left[\beta_{2(x)L}, \beta_{2(x)U}\right]$ be the neutrosophic coefficient of kurtosis for auxiliary variable X_N. The neutrosophic mean errors are denoted by $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$ and $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$, which can be computed by the following relations. Similarly, the neutrosophic sets for bias, $\mathrm{Bias}\left(\bar{y}_N\right) \in [\mathrm{Bias}_L, \mathrm{Bias}_U]$ and MSE, $\mathrm{MSE}\left(\bar{y}_N\right) \in [MSE_L, MSE_U]$, are also computed for the analysis.

$$
\begin{aligned}
&\bar{e}_{yN} = \bar{y}_N - \bar{Y}_N ; \bar{e}_{xN} = \bar{x}_N - \bar{X}_N \\
&E\left(\bar{e}_{yN}\right) = E\left(\bar{e}_{xN}\right) = 0 \\
&\mathrm{E}\left(\bar{e}_{yN}^2\right) = \phi_N \bar{Y}_N^2 C_{yN}^2 ; \quad E\left(\bar{e}_{xN}^2\right) = \phi_N \bar{X}_N^2 C_{xN}^2 \\
&E\left(\bar{e}_{yN} \bar{e}_{xN}\right) = \phi_N \bar{X}_N \bar{Y}_N C_{xN} C_{yN} \rho_{xyN}
\end{aligned}
$$

where $\bar{e}_{yN} \in \left[\bar{e}_{yL}, \bar{e}_{yU}\right] ; \bar{e}_{xN} \in \left[\bar{e}_{xL}, \bar{e}_{xU}\right] ; e_{yN} e_{xN} \in \left[\bar{e}_{xL} \bar{e}_{yL}, \bar{e}_{xU} \bar{e}_{yU}\right]$

$$
e_{yN}^2 \in \left[e_{yL}^2, e_{yU}^2\right] ; e_{xN}^2 \in \left[e_{xL}^2, e_{xU}^2\right]
$$

$$
\begin{aligned}
C_{xN}^2 &= \frac{S_{xN}^2}{\bar{X}_N^2} ; C_{xN}^2 \in \left[C_{xL}^2, C_{xU}^2\right] \\
C_{yN}^2 &= \frac{S_{yN}^2}{\bar{Y}_N^2} ; C_{yN}^2 \in \left[C_{yL}^2, C_{yU}^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\rho_{xyN} = \frac{S_{xyN}}{S_{xN} S_{yN}} ; \rho_{xyN} \in \left[\rho_{xyL}, \rho_{xyU}\right] \\
&\rho_{xr_xN} = \frac{S_{xr_xN}}{S_{xN} S_{r_xN}} ; \rho_{xr_xN} \in \left[\rho_{xr_xL}, \rho_{xr_xU}\right] \\
&\therefore \phi_N = \frac{1 - f_N}{n_N} ; \phi_N \in \left[\phi_L, \phi_U\right] ; n_N \in \left[n_L, n_U\right] \\
&S_{xN}^2 \in \left[S_{xL}^2, S_{xU}^2\right] ; S_{yN}^2 \in \left[S_{yL}^2, S_{yU}^2\right] ; S_{xyN} \in \left[S_{xyL}, S_{xyU}\right]
\end{aligned}
$$

The unbiased Neutrosophic estimator can be expressed as

$$
\eta_{0N} = \frac{1}{n} \sum_{i=1}^{n} y_{Ni} = \bar{y}_N \tag{1}
$$

The variance of (1) is given by

$$Var(\eta_{0N}) = \phi_N \bar{Y}_N^2 C_{yN}^2 \tag{2}$$

[13] proposed neutrosophic modified estimators of $\bar{Y}_N$ is given by

$$\eta_{1N} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N}\right) \tag{3}$$

$$\eta_{2N} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N}\right) \tag{4}$$

$$\eta_{3N} = \bar{y}_N \exp\left[\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right] \tag{5}$$

$$\eta_{4N} = \bar{y}_N \exp\left[\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right] \tag{6}$$

$$\eta_{5N} = \bar{y}_N \exp\left[\alpha\left(\frac{\bar{X}_N^{\frac{1}{h}} - \bar{x}_N^{\frac{1}{h}}}{\bar{X}_N^{\frac{1}{h}} + (a-1)\bar{x}_N^{\frac{1}{h}}}\right)\right] \tag{7}$$

The bias and MSE of Eqn(3-7) up to first-degree approximation is given by

$$\text{Bias}(\eta_{1N}) = \phi_N \bar{Y}_N \left[C_{xN}^2 - \rho_{xyN} C_{xN} C_{yN}\right] \tag{8}$$

$$\text{Bias}(\eta_{2N}) = \phi_N \bar{Y}_N \left[C_{xN}^2 + \rho_{xyN} C_{xN} C_{yN}\right] \tag{9}$$

$$\text{Bias}(\eta_{3N}) = \phi_N \bar{Y}_N \left[\frac{3}{8}C_{xN}^2 - \frac{1}{2}C_{xN} C_{yN} \rho_{xyN}\right] \tag{10}$$

$$\text{Bias}(\eta_{4N}) = \phi_N \bar{Y}_N \left[\frac{3}{8}C_{xN}^2 + \frac{1}{2}C_{xN} C_{yN} \rho_{xyN}\right] \tag{11}$$

$$\text{Bias}(\eta_{5N}) = \phi_N \bar{Y}_N \left[\frac{\alpha C_{xN}^2}{ah^2} - \frac{\alpha C_{xN}^2}{a^2h^2} + \frac{\alpha^2 C_{xN}^2}{2a^2h^2} - \frac{\alpha C_{xN} C_{yN} \rho_{xyN}}{ah}\right] \tag{12}$$

$$\text{MSE}(\eta_{1N}) = \phi_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{xN}^2 - 2\rho_{xyN} C_{xN} C_{yN}\right] \tag{13}$$

$$\text{MSE}(\eta_{2N}) = \phi_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{xN}^2 + 2\rho_{xyN} C_{xN} C_{yN}\right] \tag{14}$$

$$\text{MSE}(\eta_{3N}) = \phi_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4}C_{xN}^2 - C_{xN} C_{yN} \rho_{xyN}\right] \tag{15}$$

$$\text{MSE}(\eta_{4N}) = \phi_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4}C_{xN}^2 + C_{xN} C_{yN} \rho_{xyN}\right] \tag{16}$$

$$\text{MSE}(\eta_{5N})_{\min} = \phi_N \bar{Y}_N^2 C_{yN}^2 \left(1 - \rho_{xyN}^2\right) \tag{17}$$

In the lines of [4] the neutrosophic regression estimator can be expressed as

$$\eta_{6N} = \bar{y}_N + \beta_N \left(\bar{X}_N - \bar{x}_N\right) \tag{18}$$

The bias and MSE of Eqn(18) up to first-degree approximation is given by

$$\text{Bias}(\eta_{6N}) = 0 \tag{19}$$

$$\text{MSE}(\eta_{6N})_{\min} = \phi_N \bar{Y}_N^2 C_{yN}^2 \left(1 - \rho_{xyN}^2\right) \tag{20}$$

