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Determinant-Theoretic Model for MCDM under Neutrosophic Hyper Soft Rough Matrices

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Abstract. In this paper, we explore the integration of determinant theory within the framework of Neutrosophic Hyper Soft Rough Matrix (NHSRM). We introduce a formal definition of the determinant for NHSRMs, establish its mathematical properties, and develop a determinant-based algorithm tailored for Multi-Criteria Decision-Making (MCDM). To highlight its practical utility, we conduct a comparative analysis with an existing score-based approach. To demonstrate the effectiveness of the proposed approach, an example is provided in which the determinant is computed and subsequently used to rank alternatives and assess the influence of different parameters. Additionally, a sensitivity analysis is performed to examine the robustness of the decision-making strategy under varying weight parameters. The results affirm that the determinant-based approach offers a more structured and reliable decision-making mechanism compared to score-based methods.

Keywords: Neutrosophic Hyper Soft Rough Matrix, Determinant Theory, Multi-Criteria Decision Making, Score-based Algorithm, Determinant-based Algorithm.

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1. **Introduction**

Decision-making in complex real-world scenarios frequently involves navigating through layers of uncertainty, imprecision, and incomplete information. These uncertainties may stem from ambiguous data sources, conflicting expert opinions, rapidly changing environments, or limitations in measurement and observation. Over the decades, several mathematical models have developed to address these challenges. The foundational work began with Zadeh's fuzzy set theory [34], which pioneered the formalization of uncertainty by introducing degrees of membership instead of binary classifications, offering a flexible tool for representing imprecise information. Building upon this, Pawlak's rough set theory [23] enriched the foundational work by providing a formal mechanism to handle incomplete information through lower and upper approximations, addressing indiscernibility and approximation in data classification, complementing fuzzy set theory with a rule-based perspective.

Recognizing the need for even more expressive representations of uncertainty, Atanassov introduced intuitionistic fuzzy sets [1], [2], extending fuzzy sets by incorporating non-membership and hesitation margins. This provided a more nuanced approach, especially useful in scenarios involving incomplete or conflicting information. Kim et al. [16] were among the pioneering researchers to formalize the concept of a determinant for fuzzy matrices. They redefined the classical determinant operation by incorporating fuzzy set theory principles, enabling computations based on degrees of membership rather than binary values. Ragab and Eman [24] made a notable contribution by introducing definitions for both the determinant and the adjoint of fuzzy matrices. This advancement is significant, as the adjoint plays a major role in determining the inverse of a matrix, which is fundamental to applications in system analysis, control theory, and decision-making. Dhar [8] provided a comprehensive examination of the algebraic properties of fuzzy matrix determinants. He explored conditions for invertibility, rank, and linear independence within fuzzy matrix systems, offering theoretical insights that paralleled classical algebra while addressing the added complexity of fuzziness.

Mondal and Pal [20] were among the earliest to introduce a structured determinant theory for intuitionistic fuzzy matrices. Their work focused on generalizing the classical determinant to accommodate the dual components of intuitionistic fuzzy sets. By redefining matrix operations and determinant properties within this dual-parameter framework, they established conditions under which the determinant of an intuitionistic fuzzy matrix could be computed. Im et al. [9] expanded on this by introducing novel algorithms for computing the determinant of intuitionistic fuzzy matrices, including matrix transformation rules that preserve intuitionistic properties during row and column operations. Their work emphasized the operational integrity of the matrices during determinant computation, ensuring consistency with intuitionistic fuzzy logic principles. They also explored applications in solving intuitionistic fuzzy linear systems, demonstrating the practical importance of determinant-based matrix manipulation in modeling real-world problems under uncertainty. Riyaz and Murugadas [25] contributed to the field by investigating the determinant properties in the context of intuitionistic fuzzy relational matrices. They introduced new types of matrix transformations and established determinant formulas that aligned with intuitionistic fuzzy relational algebra. Their study highlighted how determinant theory could be used to analyze the structure and consistency of intuitionistic fuzzy relations, especially in areas such as decision-making, clustering, and system modeling.

The trajectory of research has continued to evolve towards richer models, most notably with the advent of neutrosophic set theory by Smarandache [27]. Neutrosophic sets introduce a triadic structure encompassing truth, indeterminacy, and falsity degrees, thereby offering a more comprehensive framework for modeling uncertainty, inconsistency, and incomplete information.

Ye et al. [33] developed a slope stability classification model using single-valued neutrosophic matrix energy to handle uncertainty in geotechnical analysis. The approach effectively addressed imprecision and indeterminacy, enhancing decision-making under uncertain conditions. Alabdullah and Aouira [4] initiated this line of research by defining determinant operations for single-valued neutrosophic matrices, thereby extending traditional matrix theory to incorporate the triadic structure of neutrosophic sets. Their work focused on establishing basic algebraic operations such as matrix multiplication, addition, and inversion, and defining a consistent determinant function that captured the neutrosophic characteristics of each matrix entry. Expanding on this foundation, Karaaslan et al. [14] extended the theory to include interval-valued neutrosophic matrices, in which the truth, indeterminacy, and falsity components are expressed as intervals instead of fixed scalars. The authors proposed methods to compute determinants under this extended setting and demonstrated their usefulness in analyzing systems where data is not only imprecise but also subject to fluctuating degrees of belief and contradiction.

Uma et al. [30], [31] further expanded the scope of determinant theory by integrating it into the structure of fuzzy neutrosophic soft matrices. This hybrid framework combines concepts from fuzzy sets, neutrosophic sets, and soft sets, offering a robust approach to handle multi-layered uncertainty and context-dependent decision parameters. Uma and collaborators formalized determinant operations for these matrices, ensuring that both the softness and the neutrosophic fuzziness were preserved during computation. Their work provided a concrete algebraic foundation for fuzzy-neutrosophic soft decision-making models and was particularly impactful in contexts such as information systems, robotics, and social decision problems where multi-faceted uncertainty is intrinsic. Vijayabalaji et al. [32] introduced the concept of Neutrosophic Soft-Rough Matrices, combining the strengths of neutrosophic, soft, and rough set theories to handle multi-layered uncertainty. Their work laid a theoretical foundation for advanced decision-making frameworks under incomplete, indeterminate, and inconsistent information. Notably, Jafar and Saeed [10], [11] introduced frameworks for Neutrosophic Hypersoft matrix (NHSM), facilitating structured data analysis across complex attribute-value domains. Their work laid the foundation for modeling highly parameterized information under uncertainty.

The fusion of neutrosophic and soft set theories has produced powerful tools like Neutrosophic Soft Sets and Neutrosophic Rough Matrices. However, these models fall short in capturing multi-parameter hierarchies and deep uncertainty. To address the limitations of existing models in handling layered and indeterminate data, Boobalan and Mathivadhana [5] presented the first structured approach to Neutrosophic Hyper Soft Rough Matrices (NHSRM),a matrix-based model integrating neutrosophic sets, hyper soft sets, and rough approximations. They developed weighted aggregation operators such as Weighted Arithmetic Mean (WAM), Weighted Geometric Mean (WGM), and Weighted Harmonic Mean (WHM), to consolidate multi-attribute evaluations under neutrosophic uncertainty. Motivated by the need to address incomplete, imprecise, and inconsistent information, this work further extends NHSRM by developing a determinant theory for assessing consistency, ranking alternatives, and improving decision-making accuracy.

In this proposed work, we first introduce the determinant theory of NHSRM as a foundational analytical tool designed to distill complex multi-dimensional, uncertain information into a single scalar value, thereby enhancing interpretability and facilitating robust decision-making. A well-defined determinant within the NHSRM framework enables the quantification of systemic uncertainty, identification of contradictions or inconsistencies among input parameters, and the ranking of alternatives using score-based evaluation and determinant-based evaluation in MCDM scenarios. To support this, we design and implement efficient algorithms for computing the determinant of NHSRM, leveraging matrix transformation rules and determinant-based evaluation techniques that respect the inherent neutrosophic uncertainty and rough boundary approximations. The need for such a framework arises from the limitations of classical and existing fuzzy or soft matrix models, which often fail to simultaneously capture the layered uncertainties across multiple attributes, particularly when data are drawn from diverse, context-sensitive sources such as sensor fusion, intelligent control, or diagnostics.

1.1. *Objectives of the proposed work*

- To formalize the structure of NHSRM: Establish a rigorous mathematical foundation and representation of NHSRMs by integrating neutrosophic, soft, and rough set theories.
- To develop determinant theory within the NHSRM framework: Construct a novel determinant-based algebraic method to analyze the structural properties of NHSRMs.
- To design computational techniques for NHSRM: Develop efficient algorithms for computing determinants in the context of NHSRMs to support their applicability in complex and uncertain systems.
- To apply the determinant-based NHSRM model to real-world decision-making: Demonstrate the practical utility of the proposed model in autonomous systems, specifically in multi-sensor data integration and conflict resolution scenarios such as self-driving vehicles.

- To evaluate the model's effectiveness in MCDM: Use determinant values to guide preference rankings, assess attribute independence, and enhance decision robustness under uncertainty.

1.2. *Research Gap*

Significant progress has been made in the foundational areas of fuzzy sets, pioneered by Zadeh [34], rough sets by Pawlak [23], and intuitionistic fuzzy sets by Atanassov [1], [2]. Their extensions to matrix-based models have also been studied extensively, including fuzzy soft matrices by Cagman and Enginoglu [6], intuitionistic fuzzy soft matrices by Chetia and Das [7], and determinant theories for fuzzy matrices by Kim et al. [16] and Ragab and Eman [24]. Nevertheless, the field of NHSRM is still emerging. Research on related neutrosophic matrix structures has been conducted by Alabdullah and Aouira [4] on determinants of neutrosophic matrices, Karaaslan et al. [14] on interval-valued neutrosophic matrices, and Uma et al. [30] on fuzzy neutrosophic soft matrices.

The foundational work of Boobalan and Mathivadhana [5] introduced NHSRMs as a tool for modeling uncertainty, imprecision, and indeterminacy. Building on this, there is a clear opportunity to develop a determinant theory within the NHSRM framework to support ranking, consistency evaluation, and decision-making. Efficient computational methods and algebraic structures for NHSRM determinants will enhance their applicability in real-time systems such as autonomous control, intelligent diagnostics, and risk assessment. This research aims to advance both the theoretical depth and practical utility of NHSRMs in complex uncertain environments.

1.3. *Motivation and Contribution*

To bridge the identified gaps, the present research proposes a novel framework for constructing and analyzing Neutrosophic Hyper Soft Rough Matrices (NHSRM) integrated with determinant theory, aimed at enhancing multi-sensor and multi-criteria decision-making under conditions of high uncertainty. Specifically, this work:

- Formalizes the structure and properties of NHSRMs.
- Introduces a determinant-based approach assess NHSRM for consistency, attribute significance, and decision support.
- Demonstrates the real-world deployment of the proposed model in scenarios where decisions like braking or accelerating are made based on conflicting sensor inputs and contextual conditions.
- Integrates neutrosophic logic with soft and rough set granularity in a unified matrix algebra framework.

- Explores the incorporation of determinant theory within NHSRMs can reveal attribute independence and optimize decision rankings for more reliable and interpretable outcomes.
- Implementation of sensitivity analysis to evaluate the impact of weight parameter variations and uncertainties on decision outcomes, enhancing model robustness.

To enhance the practical visibility of the proposed model, the methodology based on determinant theory is illustrated in Figure 1.1, and a comparative analysis is presented in Table 1.1.

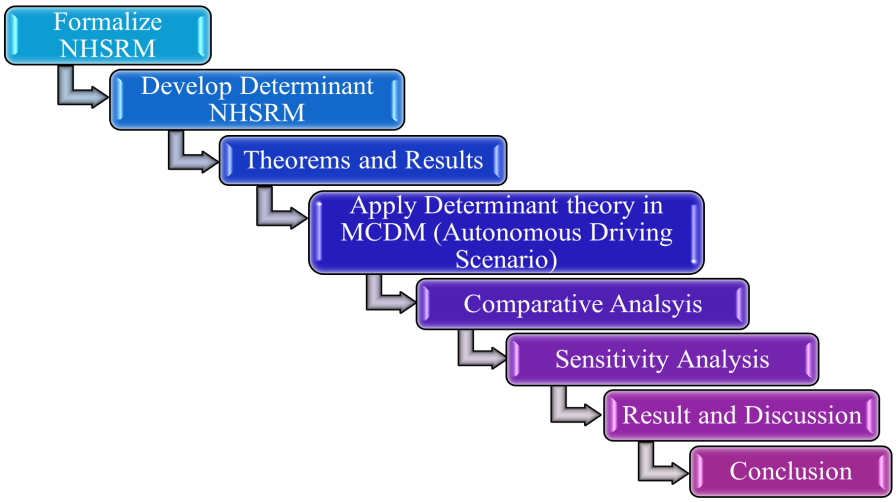

Figure: 1.1 Methodology of Determinant Theory

1.4. *Novelty*

Determinant theory has evolved from classical matrices to uncertain environments, including fuzzy [34], rough [23], and intuitionistic fuzzy sets [1, 2], along with their soft, neutrosophic, and hypersoft extensions. Foundational works on fuzzy matrices [8, 16, 24] and subsequent developments in determinant operations for intuitionistic fuzzy matrices [9, 20, 25], intuitionistic fuzzy soft matrices [7, 28], and hybrid rough models [17] established the groundwork for handling imprecision in matrix theory.

The emergence of neutrosophic and hypersoft structures [26, 27] introduced additional complexity, leading to newer matrix formulations [4, 10, 12]. Recent literature [5] has proposed NHSRM, a novel hybrid structure that captures layered uncertainty across multiple attributes, including truthiness, indeterminacy, and falsity. However, the determinant theory within this framework remains underexplored.

This study is the first to formalize determinant computation in NHSRMs and apply it to real-time, multi-criteria decision-making, specifically in autonomous systems involving sensor conflict resolution. It offers a novel contribution to both the theoretical landscape and applied decision science.

Model	Uncertainty Handling	Parametric Flexibility	Rough Approximation	Determinant Theory	Real-world Applicability
Fuzzy Matrix [16,24]	Single-valued vagueness	Limited	✗	✓	Moderate (Basic decision problems)
Intuitionistic Fuzzy Matrix [9,25]	Membership, non-membership, hesitation	Limited	✗	✓(Extended)	Moderate (Uncertain but structured data)
Neutrosophic Matrix [4,14]	Truth, Indeterminacy, Falsity	Limited	✗	✓	Moderate to High
Soft Matrix [6]	Binary uncertainty via parameter sets	✓	✗	✗	High (Simple parameter-based models)
Rough Neutrosophic Matrix [19]	Neutrosophic uncertainty with lower/upper bounds	✗	✓	✗	Moderate to High
Fuzzy Soft Rough Matrix [21]	Fuzzy + parametric + approximation-based	✓	✓	✗	High
Neutrosophic Hypersoft Matrix (NHSM) [12]	Multi-dimensional neutrosophic modeling	✓	✗	✗	High (Multilayer decision systems)
Neutrosophic Hyper Soft Rough Matrix (NHSRM) (Proposed)	Full-spectrum uncertainty with granularity (T, I, F, approx., param.)	✓	✓	✓	Very High (e.g., autonomous systems, diagnostics)

Table: 1.1 Comparison of NHSRM with the Existing Methods

The structure of the proposed work is outlined as follows:

- Section 2 defines the preliminary concepts and mathematical foundations necessary for understanding NHSRM.
- Section 3 introduces the determinant theory and mathematical structure of NHSRM.

- Section 4 presents relevant theorems, lemmas, and proofs that establish the mathematical validity of the determinant-based NHSRM model.
- Section 5 demonstrates the real-time application of both score-based and determinant-based algorithms through a comprehensive example. It also includes a sensitivity analysis to assess the robustness of decision outcomes under varying parameter weights.
- Section 6 provides the conclusion and summarizing the major findings, theoretical advancements, and potential implications of the study.

2. **Preliminaries**

Definition 2.1. (Neutrosophic set) [27]

A Neutrosophic set P on a universe U is defined by $P = \{\langle x, T^P(x), I^P(x), F^P(x)\rangle : x \in U\}$ where $T^P(x), I^P(x), F^P(x) : U \to [0,1]$ and $0 \leq T^P(x) + I^P(x) + F^P(x) \leq 3$. Here, $T^P(x)$ is the degree of truth, $I^P(x)$ is the degree of indeterminacy and $F^P(x)$ is the degree of falsity respectively.

Definition 2.2. (Soft rough set) [21]

Let $R = (\varphi, A)$ be a soft set over U the pair $S = (U, R)$ is called a soft approximation space. Based on the soft approximation space S, define the following two operations:

$$\underline{apr}_S(P) = \{u \in U; \exists a \in A, u \in \varphi(a) \subseteq P\}$$

$$\overline{apr}_S(P) = \{u \in U; \exists a \in A, u \in \varphi(a), \varphi(a) \cap P \neq \varnothing\}$$

Assigning to every subset $P \subseteq U$ two sets $\underline{apr}_S(P)$ and $\overline{apr}_S(P)$, which are called the soft S lower approximation and the soft S upper approximation of P respectively. In general, $\underline{apr}_S(P)$ and $\overline{apr}_S(P)$ as soft rough approximation of P with respect to S. Moreover the sets
$Pos_S(P) = \underline{apr}_S(P)$
$Neg_S(P) = U - \overline{apr}_S(P)$
$Bnd_S(P) = \overline{apr}_S(P) - \underline{apr}_S(P)$ are called the soft S - positive region, the soft S - negative region and the soft S - boundary region of P respectively.

Definition 2.3. (Hyper soft rough set) [13]

Let U be the universe set and P(U) be the power set of U. Suppose $a_1, a_2, a_3, ..., a_n$ where $n \geqslant 1$ be n distinct attributes whose corresponding attribute values $A_1, A_2...., A_n$ respectively. Let $S_j \subseteq A_j, j \in \{1, 2, ..., n\}$then $\prod_{j=1}^{n} S_j \subseteq \prod_{j=1}^{n} A_j$. The pair $(\varphi, \prod_{j=1}^{n} S_j) = P(U)$ where φ is a mapping defined by $\varphi : \prod_{j=1}^{n} S_j \to P(U)$ is called hyper soft rough set. The lower and upper neutrosophic hyper soft approximation spaces of $X \in P(U)$ with respect to $(\varphi, \prod_{j=1}^{n} S_j^k)$ are denoted by $\underline{apr}(X)$ and $\overline{apr}(X)$ respectively, defined by

$$\underline{apr}_S(X) = \{u \in U; \exists a_1, a_2, ..., a_n \in A, u \in \varphi(a_1, a_2, ..., a_n) \subseteq X)$$

$$\overline{apr}_S(X) = \{u \in U; \exists a_1, a_2, ..., a_n \in A, u \in \varphi(a_1, a_2, ..., a_n), \varphi(a_1, a_2, ..., a_n) \cap X \neq \varnothing)$$

If $\overline{apr}(X) \neq \underline{apr}(X)$, then X is hyper soft rough set, otherwise it is called as hyper soft rough definable set.

