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Cubic Spherical Neutrosophic Weighted Geometric Heronian Mean Operator for MCDM with Application to Fertilizer Selection

S. Gomathi[1], A. Kalamani[2], L. Senthil Kumar[3] and Rahul Shukla[4*]

[1] Department of Mathematics, Rathinam Technical Campus, Coimbatore, India; gomathiprakash2013@gmail.com
[2] Department of Mathematics, Karpagam College of Engineering, Coimbatore, India; 20kalamani@gmail.com
[3] Department of Mathematics, Dr. Mahalingam College of Engineering and Technology, Pollachi, India; senthilmaths29@gmail.com
[4] Department of Mathematical Sciences & Computing, Walter Sisulu University, Mthatha Campus, South Africa.
* Correspondence: rshukla@wsu.ac.za

Abstract: This paper introduces the Cubic Spherical Neutrosophic Weighted Geometric Heronian Mean (CSNWGHM) operator to aggregating the Cubic Spherical Neutrosophic Set (CSNS) values. The fundamental properties of idempotency, monotonicity and boundedness are rigorously verified for this operator. This aggregation operator is designed to effectively handle uncertain or imprecise information, incorporating a radius parameter for enhanced flexibility. To demonstrate the practical application, this operator is integrated into Multi - Criteria Decision-Making (MCDM) methods. The Step – Wise Weight Assessment Ratio Analysis (SWARA) method is employed to calculate the weights of the criteria. Subsequently, the proposed aggregation operator, in conjunction with a score function is utilized to identify the best and worst alternatives. An illustrative example showcases the efficacy in selecting the most appropriate fertilizer for guava and banana trees, using propositions derived from Neutrosophic SuperHyperSoft Sets (NSHSSs). Furthermore, a sensitivity and comparative analysis is conducted to establish the stability of this approach and determine the perfect correlation factor by evaluating various parameter values.

Keywords: Neutrosophic set; Superhypersoft set; Cubic spherical neutrosophic set; Heronian Mean; MCDM.

1. Introduction

The MCDM methodology is used to facilitate decision-making through a simplified approach that incorporates the opinions of multiple experts. This methodology was first introduced by Triantaphyllou in 2000 [24]. It includes various techniques to support final decision-making. Since 2000 it helps to reach the final decisions in many situations. Our study, the SWARA method is employed to determine the weights of the criteria. It is one of the easy approach and getting the weights perfectly among others. This weight evaluate strategy was introduced by Keršuliene et al and it was initially applied to the selection of alternative dispute resolution methods in virtual environments. Since its introduction, the SWARA method applied in various domains to assign weights and determine the importance of each criterion. Recently, it has been used in contexts involving complex hesitant fuzzy sets [2] to selecting the best supplier for electronic goods, Fermatean sets[3] for selecting effective poverty alleviation strategies and single valued neutrosophic sets [18] for prioritizing renewable energy systems, spray painting robot selection [14] and etc.

Smarandache was firstly initiated the concept of neutrosophic logic and Neutrosophic Sets (NSs) [19] to describe uncertain information, inconsistent data, vagueness and indeterminate answers in various contexts. NSs are characterized by three components: truth, indeterminacy and falsity. In any situation of real time, we have little confusions, unknown the reality and unsure data to measure this state we use the different types NSs in any field. Since 1999, this set had been extended into various forms. Each and every set have uniqueness and different abilities. The concept of CSNS was introduced by Gomathi et al in 2024 [7], which incorporating cube and sphere representations. To finding the center of the NSs then calculating the distances from the center. Maximum distance is covering all points which taken the radius of the sphere. CSNS is an advanced extension of NSs, incorporating neutrosophic terms with a spherical radius. Recently, CSNSs has been applied in MCDM problems as electric truck selection using cosine similarity measures [13].

In 2025, Smarandache introduced the NSHSSs, which has gained popularity in handling multiple propositions based on Cartesian products. NSHSSs is the extended version of superhyper soft sets which mappings from Cartesian product of the power set to power set of the universal set. The neutrosophic characteristics (truth, indeterminacy, falsity) collaborating with superhyper soft sets which is called NSHSSs. The great feature of using NSHSSs, there is many possibilities to take multiple propositions and accuracy in final values. Recent research shows growing interest in the NSHSSs, with applications in various selections such as electric vehicle selection and evaluating the quality of cultural industry development in the digital economy.

Fertilizer is the chemical mixer or organic substance which gives the boosting energy of the trees/planets. It supports the growth, crop yield and supply essential nutrients. Selecting the right fertilizer is the one of the crucial task for the farmers. There are numerous organic and inorganic fertilizer in the agricultural environment. Usage of the wrong fertilizer leads, reduce the fruit cultivation, degrade the soil health and wastage of the farmers time and cost. To avoid like these impact, we need to find the appropriate fertilizer. Some fertilizer needs high water source, less sol impact and etc. According to the characteristic of the fertilizer as nutrient content, soil health impact, cost and effect, water allocation, environmental friendliness and yield of the specific tree. To convert this problem into a systematic approach, consider this characteristic are the criteria and fertilizer as the alternatives.