[5] proposed Neutrosophic exponential-type estimator is given by

$$\eta_{7N} = \bar{y}_N \left(\frac{\bar{X}_N^{\frac{1}{h}} + A^{\frac{1}{h}}}{\bar{x}_N^{\frac{1}{h}} + A^{\frac{1}{h}}} \right) \left[\alpha \exp\left(\frac{\bar{X}_N^{\frac{1}{h}} - \bar{x}_N^{\frac{1}{h}}}{\bar{X}_N^{\frac{1}{h}} + \bar{x}_N^{\frac{1}{h}}} \right) + (1-\alpha) \exp\left(\frac{\bar{x}_N^{\frac{1}{h}} - \bar{X}_N^{\frac{1}{h}}}{\bar{x}_N^{\frac{1}{h}} + \bar{X}_N^{\frac{1}{h}}} \right) \right] \tag{21}$$

The bias and MSE of Eqn(21) up to first-degree approximation is given by

$$\text{Bias}\left(\eta_{7N}\right) = \phi_N \bar{Y}_N \left((B_1 - 1) - \frac{C_{xN}^2}{4h^2} B_3 + \frac{\rho_{xyN} C_{xN} C_{yN}}{2h} B_2 \right) \tag{22}$$

$$\text{MSE}\left(\eta_{7N}\right)_{\min} = \phi_N \bar{Y}_N^2 \left((B_1 - 1)^2 + C_{yN}^2 B_1^2 + \frac{C_{xN}^2}{4h^2} B_{2\text{opt}}^2 + \frac{\rho_{xyN} C_{xN} C_{yN}}{h} B_{2\text{opt}} B_1 \right) \tag{23}$$

where

$$\begin{aligned}
B_1 &= 1 - \frac{A}{h} + \frac{A}{h\bar{X}_N} - \frac{A^2}{h^2 \bar{X}_N} \\
B_2 &= -1 - 2\alpha - \frac{A}{h}(1-2\alpha) + \frac{A}{h\bar{X}_N}(1-2\alpha) - \frac{A^2}{h^2 \bar{X}_N}(1-2\alpha) \\
B_3 &= 3 - 6\alpha - \frac{A}{h}(1-2\alpha) + \frac{A}{h\bar{X}_N}(3-6\alpha) - \frac{A^2}{h^2 \bar{X}_N}(1-2\alpha) \\
B_{2\text{opt}} &= -1 - 2\alpha_{\text{opt}} - \frac{A}{h}(1-2\alpha_{\text{opt}}) + \frac{A}{hX_N}(1-2\alpha_{\text{opt}}) - \frac{A^2}{h^2 X_N}(1-2\alpha_{\text{opt}}) \\
\alpha_{\text{opt}} &= \frac{\rho_{xyN} C_{xN} C_{yN} h}{C_{xN}^2}.
\end{aligned}$$

[11] proposed Neutrosophic regression cum ratio estimators can be rewritten in general form as

$$\eta_{8N} = \bar{y}_N + b_N \left(\bar{X}_N - \bar{x}_N \right) \left(\frac{a\bar{X}_N + b}{a\bar{x}_N + b} \right) \tag{24}$$

The bias and MSE of Eqn(24) up to first-degree approximation is given by

$$Bias\left(\eta_{8N}\right) = \phi_N \left[\left(\bar{Y}_N \Delta_N^2 + \bar{X}_N b_N \Delta_N \right) C_{xN}^2 - \bar{Y}_N \rho_{xyN} C_{xN} C_{yN} \right] \tag{25}$$

$$MSE\left(\eta_{8N}\right)_{min} = \phi_N \bar{Y}_N^2 \left[\Delta_N^2 C_{xN}^2 + C_{yN}^2 \left(1 - \rho_{yxN}^2 \right) \right] \tag{26}$$

where $\Delta_N = \frac{a\bar{X}_N}{a\bar{X}_N + b}$ and we have chosen $a = \beta_{2(x)N}, b = C_{xN}$ (*see* Singh et al. (2023)) for optimal version of η_{7N}.

[14] proposed Neutrosophic ratio cum exponential estimator is given by

$$\eta_{9N} = \kappa_1 \bar{y}_N \left(\frac{a\bar{X}_N + b}{a\bar{x}_N + b} \right) + \kappa_2 \bar{y}_N \exp\left(\frac{\left(a\bar{X}_N + b\right) - \left(a\bar{x}_N + b\right)}{\left(a\bar{X}_N + b\right) + \left(a\bar{x}_N + b\right)} \right) \tag{27}$$

The bias and MSE of Eqn(22) up to first-degree approximation is given by

$$\text{Bias}(\eta_{9N}) = \bar{Y}_N \left[\kappa_1 \left(1 - \lambda \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN} + \lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 \right) \right. \tag{28}$$

$$\left. + \kappa_2 \left(1 - \frac{\lambda}{2} \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN} + \frac{3}{8} \lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 \right) - 1 \right] \tag{29}$$

$$\text{MSE}(\eta_{9N})_{\min} = \bar{Y}_N^2 \left\{ 1 - \frac{P}{Q} \right\} \tag{30}$$

where,

$$\lambda = \frac{a\bar{X}_N}{2\left(a\bar{X}_N + b\right)}$$

$$P = \begin{bmatrix} 2C(DF-BC)\left(F^2-AB\right) + 2D(CF-AD)\left(F^2-AB\right) \\ -2F(DF-BC)(CF-AD) - (DF-BC) - (CF-AD) \end{bmatrix}$$

$$Q = \left(F^2 - AB\right)^2$$

$$A = \left(1 + \phi_N \bar{Y}_N^2 C_{yN}^2 + 3\lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 - 4\lambda \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN}\right)$$

$$B = \left(1 + \phi_N \bar{Y}_N^2 C_{yN}^2 + \lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 - 2\lambda \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN}\right)$$

$$C = \left(1 + \lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 - \lambda \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN}\right)$$

$$D = \left(1 + \frac{3}{8}\lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 - \frac{\lambda}{2} \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN}\right)$$

$$F = \left(1 + \phi_N \bar{Y}_N^2 C_{yN}^2 + \frac{15}{8}\lambda^2 \phi_N \bar{X}_N^2 C_{xN}^2 - 3\lambda \phi_N \bar{Y}_N \bar{X}_N C_{xN} C_{yN} \rho_{xyN}\right)$$

The above researchers have suggested the estimation procedures based on the original information available on an auxiliary variable. To the best of our knowledge, so far no one has studied the use of dual of an auxiliary variable. In the next section, we have proposed a new estimation procedure utilizing the dual of an auxiliary variable for the estimation of population mean in case of neutrosophic data.

2. **Proposed Neutrosophic estimator**

In this section, we have introduced a new log-type neutrosophic estimator in simple random sampling by utilizing the rank of an auxiliry variable. The proposed difference to log-type neutrosophic estimator for estimation of population mean $\bar{Y}$ is given by

$$\eta^*_{10N} = \left[\alpha_1 \bar{y}_N + \beta_1\left(\bar{X}_N - \bar{x}_N\right) + \beta_2(\bar{R}_{xN} - \bar{r}_{xN})\right]\left(1 + \log\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right) \tag{31}$$

where α_1, β_1 and β_2 are real constants can be determined later, $\bar{R}_{xN}$ and $\bar{r}_{xN}$ are the rank of an auxiliary variable for population and sample mean respectively.