Definition 2.4. (NHSM) [10]

Consider a universe of discourse U with P(U) representing the set of all its possible subsets. Let $A_1, A_2...., A_\beta$ be a collection of well-defined attributes, where $\beta \geq 1$. Each attribute A_i is associated with a set of specific values denoted by $A_i^{q_i}$, where $q_i \in 1, 2, ..., n_i$ for $i = 1, 2, ..., \beta$. If the cartesian product of these attribute values is $A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta}$, then the pair $(\varphi, A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta})$ is said to be a neutrosophic hyper soft set over U, where $\varphi : (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta}) \to P(U)$ is defined as $\varphi(A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta}) = \{\langle u, T^\lambda(u), I^\lambda(u), F^\lambda(u)\rangle \in U, \lambda \in (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta})\}$ where T is the membership value of truth, I is the membership value of indeterminacy and F is the membership value of falsity. If $B_{ij} = X(u_i, A_j^k)$, $i = 1, 2, 3, ..., \alpha$, $j = 1, 2, 3, ..., \beta$ and $k = q_1, q_2, q_3, ..., q_\beta$ then the corresponding NHSM is defined as

$$[B_{ij}]_{\alpha\times\beta} = \begin{bmatrix} B_{11} & B_{12} & ... & B_{1\beta} \\ B_{21} & B_{22} & ... & B_{2\beta} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\alpha 1} & B_{\alpha 2} & ... & B_{\alpha\beta} \end{bmatrix}$$

where $B_{ij} = (T^{A_j^k}(u_i), I^{A_j^k}(u_i), F^{A_j^k}(u_i), u_i \in U, A_j^k \in (A_1^{q_1} \times A_2^{q_2} \times .... \times A_\beta^{q_\beta})) = (T_{ij}^B, I_{ij}^B, F_{ij}^B)$

Definition 2.5. (NHSRM) [5]

Consider a non-empty universe set $U = \{u_1, u_2, ..., u_\alpha\}$ with P(U) represent the set of all neutrosophic subsets of U. Let $E = \{A_1, A_2, ...A_n\}$ be the collection of parameters, where each pair of parameters is disjoint, i.e., $A_i \bigcap A_j = \varnothing \, for \, i \neq j$. For $j \in \{1, 2, ..., n\}$, let $S_j \subseteq A_j$, then the cartesian product $\prod_{j=1}^n S_j^k \subseteq \prod_{j=1}^n A_j^k$. The pair $(\varphi, \prod_{j=1}^n S_j^k) = P(U)$, where φ is a mapping defined by $\varphi : \prod_{j=1}^n S_j^k \to P(U)$ is called neutrosophic hyper soft rough set. Each element $u \in U$ is associated with the values determined by the hyper soft set, where each parameter can take multiple values. For each element $u \in U$ related with a parameter A_j is represented by the triplet (T_{ij}, I_{ij}, F_{ij}) where T_{ij} is the truth membership function, I_{ij} is the indeterminacy membership function and F_{ij} is the falsity membership function, $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$. If $P_{ij} = \gamma(u_i, A_j^k)$, where $i = 1, 2, 3, ..., \alpha$, $j = 1, 2, 3, ..., \beta$ and $k = a, b, c, ..., z$ then a NHSRM is defined as

$$P = [P_{ij}] = \begin{bmatrix} \langle \underline{P}_{11}; \overline{P}_{11}\rangle & \langle \underline{P}_{12}; \overline{P}_{12}\rangle & ... & \langle \underline{P}_{1n}; \overline{P}_{1n}\rangle \\ \langle \underline{P}_{21}; \overline{P}_{21}\rangle & \langle \underline{P}_{22}; \overline{P}_{22}\rangle & ... & \langle \underline{P}_{2n}; \overline{P}_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \underline{P}_{n1}; \overline{P}_{n1}\rangle & \langle \underline{P}_{n2}; \overline{P}_{n2}\rangle & ... & \langle \underline{P}_{nn}; \overline{P}_{nn}\rangle \end{bmatrix}$$

$$= \left[\left\langle \left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\overline{T}_{ij}^{P}, \overline{I}_{ij}^{P}, \overline{F}_{ij}^{P}\right)\right\rangle\right]$$

Lower Approximation matrix and Upper Approximation matrix are denoted by $\underline{P}_{ij} = (\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P})$, $0 \leq \underline{T}_{ij}^{P} + \underline{I}_{ij}^{P} + \underline{F}_{ij}^{P} \leq 3$ and $\overline{P}_{ij} = (\overline{T}_{ij}^{P}, \overline{I}_{ij}^{P}, \overline{F}_{ij}^{P})$, $0 \leq \overline{T}_{ij}^{P} + \overline{I}_{ij}^{P} + \overline{F}_{ij}^{P} \leq 3$ respectively. Thus, we can represent any Neutrosophic Hyper Soft Rough Set in term of Neutrosophic Fuzzy Hyper Soft Rough Matrix.

Example 2.1

The NHSRM P of order 3×3 is written as

$$P = \begin{bmatrix} \langle(0.7, 0.2, 0.2); (0.8, 0.4, 0.1)\rangle & \langle(0.6, 0.1, 0.3); (0.5, 0.3, 0.1)\rangle & \langle(0.5, 0.4, 0.2); (0.7, 0.5, 0.1)\rangle \\ \langle(0.6, 0.4, 0.1); (0.9, 0.3, 0.2)\rangle & \langle(0.8, 0.6, 0.3); (0.9, 0.0, 0.4)\rangle & \langle(1.0, 0.4, 0.2); (0.5, 0.1, 0.0)\rangle \\ \langle(1.0, 0.1, 0.1); (0.6, 0.3, 0.0)\rangle & \langle(0.7, 0.3, 0.1); (0.6, 0.2, 0.0)\rangle & \langle(0.9, 0.4, 0.3); (0.3, 0.1, 0.5)\rangle \end{bmatrix}$$

Definition 2.6. (Transpose of NHSRM) [5]

The transpose of a square NHSRM is formed by switching its rows and columns, similar to the transpose operation in standard matrices. This operation allows us to examine relationships from a different perspective, effectively interchanging the roles of elements and hierarchical parameters.

$$[P_{ij}]^{T} = \left[\left\langle \left(\underline{T}_{ji}^{P}, \underline{I}_{ji}^{P}, \underline{F}_{ji}^{P}\right); \left(\overline{T}_{ji}^{P}, \overline{I}_{ji}^{P}, \overline{F}_{ji}^{P}\right)\right\rangle\right]$$

Example 2.2

Consider Example 2.1, the transpose of P is given by

$$P^{T} = \begin{bmatrix} \langle(0.7, 0.2, 0.2); (0.8, 0.4, 0.1)\rangle & \langle(0.6, 0.4, 0.1); (0.9, 0.3, 0.2)\rangle & \langle(1.0, 0.1, 0.1); (0.6, 0.3, 0.0)\rangle \\ \langle(0.6, 0.1, 0.3); (0.5, 0.3, 0.1)\rangle & \langle(0.8, 0.6, 0.3); (0.9, 0.0, 0.4)\rangle & \langle(0.7, 0.3, 0.1); (0.6, 0.2, 0.0)\rangle \\ \langle(0.5, 0.4, 0.2); (0.7, 0.5, 0.1)\rangle & \langle(1.0, 0.4, 0.2); (0.5, 0.1, 0.0)\rangle & \langle(0.9, 0.4, 0.3); (0.3, 0.1, 0.5)\rangle \end{bmatrix}$$

3. Determinant of NHSRM

This section presents a comprehensive discussion on the concept of the definition of determinant, along with several properties related to NHSRM.

Throughout this paper, we use the following notation

$$P = \left[\left\langle \left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\overline{T}_{ij}^{P}, \overline{I}_{ij}^{P}, \overline{F}_{ij}^{P}\right)\right\rangle\right],$$

$$Q = \left[\left\langle \left(\underline{T}_{ij}^{Q}, \underline{I}_{ij}^{Q}, \underline{F}_{ij}^{Q}\right); \left(\overline{T}_{ij}^{Q}, \overline{I}_{ij}^{Q}, \overline{F}_{ij}^{Q}\right)\right\rangle\right]$$

and

$$R = \left[\left\langle \left(\underline{T}_{ij}^{R}, \underline{I}_{ij}^{R}, \underline{F}_{ij}^{R}\right); \left(\overline{T}_{ij}^{R}, \overline{I}_{ij}^{R}, \overline{F}_{ij}^{R}\right)\right\rangle\right]$$

Definition 3.1. (Component wise Addition and Multiplication of NHSRM)

Let P and Q be two NHSRMs, then their component wise addition and multiplication are defined as

$$P \oplus Q = \left[\left\langle \begin{matrix} \left(sup\left(\underline{T}_{ij}^{P}, \underline{T}_{ij}^{Q}\right), \sup\left(\underline{I}_{ij}^{P}, \underline{I}_{ij}^{Q}\right), \inf\left(\underline{F}_{ij}^{P}, \underline{F}_{ij}^{Q}\right)\right); \\ \left(sup\left(\overline{T}_{ij}^{P}, \overline{T}_{ij}^{Q}\right), \sup\left(\overline{I}_{ij}^{P}, \overline{I}_{ij}^{Q}\right), \inf\left(\overline{F}_{ij}^{P}, \overline{F}_{ij}^{Q}\right)\right) \end{matrix} \right\rangle\right]$$

$$P \odot Q = \left[\left\langle \begin{matrix} \left(\inf\left(\underline{T}_{ij}^{P}, \underline{T}_{ij}^{Q}\right), \inf\left(\underline{I}_{ij}^{P}, \underline{I}_{ij}^{Q}\right), \sup\left(\underline{F}_{ij}^{P}, \underline{F}_{ij}^{Q}\right)\right); \\ \left(\inf\left(\overline{T}_{ij}^{P}, \overline{T}_{ij}^{Q}\right), \inf\left(\overline{I}_{ij}^{P}, \overline{I}_{ij}^{Q}\right), \sup\left(\overline{F}_{ij}^{P}, \overline{F}_{ij}^{Q}\right)\right) \end{matrix} \right\rangle \right]$$

Example 3.1

Let P and Q be two NHSRMs of order 2×23 be

$$P = \begin{bmatrix} \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1) \rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7) \rangle \\ \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2) \rangle & \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2) \rangle \end{bmatrix}$$

$$Q = \begin{bmatrix} \langle (0.9, 0.5, 0.4); (0.8, 0.2, 0.1) \rangle & \langle (0.7, 0.3, 0.0); (0.6, 0.5, 0.4) \rangle \\ \langle (0.6, 0.8, 0.1); (1.0, 0.3, 0.2) \rangle & \langle (0.6, 0.2, 0.1); (0.4, 0.2, 0.0) \rangle \end{bmatrix}$$

Component wise Addition is written as

$$P \oplus Q = \begin{bmatrix} \langle (0.9, 0.6, 0.3); (0.8, 0.8, 0.1) \rangle & \langle (0.7, 0.3, 0.0); (0.6, 0.5, 0.4) \rangle \\ \langle (0.7, 0.8, 0.1); (1.0, 0.7, 0.2) \rangle & \langle (0.8, 0.2, 0.1); (0.6, 0.3, 0.0) \rangle \end{bmatrix}$$

Component wise Multiplication is written as

$$P \odot Q = \begin{bmatrix} \langle (0.5, 0.5, 0.4); (0.3, 0.2, 0.1) \rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7) \rangle \\ \langle (0.6, 0.4, 0.9); (0.4, 0.3, 0.2) \rangle & \langle (0.6, 0.2, 0.7); (0.4, 0.2, 0.2) \rangle \end{bmatrix}$$

Definition 3.2. (Composition of NHSRM)

If P and Q are two NHSRMs then their composition is denoted by $P \cdot Q$ and is defined as

$$P \cdot Q = \left[\left\langle \begin{matrix} \left(\sum_{k=1}^{n} \left(\underline{T}_{ij}^{P} \cdot \underline{T}_{ij}^{Q}\right), \sum_{k=1}^{n} \left(\underline{I}_{ij}^{P} \cdot \underline{I}_{ij}^{Q}\right), \prod_{k=1}^{n} \left(\underline{F}_{ij}^{P} + \underline{F}_{ij}^{Q}\right) \right); \\ \left(\sum_{k=1}^{n} \left(\overline{T}_{ij}^{P} \cdot \overline{T}_{ij}^{Q}\right), \sum_{k=1}^{n} \left(\overline{I}_{ij}^{P} \cdot \overline{I}_{ij}^{Q}\right), \prod_{k=1}^{n} \left(\overline{F}_{ij}^{P} + \overline{F}_{ij}^{Q}\right) \right) \end{matrix} \right\rangle \right].$$

Note: Matrix composition is possible only if the number of columns in matrix P is equal to the number of rows in matrix Q. In such cases, P and Q are termed conformable for multiplication. The resulting product is denoted as $P \cdot Q$

Definition 3.3. (Determinant of NHSRM)

Let P be a NHSRM then the determinant $|P|$ is defined as

$$|P| = \sum_{\sigma \in S_n} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right) \right\rangle \cdots \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right) \right\rangle \end{bmatrix}.$$

We can also write,

$$|P| = \left[\left\langle \begin{matrix} \left(\bigcup_{\sigma \in S_n} \bigwedge_{k=1}^{n} \underline{T}_{k\,\sigma(k)}^{P}, \bigcup_{\sigma \in S_n} \bigwedge_{k=1}^{n} \underline{I}_{k\,\sigma(k)}^{P}, \bigcap_{\sigma \in S_n} \bigvee_{k=1}^{n} \underline{F}_{k\,\sigma(k)}^{P} \right); \\ \left(\bigcup_{\sigma \in S_n} \bigwedge_{k=1}^{n} \overline{T}_{k\,\sigma(k)}^{P}, \bigcup_{\sigma \in S_n} \bigwedge_{k=1}^{n} \overline{I}_{k\,\sigma(k)}^{P}, \bigcap_{\sigma \in S_n} \bigvee_{k=1}^{n} \overline{F}_{k\,\sigma(k)}^{P} \right) \end{matrix} \right\rangle \right],$$

where S_n comprises all distinct arrangements of the indices (1,2,. . . ,n).

Example 3.2

Let us consider

$$P = \begin{bmatrix} \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1) \rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7) \rangle \\ \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2) \rangle & \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2) \rangle \end{bmatrix}$$

be a NHSRM of order 2×2 then the determinant of P can be written as

$$\begin{aligned} |P| &= \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1) \rangle \cdot \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2) \rangle \\ &\quad + \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7) \rangle \cdot \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2) \rangle \\ &= \langle (0.5, 0.2, 0.7); (0.3, 0.3, 0.2) \rangle + \langle (0.1, 0.3, 0.9); (0.4, 0.5, 0.7) \rangle \\ &= \langle (0.5, 0.3, 0.7); (0.4, 0.5, 0.2) \rangle . \end{aligned}$$

Definition 3.4. (Second order Submatrix Determinant of NHSRM)

Consider a NHSRM P. Let Q be a matrix obtained from P by deleting row e, row f, column g and column h where $e \neq f$ and $g \neq h$ then the determinant of Q is defined as $|Q| = P \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, which is a second order submatrix determinant of P.

Example 3.3

Let us consider

$$P = \begin{bmatrix} \left\langle \begin{matrix} (1,0,0); \\ (1,0,0) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.8,0.2,0.1); \\ (0.7,0.3,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.6,0.1,0.2); \\ (0.5,0.1,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.4,0.4,0.2); \\ (0.3,0.5,0.2) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} (0.7,0.1,0.2); \\ (0.6,0.1,0.3) \end{matrix} \right\rangle & \left\langle \begin{matrix} (1,0,0); \\ (1,0,0) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.5,0.3,0.2); \\ (0.4,0.3,0.3) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.3,0.5,0.2); \\ (0.2,0.6,0.2) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} (0.6,0.3,0.1); \\ (0.5,0.3,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.5,0.4,0.1); \\ (0.4,0.4,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (1,0,0); \\ (1,0,0) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.2,0.6,0.2); \\ (0.1,0.7,0.2) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} (0.3,0.5,0.2); \\ (0.2,0.6,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.2,0.4,0.2); \\ (0.1,0.5,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (0.4,0.6,0.2); \\ (0.3,0.7,0.2) \end{matrix} \right\rangle & \left\langle \begin{matrix} (1,0,0); \\ (1,0,0) \end{matrix} \right\rangle \end{bmatrix}$$

be a NHSRM of order 4×4 then the matrix of Q can be obtained by deleting row 2 & 3 and column 1 & 4, we get

$$Q = \begin{bmatrix} \langle (0.8, 0.2, 0.1); (0.7, 0.3, 0.2) \rangle & \langle (0.6, 0.1, 0.2); (0.5, 0.1, 0.2) \rangle \\ \langle (0.2, 0.4, 0.2); (0.1, 0.5, 0.2) \rangle & \langle (0.4, 0.6, 0.2); (0.3, 0.7, 0.2) \rangle \end{bmatrix}$$

The determinant of Q can be written as

$$\begin{aligned} |Q| &= \langle (0.4, 0.2, 0.2); (0.3, 0.3, 0.2) \rangle + \langle (0.2, 0.1, 0.2); (0.1, 0.1, 0.2) \rangle \\ &= \langle (0.4, 0.2, 0.2); (0.3, 0.3, 0.2) \rangle . \end{aligned}$$

Definition 3.5. (k^{th} order Submatrix Determinant of NHSRM)

Consider a NHSRM P. Let Q be a matrix obtained from P by deleting row e_1, row e_2, ..., row e_k and column f_1, column f_2,..., column f_k then the determinant of Q is defined as $|Q| = P \begin{bmatrix} e_1 & e_2 & \dots & e_k \\ f_1 & f_2 & \dots & f_k \end{bmatrix}$, which is a k^{th} order submatrix determinant of P.

Definition 3.6. (Diagonal Dominance of NHSRM)

Let P be a NHSRM then it satisfy the diagonal dominance, if the diagonal element dominates the corresponding off-diagonal elements under the neutrosophic ordering.

$$i.e., \left\langle \left(\underline{T}_{ii}^{P}, \underline{I}_{ii}^{P}, \underline{F}_{ii}^{P}\right); \left(\overline{T}_{ii}^{P}, \overline{I}_{ii}^{P}, \overline{F}_{ii}^{P}\right) \right\rangle \geqslant \left\langle \left(\underline{T}_{i\sigma(i)}^{P}, \underline{I}_{i\sigma(i)}^{P}, \underline{F}_{i\sigma(i)}^{P}\right); \left(\overline{T}_{i\sigma(i)}^{P}, \overline{I}_{i\sigma(i)}^{P}, \overline{F}_{i\sigma(i)}^{P}\right) \right\rangle; \forall i = 1, 2, ..., n$$

for all permutation $\sigma$of {1,2,...,n}, where the ordering is defined component-wise as

$$\underline{T}_{ii} \geq \underline{T}_{i\sigma(i)}, \overline{T}_{ii} \geq \overline{T}_{i\sigma(i)}$$
$$\underline{I}_{ii} \geq \underline{I}_{i\sigma(i)}, \overline{I}_{ii} \geq \overline{I}_{i\sigma(i)}$$
$$\underline{F}_{ii} \leq \underline{F}_{i\sigma(i)}, \overline{F}_{ii} \leq \overline{F}_{i\sigma(i)}$$

Definition 3.7. (Zero NHSRM)

Let O be a NHSRM then the zero NHSRM is defined as $O = [\langle (0, 0, 1) ; (0, 0, 1) \rangle]$.

Example 3.4

Let a zero NHSRM of order 2×2 be

$$O = \begin{bmatrix} \langle (0,0,1);(0,0,1)\rangle & \langle (0,0,1);(0,0,1)\rangle \\ \langle (0,0,1);(0,0,1)\rangle & \langle (0,0,1);(0,0,1)\rangle \end{bmatrix}$$

3.1. *Properties of NHSRM*

1. The determinant of a square NHSRM exists if the matrix is invertible in the defined algebraic structure.
2. Swapping any two rows (or columns) in a NHSRM does not affect the value of its determinant.
3. In a NHSRM, the determinant remains unchanged when the elements of one row (or column) are added to the corresponding elements of another row (or column).
4. For a NHSRM, the determinant value is same when its rows and columns are transposed.
5. For a (upper or lower) triangular or diagonal NHSRM, the determinant is simply the product of the diagonal entries.

4. Theorems and results

This section outlines the fundamental theorems and results of NHSRM, forming the basis for its analysis and applications.

Theorem 4.1

Let P be a NHSRM. If $\left[\left\langle \left(\underline{T}_{ii}^{P}, \underline{I}_{ii}^{P}, \underline{F}_{ii}^{P}\right); \left(\overline{T}_{ii}^{P}, \overline{I}_{ii}^{P}, \overline{F}_{ii}^{P}\right) \right\rangle\right] \geqslant \left[\left\langle \left(\underline{T}_{ik}^{P}, \underline{I}_{ik}^{P}, \underline{F}_{ik}^{P}\right); \left(\overline{T}_{ik}^{P}, \overline{I}_{ik}^{P}, \overline{F}_{ik}^{P}\right) \right\rangle\right];\ k = 1, 2, ...n,\ \forall 1 \leqslant i \leqslant n$ then

$$|P| = \left\langle \left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right); \left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right) \right\rangle ... \left\langle \left(\underline{T}_{nn}^{P}, \underline{I}_{nn}^{P}, \underline{F}_{nn}^{P}\right); \left(\overline{T}_{nn}^{P}, \overline{I}_{nn}^{P}, \overline{F}_{nn}^{P}\right) \right\rangle$$

Proof:

Let P be NHSRM , where each element is of the form $P = \left[\left\langle \left(\underline{T}_{ij}^{P}, \underline{I}_{ij}^{P}, \underline{F}_{ij}^{P}\right); \left(\overline{T}_{ij}^{P}, \overline{I}_{ij}^{P}, \overline{F}_{ij}^{P}\right) \right\rangle\right]$,

Then by the definition 3.3, we can be written as

$$|P| \geqslant \left\langle \left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right); \left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right) \right\rangle ... \left\langle \left(\underline{T}_{nn}^{P}, \underline{I}_{nn}^{P}, \underline{F}_{nn}^{P}\right); \left(\overline{T}_{nn}^{P}, \overline{I}_{nn}^{P}, \overline{F}_{nn}^{P}\right) \right\rangle \tag{1}$$

For any permutation $\sigma \in S_n$, by definition 3.6, we have

$$\left\langle\left(\underline{T}_{ii}^{P}, \underline{I}_{ii}^{P}, \underline{F}_{ii}^{P}\right);\left(\overline{T}_{ii}^{P}, \overline{I}_{ii}^{P}, \overline{F}_{ii}^{P}\right)\right\rangle \geqslant \left\langle\left(\underline{T}_{i\sigma(i)}^{P}, \underline{I}_{i\sigma(i)}^{P}, \underline{F}_{i\sigma(i)}^{P}\right);\left(\overline{T}_{i\sigma(i)}^{P}, \overline{I}_{i\sigma(i)}^{P}, \overline{F}_{i\sigma(i)}^{P}\right)\right\rangle; \forall i = 1, 2, ..., n \quad (2)$$

Since $\left[\left\langle\left(\underline{T}_{ii}^{P}, \underline{I}_{ii}^{P}, \underline{F}_{ii}^{P}\right);\left(\overline{T}_{ii}^{P}, \overline{I}_{ii}^{P}, \overline{F}_{ii}^{P}\right)\right\rangle\right] \geqslant \left[\left\langle\left(\underline{T}_{ik}^{P}, \underline{I}_{ik}^{P}, \underline{F}_{ik}^{P}\right);\left(\overline{T}_{ik}^{P}, \overline{I}_{ik}^{P}, \overline{F}_{ik}^{P}\right)\right\rangle\right]; k = 1, 2, ...n, \ \forall 1 \leqslant i \leqslant n$
Hence using Equation (1) implies

$$\left\langle\left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right);\left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right)\right\rangle ... \left\langle\left(\underline{T}_{nn}^{P}, \underline{I}_{nn}^{P}, \underline{F}_{nn}^{P}\right);\left(\overline{T}_{nn}^{P}, \overline{I}_{nn}^{P}, \overline{F}_{nn}^{P}\right)\right\rangle$$

$$\geqslant \begin{bmatrix} \left\langle\left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right);\left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle ... \\ \left\langle\left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right);\left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

By using Equation (2), the above Equation becomes

$$\geqslant \sum_{\sigma \in S_n} \begin{bmatrix} \left\langle\left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right);\left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle ... \\ \left\langle\left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right);\left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix} \geq |P| \quad (3)$$

From Equation (1) and (3), we get

$$|P| = \left\langle\left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right);\left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right)\right\rangle ... \left\langle\left(\underline{T}_{nn}^{P}, \underline{I}_{nn}^{P}, \underline{F}_{nn}^{P}\right);\left(\overline{T}_{nn}^{P}, \overline{I}_{nn}^{P}, \overline{F}_{nn}^{P}\right)\right\rangle$$

Hence the theorem.

Example 4.1

Let us consider P be 2×2 NHSRM

$$P = \begin{bmatrix} \langle(0.8, 0.1, 0.1); (0.7, 0.1, 0.2)\rangle & \langle(0.6, 0.2, 0.2); (0.5, 0.3, 0.2)\rangle \\ \langle(0.2, 0.6, 0.2); (0.1, 0.7, 0.2)\rangle & \langle(0.4, 0.4, 0.2); (0.3, 0.5, 0.2)\rangle \end{bmatrix}$$

$$\left\langle\left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right);\left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right)\right\rangle = \langle(0.8, 0.2, 0.1); (0.7, 0.3, 0.2)\rangle$$

$$\left\langle\left(\underline{T}_{12}^{P}, \underline{I}_{12}^{P}, \underline{F}_{12}^{P}\right);\left(\overline{T}_{12}^{P}, \overline{I}_{12}^{P}, \overline{F}_{12}^{P}\right)\right\rangle = \langle(0.6, 0.1, 0.2); (0.5, 0.7, 0.2)\rangle$$

Compare the elements of above terms

$$\underline{T}_{11} \geq \underline{T}_{12}, \overline{T}_{11} \geq \overline{T}_{12}$$

$$\underline{I}_{11} \geq \underline{I}_{12}, \overline{I}_{11} \geq \overline{I}_{12}$$

$$\underline{F}_{11} \leq \underline{F}_{12}, \overline{F}_{11} \leq \overline{F}_{12}$$

$$\therefore \left\langle\left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right);\left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right)\right\rangle \geq \left\langle\left(\underline{T}_{12}^{P}, \underline{I}_{12}^{P}, \underline{F}_{12}^{P}\right);\left(\overline{T}_{121}^{P}, \overline{I}_{12}^{P}, \overline{F}_{12}^{P}\right)\right\rangle$$

Similarly $\left\langle\left(\underline{T}_{22}^{P}, \underline{I}_{22}^{P}, \underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P}, \overline{I}_{22}^{P}, \overline{F}_{22}^{P}\right)\right\rangle \geq \left\langle\left(\underline{T}_{21}^{P}, \underline{I}_{21}^{P}, \underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P}, \overline{I}_{21}^{P}, \overline{F}_{21}^{P}\right)\right\rangle$,

$$|P| = \left\langle\left(\underline{T}_{11}^{P}, \underline{I}_{11}^{P}, \underline{F}_{11}^{P}\right);\left(\overline{T}_{11}^{P}, \overline{I}_{11}^{P}, \overline{F}_{11}^{P}\right)\right\rangle \cdot \left\langle\left(\underline{T}_{22}^{P}, \underline{I}_{22}^{P}, \underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P}, \overline{I}_{22}^{P}, \overline{F}_{22}^{P}\right)\right\rangle = \langle(0.4, 0.2, 0.2); (0.3, 0.3, 0.2)\rangle$$

Theorem 4.2

Let P and Q be two NHSRMs then the following statement hold:

(i) The determinant of the product is greater than or equal to the product of their individual determinants: $|P \cdot Q| \geq |P| \cdot |Q|$

(ii) The determinant of the product is less than or equal to the determinant of the sum: $|P \cdot Q| \leqslant |P + Q|$

Proof:

(i) Consider

$$P\cdot Q=\left[\left\langle\begin{array}{l}\left(\sum_{k=1}^{n}\underline{T}_{ik}^{P}\cdot\underline{T}_{kj}^{Q},\sum_{k=1}^{n}\underline{I}_{ik}^{P}\cdot\underline{I}_{kj}^{Q},\prod_{k=1}^{n}\underline{F}_{ik}^{P}+\underline{F}_{kj}^{Q}\right);\\ \left(\sum_{k=1}^{n}\overline{T}_{ik}^{P}\cdot\overline{T}_{kj}^{Q},\sum_{k=1}^{n}\overline{I}_{ik}^{P}\cdot\overline{I}_{kj}^{Q},\prod_{k=1}^{n}\overline{F}_{ik}^{P}+\overline{F}_{kj}^{Q}\right)\end{array}\right\rangle\right]$$

By applying Definition 3.3, the above expression becomes

$$|P\cdot Q|=\sum_{\sigma\in S_n}\left[\begin{array}{l}\left\langle\begin{array}{l}\left(\sum_{k=1}^{n}\underline{T}_{1k}^{P}\cdot\underline{T}_{k\sigma(1)}^{Q},\sum_{k=1}^{n}\underline{I}_{1k}^{P}\cdot\underline{I}_{k\sigma(1)}^{Q},\prod_{k=1}^{n}\underline{F}_{1k}^{P}+\underline{F}_{k\sigma(1)}^{Q}\right);\\ \left(\sum_{k=1}^{n}\overline{T}_{1k}^{P}\cdot\overline{T}_{k\sigma(1)}^{Q},\sum_{k=1}^{n}\overline{I}_{1k}^{P}\cdot\overline{I}_{k\sigma(1)}^{Q},\prod_{k=1}^{n}\overline{F}_{1k}^{P}+\overline{F}_{k\sigma(1)}^{Q}\right)\end{array}\right\rangle\cdots\\ \left\langle\begin{array}{l}\left(\sum_{k=1}^{n}\underline{T}_{nk}^{P}\cdot\underline{T}_{k\sigma(n)}^{Q},\sum_{k=1}^{n}\underline{I}_{nk}^{P}\cdot\underline{I}_{k\sigma(n)}^{Q},\prod_{k=1}^{n}\underline{F}_{nk}^{P}+\underline{F}_{k\sigma(n)}^{Q}\right);\\ \left(\sum_{k=1}^{n}\overline{T}_{nk}^{P}\cdot\overline{T}_{k\sigma(n)}^{Q},\sum_{k=1}^{n}\overline{I}_{nk}^{P}\cdot\overline{I}_{k\sigma(n)}^{Q},\prod_{k=1}^{n}\overline{F}_{nk}^{P}+\overline{F}_{k\sigma(n)}^{Q}\right)\end{array}\right\rangle\end{array}\right]$$

$$=\sum_{\sigma\in S_n}\left[\begin{array}{l}\left\langle\sum_{k_1\ldots k_n}\left(\underline{T}_{1k_1}^{P}\ldots\underline{T}_{nk_n}^{P}\right)\cdot\left(\underline{T}_{k_1\sigma(1)}^{Q}\ldots\underline{T}_{k_n\sigma(n)}^{Q}\right),\sum_{k_1\ldots k_n}\left(\underline{I}_{1k_1}^{P}\ldots\underline{I}_{nk_n}^{P}\right)\cdot\left(\underline{I}_{k_1\sigma(1)}^{Q}\ldots\underline{I}_{k_n\sigma(n)}^{Q}\right),\right.\\ \left.\prod_{k_1\ldots k_n}\left(\underline{F}_{1k_1}^{P}\ldots\underline{F}_{nk_n}^{P}\right)+\left(\underline{F}_{k_1\sigma(1)}^{Q}\ldots\underline{F}_{k_n\sigma(n)}^{Q}\right)\right\rangle;\left\langle\sum_{k_1\ldots k_n}\left(\overline{T}_{1k_1}^{P}\ldots\overline{T}_{nk_n}^{P}\right)\cdot\left(\overline{T}_{k_1\sigma(1)}^{Q}\ldots\overline{T}_{k_n\sigma(n)}^{Q}\right),\right.\\ \left.\sum_{k_1\ldots k_n}\left(\overline{I}_{1k_1}^{P}\ldots\overline{I}_{nk_n}^{P}\right)\cdot\left(\overline{I}_{k_1\sigma(1)}^{Q}\ldots\overline{I}_{k_n\sigma(n)}^{Q}\right),\prod_{k_1\ldots k_n}\left(\overline{F}_{1k_1}^{P}\ldots\overline{F}_{nk_n}^{P}\right)+\left(\overline{F}_{k_1\sigma(1)}^{Q}\ldots\overline{F}_{k_n\sigma(n)}^{Q}\right)\right\rangle\end{array}\right]$$

$$=\sum_{\sigma\in S_n}\sum_{k_1\ldots k_n}\left[\begin{array}{l}\left\langle\left(\underline{T}_{1k_1}^{P},\underline{I}_{1k_1}^{P},\underline{F}_{1k_1}^{P}\right);\left(\overline{T}_{1k_1}^{P},\overline{I}_{1k_1}^{P},\overline{F}_{1k_1}^{P}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{nk_n}^{P},\underline{I}_{nk_n}^{P},\underline{F}_{nk_n}^{P}\right);\left(\overline{T}_{nk_n}^{P},\overline{I}_{nk_n}^{P},\overline{F}_{nk_n}^{P}\right)\right\rangle\end{array}\right]$$
$$\cdot\left[\begin{array}{l}\left\langle\left(\underline{T}_{k_1\sigma(1)}^{Q},\underline{I}_{k_1\sigma(1)}^{Q},\underline{F}_{k_1\sigma(1)}^{Q}\right);\left(\overline{T}_{k_1\sigma(1)}^{Q},\overline{I}_{k_1\sigma(1)}^{Q},\overline{F}_{k_1\sigma(1)}^{Q}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{k_n\sigma(n)}^{Q},\underline{I}_{k_n\sigma(n)}^{Q},\underline{F}_{k_n\sigma(n)}^{Q}\right);\left(\overline{T}_{k_n\sigma(n)}^{Q},\overline{I}_{k_n\sigma(n)}^{Q},\overline{F}_{k_n\sigma(n)}^{Q}\right)\right\rangle\end{array}\right]$$

$$=\sum_{k_1\ldots k_n}\left[\begin{array}{l}\left\langle\left(\underline{T}_{1k_1}^{P},\underline{I}_{1k_1}^{P},\underline{F}_{1k_1}^{P}\right);\left(\overline{T}_{1k_1}^{P},\overline{I}_{1k_1}^{P},\overline{F}_{1k_1}^{P}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{nk_n}^{P},\underline{I}_{nk_n}^{P},\underline{F}_{nk_n}^{P}\right);\left(\overline{T}_{nk_n}^{P},\overline{I}_{nk_n}^{P},\overline{F}_{nk_n}^{P}\right)\right\rangle\end{array}\right]$$
$$\cdot\sum_{\sigma\in S_n}\left[\begin{array}{l}\left\langle\left(\underline{T}_{k_1\sigma(1)}^{Q},\underline{I}_{k_1\sigma(1)}^{Q},\underline{F}_{k_1\sigma(1)}^{Q}\right);\left(\overline{T}_{k_1\sigma(1)}^{Q},\overline{I}_{k_1\sigma(1)}^{Q},\overline{F}_{k_1\sigma(1)}^{Q}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{k_n\sigma(n)}^{Q},\underline{I}_{k_n\sigma(n)}^{Q},\underline{F}_{k_n\sigma(n)}^{Q}\right);\left(\overline{T}_{k_n\sigma(n)}^{Q},\overline{I}_{k_n\sigma(n)}^{Q},\overline{F}_{k_n\sigma(n)}^{Q}\right)\right\rangle\end{array}\right]$$

$$\geqslant\sum_{(k_1\ldots k_n)\in S_n}\left[\begin{array}{l}\left\langle\left(\underline{T}_{1k_1}^{P},\underline{I}_{1k_1}^{P},\underline{F}_{1k_1}^{P}\right);\left(\overline{T}_{1k_1}^{P},\overline{I}_{1k_1}^{P},\overline{F}_{1k_1}^{P}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{nk_n}^{P},\underline{I}_{nk_n}^{P},\underline{F}_{nk_n}^{P}\right);\left(\overline{T}_{nk_n}^{P},\overline{I}_{nk_n}^{P},\overline{F}_{nk_n}^{P}\right)\right\rangle\end{array}\right]$$
$$\cdot\sum_{\sigma\in S_n}\left[\begin{array}{l}\left\langle\left(\underline{T}_{1\sigma(1)}^{Q},\underline{I}_{1\sigma(1)}^{Q},\underline{F}_{1\sigma(1)}^{Q}\right);\left(\overline{T}_{1\sigma(1)}^{Q},\overline{I}_{1\sigma(1)}^{Q},\overline{F}_{1\sigma(1)}^{Q}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}_{n\sigma(n)}^{Q},\underline{I}_{n\sigma(n)}^{Q},\underline{F}_{n\sigma(n)}^{Q}\right);\left(\overline{T}_{n\sigma(n)}^{Q},\overline{I}_{n\sigma(n)}^{Q},\overline{F}_{n\sigma(n)}^{Q}\right)\right\rangle\end{array}\right]$$

$$\geqslant \sum_{(k_1\ldots k_n)\in S_n}\begin{bmatrix}\left\langle\left(\underline{T}^P_{1k_1},\underline{I}^P_{1k_1},\underline{F}^P_{1k_1}\right);\left(\overline{T}^P_{1k_1},\overline{I}^P_{1k_1},\overline{F}^P_{1k_1}\right)\right\rangle\ldots\\ \left\langle\left(\underline{T}^P_{nk_n},\underline{I}^P_{nk_n},\underline{F}^P_{nk_n}\right);\left(\overline{T}^P_{nk_n},\overline{I}^P_{nk_n},\overline{F}^P_{nk_n}\right)\right\rangle\end{bmatrix}\cdot|Q|\geqslant|P|\cdot|Q|$$