The equation (31) can be transformed in terms of $e'_N s$ as follows:

$$\eta^*_{10N} = \left(\alpha_1 \bar{Y}_N(1+e_{0N}) - \beta_1 \bar{X}_N e_{1N} - \beta_2 \bar{R}_{xN} e_{2N}\right)\left(1 + e_{1N} - \frac{e_{1N}^2}{2}\right) \tag{32}$$

Simplifying Eqn (32) and taking expectations up to the first degree of approximation, we have

$$\begin{aligned} \eta^*_{10N} - \bar{Y}_N =& \bar{Y}_N\alpha_1 + e_{0N}\bar{Y}_N\alpha_1 + e_{0N}e_{1N}\bar{Y}_N\alpha_1 - e_{1N}^2\left(\frac{\bar{Y}_N\alpha_1}{2} + \bar{X}_N\beta_1\right) \\ & + e_{1N}\left(\bar{Y}_N\alpha_1 - \bar{X}_N\beta_1\right) - e_{2N}\bar{R}_{xN}\beta_2 - e_{1N}e_{2N}\bar{R}_{xN}\beta_2 - \bar{Y}_N \end{aligned} \tag{33}$$

Taking expectations on both side, we obtain bias of η^*_{10N} as

$$\begin{aligned}\text{Bias}(\eta^*_{10N}) =&\bar{Y}_N \left(\alpha_1 + \alpha_1\phi_N\rho_{yxN}C_{xN}C_{yN} - \left(\frac{\alpha_1}{2} + \frac{\bar{X}_N}{\bar{Y}_N}\beta_1\right)\phi_N C^2_{xN} - 1\right) \\ &- \phi_N\rho_{xr_xN}C_{r_xN}C_{xN}\bar{R}_{xN}\beta_2 \end{aligned} \tag{34}$$

To drive the MSE of η^*_{10N}, squaring on both side of equation (33) up to power two, we get

$$\begin{aligned}\text{MSE}(\eta^*_{10N}) = &\big[\bar{Y}^2_N - 2\bar{Y}^2_N\alpha_1 + \bar{Y}^2_N\alpha^2_1 + C^2_{yN}\bar{Y}^2_N\alpha^2_1\phi_N \\ &+ C^2_{xN}\left(\bar{Y}^2_N\alpha_1 + 2\bar{X}_N\bar{Y}_N\beta_1 - 4\bar{X}_N\bar{Y}_N\alpha_1\beta_1 + \bar{X}^2_N\beta^2_1\right)\phi_N \\ &+ C_{r_xN}C_{xN}\left(2\bar{R}_{xN}\bar{Y}_N\beta_2 - 4\bar{R}_{xN}\bar{Y}_N\alpha_1\beta_2 + 2\bar{R}_{xN}\bar{X}_N\beta_1\beta_2\right)\rho_{xr_xN}\phi_N \\ &- 2C_{r_xN}C_{yN}\bar{R}_{xN}\bar{Y}_N\alpha_1\beta_2\rho_{yr_xN}\phi_N + C^2_{r_xN}\bar{R}^2_{xN}\beta^2_2\phi_N \\ &+ C_{xN}C_{yN}\left(-2\bar{Y}^2_N\alpha_1 + 4\bar{Y}^2_N\alpha^2_1 - 2\bar{X}_N\bar{Y}_N\alpha_1\beta_1\right)\rho_{yxN}\phi_N\ \big] \end{aligned} \tag{35}$$

The optimum values of α_1, β_1 and β_2 can be determine by setting $\dfrac{\partial(\text{MSE}(\eta^*_{10N}))}{\partial z} = 0$, $(z = \alpha_1, \beta_1, \beta_2)$ then solving system of linear equations, we obtain

$$\alpha_{1opt} = -\frac{B^*}{2A^*} \tag{36}$$

$$\beta_{1opt} = -\frac{C^*}{2C_{xN}\bar{X}_N A^*} \tag{37}$$

$$\beta_{2opt} = -\frac{D^*(B^* + E^*)}{2C_{r_xN}\bar{R}_x A^*} \tag{38}$$

where

$$\begin{aligned} A^* =& 1 - \rho^2_{xr_xN} - 4C^2_{xN}\phi_N + C^2_{yN}\phi_N + 4C^2_{xN}\rho^2_{xr_xN}\phi_N - C^2_{yN}\rho^2_{xr_xN}\phi_N - C^2_{yN}\rho^2_{yr_xN}\phi_N \\ &+ 2C^2_{yN}\rho_{xr_xN}\rho_{yr_xN}\rho_{yxN}\phi_N - C^2_{yN}\rho^2_{yxN}\phi_N, \\ B^* =& -2 + 2\rho^2_{xr_xN} + 5C^2_{xN}\phi_N - 5C^2_{xN}\rho^2_{xr_xN}\phi_N, \\ C^* =& -2C_{xN}\bar{Y}_N + 2C_{xN}\bar{Y}_N\rho^2_{xr_xN} + 2C_{yN}\bar{Y}_N\rho_{xr_xN}\rho_{yr_xN} - 2C_{yN}\bar{Y}_N\rho_{yxN} \\ &+ 2C^3_{xN}\bar{Y}_N\phi_N + 2C_{xN}C^2_{yN}\bar{Y}_N\phi_N - 2C^3_{xN}\bar{Y}_N\rho^2_{xr_xN}\phi_N - 2C_{xN}C^2_{yN}\bar{Y}_N\rho^2_{xr_xN}\phi_N \\ &- 5C^2_{xN}C_{yN}\bar{Y}_N\rho_{xr_xN}\rho_{yr_xN}\phi_N - 2C_{xN}C^2_{yN}\bar{Y}_N\rho^2_{yr_xN}\phi_N + 5C^2_{xN}C_{yN}\bar{Y}_N\rho_{yxN}\phi_N \\ &+ 4C_{xN}C^2_{yN}\bar{Y}_N\rho_{xr_xN}\rho_{yr_xN}\rho_{yxN}\phi_N - 2C_{xN}C^2_{yN}\bar{Y}_N\rho^2_{yxN}\phi_N, \\ D^* =& -C_{yN}\bar{Y}_N\rho_{yr_xN} + C_{yN}\bar{Y}_N\rho_{xr_xN}\rho_{yxN}, \qquad E^* = 5C^2_{xN}\phi_N. \end{aligned}$$

Now, putting Equation (36-38) in (35), we obtain minimum MSE of η^*_{8N}, which can be expressed as

$$\text{MSE}(\eta^*_{10N})_{\min} = \frac{\bar{Y}^2_N\phi_N\left(9C^4_{xN}(1-\rho^2_{xr_xN})\phi_N + 4C^2_{yN}Z^*(-1 + C^2_{xN}\phi_N)\right)}{4 - 4\rho^2_{xr_xN} + 16C^2_{xN}(1-\rho^2_{xr_xN})\phi_N - 4C^2_{yN}Z^*\phi_N} \tag{39}$$

where $Z^* = (-1 + \rho^2_{xr_xN} + \rho^2_{yr_xN} + \rho^2_{yxN} - 2\rho_{yxN}\rho_{xr_xN}\rho_{yr_xN})$.
(or)

$$\mathrm{MSE}(\eta^*_{10N})_{\min} = \bar{Y}^2_N \left(\frac{\Delta_1}{\Delta_2}\right) \tag{40}$$

where

$$\Delta_1 = \phi_N \left(9C^4_{xN}(1-\rho^2_{xr_xN})\phi_N + 4C^2_{yN}Z^*(-1 + C^2_{xN}\phi_N)\right)$$
$$\Delta_2 = 4 - 4\rho^2_{xr_xN} + 16C^2_{xN}(1-\rho^2_{xr_xN})\phi_N - 4C^2_{yN}Z^*\phi_N$$