(ii) Consider

$$P\cdot Q=\begin{bmatrix}\left\langle\begin{matrix}\left(\sum_{k=1}^{n}\underline{T}^P_{ik}\cdot\underline{T}^Q_{kj},\sum_{k=1}^{n}\underline{I}^P_{ik}\cdot\underline{I}^Q_{kj},\prod_{k=1}^{n}\underline{F}^P_{ik}+\underline{F}^Q_{kj}\right);\\ \left(\sum_{k=1}^{n}\overline{T}^P_{ik}\cdot\overline{T}^Q_{kj},\sum_{k=1}^{n}\overline{I}^P_{ik}\cdot\overline{I}^Q_{kj},\prod_{k=1}^{n}\overline{F}^P_{ik}+\overline{F}^Q_{kj}\right)\end{matrix}\right\rangle\end{bmatrix}$$

By applying Definition 3.3, the above expression becomes

$$|P\cdot Q|=\sum_{\sigma\in S_n}\begin{bmatrix}\left\langle\begin{matrix}\left(\sum_{k=1}^{n}\underline{T}^P_{1k}\cdot\underline{T}^Q_{k\sigma(1)},\sum_{k=1}^{n}\underline{I}^P_{1k}\cdot\underline{I}^Q_{k\sigma(1)},\prod_{k=1}^{n}\underline{F}^P_{1k}+\underline{F}^Q_{k\sigma(1)}\right);\\ \left(\sum_{k=1}^{n}\overline{T}^P_{1k}\cdot\overline{T}^Q_{k\sigma(1)},\sum_{k=1}^{n}\overline{I}^P_{1k}\cdot\overline{I}^Q_{k\sigma(1)},\prod_{k=1}^{n}\overline{F}^P_{1k}+\overline{F}^Q_{k\sigma(1)}\right)\end{matrix}\right\rangle\ldots\\ \left\langle\begin{matrix}\left(\sum_{k=1}^{n}\underline{T}^P_{nk}\cdot\underline{T}^Q_{k\sigma(n)},\sum_{k=1}^{n}\underline{I}^P_{nk}\cdot\underline{I}^Q_{k\sigma(n)},\prod_{k=1}^{n}\underline{F}^P_{nk}+\underline{F}^Q_{k\sigma(n)}\right);\\ \left(\sum_{k=1}^{n}\overline{T}^P_{nk}\cdot\overline{T}^Q_{k\sigma(n)},\sum_{k=1}^{n}\overline{I}^P_{nk}\cdot\overline{I}^Q_{k\sigma(n)},\prod_{k=1}^{n}\overline{F}^P_{nk}+\overline{F}^Q_{k\sigma(n)}\right)\end{matrix}\right\rangle\end{bmatrix}$$

$$=\sum_{\sigma\in S_n}\begin{bmatrix}\left\langle\begin{matrix}\left(\bigcap_{1\leqslant s,t\leqslant n}\underline{T}^P_{1k}\cdot\underline{T}^Q_{k\sigma(1)},\bigcap_{1\leqslant s,t\leqslant n}\underline{I}^P_{1k}\cdot\underline{I}^Q_{k\sigma(1)},\bigcup_{1\leqslant s,t\leqslant n}\underline{F}^P_{1k}+\underline{F}^Q_{k\sigma(1)}\right);\\ \left(\bigcap_{1\leqslant s,t\leqslant n}\overline{T}^P_{1k}\cdot\overline{T}^Q_{k\sigma(1)},\bigcap_{1\leqslant s,t\leqslant n}\overline{I}^P_{1k}\cdot\overline{I}^Q_{k\sigma(1)},\bigcup_{1\leqslant s,t\leqslant n}\overline{F}^P_{1k}+\overline{F}^Q_{k\sigma(1)}\right)\end{matrix}\right\rangle\ldots\\ \left\langle\begin{matrix}\left(\bigcap_{1\leqslant s,t\leqslant n}\underline{T}^P_{nk}\cdot\underline{T}^Q_{k\sigma(n)},\bigcap_{1\leqslant s,t\leqslant n}\underline{I}^P_{nk}\cdot\underline{I}^Q_{k\sigma(n)},\bigcup_{1\leqslant s,t\leqslant n}\underline{F}^P_{nk}+\underline{F}^Q_{k\sigma(n)}\right);\\ \left(\bigcap_{1\leqslant s,t\leqslant n}\overline{T}^P_{nk}\cdot\overline{T}^Q_{k\sigma(n)},\bigcap_{1\leqslant s,t\leqslant n}\overline{I}^P_{nk}\cdot\overline{I}^Q_{k\sigma(n)},\bigcup_{1\leqslant s,t\leqslant n}\overline{F}^P_{nk}+\overline{F}^Q_{k\sigma(n)}\right)\end{matrix}\right\rangle\end{bmatrix}$$

$$\leqslant\sum_{\sigma\in S_n}\begin{bmatrix}\left\langle\begin{matrix}\left(\underline{T}^P_{1k}+\underline{T}^Q_{k\sigma(1)},\underline{I}^P_{1k}+\underline{I}^Q_{k\sigma(1)},\underline{F}^P_{1k}\cdot\underline{F}^Q_{k\sigma(1)}\right);\\ \left(\overline{T}^P_{1k}+\overline{T}^Q_{k\sigma(1)},\overline{I}^P_{1k}+\overline{I}^Q_{k\sigma(1)},\overline{F}^P_{1k}\cdot\overline{F}^Q_{k\sigma(1)}\right)\end{matrix}\right\rangle\ldots\\ \left\langle\begin{matrix}\left(\underline{T}^P_{nk}+\underline{T}^Q_{k\sigma(n)},\underline{I}^P_{nk}+\underline{I}^Q_{k\sigma(n)},\underline{F}^P_{nk}\cdot\underline{F}^Q_{k\sigma(n)}\right);\\ \left(\overline{T}^P_{nk}+\overline{T}^Q_{k\sigma(n)},\overline{I}^P_{nk}+\overline{I}^Q_{k\sigma(n)},\overline{F}^P_{nk}\cdot\overline{F}^Q_{k\sigma(n)}\right)\end{matrix}\right\rangle\end{bmatrix}$$

$$\leqslant\left|\left\langle\left(\underline{T}^P_{ij},\underline{I}^P_{ij},\underline{F}^P_{ij}\right);\left(\overline{T}^P_{ij},\overline{I}^P_{ij},\overline{F}^P_{ij}\right)\right\rangle+\left\langle\left(\underline{T}^Q_{ij},\underline{I}^Q_{ij},\underline{F}^Q_{ij}\right);\left(\overline{T}^Q_{ij},\overline{I}^Q_{ij},\overline{F}^Q_{ij}\right)\right\rangle\right|$$

$$\leqslant|P+Q|$$

Hence the theorem is proved.

Theorem 4.3

Let P be a NHSRM then

(i) $|P| = \sum_{k=1}^{n} \left\langle \left(\underline{T}_{ik}^{P}, \underline{I}_{ik}^{P}, \underline{F}_{ik}^{P}\right); \left(\overline{T}_{ik}^{P}, \overline{I}_{ik}^{P}, \overline{F}_{ik}^{P}\right)\right\rangle \cdot |P_{ik}|; i = 1, 2, ..., n$

$$(ii)\, |P| = \sum_{e<f} \begin{vmatrix} \left\langle \left(\underline{T}_{1e}^{P}, \underline{I}_{1e}^{P}, \underline{F}_{1e}^{P}\right); \left(\overline{T}_{1e}^{P}, \overline{I}_{1e}^{P}, \overline{F}_{1e}^{P}\right)\right\rangle & \left\langle \left(\underline{T}_{1f}^{P}, \underline{I}_{1f}^{P}, \underline{F}_{1f}^{P}\right); \left(\overline{T}_{1f}^{P}, \overline{I}_{1f}^{P}, \overline{F}_{1f}^{P}\right)\right\rangle \\ \left\langle \left(\underline{T}_{2e}^{P}, \underline{I}_{2e}^{P}, \underline{F}_{2e}^{P}\right); \left(\overline{T}_{2e}^{P}, \overline{I}_{2e}^{P}, \overline{F}_{2e}^{P}\right)\right\rangle & \left\langle \left(\underline{T}_{2f}^{P}, \underline{I}_{2f}^{P}, \underline{F}_{2f}^{P}\right); \left(\overline{T}_{2f}^{P}, \overline{I}_{2f}^{P}, \overline{F}_{2f}^{P}\right)\right\rangle \end{vmatrix}$$

$$\cdot P \begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix}$$

where the summation includes all indices e and f in {1, 2, . . . , n} such that $e < f$.

Proof:

(i) By Definition 3.3, we can write

$$|P| = \sum_{\sigma\in S_n} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle ... \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

$$= \sum_{k=1}^{n} \sum_{\substack{\sigma\in S_n \\ \sigma(i)=k}} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle ... \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

$$= \sum_{k=1}^{n} \left[\left\langle \left(\underline{T}_{ik}^{P}, \underline{I}_{ik}^{P}, \underline{F}_{ik}^{P}\right); \left(\overline{T}_{ik}^{P}, \overline{I}_{ik}^{P}, \overline{F}_{ik}^{P}\right)\right\rangle\right] \cdot |P_{ik}|$$

Hence, by applying Definition 3.3, we can expand $|P_{ik}|$ as

$$|P_{ik}| = \sum_{\beta\in S_{n_i n_k}} \begin{bmatrix} \left\langle \left(\underline{T}_{1\beta(1)}^{P}, \underline{I}_{1\beta(1)}^{P}, \underline{F}_{1\beta(1)}^{P}\right); \left(\overline{T}_{1\beta(1)}^{P}, \overline{I}_{1\beta(1)}^{P}, \overline{F}_{1\beta(1)}^{P}\right)\right\rangle ... \\ \left\langle \left(\underline{T}_{i-1\beta(i-1)}^{P}, \underline{I}_{i-1\beta(i-1)}^{P}, \underline{F}_{i-1\beta(i-1)}^{P}\right); \left(\overline{T}_{i-1\beta(i-1)}^{P}, \overline{I}_{i-1\beta(i-1)}^{P}, \overline{F}_{i-1\beta(i-1)}^{P}\right)\right\rangle \\ \cdot \left\langle \left(\underline{T}_{i+1\beta(i+1)}^{P}, \underline{I}_{i+1\beta(i+1)}^{P}, \underline{F}_{i+1\beta(i+1)}^{P}\right); \left(\overline{T}_{i+1\beta(i+1)}^{P}, \overline{I}_{i+1\beta(i+1)}^{P}, \overline{F}_{i+1\beta(i+1)}^{P}\right)\right\rangle \\ ... \left\langle \left(\underline{T}_{n\beta(n)}^{P}, \underline{I}_{n\beta(n)}^{P}, \underline{F}_{n\beta(n)}^{P}\right); \left(\overline{T}_{n\beta(n)}^{P}, \overline{I}_{n\beta(n)}^{P}, \overline{F}_{n\beta(n)}^{P}\right)\right\rangle \end{bmatrix}$$

where $n_i = \{1, 2, ..., n\}\backslash\{i\}$ and $S_{n_i n_k}$ refers to the collection of all permutations mapping elements from the set n_i to the set n_k

(ii) By Definition 3.3, we can write

$$|P| = \sum_{\sigma\in S_n} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle ... \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

$$= \sum_{\substack{\sigma\in S_n \\ \sigma(1,2)=(1,f),\, f>1}} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle \dots \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

$$+ \cdots + \sum_{\substack{\sigma\in S_n \\ \sigma(1,2)=(n-1,f),\, f>n-1}} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle \dots \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix}$$

$$=\sum_{e<f}\sum_{\substack{\sigma\in S_n\\ \sigma(1,2)=(e,f)}}\begin{bmatrix}\left\langle\left(\underline{T}^P_{1\sigma(1)},\underline{I}^P_{1\sigma(1)},\underline{F}^P_{1\sigma(1)}\right);\left(\overline{T}^P_{1\sigma(1)},\overline{I}^P_{1\sigma(1)},\overline{F}^P_{1\sigma(1)}\right)\right\rangle\cdots\\ \left\langle\left(\underline{T}^P_{n\sigma(n)},\underline{I}^P_{n\sigma(n)},\underline{F}^P_{n\sigma(n)}\right);\left(\overline{T}^P_{n\sigma(n)},\overline{I}^P_{n\sigma(n)},\overline{F}^P_{n\sigma(n)}\right)\right\rangle\end{bmatrix}$$

Let $S(p,q)=[\sigma\{1,2\}\to\{p,q\},\sigma\, is\, bijection]$ then

$$=\sum_{e<f}\sum_{\beta\in S(e,f)}\begin{bmatrix}\left\langle\left(\underline{T}^P_{1\beta(1)},\underline{I}^P_{1\beta(1)},\underline{F}^P_{1\beta(1)}\right);\left(\overline{T}^P_{1\beta(1)},\overline{I}^P_{1\beta(1)},\overline{F}^P_{1\beta(1)}\right)\right\rangle\cdots\\ \left\langle\left(\underline{T}^P_{n\beta(n)},\underline{I}^P_{n\beta(n)},\underline{F}^P_{n\beta(n)}\right);\left(\overline{T}^P_{n\beta(n)},\overline{I}^P_{n\beta(n)},\overline{F}^P_{n\beta(n)}\right)\right\rangle\end{bmatrix}$$

$$|P|=\sum_{e<f}\begin{vmatrix}\left\langle(\underline{T}^P_{1e},\underline{I}^P_{1e},\underline{F}^P_{1e});\left(\overline{T}^P_{1e},\overline{I}^P_{1e},\overline{F}^P_{1e}\right)\right\rangle & \left\langle(\underline{T}^P_{1f},\underline{I}^P_{1f},\underline{F}^P_{1f});\left(\overline{T}^P_{1f},\overline{I}^P_{1f},\overline{F}^P_{1f}\right)\right\rangle\\ \left\langle(\underline{T}^P_{2e},\underline{I}^P_{2e},\underline{F}^P_{2e});\left(\overline{T}^P_{2e},\overline{I}^P_{2e},\overline{F}^P_{2e}\right)\right\rangle & \left\langle(\underline{T}^P_{2f},\underline{I}^P_{2f},\underline{F}^P_{2f});\left(\overline{T}^P_{2f},\overline{I}^P_{2f},\overline{F}^P_{2f}\right)\right\rangle\end{vmatrix}$$
$$\cdot P\begin{bmatrix}1&2\\ e&f\end{bmatrix}$$

Hence the theorem.

Theorem 4.4

Let P be a NHSRM then

$$|P|=\sum_{f_1<f_2<\cdots<f_k}\begin{vmatrix}\left\langle(\underline{T}^P_{1f_1},\underline{I}^P_{1f_1},\underline{F}^P_{1f_1});\left(\overline{T}^P_{1f_1},\overline{I}^P_{1f_1},\overline{F}^P_{1f_1}\right)\right\rangle & \cdots & \left\langle(\underline{T}^P_{1f_k},\underline{I}^P_{1f_k},\underline{F}^P_{1f_k});\left(\overline{T}^P_{1f_k},\overline{I}^P_{1f_k},\overline{F}^P_{1f_k}\right)\right\rangle\\ \left\langle(\underline{T}^P_{2f_1},\underline{I}^P_{2f_1},\underline{F}^P_{2f_1});\left(\overline{T}^P_{2f_1},\overline{I}^P_{2f_1},\overline{F}^P_{2f_1}\right)\right\rangle & \cdots & \left\langle(\underline{T}^P_{2f_k},\underline{I}^P_{2f_k},\underline{F}^P_{2f_k});\left(\overline{T}^P_{2f_k},\overline{I}^P_{2f_k},\overline{F}^P_{2f_k}\right)\right\rangle\\ \vdots & \ddots & \vdots\\ \left\langle(\underline{T}^P_{kf_1},\underline{I}^P_{kf_1},\underline{F}^P_{kf_1});\left(\overline{T}^P_{kf_1},\overline{I}^P_{kf_1},\overline{F}^P_{kf_1}\right)\right\rangle & \cdots & \left\langle(\underline{T}^P_{kf_k},\underline{I}^P_{kf_k},\underline{F}^P_{kf_k});\left(\overline{T}^P_{kf_k},\overline{I}^P_{kf_k},\overline{F}^P_{kf_k}\right)\right\rangle\end{vmatrix}$$
$$\cdot P\begin{bmatrix}1&2&\dots&k\\ f_1&f_2&\dots&f_k\end{bmatrix}$$

where the summation is taken over all $f_1,f_2,\,...,\,f_k\in\{1,2,...,k\}$ such that $f_1<f_2<...<f_k$

Proof:

The proof of Theorem 4.4 is similar to that of Theorem 4.3.