3. Efficiency comparisons

In this section, the effectiveness of the proposed neutrosophic estimator η^*_{10N} has been compared with conventional estimators (η_{0N} to η_{9N}) theoretically as follows:
The proposed neutrosophic estimator η^*_{10N} is more efficient than η_{0N} to η_{9N} if,

$$\text{(i) } Var(\eta_{0N}) - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N C^2_{yN} - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{41}$$

$$\text{(ii) } MSE(\eta_{1N}) - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N \left[C^2_{yN} + C^2_{xN} \ -2\rho_{xyN}C_{xN}C_{yN}\right] - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{42}$$

$$\text{(iii) } MSE(\eta_{2N}) - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N \left[C^2_{yN} + C^2_{xN} \ +2\rho_{xyN}C_{xN}C_{yN}\right] - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{43}$$

$$\text{(iv) } MSE(\eta_{3N}) - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N \left[C^2_{yN} + \frac{1}{4}C^2_{xN} \ -\rho_{xyN}C_{xN}C_{yN}\right] - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{44}$$

$$\text{(v) } MSE(\eta_{4N}) - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N \left[C^2_{yN} + \frac{1}{4}C^2_{xN} \ +\rho_{xyN}C_{xN}C_{yN}\right] - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{45}$$

$$\text{(vi) } MSE(\eta_{5N})_{min} - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N C^2_{yN} \left(1 - \rho^2_{xyN}\right) - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{46}$$

$$\text{(vii) } MSE(\eta_{6N})_{min} - MSE(\eta^*_{10N})_{min} > 0 \ or,$$
$$\phi_N C^2_{yN} \left(1 - \rho^2_{xyN}\right) - \left(\frac{\Delta_1}{\Delta_2}\right) > 0 \tag{47}$$

(viii) $MSE(\eta_{7N})_{min} - MSE(\eta^{*}_{10N})_{min} > 0$ $or,$

$$\phi_N \bar{Y}_N^2 \left((B_1-1)^2 + C_{yN}^2 B_1^2 + \frac{C_{xN}^2}{4h^2} B_{2\text{opt}}^2 + \frac{\rho_{xyN} C_{xN} C_{yN}}{h} B_{2\text{opt}} B_1 \right) - \left(\frac{\Delta_1}{\Delta_2} \right) > 0 \tag{48}$$

(ix) $MSE(\eta_{8N}) - MSE(\eta^{*}_{10N})_{min} > 0$ $or,$

$$\phi_N \left[\Delta_N^2 C_{xN}^2 + C_{yN}^2 \left(1 - \rho_{yxN}^2 \right) \right] - \left(\frac{\Delta_1}{\Delta_2} \right) > 0 \tag{49}$$

(x) $MSE(\eta_{9N}) - MSE(\eta^{*}_{10N})_{min} > 0$ $or,$

$$\left\{ 1 - \frac{P}{Q} \right\} - \left(\frac{\Delta_1}{\Delta_2} \right) > 0 \tag{50}$$

From (i) to (x) denotes the conventional estimators (η_{0N} to η_{9N}) respectively,theoretically compared with proposed neutrosophic estimator.

4. **Empirical Study**

In this section, we have conducted numerical analysis using two real interval-based temperature datasets to appraise the merits of the proposed estimator. The data sets are given below:
Population 1 [Source:https://www.kaggle.com/datasets/abdullah0a/urban-air-quality-and-health-impact-dataset/data]: The data describes the average, minimum and maximum temperatures of U.S. city recorded between 07/09/24 to 21/09/24. Let the average temperature be the study variable y_N and (y_L, y_U) be the minimum and maximum temperatures. Let x_N be the heat index value of each day.
Population 2 [Source: https://www.kaggle.com/datasets/abdullah0a/urban-air-quality-and-health-impact-dataset/data]: The dataset outlines the average, minimum and maximum temperatures recorded in a U.S. city from 07/09/24 to 21/09/24. Let the average temperature as the study variable y_N, with (y_L, y_U) denotes the minimum and maximum temperatures, respectively. Let x_N denote the atmospheric pressure (in hPa).
To evaluate the performance of the proposed estimator η^{*}_{10N} compared to conventional neutrosophic estimators (η_{0N} to η_{9N}) on the basis of mean squared error (MSE) and percentage relative efficiency (PRE), Which can be computed using the formula given by

$$\text{MSE}(\Theta) = \frac{1}{L} \sum_{z=1}^{L} \left(\Theta_z - \overline{Y}_N \right)^2 \tag{51}$$

$$\text{PRE}(\Theta, \eta_{1N}) = \frac{\text{MSE}(\eta_{1N})}{\text{MSE}(\Theta)} \times 100\% \tag{52}$$

where $\Theta = \eta_1$ to η^{*}_{10N}. The results are presented in Tables 2 and 3 for population 1 and 2.

Table 1: Datasets description of classical and Neutrosophic Values

Parameters	Neutrosophic Values	Classical Values	Neutrosophic Values	Classical Values
	Population 1		**Population 2**	
N_N	$[15, 15]$	15	$[15, 15]$	15
n_N	$[6, 6]$	6	$[6, 6]$	6
$\overline{X}_N$	$[88.81221, 88.81221]$	88.81221	$[1006.873, 1006.873]$	1006.873
$\overline{Y}_N$	$[82.76, 101.0067]$	91.71333	$[82.76, 101.0067]$	91.71333
C_{yN}	$[0.04439109, 0.03994192]$	0.03578257	$[0.04439109, 0.03994192]$	0.03578257
C_{xN}	$[0.04288694, 0.04288694]$	0.04288694	$[0.00252778, 0.00252778]$	0.00252778
C_{r_xN}	$[0.559017, 0.559017]$	0.559017	$[0.5585176, 0.5585176]$	0.5585176
ρ_{yxN}	$[0.8159744, 0.6993929]$	0.903539	$[0.1310409, -0.2716254]$	-0.02646475
ρ_{yr_xN}	$[0.8051589, 0.6571827]$	0.8663093	$[0.1359812, -0.190793]$	0.03482949
ρ_{xr_xN}	$[0.9876591, 0.9876591]$	0.9876591	$[0.8868817, 0.8868817]$	0.8868817
$\beta_{2(x)N}$	$[2.06003, 2.06003]$	2.06003	$[3.932583, 3.932583]$	3.932583

Table 2: Comparaison of proposed η^*_{10N} neutrosophic and conventional η_{0N} to η_{9N} neutrosophic estimator based on MSE and PRE values for population 1

Estimators	MSE Neutrosophic	PRE Neutrosophic	MSE Classical	PRE Classical
η_{0N}	[1.34969, 1.62764]	[100, 100]	1.07698	100
η_{1N}	[0.45105, 0.83148]	[299.23482, 195.75]	0.19775	544.6111422
η_{2N}	[0.48147, 1.05956]	[280.32552, 153.61]	0.29148	369.4885283
η_{3N}	[4.73744, 5.94873]	[28.48977, 27.36]	4.95666	21.7279614
η_{4N}	[0.60064, 0.87447]	[224.70944, 186.13]	0.29746	362.061626
η_{5N}	[2.72862, 3.31906]	[49.46404, 49.04]	2.63005	40.949099
η_{6N}	[0.45105, 0.83148]	[299.23482, 195.75]	0.19775	544.6111422
η_{7N}	[0.83034, 1.39333]	[162.5452, 116.82]	0.67057	160.6072094
η_{8N}	[1.26863, 2.04932]	[106.38947, 79.42]	1.20180	89.6138763
η_{9N}	[0.45104, 0.83143]	[299.23653, 195.76]	0.19775	544.6121402
η^*_{10N}	**[0.45071, 0.75624]**	**[299.45589, 215.23]**	**0.16734**	**643.591064**