Lemma 4.1

Let $X=\begin{bmatrix}\left\langle(\underline{T}^P,\underline{I}^P,\underline{F}^P);\left(\overline{T}^P,\overline{I}^P,\overline{F}^P\right)\right\rangle & \left\langle(\underline{T}^Q,\underline{I}^Q,\underline{F}^Q);\left(\overline{T}^Q,\overline{I}^Q,\overline{F}^Q\right)\right\rangle\\ \left\langle(\underline{T}^R,\underline{I}^R,\underline{F}^R);\left(\overline{T}^R,\overline{I}^R,\overline{F}^R\right)\right\rangle & \left\langle(\underline{T}^S,\underline{I}^S,\underline{F}^S);\left(\overline{T}^S,\overline{I}^S,\overline{F}^S\right)\right\rangle\end{bmatrix}$ be a NHSRM then

$$\begin{vmatrix}\left\langle(\underline{T}^P,\underline{I}^P,\underline{F}^P);\left(\overline{T}^P,\overline{I}^P,\overline{F}^P\right)\right\rangle & \left\langle(\underline{T}^Q,\underline{I}^Q,\underline{F}^Q);\left(\overline{T}^Q,\overline{I}^Q,\overline{F}^Q\right)\right\rangle\\ \left\langle(\underline{T}^P,\underline{I}^P,\underline{F}^P);\left(\overline{T}^P,\overline{I}^P,\overline{F}^P\right)\right\rangle & \left\langle(\underline{T}^Q,\underline{I}^Q,\underline{F}^Q);\left(\overline{T}^Q,\overline{I}^Q,\overline{F}^Q\right)\right\rangle\end{vmatrix}$$
$$\cdot\begin{vmatrix}\left\langle(\underline{T}^R,\underline{I}^R,\underline{F}^R);\left(\overline{T}^R,\overline{I}^R,\overline{F}^R\right)\right\rangle & \left\langle(\underline{T}^S,\underline{I}^S,\underline{F}^S);\left(\overline{T}^S,\overline{I}^S,\overline{F}^S\right)\right\rangle\\ \left\langle(\underline{T}^R,\underline{I}^R,\underline{F}^R);\left(\overline{T}^R,\overline{I}^R,\overline{F}^R\right)\right\rangle & \left\langle(\underline{T}^S,\underline{I}^S,\underline{F}^S);\left(\overline{T}^S,\overline{I}^S,\overline{F}^S\right)\right\rangle\end{vmatrix}\le|X|$$

Proof:

Consider

$$\begin{vmatrix}\left\langle(\underline{T}^P,\underline{I}^P,\underline{F}^P);\left(\overline{T}^P,\overline{I}^P,\overline{F}^P\right)\right\rangle & \left\langle(\underline{T}^Q,\underline{I}^Q,\underline{F}^Q);\left(\overline{T}^Q,\overline{I}^Q,\overline{F}^Q\right)\right\rangle\\ \left\langle(\underline{T}^P,\underline{I}^P,\underline{F}^P);\left(\overline{T}^P,\overline{I}^P,\overline{F}^P\right)\right\rangle & \left\langle(\underline{T}^Q,\underline{I}^Q,\underline{F}^Q);\left(\overline{T}^Q,\overline{I}^Q,\overline{F}^Q\right)\right\rangle\end{vmatrix}$$
$$\cdot\begin{vmatrix}\left\langle(\underline{T}^R,\underline{I}^R,\underline{F}^R);\left(\overline{T}^R,\overline{I}^R,\overline{F}^R\right)\right\rangle & \left\langle(\underline{T}^S,\underline{I}^S,\underline{F}^S);\left(\overline{T}^S,\overline{I}^S,\overline{F}^S\right)\right\rangle\\ \left\langle(\underline{T}^R,\underline{I}^R,\underline{F}^R);\left(\overline{T}^R,\overline{I}^R,\overline{F}^R\right)\right\rangle & \left\langle(\underline{T}^S,\underline{I}^S,\underline{F}^S);\left(\overline{T}^S,\overline{I}^S,\overline{F}^S\right)\right\rangle\end{vmatrix}$$

$$= \begin{bmatrix} \left\langle \left(\underline{T}^{P},\underline{I}^{P},\underline{F}^{P}\right);\left(\overline{T}^{P},\overline{I}^{P},\overline{F}^{P}\right)\right\rangle \left\langle \left(\underline{T}^{Q},\underline{I}^{Q},\underline{F}^{Q}\right);\left(\overline{T}^{Q},\overline{I}^{Q},\overline{F}^{Q}\right)\right\rangle \\ +\left\langle \left(\underline{T}^{P},\underline{I}^{P},\underline{F}^{P}\right);\left(\overline{T}^{P},\overline{I}^{P},\overline{F}^{P}\right)\right\rangle \left\langle \left(\underline{T}^{Q},\underline{I}^{Q},\underline{F}^{Q}\right);\left(\overline{T}^{Q},\overline{I}^{Q},\overline{F}^{Q}\right)\right\rangle \end{bmatrix}$$

$$\cdot \begin{bmatrix} \left\langle \left(\underline{T}^{R},\underline{I}^{R},\underline{F}^{R}\right);\left(\overline{T}^{R},\overline{I}^{R},\overline{F}^{R}\right)\right\rangle \left\langle \left(\underline{T}^{S},\underline{I}^{S},\underline{F}^{S}\right);\left(\overline{T}^{S},\overline{I}^{S},\overline{F}^{S}\right)\right\rangle \\ +\left\langle \left(\underline{T}^{R},\underline{I}^{R},\underline{F}^{R}\right);\left(\overline{T}^{R},\overline{I}^{R},\overline{F}^{R}\right)\right\rangle \left\langle \left(\underline{T}^{S},\underline{I}^{S},\underline{F}^{S}\right);\left(\overline{T}^{S},\overline{I}^{S},\overline{F}^{S}\right)\right\rangle \end{bmatrix}$$

$$\begin{aligned} &= \left[\left\langle \left(\underline{T}^{P},\underline{I}^{P},\underline{F}^{P}\right);\left(\overline{T}^{P},\overline{I}^{P},\overline{F}^{P}\right)\right\rangle \left\langle \left(\underline{T}^{Q},\underline{I}^{Q},\underline{F}^{Q}\right);\left(\overline{T}^{Q},\overline{I}^{Q},\overline{F}^{Q}\right)\right\rangle\right] \\ &\cdot \left[\left\langle \left(\underline{T}^{R},\underline{I}^{R},\underline{F}^{R}\right);\left(\overline{T}^{R},\overline{I}^{R},\overline{F}^{R}\right)\right\rangle \left\langle \left(\underline{T}^{S},\underline{I}^{S},\underline{F}^{S}\right);\left(\overline{T}^{S},\overline{I}^{S},\overline{F}^{S}\right)\right\rangle\right] \end{aligned}$$

$$\leqslant \begin{vmatrix} \left\langle \left(\underline{T}^{P},\underline{I}^{P},\underline{F}^{P}\right);\left(\overline{T}^{P},\overline{I}^{P},\overline{F}^{P}\right)\right\rangle & \left\langle \left(\underline{T}^{Q},\underline{I}^{Q},\underline{F}^{Q}\right);\left(\overline{T}^{Q},\overline{I}^{Q},\overline{F}^{Q}\right)\right\rangle \\ \left\langle \left(\underline{T}^{R},\underline{I}^{R},\underline{F}^{R}\right);\left(\overline{T}^{R},\overline{I}^{R},\overline{F}^{R}\right)\right\rangle & \left\langle \left(\underline{T}^{S},\underline{I}^{S},\underline{F}^{S}\right);\left(\overline{T}^{S},\overline{I}^{S},\overline{F}^{S}\right)\right\rangle \end{vmatrix} \leqslant |X|$$

Hence the lemma.

Remark 4.1

Let P be a NHSRM and $P(e \Rightarrow f)$ denote the matrix formed by replacing the f^{th} row of P with its e^{th} row.

Theorem 4.5

Let P be a NHSRM then

(i) $|P(2 \Rightarrow 1)|\ |P(1 \Rightarrow 2)| \leqslant |P|$

(ii) $|P(2 \Rightarrow 1)|\ |P(3 \Rightarrow 2)| \leqslant |P|$

(iii) $|P(r \Rightarrow s)|\ |P(s \Rightarrow t)| \leqslant |P|$

Proof:

(i) Consider L.H.S, by Theorem 4.3, we can write $|P(2 \Rightarrow 1)|\ |P(1 \Rightarrow 2)|$

$$= \begin{bmatrix} \begin{vmatrix} \left\langle \left(\underline{T}^{P}_{11},\underline{I}^{P}_{11},\underline{F}^{P}_{11}\right);\left(\overline{T}^{P}_{11},\overline{I}^{P}_{11},\overline{F}^{P}_{11}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{12},\underline{I}^{P}_{12},\underline{F}^{P}_{12}\right);\left(\overline{T}^{P}_{12},\overline{I}^{P}_{12},\overline{F}^{P}_{12}\right)\right\rangle \\ \left\langle \left(\underline{T}^{P}_{11},\underline{I}^{P}_{11},\underline{F}^{P}_{11}\right);\left(\overline{T}^{P}_{11},\overline{I}^{P}_{11},\overline{F}^{P}_{11}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{12},\underline{I}^{P}_{12},\underline{F}^{P}_{12}\right);\left(\overline{T}^{P}_{12},\overline{I}^{P}_{12},\overline{F}^{P}_{12}\right)\right\rangle \end{vmatrix} \cdot P\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \\ +\ldots+ \begin{vmatrix} \left\langle \left(\underline{T}^{P}_{1e},\underline{I}^{P}_{1e},\underline{F}^{P}_{1e}\right);\left(\overline{T}^{P}_{1e},\overline{I}^{P}_{1e},\overline{F}^{P}_{1e}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{1f},\underline{I}^{P}_{1f},\underline{F}^{P}_{1f}\right);\left(\overline{T}^{P}_{1f},\overline{I}^{P}_{1f},\overline{F}^{P}_{1f}\right)\right\rangle \\ \left\langle \left(\underline{T}^{P}_{1e},\underline{I}^{P}_{1e},\underline{F}^{P}_{1e}\right);\left(\overline{T}^{P}_{1e},\overline{I}^{P}_{1e},\overline{F}^{P}_{1e}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{1f},\underline{I}^{P}_{1f},\underline{F}^{P}_{1f}\right);\left(\overline{T}^{P}_{1f},\overline{I}^{P}_{1f},\overline{F}^{P}_{1f}\right)\right\rangle \end{vmatrix} \\ \cdot P\begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix} \\ +\ldots+ \begin{vmatrix} \left\langle \left(\underline{T}^{P}_{1n-1},\underline{I}^{P}_{1n-1},\underline{F}^{P}_{1n-1}\right);\left(\overline{T}^{P}_{1n-1},\overline{I}^{P}_{1n-1},\overline{F}^{P}_{1n-1}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{1n},\underline{I}^{P}_{1n},\underline{F}^{P}_{1n}\right);\left(\overline{T}^{P}_{1n},\overline{I}^{P}_{1n},\overline{F}^{P}_{1n}\right)\right\rangle \\ \left\langle \left(\underline{T}^{P}_{1n-1},\underline{I}^{P}_{1n-1},\underline{F}^{P}_{1n-1}\right);\left(\overline{T}^{P}_{1n-1},\overline{I}^{P}_{1n-1},\overline{F}^{P}_{1n-1}\right)\right\rangle & \left\langle \left(\underline{T}^{P}_{1n},\underline{I}^{P}_{1n},\underline{F}^{P}_{1n}\right);\left(\overline{T}^{P}_{1n},\overline{I}^{P}_{1n},\overline{F}^{P}_{1n}\right)\right\rangle \end{vmatrix} \\ \cdot P\begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix} \end{bmatrix}$$

$$
\cdot\left[\begin{array}{l}
\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{21},\underline{I}^P_{21},\underline{F}^P_{21}\right);\left(\overline{T}^P_{21},\overline{I}^P_{21},\overline{F}^P_{21}\right)\right\rangle & \left\langle\left(\underline{T}^P_{22},\underline{I}^P_{22},\underline{F}^P_{22}\right);\left(\overline{T}^P_{22},\overline{I}^P_{22},\overline{F}^P_{22}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{21},\underline{I}^P_{21},\underline{F}^P_{21}\right);\left(\overline{T}^P_{21},\overline{I}^P_{21},\overline{F}^P_{21}\right)\right\rangle & \left\langle\left(\underline{T}^P_{22},\underline{I}^P_{22},\underline{F}^P_{22}\right);\left(\overline{T}^P_{22},\overline{I}^P_{22},\overline{F}^P_{22}\right)\right\rangle
\end{array}\right| \cdot P\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \\
+\ldots+\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{2e},\underline{I}^P_{2e},\underline{F}^P_{2e}\right);\left(\overline{T}^P_{2e},\overline{I}^P_{2e},\overline{F}^P_{2e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2f},\underline{I}^P_{2f},\underline{F}^P_{2f}\right);\left(\overline{T}^P_{2f},\overline{I}^P_{2f},\overline{F}^P_{2f}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{2e},\underline{I}^P_{2e},\underline{F}^P_{2e}\right);\left(\overline{T}^P_{2e},\overline{I}^P_{2e},\overline{F}^P_{2e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2f},\underline{I}^P_{2f},\underline{F}^P_{2f}\right);\left(\overline{T}^P_{2f},\overline{I}^P_{2f},\overline{F}^P_{2f}\right)\right\rangle
\end{array}\right| \\
\cdot P\begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix} \\
+\ldots+\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{2n-1},\underline{I}^P_{2n-1},\underline{F}^P_{2n-1}\right);\left(\overline{T}^P_{2n-1},\overline{I}^P_{2n-1},\overline{F}^P_{2n-1}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2n},\underline{I}^P_{2n},\underline{F}^P_{2n}\right);\left(\overline{T}^P_{2n},\overline{I}^P_{2n},\overline{F}^P_{2n}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{2n-1},\underline{I}^P_{2n-1},\underline{F}^P_{2n-1}\right);\left(\overline{T}^P_{2n-1},\overline{I}^P_{2n-1},\overline{F}^P_{2n-1}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2n},\underline{I}^P_{2n},\underline{F}^P_{2n}\right);\left(\overline{T}^P_{2n},\overline{I}^P_{2n},\overline{F}^P_{2n}\right)\right\rangle
\end{array}\right| \\
\cdot P\begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix}
\end{array}\right]
$$

$$
=\sum_{e<f}\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{1e},\underline{I}^P_{1e},\underline{F}^P_{1e}\right);\left(\overline{T}^P_{1e},\overline{I}^P_{1e},\overline{F}^P_{1e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{1f},\underline{I}^P_{1f},\underline{F}^P_{1f}\right);\left(\overline{T}^P_{1f},\overline{I}^P_{1f},\overline{F}^P_{1f}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{1e},\underline{I}^P_{1e},\underline{F}^P_{1e}\right);\left(\overline{T}^P_{1e},\overline{I}^P_{1e},\overline{F}^P_{1e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{1f},\underline{I}^P_{1f},\underline{F}^P_{1f}\right);\left(\overline{T}^P_{1f},\overline{I}^P_{1f},\overline{F}^P_{1f}\right)\right\rangle
\end{array}\right| P\begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix}
$$

$$
\cdot\sum_{g<h}\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{2g},\underline{I}^P_{2g},\underline{F}^P_{2e}\right);\left(\overline{T}^P_{2g},\overline{I}^P_{2g},\overline{F}^P_{2g}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2h},\underline{I}^P_{2h},\underline{F}^P_{2h}\right);\left(\overline{T}^P_{2h},\overline{I}^P_{2h},\overline{F}^P_{2h}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{2g},\underline{I}^P_{2g},\underline{F}^P_{2g}\right);\left(\overline{T}^P_{2g},\overline{I}^P_{2g},\overline{F}^P_{2g}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2h},\underline{I}^P_{2h},\underline{F}^P_{2h}\right);\left(\overline{T}^P_{2h},\overline{I}^P_{2h},\overline{F}^P_{2h}\right)\right\rangle
\end{array}\right| P\begin{bmatrix} 1 & 2 \\ g & h \end{bmatrix}
$$

$$
\leqslant\sum_{\substack{e<f\\ g<h}}\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{1e},\underline{I}^P_{1e},\underline{F}^P_{1e}\right);\left(\overline{T}^P_{1e},\overline{I}^P_{1e},\overline{F}^P_{1e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{1f},\underline{I}^P_{1f},\underline{F}^P_{1f}\right);\left(\overline{T}^P_{1f},\overline{I}^P_{1f},\overline{F}^P_{1f}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{2g},\underline{I}^P_{2g},\underline{F}^P_{2g}\right);\left(\overline{T}^P_{2g},\overline{I}^P_{2g},\overline{F}^P_{2g}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2h},\underline{I}^P_{2h},\underline{F}^P_{2h}\right);\left(\overline{T}^P_{2h},\overline{I}^P_{2h},\overline{F}^P_{2h}\right)\right\rangle
\end{array}\right|
$$

$$
\cdot P\begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix}\cdot P\begin{bmatrix} 1 & 2 \\ g & h \end{bmatrix}
$$

By lemma 4.1, we introduce symbols $\gamma_1, \gamma_2, \gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and γ
Define

$$
\gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{1e},\underline{I}^P_{1e},\underline{F}^P_{1e}\right);\left(\overline{T}^P_{1e},\overline{I}^P_{1e},\overline{F}^P_{1e}\right)\right\rangle & \left\langle\left(\underline{T}^P_{1f},\underline{I}^P_{1f},\underline{F}^P_{1f}\right);\left(\overline{T}^P_{1f},\overline{I}^P_{1f},\overline{F}^P_{1f}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{2g},\underline{I}^P_{2g},\underline{F}^P_{2g}\right);\left(\overline{T}^P_{2g},\overline{I}^P_{2g},\overline{F}^P_{2g}\right)\right\rangle & \left\langle\left(\underline{T}^P_{2h},\underline{I}^P_{2h},\underline{F}^P_{2h}\right);\left(\overline{T}^P_{2h},\overline{I}^P_{2h},\overline{F}^P_{2h}\right)\right\rangle
\end{array}\right|
$$

$$
\cdot P\begin{bmatrix} 1 & 2 \\ e & f \end{bmatrix} P\begin{bmatrix} 1 & 2 \\ g & h \end{bmatrix}
$$

$$
\gamma_1=\sum_{(e,f)=(g,h)}\gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\sum_{e<f}\gamma\begin{bmatrix} e & f \\ e & f \end{bmatrix}
$$

$$
\gamma_2=\sum_{(e,f)\neq(g,h)}\gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix} \text{ and } \gamma=\gamma_1+\gamma_2
$$

Then, we observe that

$$
\gamma\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}=\left|\begin{array}{cc}
\left\langle\left(\underline{T}^P_{11},\underline{I}^P_{11},\underline{F}^P_{11}\right);\left(\overline{T}^P_{11},\overline{I}^P_{11},\overline{F}^P_{11}\right)\right\rangle & \left\langle\left(\underline{T}^P_{12},\underline{I}^P_{12},\underline{F}^P_{12}\right);\left(\overline{T}^P_{12},\overline{I}^P_{12},\overline{F}^P_{12}\right)\right\rangle \\
\left\langle\left(\underline{T}^P_{21},\underline{I}^P_{21},\underline{F}^P_{21}\right);\left(\overline{T}^P_{21},\overline{I}^P_{21},\overline{F}^P_{21}\right)\right\rangle & \left\langle\left(\underline{T}^P_{22},\underline{I}^P_{22},\underline{F}^P_{22}\right);\left(\overline{T}^P_{22},\overline{I}^P_{22},\overline{F}^P_{22}\right)\right\rangle
\end{array}\right|
$$

$$
\cdot P\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}
$$

$\gamma_1 = |P|$ (By Theorem 4.3)
$|P(2 \Rightarrow 1)|\ |P(1 \Rightarrow 2)| \leqslant \gamma = |P| + \gamma_2$
Now we show that $\gamma_2 \leqslant |P|$, There are two cases to be considered.

(1) $(e, f) = (g, h)$
(2) $(e, f) \neq (g, h)$

Case 1: We consider $p = \gamma \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, a term of γ_2.

$$\begin{aligned} \text{Let } p_1 &= \left\langle \left(\underline{T}^P_{11}, \underline{I}^P_{11}, \underline{F}^P_{11}\right); \left(\overline{T}^P_{11}, \overline{I}^P_{11}, \overline{F}^P_{11}\right)\right\rangle \left\langle \left(\underline{T}^P_{23}, \underline{I}^P_{23}, \underline{F}^P_{23}\right); \left(\overline{T}^P_{23}, \overline{I}^P_{23}, \overline{F}^P_{23}\right)\right\rangle \\ &\cdot P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} P \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \\ \text{and } p_2 &= \left\langle \left(\underline{T}^P_{11}, \underline{I}^P_{11}, \underline{F}^P_{11}\right); \left(\overline{T}^P_{11}, \overline{I}^P_{11}, \overline{F}^P_{11}\right)\right\rangle \left\langle \left(\underline{T}^P_{21}, \underline{I}^P_{21}, \underline{F}^P_{21}\right); \left(\overline{T}^P_{21}, \overline{I}^P_{21}, \overline{F}^P_{21}\right)\right\rangle \\ &\cdot P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} P \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

Then $p = p_1 + p_2$

$$p_1 \leqslant \left| \begin{matrix} \left\langle \left(\underline{T}^P_{11}, \underline{I}^P_{11}, \underline{F}^P_{11}\right); \left(\overline{T}^P_{11}, \overline{I}^P_{11}, \overline{F}^P_{11}\right)\right\rangle & \left\langle \left(\underline{T}^P_{13}, \underline{I}^P_{13}, \underline{F}^P_{13}\right); \left(\overline{T}^P_{13}, \overline{I}^P_{13}, \overline{F}^P_{13}\right)\right\rangle \\ \left\langle \left(\underline{T}^P_{21}, \underline{I}^P_{21}, \underline{F}^P_{21}\right); \left(\overline{T}^P_{21}, \overline{I}^P_{21}, \overline{F}^P_{21}\right)\right\rangle & \left\langle \left(\underline{T}^P_{23}, \underline{I}^P_{23}, \underline{F}^P_{23}\right); \left(\overline{T}^P_{23}, \overline{I}^P_{23}, \overline{F}^P_{23}\right)\right\rangle \end{matrix} \right| P \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \leqslant |P|$$

$$p_2 \leqslant \left| \begin{matrix} \left\langle \left(\underline{T}^P_{11}, \underline{I}^P_{11}, \underline{F}^P_{11}\right); \left(\overline{T}^P_{11}, \overline{I}^P_{11}, \overline{F}^P_{11}\right)\right\rangle & \left\langle \left(\underline{T}^P_{12}, \underline{I}^P_{12}, \underline{F}^P_{12}\right); \left(\overline{T}^P_{12}, \overline{I}^P_{12}, \overline{F}^P_{12}\right)\right\rangle \\ \left\langle \left(\underline{T}^P_{21}, \underline{I}^P_{21}, \underline{F}^P_{21}\right); \left(\overline{T}^P_{21}, \overline{I}^P_{21}, \overline{F}^P_{21}\right)\right\rangle & \left\langle \left(\underline{T}^P_{22}, \underline{I}^P_{22}, \underline{F}^P_{22}\right); \left(\overline{T}^P_{22}, \overline{I}^P_{22}, \overline{F}^P_{22}\right)\right\rangle \end{matrix} \right| P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \leqslant |P|$$

$$\text{and } \gamma \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \leqslant |P|$$

Case 2: We consider $q = \gamma \begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix}$, a term of γ_2

$$\begin{aligned} \text{Let } q_1 &= \left\langle \left(\underline{T}^P_{11}, \underline{I}^P_{11}, \underline{F}^P_{11}\right); \left(\overline{T}^P_{11}, \overline{I}^P_{11}, \overline{F}^P_{11}\right)\right\rangle \left\langle \left(\underline{T}^P_{2n}, \underline{I}^P_{2n}, \underline{F}^P_{2n}\right); \left(\overline{T}^P_{2n}, \overline{I}^P_{2n}, \overline{F}^P_{2n}\right)\right\rangle \\ &\cdot P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} P \begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix} \\ \text{and } q_2 &= \left\langle \left(\underline{T}^P_{12}, \underline{I}^P_{12}, \underline{F}^P_{12}\right); \left(\overline{T}^P_{12}, \overline{I}^P_{12}, \overline{F}^P_{12}\right)\right\rangle \left\langle \left(\underline{T}^P_{2n-1}, \underline{I}^P_{2n-1}, \underline{F}^P_{2n-1}\right); \left(\overline{T}^P_{2n-1}, \overline{I}^P_{2n-1}, \overline{F}^P_{2n-1}\right)\right\rangle \\ &\cdot P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} P \begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix} \end{aligned}$$

Then $q = q_1 + q_2$. To show that $q_1 \leqslant |P|$ and $q_2 = |P|$

We observe that all coordinates of the element p_{ij} involved in $P \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} P \begin{bmatrix} 1 & 2 \\ n-1 & n \end{bmatrix}$. All the coordinates of the element p_{ij} appearing in these determinants are drawn from the k^{th} row of p_k in P for $k \geqslant 3$. Therefore, if we define $r = p_{3n-1}\ p_{4n-2}\ \ldots\ p_{k+2n-k}\ \ldots\ p_{n2}$, then it follows that

$$q_1 \leqslant \left[\left\langle\left(\underline{T}_{11}^P, \underline{I}_{11}^P, \underline{F}_{11}^P\right); \left(\overline{T}_{11}^P, \overline{I}_{11}^P, \overline{F}_{11}^P\right)\right\rangle \left\langle\left(\underline{T}_{2n}^P, \underline{I}_{2n}^P, \underline{F}_{2n}^P\right); \left(\overline{T}_{2n}^P, \overline{I}_{2n}^P, \overline{F}_{2n}^P\right)\right\rangle\right] . r \leqslant |P|$$

For q_2, let $s = p_{3n}\ p_{4n-1}\ p_{5n-2}\ \ldots\ p_{n3}\ p_{2n-1}$, then we see that

$$q_2 \leqslant \begin{bmatrix} \left\langle\left(\underline{T}_{12}^P, \underline{I}_{12}^P, \underline{F}_{12}^P\right); \left(\overline{T}_{12}^P, \overline{I}_{12}^P, \overline{F}_{12}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{2n-1}^P, \underline{I}_{2n-1}^P, \underline{F}_{2n-1}^P\right); \left(\overline{T}_{2n-1}^P, \overline{I}_{2n-1}^P, \overline{F}_{2n-1}^P\right)\right\rangle \end{bmatrix} . s \leqslant |P|$$

For any $q = \gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix}_{(e,f)\neq(g,h)}$, we apply either Case 1 or Case 2 method and we can deduce that $q = \gamma\begin{bmatrix} e & f \\ g & h \end{bmatrix}_{(e,f)\neq(g,h)} \leqslant |P|$

Hence the result (i) is proved.

(ii) First we consider $|P(2 \Rightarrow 1)|\ |P(3 \Rightarrow 2)|$

$$= \begin{vmatrix} \left\langle\left(\underline{T}_{11}^P, \underline{I}_{11}^P, \underline{F}_{11}^P\right); \left(\overline{T}_{11}^P, \overline{I}_{11}^P, \overline{F}_{11}^P\right)\right\rangle & \left\langle\left(\underline{T}_{12}^P, \underline{I}_{12}^P, \underline{F}_{12}^P\right); \left(\overline{T}_{12}^P, \overline{I}_{12}^P, \overline{F}_{12}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{31}^P, \underline{I}_{31}^P, \underline{F}_{31}^P\right); \left(\overline{T}_{31}^P, \overline{I}_{31}^P, \overline{F}_{31}^P\right)\right\rangle & \left\langle\left(\underline{T}_{32}^P, \underline{I}_{32}^P, \underline{F}_{32}^P\right); \left(\overline{T}_{32}^P, \overline{I}_{32}^P, \overline{F}_{32}^P\right)\right\rangle \end{vmatrix}$$

$$. \begin{vmatrix} \left\langle\left(\underline{T}_{21}^P, \underline{I}_{21}^P, \underline{F}_{21}^P\right); \left(\overline{T}_{21}^P, \overline{I}_{21}^P, \overline{F}_{21}^P\right)\right\rangle & \left\langle\left(\underline{T}_{22}^P, \underline{I}_{22}^P, \underline{F}_{22}^P\right); \left(\overline{T}_{22}^P, \overline{I}_{22}^P, \overline{F}_{22}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{21}^P, \underline{I}_{21}^P, \underline{F}_{21}^P\right); \left(\overline{T}_{21}^P, \overline{I}_{21}^P, \overline{F}_{21}^P\right)\right\rangle & \left\langle\left(\underline{T}_{22}^P, \underline{I}_{22}^P, \underline{F}_{22}^P\right); \left(\overline{T}_{22}^P, \overline{I}_{22}^P, \overline{F}_{22}^P\right)\right\rangle \end{vmatrix}$$

$$\leqslant \begin{vmatrix} \left\langle\left(\underline{T}_{21}^P, \underline{I}_{21}^P, \underline{F}_{21}^P\right); \left(\overline{T}_{21}^P, \overline{I}_{21}^P, \overline{F}_{21}^P\right)\right\rangle & \left\langle\left(\underline{T}_{22}^P, \underline{I}_{22}^P, \underline{F}_{22}^P\right); \left(\overline{T}_{22}^P, \overline{I}_{22}^P, \overline{F}_{22}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{31}^P, \underline{I}_{31}^P, \underline{F}_{31}^P\right); \left(\overline{T}_{31}^P, \overline{I}_{31}^P, \overline{F}_{31}^P\right)\right\rangle & \left\langle\left(\underline{T}_{32}^P, \underline{I}_{32}^P, \underline{F}_{32}^P\right); \left(\overline{T}_{32}^P, \overline{I}_{32}^P, \overline{F}_{32}^P\right)\right\rangle \end{vmatrix}$$

We define a symbol ϑ

$$\vartheta\begin{bmatrix} g & h \\ e & f \end{bmatrix} = \begin{vmatrix} \left\langle\left(\underline{T}_{2g}^P, \underline{I}_{2g}^P, \underline{F}_{2g}^P\right); \left(\overline{T}_{2g}^P, \overline{I}_{2g}^P, \overline{F}_{2g}^P\right)\right\rangle & \left\langle\left(\underline{T}_{2h}^P, \underline{I}_{2h}^P, \underline{F}_{2h}^P\right); \left(\overline{T}_{2h}^P, \overline{I}_{2h}^P, \overline{F}_{2h}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{3e}^P, \underline{I}_{3e}^P, \underline{F}_{3e}^P\right); \left(\overline{T}_{3e}^P, \overline{I}_{3e}^P, \overline{F}_{3e}^P\right)\right\rangle & \left\langle\left(\underline{T}_{3f}^P, \underline{I}_{3f}^P, \underline{F}_{3f}^P\right); \left(\overline{T}_{3f}^P, \overline{I}_{3f}^P, \overline{F}_{3f}^P\right)\right\rangle \end{vmatrix}$$

$$\cdot P\begin{bmatrix} 2 & 3 \\ g & h \end{bmatrix} P\begin{bmatrix} 2 & 3 \\ e & f \end{bmatrix}$$

Then it is true that $|P(2 \Rightarrow 1)|\ |P(1 \Rightarrow 2)|$

$$= \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} \left\langle\left(\underline{T}_{1e}^P, \underline{I}_{1e}^P, \underline{F}_{1e}^P\right); \left(\overline{T}_{1e}^P, \overline{I}_{1e}^P, \overline{F}_{1e}^P\right)\right\rangle & \left\langle\left(\underline{T}_{1f}^P, \underline{I}_{1f}^P, \underline{F}_{1f}^P\right); \left(\overline{T}_{1f}^P, \overline{I}_{1f}^P, \overline{F}_{1f}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{3e}^P, \underline{I}_{3e}^P, \underline{F}_{3e}^P\right); \left(\overline{T}_{3e}^P, \overline{I}_{3e}^P, \overline{F}_{3e}^P\right)\right\rangle & \left\langle\left(\underline{T}_{3f}^P, \underline{I}_{3f}^P, \underline{F}_{3f}^P\right); \left(\overline{T}_{3f}^P, \overline{I}_{3f}^P, \overline{F}_{3f}^P\right)\right\rangle \end{vmatrix} P\begin{bmatrix} 2 & 3 \\ e & f \end{bmatrix}$$

$$. \begin{vmatrix} \left\langle\left(\underline{T}_{2g}^P, \underline{I}_{2g}^P, \underline{F}_{2g}^P\right); \left(\overline{T}_{2g}^P, \overline{I}_{2g}^P, \overline{F}_{2g}^P\right)\right\rangle & \left\langle\left(\underline{T}_{2h}^P, \underline{I}_{2h}^P, \underline{F}_{2h}^P\right); \left(\overline{T}_{2h}^P, \overline{I}_{2h}^P, \overline{F}_{2h}^P\right)\right\rangle \\ \left\langle\left(\underline{T}_{2g}^P, \underline{I}_{2g}^P, \underline{F}_{2g}^P\right); \left(\overline{T}_{2g}^P, \overline{I}_{2g}^P, \overline{F}_{2g}^P\right)\right\rangle & \left\langle\left(\underline{T}_{2h}^P, \underline{I}_{2h}^P, \underline{F}_{2h}^P\right); \left(\overline{T}_{2h}^P, \overline{I}_{2h}^P, \overline{F}_{2h}^P\right)\right\rangle \end{vmatrix} P\begin{bmatrix} 2 & 3 \\ g & h \end{bmatrix}$$

$$\leqslant \sum_{\substack{e<f\\ g<h}} \left| \begin{array}{cc} \left\langle\left(\underline{T}_{2g}^{P},\underline{I}_{2g}^{P},\underline{F}_{2g}^{P}\right);\left(\overline{T}_{2g}^{P},\overline{I}_{2g}^{P},\overline{F}_{2g}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{2h}^{P},\underline{I}_{2h}^{P},\underline{F}_{2h}^{P}\right);\left(\overline{T}_{2h}^{P},\overline{I}_{2h}^{P},\overline{F}_{2h}^{P}\right)\right\rangle \\ \left\langle\left(\underline{T}_{3e}^{P},\underline{I}_{3e}^{P},\underline{F}_{3e}^{P}\right);\left(\overline{T}_{3e}^{P},\overline{I}_{3e}^{P},\overline{F}_{3e}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{3f}^{P},\underline{I}_{3f}^{P},\underline{F}_{3f}^{P}\right);\left(\overline{T}_{3f}^{P},\overline{I}_{3f}^{P},\overline{F}_{3f}^{P}\right)\right\rangle \end{array} \right|$$

$$\cdot P\left[\begin{array}{cc} 2 & 3 \\ g & h \end{array}\right] P\left[\begin{array}{cc} 2 & 3 \\ e & f \end{array}\right]$$

$$\leqslant \sum_{\substack{e<f\\ g<h}} \vartheta\left[\begin{array}{cc} g & h \\ e & f \end{array}\right]$$

$$= \sum_{(e,f)=(g,h)} \vartheta\left[\begin{array}{cc} g & h \\ e & f \end{array}\right] + \sum_{(e,f)\neq(g,h)} \vartheta\left[\begin{array}{cc} g & h \\ e & f \end{array}\right]$$

We have to show that $\vartheta\left[\begin{array}{cc} g & h \\ e & f \end{array}\right]_{(e,f)\neq(g,h)} \leqslant |P|$

There are two cases arise

Case 1 : We take

$$\vartheta\left[\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array}\right] = \left| \begin{array}{cc} \left\langle\left(\underline{T}_{21}^{P},\underline{I}_{21}^{P},\underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P},\overline{I}_{21}^{P},\overline{F}_{21}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{22}^{P},\underline{I}_{22}^{P},\underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P},\overline{I}_{22}^{P},\overline{F}_{22}^{P}\right)\right\rangle \\ \left\langle\left(\underline{T}_{31}^{P},\underline{I}_{31}^{P},\underline{F}_{31}^{P}\right);\left(\overline{T}_{31}^{P},\overline{I}_{31}^{P},\overline{F}_{31}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{33}^{P},\underline{I}_{33}^{P},\underline{F}_{33}^{P}\right);\left(\overline{T}_{33}^{P},\overline{I}_{33}^{P},\overline{F}_{33}^{P}\right)\right\rangle \end{array} \right|$$

$$\cdot P\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right] P\left[\begin{array}{cc} 2 & 3 \\ 1 & 3 \end{array}\right]$$

$$= \left[\begin{array}{l} \left\langle\left(\underline{T}_{21}^{P},\underline{I}_{21}^{P},\underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P},\overline{I}_{21}^{P},\overline{F}_{21}^{P}\right)\right\rangle \left\langle\left(\underline{T}_{33}^{P},\underline{I}_{33}^{P},\underline{F}_{33}^{P}\right);\left(\overline{T}_{33}^{P},\overline{I}_{33}^{P},\overline{F}_{33}^{P}\right)\right\rangle \\ + \left\langle\left(\underline{T}_{22}^{P},\underline{I}_{22}^{P},\underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P},\overline{I}_{22}^{P},\overline{F}_{22}^{P}\right)\right\rangle \left\langle\left(\underline{T}_{31}^{P},\underline{I}_{31}^{P},\underline{F}_{31}^{P}\right);\left(\overline{T}_{31}^{P},\overline{I}_{31}^{P},\overline{F}_{31}^{P}\right)\right\rangle \end{array}\right] \cdot P\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right] P\left[\begin{array}{cc} 2 & 3 \\ 1 & 3 \end{array}\right]$$

$$= \left\langle\left(\underline{T}_{21}^{P},\underline{I}_{21}^{P},\underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P},\overline{I}_{21}^{P},\overline{F}_{21}^{P}\right)\right\rangle \left\langle\left(\underline{T}_{33}^{P},\underline{I}_{33}^{P},\underline{F}_{33}^{P}\right);\left(\overline{T}_{33}^{P},\overline{I}_{33}^{P},\overline{F}_{33}^{P}\right)\right\rangle P\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right] P\left[\begin{array}{cc} 2 & 3 \\ 1 & 3 \end{array}\right]$$

$$+ \left\langle\left(\underline{T}_{22}^{P},\underline{I}_{22}^{P},\underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P},\overline{I}_{22}^{P},\overline{F}_{22}^{P}\right)\right\rangle \left\langle\left(\underline{T}_{31}^{P},\underline{I}_{31}^{P},\underline{F}_{31}^{P}\right);\left(\overline{T}_{31}^{P},\overline{I}_{31}^{P},\overline{F}_{31}^{P}\right)\right\rangle P\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right] P\left[\begin{array}{cc} 2 & 3 \\ 1 & 3 \end{array}\right]$$

$$\leqslant \left| \begin{array}{cc} \left\langle\left(\underline{T}_{21}^{P},\underline{I}_{21}^{P},\underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P},\overline{I}_{21}^{P},\overline{F}_{21}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{23}^{P},\underline{I}_{23}^{P},\underline{F}_{23}^{P}\right);\left(\overline{T}_{23}^{P},\overline{I}_{23}^{P},\overline{F}_{23}^{P}\right)\right\rangle \\ \left\langle\left(\underline{T}_{31}^{P},\underline{I}_{31}^{P},\underline{F}_{31}^{P}\right);\left(\overline{T}_{31}^{P},\overline{I}_{31}^{P},\overline{F}_{31}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{33}^{P},\underline{I}_{33}^{P},\underline{F}_{33}^{P}\right);\left(\overline{T}_{33}^{P},\overline{I}_{33}^{P},\overline{F}_{33}^{P}\right)\right\rangle \end{array} \right| P\left[\begin{array}{cc} 2 & 3 \\ 1 & 3 \end{array}\right]$$