Table 3: Comparaison of proposed η_{10N}^* neutrosophic and conventional η_{0N} to η_{9N} neutrosophic estimator based on MSE and PRE values for population 2

Estimators	MSE Neutrosophic	PRE Neutrosophic	MSE Classical	PRE Classical
η_{0N}	[1.34969, 1.62764]	[100, 100]	1.07698	100
η_{1N}	[1.32651, 1.50755]	[101.74717, 107.97]	1.07623	100.0701
η_{2N}	[1.33392, 1.69012]	[101.18193, 96.3]	1.08638	99.1346
η_{3N}	[1.3742, 1.5782]	[98.21577, 103.13]	1.07833	99.875
η_{4N}	[1.34071, 1.65725]	[100.66958, 98.21]	1.08034	99.6893
η_{5N}	[1.36085, 1.60129]	[99.17953, 101.65]	1.07631	100.0622
η_{6N}	[1.32651, 1.50755]	[101.74717, 107.97]	1.07623	100.0701
η_{7N}	[1.36961, 1.58106]	[98.54517, 102.95]	1.14015	94.4596
η_{8N}	[1.32823, 1.51011]	[101.61545, 107.78]	1.07834	99.8741
η_{9N}	[1.32626, 1.50737]	[101.76637, 107.98]	1.07609	100.0828
η_{10N}^*	**[1.32379, 1.48819]**	**[101.95649, 109.37]**	**1.05894**	**101.7032**

Comments on the results:

From Tables 2 and 3, it is found that:

(i) The MSE values of the proposed neutrosophic estimator η_{10N}^* are the smallest among all available neutrosophic estimators for populations 1 and 2.

(ii) The PRE values of η_{10N}^* are the highest among all available neutrosophic estimators for populations 1 and 2.

(iii) Among all conventional neutrosophic estimators η_{9N} shows good performance populations 1 and 2, while the proposed neutrosophic estimator η_{10N}^* has minimum MSE and maximum PRE values over η_{9N}.

(iv) The proposed estimator η_{10N}^* consistently achieves better performance compared to all conventional estimators, which have minimum MSE and maximum PRE values for populations 1 and 2.

(v) Overall, the results validate the efficiency of the proposed estimator η_{10N}^*. For a closer view of the results obtained in Tables 2 and 3, we have presented in Figures 2-5.