$$+ \left| \begin{array}{cc} \left\langle\left(\underline{T}_{21}^{P},\underline{I}_{21}^{P},\underline{F}_{21}^{P}\right);\left(\overline{T}_{21}^{P},\overline{I}_{21}^{P},\overline{F}_{21}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{22}^{P},\underline{I}_{22}^{P},\underline{F}_{22}^{P}\right);\left(\overline{T}_{22}^{P},\overline{I}_{22}^{P},\overline{F}_{22}^{P}\right)\right\rangle \\ \left\langle\left(\underline{T}_{31}^{P},\underline{I}_{31}^{P},\underline{F}_{31}^{P}\right);\left(\overline{T}_{31}^{P},\overline{I}_{31}^{P},\overline{F}_{31}^{P}\right)\right\rangle & \left\langle\left(\underline{T}_{32}^{P},\underline{I}_{32}^{P},\underline{F}_{32}^{P}\right);\left(\overline{T}_{32}^{P},\overline{I}_{32}^{P},\overline{F}_{32}^{P}\right)\right\rangle \end{array} \right| P\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right]$$

$$\leqslant |P| + |P| = |P|$$

Case 2: We consider $\vartheta\begin{bmatrix} n-1 & n \\ 1 & 2 \end{bmatrix}$

$$
\begin{aligned}
&= \begin{bmatrix} \left\langle \left(\underline{T}^P_{2n-1}, \underline{I}^P_{2n-1}, \underline{F}^P_{2n-1}\right); \left(\overline{T}^P_{2n-1}, \overline{I}^P_{2n-1}, \overline{F}^P_{2n-1}\right)\right\rangle \left\langle \left(\underline{T}^P_{32}, \underline{I}^P_{32}, \underline{F}^P_{32}\right); \left(\overline{T}^P_{32}, \overline{I}^P_{32}, \overline{F}^P_{32}\right)\right\rangle \\ + \left\langle \left(\underline{T}^P_{2n}, \underline{I}^P_{2n}, \underline{F}^P_{2n}\right); \left(\overline{T}^P_{2n}, \overline{I}^P_{2n}, \overline{F}^P_{2n}\right)\right\rangle \left\langle \left(\underline{T}^P_{31}, \underline{I}^P_{31}, \underline{F}^P_{31}\right); \left(\overline{T}^P_{31}, \overline{I}^P_{31}, \overline{F}^P_{31}\right)\right\rangle \end{bmatrix} \\
&\quad \cdot P\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} P\begin{bmatrix} 2 & 3 \\ n-1 & n \end{bmatrix} \\
&= \left\langle \left(\underline{T}^P_{2n-1}, \underline{I}^P_{2n-1}, \underline{F}^P_{2n-1}\right); \left(\overline{T}^P_{2n-1}, \overline{I}^P_{2n-1}, \overline{F}^P_{2n-1}\right)\right\rangle \left\langle \left(\underline{T}^P_{32}, \underline{I}^P_{32}, \underline{F}^P_{32}\right); \left(\overline{T}^P_{32}, \overline{I}^P_{32}, \overline{F}^P_{32}\right)\right\rangle \\
&\quad \cdot P\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} P\begin{bmatrix} 2 & 3 \\ n-1 & n \end{bmatrix} \\
&+ \left\langle \left(\underline{T}^P_{2n}, \underline{I}^P_{2n}, \underline{F}^P_{2n}\right); \left(\overline{T}^P_{2n}, \overline{I}^P_{2n}, \overline{F}^P_{2n}\right)\right\rangle \left\langle \left(\underline{T}^P_{31}, \underline{I}^P_{31}, \underline{F}^P_{31}\right); \left(\overline{T}^P_{31}, \overline{I}^P_{31}, \overline{F}^P_{31}\right)\right\rangle \\
&\quad \cdot P\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} P\begin{bmatrix} 2 & 3 \\ n-1 & n \end{bmatrix} \\
&\leqslant \begin{vmatrix} \left\langle \left(\underline{T}^P_{2n-1}, \underline{I}^P_{2n-1}, \underline{F}^P_{2n-1}\right); \left(\overline{T}^P_{2n-1}, \overline{I}^P_{2n-1}, \overline{F}^P_{2n-1}\right)\right\rangle & \left\langle \left(\underline{T}^P_{22}, \underline{I}^P_{22}, \underline{F}^P_{22}\right); \left(\overline{T}^P_{22}, \overline{I}^P_{22}, \overline{F}^P_{23}\right)\right\rangle \\ \left\langle \left(\underline{T}^P_{3n-1}, \underline{I}^P_{3n-1}, \underline{F}^P_{3n-1}\right); \left(\overline{T}^P_{3n-1}, \overline{I}^P_{3n-1}, \overline{F}^P_{3n-1}\right)\right\rangle & \left\langle \left(\underline{T}^P_{32}, \underline{I}^P_{32}, \underline{F}^P_{32}\right); \left(\overline{T}^P_{32}, \overline{I}^P_{32}, \overline{F}^P_{32}\right)\right\rangle \end{vmatrix} \\
&\quad \cdot P\begin{bmatrix} 2 & 3 \\ n-1 & n \end{bmatrix} \\
&+ \begin{vmatrix} \left\langle \left(\underline{T}^P_{21}, \underline{I}^P_{21}, \underline{F}^P_{21}\right); \left(\overline{T}^P_{21}, \overline{I}^P_{21}, \overline{F}^P_{21}\right)\right\rangle & \left\langle \left(\underline{T}^P_{2n}, \underline{I}^P_{2n}, \underline{F}^P_{2n}\right); \left(\overline{T}^P_{2n}, \overline{I}^P_{2n}, \overline{F}^P_{2n}\right)\right\rangle \\ \left\langle \left(\underline{T}^P_{31}, \underline{I}^P_{31}, \underline{F}^P_{31}\right); \left(\overline{T}^P_{31}, \overline{I}^P_{31}, \overline{F}^P_{31}\right)\right\rangle & \left\langle \left(\underline{T}^P_{3n}, \underline{I}^P_{3n}, \underline{F}^P_{3n}\right); \left(\overline{T}^P_{3n}, \overline{I}^P_{3n}, \overline{F}^P_{3n}\right)\right\rangle \end{vmatrix} \\
&\quad \cdot P\begin{bmatrix} 2 & 3 \\ n-1 & n \end{bmatrix} \\
&\leqslant |P| + |P| = |P|
\end{aligned}
$$

Hence (ii) proved. Similarly, we can prove (iii)

Example 4.2

Let us consider the NHSRM P of order 2×2 defined as

$$P = \begin{bmatrix} \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1)\rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7)\rangle \\ \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2)\rangle & \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2)\rangle \end{bmatrix}$$

Then the matrix

$$P(2 \Rightarrow 1) = \begin{bmatrix} \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1)\rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7)\rangle \\ \langle (0.5, 0.6, 0.3); (0.3, 0.8, 0.1)\rangle & \langle (0.1, 0.3, 0.2); (0.4, 0.5, 0.7)\rangle \end{bmatrix}$$

and

$$P(1 \Rightarrow 2) = \begin{bmatrix} \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2)\rangle & \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2)\rangle \\ \langle (0.7, 0.4, 0.9); (0.4, 0.7, 0.2)\rangle & \langle (0.8, 0.2, 0.7); (0.6, 0.3, 0.2)\rangle \end{bmatrix}$$

$$|P| = \langle (0.5, 0.3, 0.7); (0.4, 0.5, 0.2)\rangle$$

$$\therefore |P(2 \Rightarrow 1)|\ |P(1 \Rightarrow 2)| = \langle (0.1, 0.2, 0.9); (0.3, 0.5, 0.7)\rangle \leqslant |P|$$

Theorem 4.6

Let P, Q and R be three NHSRMs then the following results hold:

(i) $\begin{vmatrix} P & R \\ O & Q \end{vmatrix} = |P| \cdot |Q|$where $O = [\langle (0,0,1)\,;(0,0,1)\rangle]$ is the zero NHSRM of order $n \times n$.

(ii) $\left|PP^T\right| \geqslant |P|$

Proof

(i) Suppose

$$\begin{bmatrix} P & R \\ O & Q \end{bmatrix} = \left[\left\langle \left(\underline{T}^S_{ij}, \underline{I}^S_{ij}, \underline{F}^S_{ij}\right); \left(\overline{T}^S_{ij}, \overline{I}^S_{ij}, \overline{F}^S_{ij}\right)\right\rangle\right]$$

Taking determinant on both sides, we get

$$\begin{vmatrix} P & R \\ O & Q \end{vmatrix} = \sum_{\sigma \in S_{2n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\sigma(1)}, \underline{I}^S_{1\sigma(1)}, \underline{F}^S_{1\sigma(1)}\right); \left(\overline{T}^S_{1\sigma(1)}, \overline{I}^S_{1\sigma(1)}, \overline{F}^S_{1\sigma(1)}\right)\right\rangle \ldots \\ \left\langle \left(\underline{T}^S_{2n\sigma(2n)}, \underline{I}^S_{2n\sigma(2n)}, \underline{F}^S_{2n\sigma(2n)}\right); \left(\overline{T}^S_{2n\sigma(2n)}, \overline{I}^S_{2n\sigma(2n)}, \overline{F}^S_{2n\sigma(2n)}\right)\right\rangle \end{bmatrix}$$

$$= \sum_{\substack{\sigma \in S_{2n} \\ \sigma(i) \leqslant n;\, i \leqslant n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\sigma(1)}, \underline{I}^S_{1\sigma(1)}, \underline{F}^S_{1\sigma(1)}\right); \left(\overline{T}^S_{1\sigma(1)}, \overline{I}^S_{1\sigma(1)}, \overline{F}^S_{1\sigma(1)}\right)\right\rangle \ldots \\ \left\langle \left(\underline{T}^S_{2n\sigma(2n)}, \underline{I}^S_{2n\sigma(2n)}, \underline{F}^S_{2n\sigma(2n)}\right); \left(\overline{T}^S_{2n\sigma(2n)}, \overline{I}^S_{2n\sigma(2n)}, \overline{F}^S_{2n\sigma(2n)}\right)\right\rangle \end{bmatrix}$$
$$+ \sum_{\substack{\sigma \in S_{2n} \\ \sigma(i) \leqslant n;\, i > n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\sigma(1)}, \underline{I}^S_{1\sigma(1)}, \underline{F}^S_{1\sigma(1)}\right); \left(\overline{T}^S_{1\sigma(1)}, \overline{I}^S_{1\sigma(1)}, \overline{F}^S_{1\sigma(1)}\right)\right\rangle \ldots \\ \left\langle \left(\underline{T}^S_{2n\sigma(2n)}, \underline{I}^S_{2n\sigma(2n)}, \underline{F}^S_{2n\sigma(2n)}\right); \left(\overline{T}^S_{2n\sigma(2n)}, \overline{I}^S_{2n\sigma(2n)}, \overline{F}^S_{2n\sigma(2n)}\right)\right\rangle \end{bmatrix}$$

Since for any permutation $\sigma \in S_n$ such that $\exists i > n; \sigma(i) \leqslant n$ then

$$\left[\left\langle \left(\underline{T}^S_{i\sigma(i)}, \underline{I}^S_{i\sigma(i)}, \underline{F}^S_{i\sigma(i)}\right); \left(\overline{T}^S_{i\sigma(i)}, \overline{I}^S_{i\sigma(i)}, \overline{F}^S_{i\sigma(i)}\right)\right\rangle\right] = [\langle (0,0,1)\,;(0,0,1)\rangle]$$

$$\begin{vmatrix} P & R \\ O & Q \end{vmatrix} = \sum_{\substack{\sigma \in S_{2n} \\ \sigma(i) \leqslant n;\, i \leqslant n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\sigma(1)}, \underline{I}^S_{1\sigma(1)}, \underline{F}^S_{1\sigma(1)}\right); \left(\overline{T}^S_{1\sigma(1)}, \overline{I}^S_{1\sigma(1)}, \overline{F}^S_{1\sigma(1)}\right)\right\rangle \ldots \\ \left\langle \left(\underline{T}^S_{2n\sigma(2n)}, \underline{I}^S_{2n\sigma(2n)}, \underline{F}^S_{2n\sigma(2n)}\right); \left(\overline{T}^S_{2n\sigma(2n)}, \overline{I}^S_{2n\sigma(2n)}, \overline{F}^S_{2n\sigma(2n)}\right)\right\rangle \end{bmatrix}$$
$$+ \langle (0,0,1)\,;(0,0,1)\rangle$$
$$= \sum_{\substack{\sigma \in S_n \\ \sigma(i) = \sigma(i);\, i \leqslant n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\sigma(1)}, \underline{I}^S_{1\sigma(1)}, \underline{F}^S_{1\sigma(1)}\right); \left(\overline{T}^S_{1\sigma(1)}, \overline{I}^S_{1\sigma(1)}, \overline{F}^S_{1\sigma(1)}\right)\right\rangle \ldots \\ \left(\underline{T}^S_{n\sigma(n)}, \underline{I}^S_{n\sigma(n)}, \underline{F}^S_{n\sigma(n)}\right); \left(\overline{T}^S_{n\sigma(n)}, \overline{I}^S_{n\sigma(n)}, \overline{F}^S_{n\sigma(n)}\right) \end{bmatrix}$$
$$\cdot \sum_{\substack{\sigma \in S_n \\ \beta(i) = \sigma(i+n);\, i \leqslant n}} \begin{bmatrix} \left\langle \left(\underline{T}^S_{1\beta(1)}, \underline{I}^S_{1\beta(1)}, \underline{F}^S_{1\beta(1)}\right); \left(\overline{T}^S_{1\beta(1)}, \overline{I}^S_{1\beta(1)}, \overline{F}^S_{1\beta(1)}\right)\right\rangle \ldots \\ \left\langle \left(\underline{T}^S_{n\beta(n)}, \underline{I}^S_{n\beta(n)}, \underline{F}^S_{n\beta(n)}\right); \left(\overline{T}^S_{n\beta(n)}, \overline{I}^S_{n\beta(n)}, \overline{F}^S_{n\beta(n)}\right)\right\rangle \end{bmatrix}$$
$$= |P| \cdot |Q|$$

(ii) Let $PP^T = \left[\left\langle \left(\underline{T}_{ij}^{PP^T}, \underline{I}_{ij}^{PP^T}, \underline{F}_{ij}^{PP^T}\right); \left(\overline{T}_{ij}^{PP^T}, \overline{I}_{ij}^{PP^T}, \overline{F}_{ij}^{PP^T}\right)\right\rangle\right]$

$$= \begin{bmatrix} \left(\sum_{k=1}^{n} \underline{T}_{ik}^{P}.\underline{T}_{kj}^{P^T}, \sum_{k=1}^{n} \underline{I}_{ik}^{P}.\underline{I}_{kj}^{P^T}, \prod_{k=1}^{n} \underline{F}_{ik}^{P} + \underline{F}_{kj}^{P^T}\right); \\ \left(\sum_{k=1}^{n} \overline{T}_{ik}^{P}.\overline{T}_{kj}^{P^T}, \sum_{k=1}^{n} \overline{I}_{ik}^{P}.\overline{I}_{kj}^{P^T}, \prod_{k=1}^{n} \overline{F}_{ik}^{P} + \overline{F}_{kj}^{P^T}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\sum_{k=1}^{n} \underline{T}_{ik}^{P}.\underline{T}_{jk}^{P}, \sum_{k=1}^{n} \underline{I}_{ik}^{P}.\underline{I}_{jk}^{P}, \prod_{k=1}^{n} \underline{F}_{ik}^{P} + \underline{F}_{jk}^{P}\right); \\ \left(\sum_{k=1}^{n} \overline{T}_{ik}^{P}.\overline{T}_{jk}^{P}, \sum_{k=1}^{n} \overline{I}_{ik}^{P}.\overline{I}_{jk}^{P}, \prod_{k=1}^{n} \overline{F}_{ik}^{P} + \overline{F}_{jk}^{P}\right) \end{bmatrix}, \forall i, j \in \{1, 2, ..., n\}$$

If i = j, we see that $\left[\left\langle \left(\underline{T}_{ii}^{PP^T}, \underline{I}_{ii}^{PP^T}, \underline{F}_{ii}^{PP^T}\right); \left(\overline{T}_{ii}^{PP^T}, \overline{I}_{ii}^{PP^T}, \overline{F}_{ii}^{PP^T}\right)\right\rangle\right]$

$$= \left[\left\langle \left(\sum_{k=1}^{n} \underline{T}_{ik}^{P}, \sum_{k=1}^{n} \underline{I}_{ik}^{P}, \prod_{k=1}^{n} \underline{F}_{ik}^{P}\right); \left(\sum_{k=1}^{n} \overline{T}_{ik}^{P}, \sum_{k=1}^{n} \overline{I}_{ik}^{P}, \prod_{k=1}^{n} \overline{F}_{ik}^{P}\right)\right\rangle\right]$$

$$= \sum_{k=1}^{n} \left[\left\langle \left(\underline{T}_{ik}^{P}, \underline{I}_{ik}^{P}, \underline{F}_{ik}^{P}\right); \left(\overline{T}_{ik}^{P}, \overline{I}_{ik}^{P}, \overline{F}_{ik}^{P}\right)\right\rangle\right]$$

For any permutation $\sigma \in S_n$, we get

$$\geqslant \left[\left\langle \left(\underline{T}_{i\sigma(i)}^{P}, \underline{I}_{i\sigma(i)}^{P}, \underline{F}_{i\sigma(i)}^{P}\right); \left(\overline{T}_{i\sigma(i)}^{P}, \overline{I}_{i\sigma(i)}^{P}, \overline{F}_{i\sigma(i)}^{P}\right)\right\rangle\right]$$

$$\left|PP^T\right| \geqslant \sum_{\sigma \in S_n} \begin{bmatrix} \left\langle \left(\underline{T}_{1\sigma(1)}^{P}, \underline{I}_{1\sigma(1)}^{P}, \underline{F}_{1\sigma(1)}^{P}\right); \left(\overline{T}_{1\sigma(1)}^{P}, \overline{I}_{1\sigma(1)}^{P}, \overline{F}_{1\sigma(1)}^{P}\right)\right\rangle \cdots \\ \left\langle \left(\underline{T}_{n\sigma(n)}^{P}, \underline{I}_{n\sigma(n)}^{P}, \underline{F}_{n\sigma(n)}^{P}\right); \left(\overline{T}_{n\sigma(n)}^{P}, \overline{I}_{n\sigma(n)}^{P}, \overline{F}_{n\sigma(n)}^{P}\right)\right\rangle \end{bmatrix} \geqslant |P|$$

Hence the theorem.

5. Determinant-Based Multi-Criteria Decision Making Using NHSRM

This NHSRM provides a foundation for applying advanced decision-making techniques such as score functions, aggregation operators, or determinant-based analysis in the context of conflict resolution among sensors or ranking possible alternative under uncertainty.

Definition 5.1. (Score Function)
The score function for a single Neutrosophic number $A_i = (T_{ij}, I_{ij}, F_{ij})$ is defined as

$$S_{ij}(A_i) = T_{ij} + (1 - I_{ij}) - F_{ij}$$

Definition 5.2. (Neutrosophic Score Function)
The score function $S(A_i)$ of an alternative A_i is a quantitative measure obtained by aggregating the weighted average values of truth, indeterminacy and falsity corresponding to all attributes:

$$S_{ij}(A_i) = w_T\tilde{T}(A_i) - w_I\tilde{I}(A_i) - w_F\tilde{F}(A_i)$$

where w_T, w_I, w_F are the respective weights for truth, indeterminacy and falsity. $\tilde{T}(A_i), \tilde{I}(A_i), \tilde{F}(A_i)$ indicates the mean values computed from the lower and upper bounds of the T, I, and F components across every attribute for the alternative A_i.

Definition 5.3. (Euclidean Norm)
The Euclidean Norm is a measure of the length (or magnitude) of a vector in Euclidean space. For a vector $v = (x_1, x_2, ..., x_n)$, the Euclidean norm is defined as $||v|| = \sqrt{\sum x_i^2}$

Definition 5.4. (Laplace expansion for determinant of NHSRM)
Let $P = \left[\left\langle \left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\overline{T}^P_{ij}, \overline{I}^P_{ij}, \overline{F}^P_{ij}\right)\right\rangle\right]$be a NHSRM of order $n \times n$, each element $\left\langle \left(\underline{T}^P_{ij}, \underline{I}^P_{ij}, \underline{F}^P_{ij}\right); \left(\overline{T}^P_{ij}, \overline{I}^P_{ij}, \overline{F}^P_{ij}\right)\right\rangle$ is transformed into scalar element $S_{ij}(A_i)$, then $|P|$ is defined as $|P| = \sum_{j=1}^{n} (-1)^{i+j} S_{ij}(A_i)|M_{ij}|$ where $S_{ij}(A_i)$is the scalar element and M_{ij} is the minor matrix of P.

5.1. *Interpretation of the Determinant*

1. Non-Zero Determinant $\det(M) \neq 0$
   Indicates that the matrix is non-singular, and the input from each attribute contributes linearly independent information. In decision-making terms, this confirms that no attribute input is redundant when considering all alternatives.
2. Negative Determinant $\det(M) < 0$
   A negative sign often reflects conflict or contradiction in the system. For example, while one attribute may strongly support alternative, another may oppose it. This discordance is echoed in the determinant, highlighting areas where conflicting evidence needs resolution.
3. Magnitude of the Determinant $\det(M) > 0$
   The relatively small magnitude of the determinant (close to zero) suggests that while the data is independent, it is also near-singular, meaning the system could be sensitive to small changes in attribute scores, a condition to be carefully monitored in real-time decision systems.

5.2. *Algorithm for Score – Based Evaluation*

This approach evaluates each alternative by assigning scores based on its performance across multiple parameters, enabling a systematic and structured comparison.

1. Construct NHSRM.
2. Calculate aggregated score for all alternatives.
3. Find the score value using the score function by Definition 5.1.
4. Select the optimal alternatives based on the ranking results.

5.3. *Algorithm for Determinant – Based Evaluation*

This method applies the mathematical concept of determinants to decision matrices, enabling objective ranking of alternatives by capturing the influence and interactions of various criteria.

1. Construct NHSRM.
2. Calculate average bounds $\tilde{T}_{ij} = \frac{\underline{T}_{ij} + \overline{T}_{ij}}{2}, \frac{\underline{I}_{ij} + \overline{I}_{ij}}{2}, \frac{\underline{F}_{ij} + \overline{F}_{ij}}{2}$
3. Apply Neutrosophic score function of each element using Definition 5.2 to construct the Neutrosophic score matrix, where $w_T = 1.0$, $w_I = 0.5$, $w_F = 0.2$ (assumed).
4. Find the determinant value and compare the interpretation of the determinant.
5. Compute the Euclidean norm

6. Fix the rank by their norms.

5.4. *Application of NHSRM based on Determinant theory*

In this section, we present a practical implementation of a NHSRM applied to an autonomous driving scenario.
Autonomous vehicles operate in dynamic and often unpredictable environments, requiring them to make rapid and accurate decisions to ensure safety and efficiency. These decisions are based on continuous input from multiple sensors such as LIDAR, radar, cameras, and ultrasonic devices, each providing critical information about the vehicle's surroundings. However, sensor data can be affected by various factors like noise, occlusion, weather conditions, or conflicting readings. As a result, robust decision-making mechanisms are essential to interpret this complex data and ensure safe and efficient vehicle operation.
By utilizing determinant theory within the NHSRM structure, this approach enables the evaluation of multiple driving actions based on input from various sensors. Each sensor contributes neutrosophic values capturing degrees of truth, indeterminacy, and falsity for different driving alternatives such as braking, slowing down, continuing cautiously, or accelerating. The NHSRM captures these evaluations as neutrosophic approximations, accounting for both lower and upper bounds derived from expert or real-time assessments. The determinant-based analysis allows for the structured comparison of these actions by assessing the linear independence and influence of each sensor's input. This enables the autonomous vehicle to effectively prioritize alternatives and select the most appropriate course of action, even in the presence of uncertainty, imprecision, and conflicting sensor data. The objective is to assist a self-driving vehicle in selecting an appropriate driving action based on sensory input from four distinct sensors:

1. LIDAR
2. Camera
3. Radar
4. Ultrasonic

The vehicle must choose among four possible actions:

1. A1: Brake immediately
2. A2: Slow down
3. A3: Continue cautiously
4. A4: Accelerate

By applying a determinant-based algorithm along with score-based aggregation, this study demonstrates how the NHSRM model can effectively process sensor data to rank alternatives and support real-time decision-making. This example highlights the practical benefits of integrating NHSRM and determinant theory in intelligent control systems, ensuring robustness against data inconsistency and environmental unpredictability.

5.4.1. ***Score − Based Evaluation***

Step 1 Construct NHSRM where each sensor evaluates the available actions based on three components

- Truth (T): Support for the action
- Indeterminacy (I): Uncertainty
- Falsity (F): Opposition to the action

Sensor	**A1 (Brake)**	**A2 (Slow)**	**A3 (Continue)**	**A4 (Accelerate)**
LIDAR	(0.7, 0.2, 0.1);(0.9, 0.1, 0.0)	(0.5, 0.3, 0.2);(0.7, 0.2, 0.1)	(0.2, 0.2, 0.6);(0.4, 0.3, 0.3)	(0.0, 0.1, 0.9);(0.1, 0.2, 0.7)
Camera	(0.1, 0.3, 0.6);(0.3, 0.2, 0.5)	(0.4, 0.4, 0.2);(0.6, 0.3, 0.1)	(0.6, 0.3, 0.1);(0.8, 0.2, 0.0)	(0.0, 0.3, 0.7);(0.1, 0.4, 0.5)
Radar	(0.6, 0.3, 0.1);(0.8, 0.2, 0.0)	(0.3, 0.5, 0.2);(0.5, 0.4, 0.1)	(0.3, 0.4, 0.3);(0.5, 0.3, 0.2)	(0.1, 0.6, 0.3);(0.3, 0.5, 0.2)
Ultrasonic	(0.5, 0.4, 0.1);(0.7, 0.3, 0.0)	(0.4, 0.3, 0.3);(0.6, 0.2, 0.2)	(0.2, 0.5, 0.3);(0.4, 0.4, 0.2)	(0.0, 0.4, 0.6);(0.2, 0.5, 0.3)

Table 5.1 NHSRM for Sensor-Based Evaluation

Step 2 Calculate aggregated score for all actions

Sensor	**A1 (Brake)**	**A2 (Slow)**	**A3 (Continue)**	**A4 (Accelerate)**
LIDAR	(0.8, 0.15, 0.05)	(0.6, 0.25, 0.15)	(0.3, 0.25, 0.45)	(0.05, 0.15, 0.8)
Camera	(0.2, 0.25, 0.55)	(0.5, 0.35, 0.15)	(0.7, 0.25, 0.05)	(0.05, 0.35, 0.6)
Radar	(0.7, 0.25, 0.05)	(0.4, 0.45, 0.15)	(0.4, 0.35,0.25)	(0.2, 0.55, 0.25)
Ultrasonic	(0.6, 0.35, 0.05)	(0.5, 0.25, 0.25)	(0.3, 0.45, 0.25)	(0.1, 0.45, 0.45)
Average	**(0.575, 0.25, 0.175)**	**(0.5, 0.325, 0.2)**	**(0.425, 0.325, 0.25)**	**(0.1, 0.375, 0.525)**

Table 5.2 Aggregated Score

Step 3 Find the score value using the score function by Definition 5.1

Action	**Score Value**
A1 (Brake)	1.15
A2 (Slow)	0.975
A3 (Continue)	0.85
A4 (Accelerate)	0.20

Table 5.3 Score value

Step 4 Select the optimal action based on the ranking results

Action	**Score Value**	**Rank**
A1 (Brake)	1.15	1
A2 (Slow)	0.975	2
A3 (Continue)	0.85	3
A4 (Accelerate)	0.20	4

Table 5.4 Ranking

5.4.2. ***Determinant – Based Evaluation***

Step 1 Construct NHSRM in Table 5.1

Step 2 Calculate average bounds for all actions

Sensor	A1 (Brake)	A2 (Slow)	A3 (Continue)	A4 (Accelerate)
LIDAR	(0.8, 0.15, 0.05)	(0.6, 0.25, 0.15)	(0.3, 0.25, 0.45)	(0.05, 0.15, 0.8)
Camera	(0.2, 0.25, 0.55)	(0.5, 0.35, 0.15)	(0.7, 0.25, 0.05)	(0.05, 0.35, 0.6)
Radar	(0.7, 0.25, 0.05)	(0.4, 0.45, 0.15)	(0.4, 0.35,0.25)	(0.2, 0.55, 0.25)
Ultrasonic	(0.6, 0.35, 0.05)	(0.5, 0.25, 0.25)	(0.3, 0.45, 0.25)	(0.1, 0.45, 0.45)

Table 5.5 Average bounds for all actions

Step 3 We apply Neutrosophic score function using definition 5.2, to each element in Table 5.5

$$NS_{ij}(A_i) = \begin{bmatrix} 0.715 & 0.475 & 0.145 & -0.185 \\ 0.065 & 0.395 & 0.565 & -0.155 \\ 0.615 & 0.295 & 0.245 & 0.015 \\ 0.485 & 0.375 & 0.085 & -0.065 \end{bmatrix}$$

Step 4 Find the determinant value and compare the interpretation of the determinant

$$\det(NS_{ij}(A_i)) = 0.715\,.\det(NS_{11}(A_i)) - 0.445.\det(NS_{12}(A_i)) + 0.085.\det(NS_{13}(A_i))$$
$$-(-0.185).\det(NS_{14}(A_i)) = 0.0122$$

Interpretation: The positive determinant indicates that the system of sensor scores is linearly independent and has unique solution.

Step 5 Compute the Euclidean Norm for the matrix obtained in Step 3.

Action	Euclidean Norm
A1 (Brake)	1.062
A2 (Slow)	0.781
A3 (Continue)	0.638
A4 (Accelerate)	0.249

Table 5.6 Euclidean Norm

Step 6 Fix the rank by their norms

Action	Euclidean Norm	Rank
A1 (Brake)	1.062	1
A2 (Slow)	0.781	2
A3 (Continue)	0.638	3
A4 (Accelerate)	0.249	4

Table 5.7 Ranking

5.5. *Comparative Analysis*

A comparative analysis of the score-based algorithm and the determinant-based algorithm within the context of NHSRM reveals key differences in their computational approach and applicability. The score-based algorithm is simple, fast, and suitable for real-time decision-making, as it directly computes scalar scores from truth, indeterminacy, and falsity values. In contrast, the determinant-based algorithm captures deeper interdependencies among attributes through matrix structure and recursive evaluation, making it more robust in detecting contradictions and systemic uncertainty. The score-based algorithm suits smaller datasets or independent attributes but may overlook hidden conflicts when scores are uniformly high. In contrast, the determinant-based algorithm captures structural dependencies and is better suited for complex decisions with interacting, uncertain criteria. Together, both methods offer complementary perspectives, the score-based algorithm emphasizes efficiency and simplicity, while the determinant-based algorithm provides depth and structural insight.

Action	**Score-Based Evaluation**	**Determinant-Based Evaluation**	**Ranking**	**Interpretation**
A1 (Brake)	1.15	1.062	1	Strongest and most consistent support
A2 (Slow Down)	0.975	0.781	2	Good option with moderate support
A3 (Continue)	0.85	0.638	3	Weak but acceptable support
A4 (Accelerate)	0.20	0.249	4	Low support, avoid this action

Table 5.8 Comparison of Score-Based and Determinant-Based Evaluation

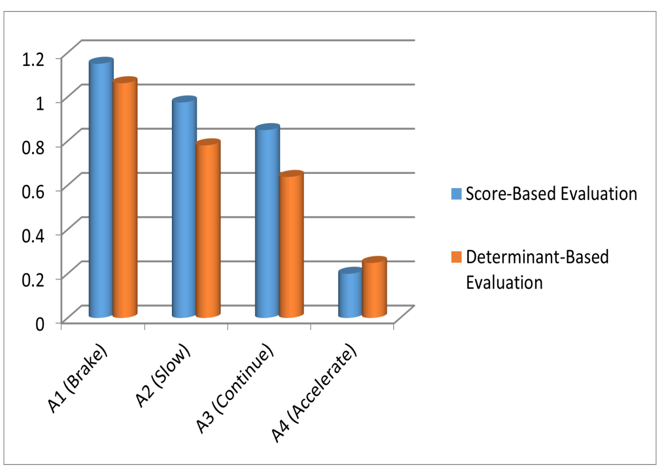

Figure 5.1 Comparative Analysis

5.6. *Sensitivity Analysis*

The sensitivity analysis investigates to changes in the weight parameters w_T, w_I and w_F affect the final action scores and their respective rankings. Increasing indeterminacy or falsity weights tends to benefit A4 while penalizing A2. A1 remains stable in most cases but can lose its top position when falsity is emphasized. A2 consistently drops in rank with higher uncertainty or reduced emphasis on truth. Overall, rankings are most sensitive to falsity weight, followed by indeterminacy, with truth weight having a stabilizing effect. The line graph in Figure 6.1 and the spider graph in Figure 6.2 illustrate the variation in action scores across different scenarios, highlighting the stability of A1 and the sensitivity of A4 to changes in weight parameters, thereby indicating its responsiveness to uncertainty.

Scenario	w_T	w_I	w_F	**A1 Score**	**A2 Score**	**A3 Score**	**A4 Score**	**Ranking**
Actual	1.0	0.5	0.2	1.062	0.781	0.638	0.249	$A1 > A2 > A3 > A4$
S1	1.0	0.8	0.2	0.89	0.49	0.51	0.62	$A1 > A4 > A3 > A2$
S2	0.9	0.5	0.8	0.91	0.36	0.54	1.1	$A4 > A1 > A3 > A2$
S3	0.7	0.5	0.4	0.65	0.29	0.37	0.66	$A4 > A1 > A3 > A2$

Table 6.1 Weight Variations

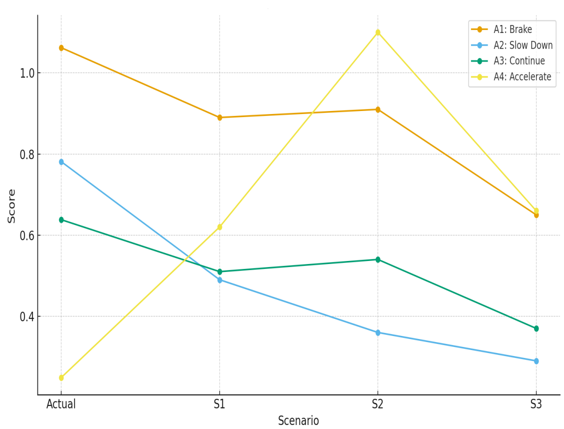

Figure 6.1 Line Graph of Action Scores

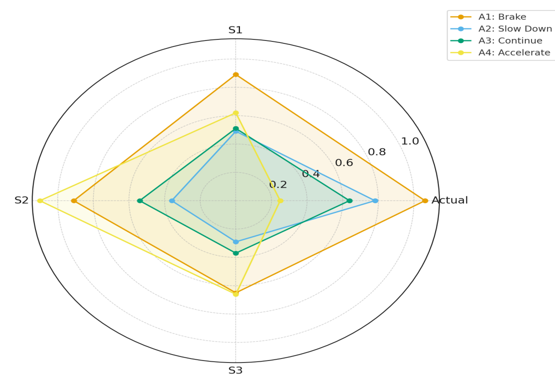

Figure 6.2 Spider Graph of Action Scores

6. **Result and Discussion**

Based on both the score-based evaluation and the determinant-based analysis, Action A1: Brake is identified as the most favorable decision under the current operational conditions. This action achieves the highest score and exhibits strong support across multiple evaluation dimensions. It is followed in preference by A2: Slow Down and A3: Continue, which also demonstrate moderate levels of support and scoring consistency.

In contrast, A4: Accelerate ranks the lowest among all available actions. Its comparatively low score and minimal determinant-based support indicate that it is the least suitable choice in the present scenario. These findings suggest that accelerating is unsuitable in the present context and could compromise the effectiveness or safety of the system. A comprehensive comparison of the actions is provided in Table 5.8 and the evaluation outcomes are visually represented in Figure 5.1. A brief sensitivity analysis shows that A1 remains stable under weight variations, confirming its robustness. A4 improves when indeterminacy or falsity is emphasized, while A2 declines, indicating sensitivity to uncertainty.

7. Conclusion and Future Scope

This paper proposes a novel integration of determinant theory into the NHSRM framework to enhance decision-making capabilities in environments characterized by uncertainty, imprecision, and incomplete information. We formally defined the determinant of NHSRMs, examined its algebraic properties, and established key theorems to support its theoretical foundation. Two decision-making approaches a score-based algorithm and a determinant-based algorithm were developed and comparatively analyzed. Through a detailed illustrative example, we demonstrated how neutrosophic triplets can be transformed into scalar values, facilitating determinant computation for the ranking of alternatives and assessing parameter influence in complex decision-making scenarios.

This work lays a foundation for future research, particularly in incorporating matrix inversion and eigenvalue analysis, which are crucial for algorithmic implementations in fields like automated control systems, intelligent diagnostics and decision-support architectures.

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