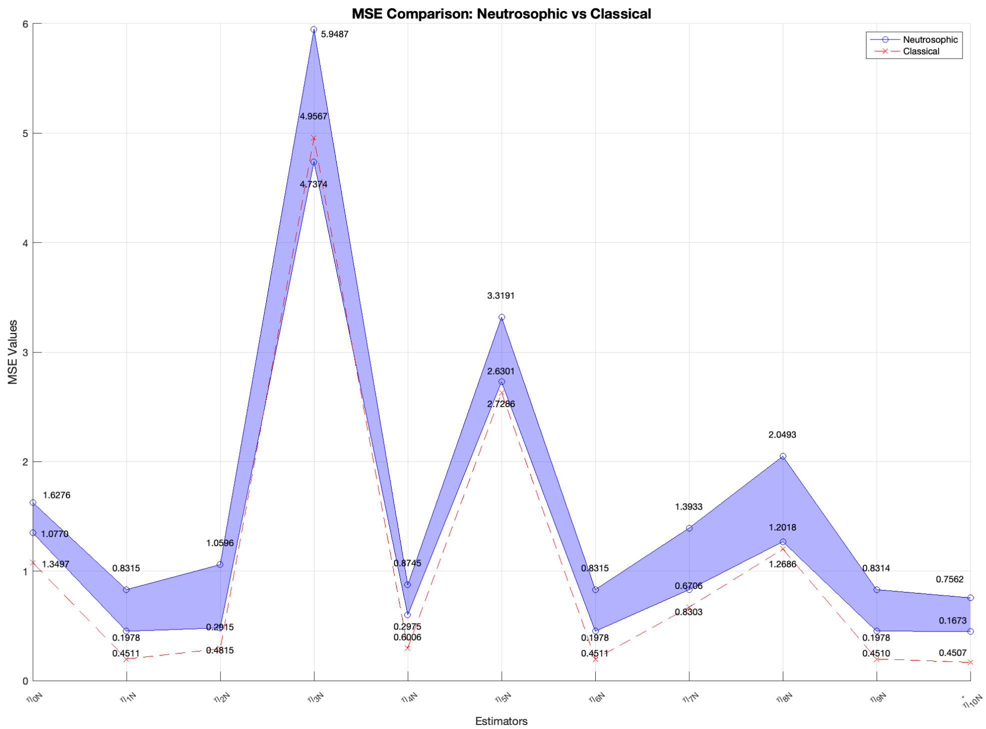

Figure 2: MSEs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 1

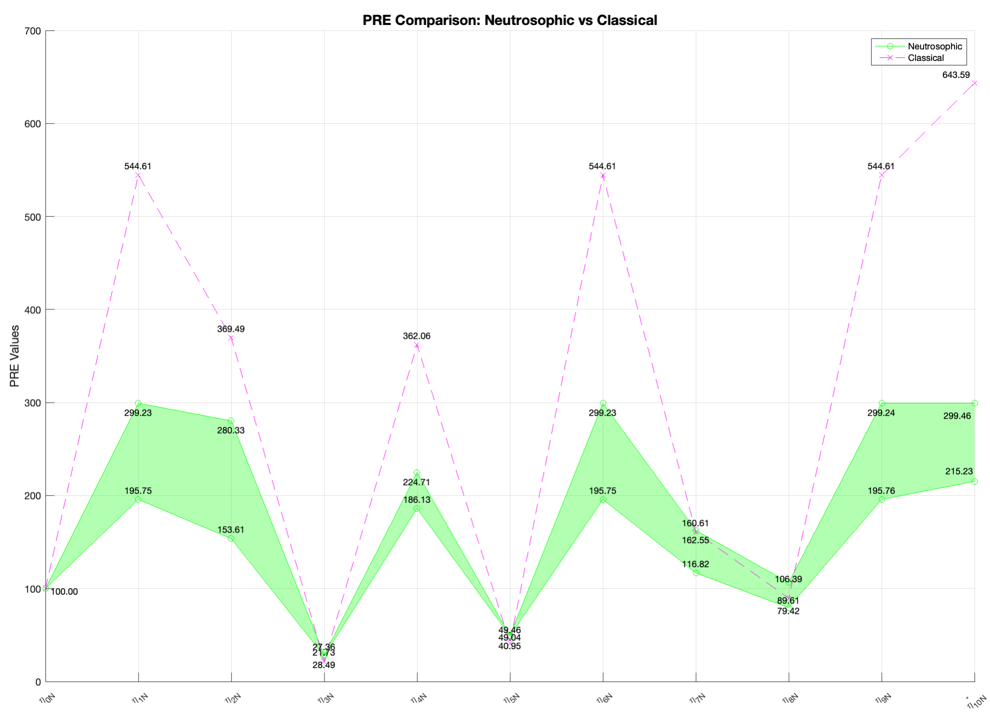

Figure 3: PREs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 1

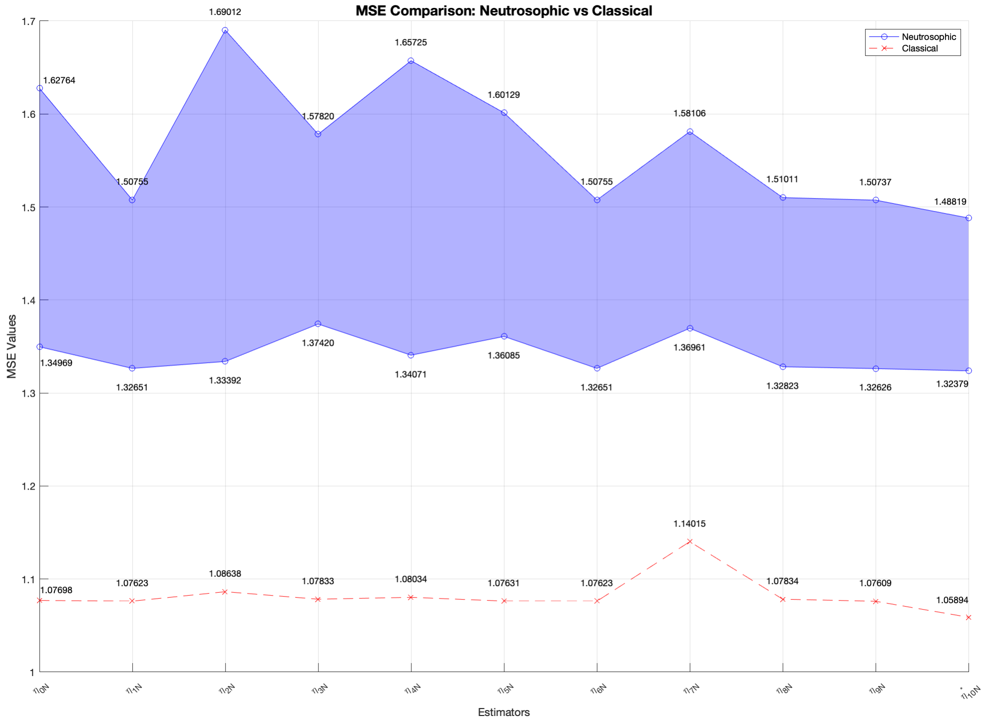

Figure 4: MSEs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 2

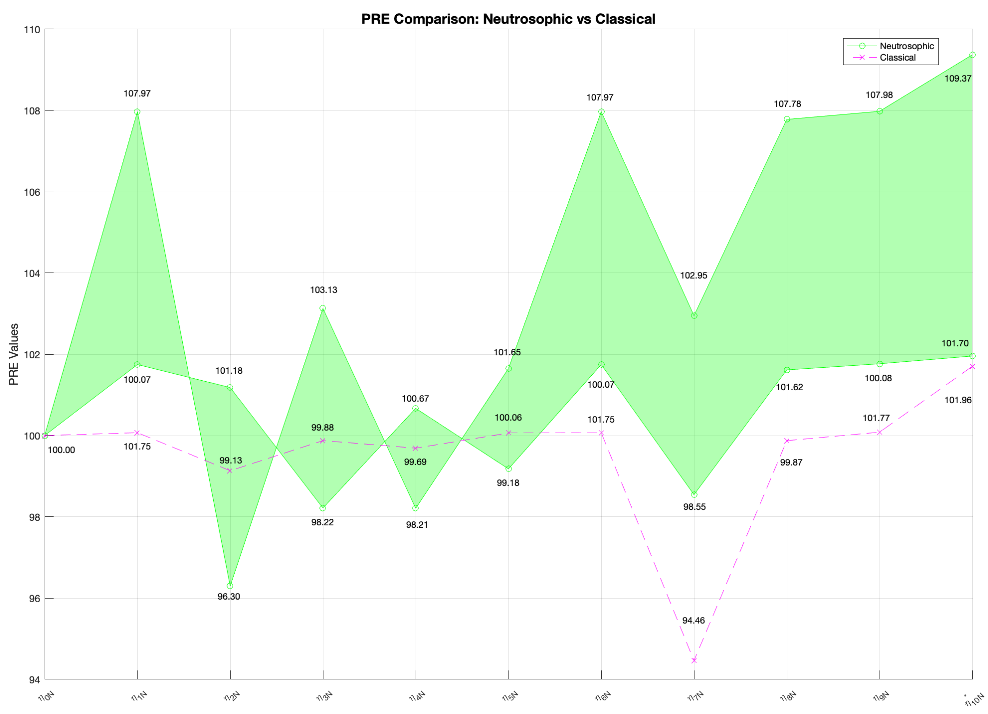

Figure 5: PREs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 2

5. Simulation study

In this section, we have conducted simulation study to validate the numerical results. For this purpose, we have artificially generated neutrosophic data sets and comparison is made based on MSEs and PREs among all the neutrosophic estimators. The study variable Y_N and auxiliary variable X_N follows the bivariate normal distribution, further simulation process is mentioned as follows:

Simulation Procedure for Neutrosophic Data:

For Population 3

(i) A covariance matrix is constructed to fix the correlation between Y_N and X_N.

$$\mu = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 5.4 \\ 5.4 & 9 \end{bmatrix} \text{ and } \rho_{YX} = 0.9.$$

(ii) A bivariate normal distribution is used to simulate correlated random variables for Y and X.

(iii) The study variable Y is transformed into $Y_N \in [Y_L, Y_U]$ using $Y_L = Y - \frac{\alpha}{2}$ and $Y_U = Y + \frac{\alpha}{2}$, where α is interval width, which can be arbitrary constant.

(iv) The auxiliary variable X remains a single-valued variable as X_N.

(v) we have generated $N = 2000, n = 40$ neutrosophic study $Y_N \in [Y_L, Y_U]$ and neutrosophic auxiliary $X_N \in [X_L, X_U]$ variables is taken for detailed analysis.

For Population 4

(i) A covariance matrix is constructed to fix the correlation between Y_N and X_N.

$$\mu = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & -5.4 \\ -5.4 & 9 \end{bmatrix} \text{ and } \rho_{YX} = -0.9.$$

(ii) A bivariate normal distribution is used to simulate correlated random variables for Y and X.

(iii) The study variable Y is transformed into $Y_N \in [Y_L, Y_U]$ using $Y_L = Y - \frac{\alpha}{2}$ and $Y_U = Y + \frac{\alpha}{2}$, where α is interval width, which can be arbitrary constant.

(iv) The auxiliary variable X remains a single-valued variable as X_N.

(v) we have generated $N = 2000, n = 40$ neutrosophic study $Y_N \in [Y_L, Y_U]$ and neutrosophic auxiliary $X_N \in [X_L, X_U]$ variables is choosen for detailed analysis.

To compute the MSE and PRE of proposed optimal difference to log-type neutrosophic estimator η^*_{10N} and conventional neutrosophic estimators (η_{0N} to η_{9N}) using formulae (40) and (41), the results are presented in table 4 and 5 for population 3 and 4.

Table 4: Comparaison of proposed η^*_{10N} neutrosophic and conventional (η_{0N} to η_{9N}) neutrosophic estimator based on MSE and PRE values for population 3 ($\rho_{yxN} = 0.9$)

Estimators	MSE Neutrosophic	PRE Neutrosophic	MSE Classical	PRE Classical
η_{0N}	[0.21988, 0.21988]	[100, 100]	0.21988	100
η_{1N}	[0.04047, 0.04047]	[543.37498, 543.37]	0.04047	543.375
η_{2N}	[0.05988, 0.1098]	[367.20136, 200.26]	0.081	271.4709
η_{3N}	[1.01363, 1.27359]	[21.69271, 17.26]	1.13976	19.2919
η_{4N}	[0.06066, 0.04689]	[362.45896, 468.96]	0.05282	416.324
η_{5N}	[0.53754, 0.62878]	[40.90565, 34.97]	0.5822	37.7676
η_{6N}	[0.04047, 0.04047]	[543.37498, 543.37]	0.04047	543.375
η_{7N}	[0.12045, 0.15987]	[182.55753, 137.54]	0.13918	157.984
η_{8N}	[0.16956, 0.23268]	[129.68167, 94.5]	0.19955	110.1888
η_{9N}	[0.04046, 0.04047]	[543.5057, 543.38]	0.04046	543.417
η^*_{10N}	**[0.03813, 0.03678]**	**[576.62201, 597.85]**	**0.03749**	**586.5058**

Table 5: Comparaison of proposed η^*_{10N} neutrosophic and conventional (η_{0N} to η_{9N}) neutrosophic estimator based on MSE and PRE values for population 4 ($\rho_{yxN} = -0.9$)

Estimators	MSE Neutrosophic	PRE Neutrosophic	MSE Classical	PRE Classical
η_{0N}	[0.22137, 0.22137]	[100, 100]	0.22137	100
η_{1N}	[0.04109, 0.04109]	[538.78439, 538.78]	0.04109	538.7844
η_{2N}	[1.03416, 1.3007]	[21.40616, 17.02]	1.16347	19.0269
η_{3N}	[0.06279, 0.11568]	[352.54503, 191.36]	0.08528	259.5772
η_{4N}	[0.54599, 0.63933]	[40.54532, 34.63]	0.59167	37.4149
η_{5N}	[0.06031, 0.04682]	[367.07322, 472.78]	0.05258	421.048
η_{6N}	[0.04109, 0.04109]	[538.78439, 538.78]	0.04109	538.7844
η_{7N}	[0.12365, 0.16429]	[179.02978, 134.75]	0.14296	154.8474
η_{8N}	[0.17166, 0.23541]	[128.96199, 94.04]	0.20196	109.6146
η_{9N}	[0.04013, 0.04028]	[551.65676, 549.64]	0.04021	550.5098
η^*_{10N}	**[0.03859, 0.03715]**	**[573.68999, 595.89]**	**0.03791**	**584.0203**

Comments on the results:

From the Tables 4 and 5, it is observed that:

(i) The MSE values of the proposed neutrosophic estimator η^*_{10N} are the smallest among all the conventional estimators for populations 3 and 4.

(ii) The PRE values of η^*_{10N} are significantly higher than all conventional estimators.

(iii) Among all conventional neutrosophic estimators η_{9N} shows good performance populations 3 and 4. While the proposed neutrosophic estimator η^*_{10N} has minimum MSE and maximum PRE values over η_{9N}.

(iv) The proposed estimator η^*_{10N} gives superior performances over conventional neutrosophic estimators in terms of achieving the lowest MSE and the highest PRE values in populations 3 and 4.

(v) Overall, the results validate that η^*_{10N} is efficient neutrosophic estimator under different population structures ($\rho_{yxN} = 0.9$ and $\rho_{yxN} = -0.9$). For an instant view of the results obtained in Tables 4 and 5, we have presented in Figures 6-9.

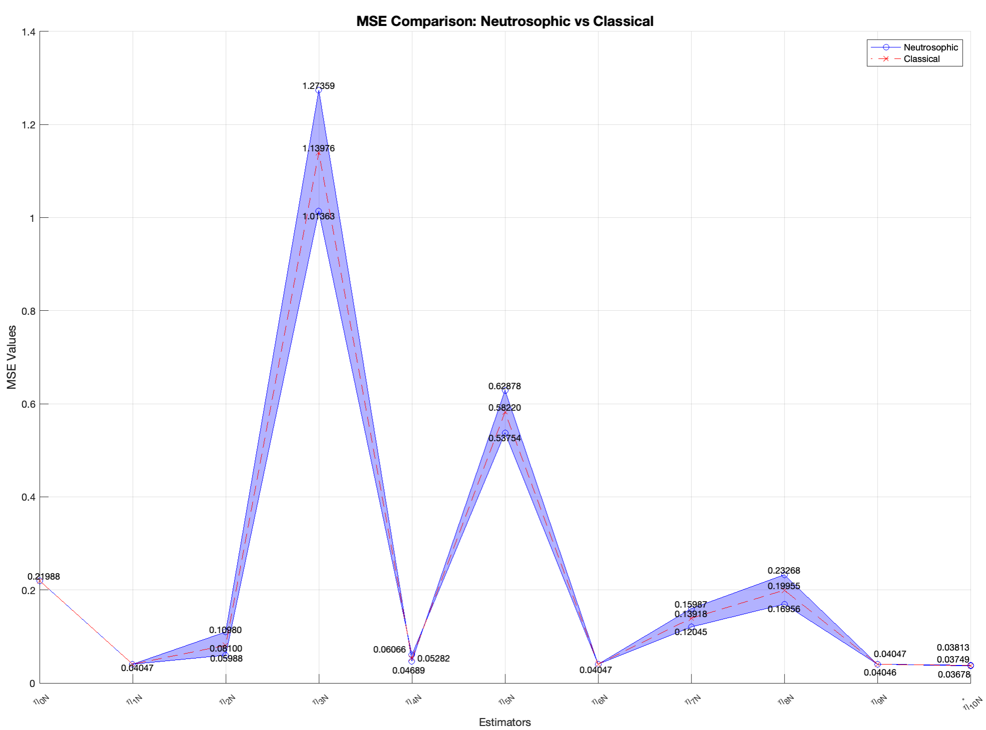

Figure 6: MSEs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 3

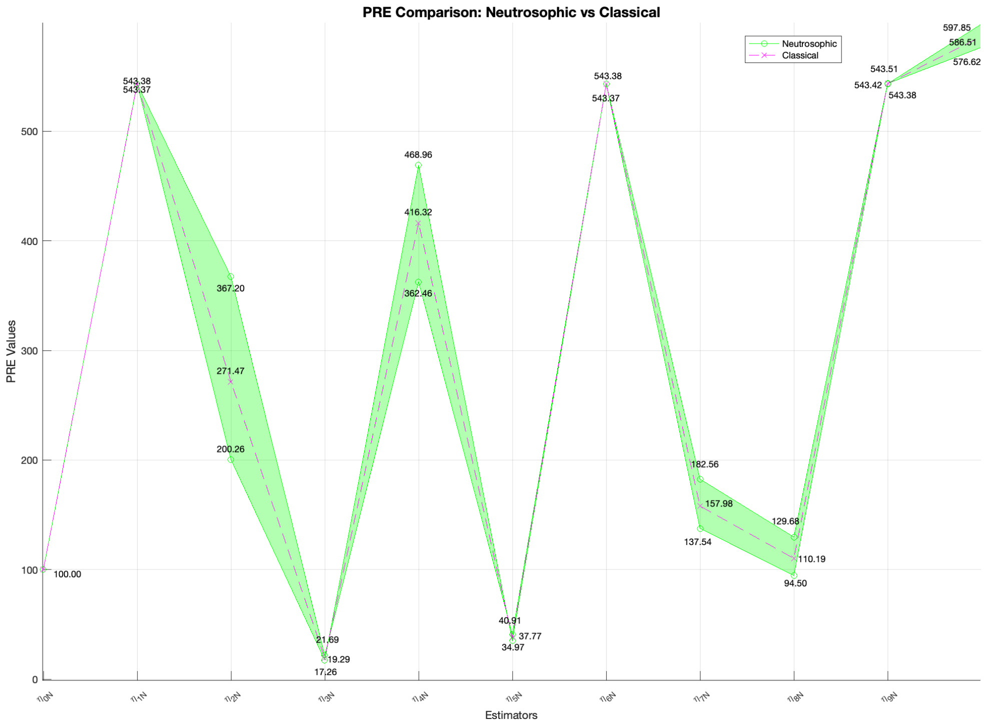

Figure 7: PREs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 3

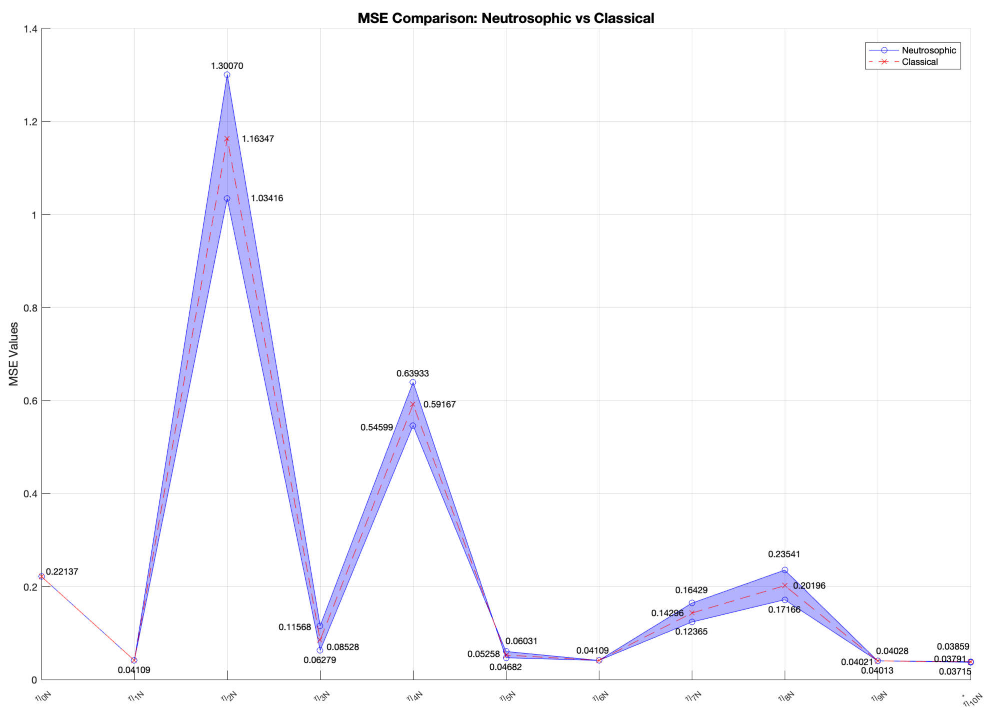

Figure 8: MSEs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η^*_{10N} neutrosophic estimator for population 4

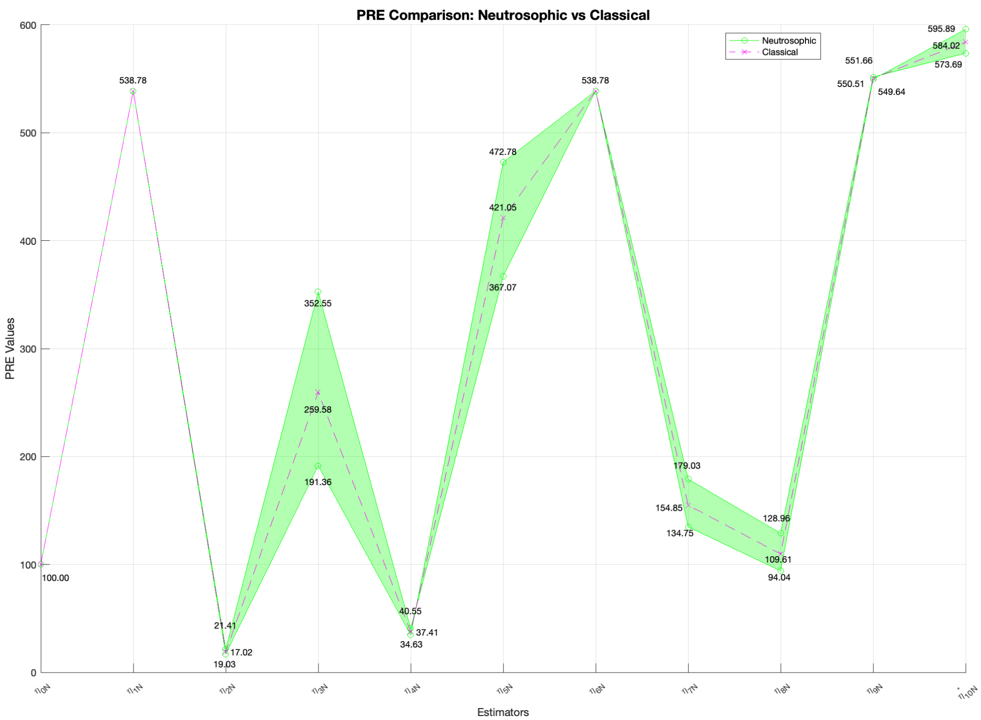

Figure 9: PREs comparison neutrosophic vs classical of existing (η_{0N} to η_{9N}) and proposed η_{10N}^{*} neutrosophic estimator for population 4

6. Conclusion

In the present study, we introduced an optimal neutrosophic difference to log-type estimator under simple random sampling. The effectiveness and performance of the proposed estimator, η_{10N}^{*}, have been evaluated through numerical and simulation studies using two real and two artificially generated datasets, respectively. The results have been compared with the usual unbiased, ratio, product, regression and other existing estimators discussed above as $\eta_{0N} - \eta_{9N}$. From the results in Tables 2-5, it has been found that the proposed estimator, η_{10N}^{*}, is more efficient than all existing neutrosophic estimators considered in this study, as evidenced by lower MSE values and higher PRE values.

Thus, the present study reveals the nice behavior of the proposed estimator for estimating the population mean in case of indeterminate or neutrosophic data. Hence, survey practitioners may be motivated to use it for their practical applications. Moreover, this work may be extended to multi-auxiliary variables in the presence of non-response using an EWMA statistic under stratified random sampling, ranked set sampling and machine learning techniques. The proposed estimator may be applied in real life situation when the population mean of the auxiliary variable is known. It may provide attractive results for highly correlated data sets when the simple random sampling scheme is used.

Acknowledgements

Authors are highly thankful to Vellore Institute of Technology for providing financial assistance and the necessary infrastructure to accomplish the present work, authors are thankful for VIT SEED GRANT (RGEMS) Sanction Order No.SG20230035 for technical assistance.

Conflict of interest

There is no conflict of interest associated with the present article.

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Received: June 6, 2025. Accepted: Sep 9, 2025