In this study, we take the CSNSs and introducing the aggregation operator called CSNWGHM. The fundamental properties also verified. To show the application of this method we give the illustrative example. To convert cubic spherical neutrosophic value into neutrosophic value using the proposed aggregation operator. To get the final value using the score value which given the accuracy in values. Ranking the alternatives (highest value is best).

The outline of the paper is as follows:

- The literature review section gives a detailed overview of relevant prior works.
- The preliminaries section presents the fundamental definitions used in this paper.
- The proposed work section introduces new aggregation operators along with their theorems and properties.
- The algorithm section outlines the procedure of the entire methodology.
- The illustrative example section shows the practical application of the proposed framework and includes sensitivity and comparative analysis.
- Finally, the conclusion section summarizes the results and highlights the proposed method.

Here is an acronym table listing the abbreviations along with their full forms.

Table 1. Acronym and expansions

Acronym	Expansion
MCDM	Multi-Criteria Decision-Making
HM	Heronian Mean
SWARA	Step-wise Weight Assessment Ratio Analysis
NS	Neutrosophic Set
NSHSS	Neutrosophic SuperHyperSoft Set
CSNS	Cube Spherical Neutrosophic Set
CSNWGHM	Cubic Spherical Neutrosophic Weighted Geometric Heronian Mean

1.1. Research gap

Many researchers have published their work using the Heronian Mean (HM) in the context of fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, NSs and single-valued neutrosophic sets. Some of these studies are discussed in the section 1.2. In the case of CSNS, certain extensions such as geometric and arithmetic cubic spherical neutrosophic aggregate operators have been developed. However, to the best of our knowledge there no existing study has applied the HM operator within the CSNS framework. To fill this gap, the present study introduces new aggregation operator is called CSNWGHM. This proposed operator is applied to a MCDM problem for aggregating decision values. Their application is demonstrated in the illustrative example section.

1.2.Literature review

Many types of mean operators are widely used in the mathematical field. This study, we focus on the HM operator [22] which was introduced by Sykora in 2009. In recent years, the HM has been extended to various aggregation operators and has found diverse applications. Originally, the HM was developed to identify interrelationships between two given arguments. Over time, it has become popular in MCDM problems to aggregate multiple values into a single representative term. It has been effectively applied within various environments including fuzzy sets, intuitionistic fuzzy sets, and NSs.

The following table summarizes recent applications of the HM in decision-making problems using various types of fuzzy and NSs.

Table 2. Application of HM in various sets

Author(s)	Set type	Application area
Smarandache et al. [20]	NSHSSs	Optimizing electric vehicle selection
Abdalla et al. [1]	Interval – valued Fermatean neutrosophic superhyper soft sets	Prioritization of patients awaiting organ transplantation
Ramasamy et al [17]	NSHSSs	Selecting optimal mobile phones
Smarandache et al. [21]	NSHSSs	Selecting projects with the highest potential for positive environmental

		impact
Kiruthika et al [12]	NSHSSs	Selecting an appropriate Q1 journal
Zhao [27]	Superhypersoft sets	Strategies selection in artificial intelligence
Wu et al [26]	Superhypersoft sets	Assessing the fitness management
Wang [25]	Neutrosophic cubic spherical fuzzy sets	Supply chains performance evaluation
Lu [16]	CSNSs	Communication effectiveness evaluation of intangible cultural heritage traditional music
Gomathi et al [9]	CSNSs	Green supplier selection
Krishnaprakash et al. [13]	CSNSs	Selection of electric truck
Gomathi et al [8]	CSNSs	Evaluating fertilizer brands for coconut farming
Tian et al [23]	Single - valued neutrosophic sets	Evaluating the quality in digital media art design powered by computational tools
Fan et al. [6]	Bipolar neutrosophic sets	Selecting the best car
Gulistan et al. [10]	Neutrosophic cubic sets	Choosing the best investment company
Fan & Ye [5]	Linguistic neutrosophic cubic sets	Selecting the design scheme
Li et al. [15]	Single - valued neutrosophic sets	Air quality evaluation
Zhong et al. [28]	Bipolar Pythagorean neutrosophic sets	Evaluating hotel management service quality through customer feedback

2. Preliminaries

Definition 2.1[7] Let $\underset{..}{\mathrm{U}}$ be universal set. The CSNS is defined as $\Delta=\{\langle \mathbb{x}, \breve{\boldsymbol{\delta}}(\mathbb{x}), \check{\boldsymbol{\tau}}(\mathbb{x}), \dot{\boldsymbol{\gamma}}(\mathbb{x}); \breve{\boldsymbol{\mu}}\rangle : \mathbb{x} \in \underset{..}{\mathrm{U}}\}$ where $\breve{\boldsymbol{\delta}}(\mathbb{x}), \check{\boldsymbol{\tau}}(\mathbb{x}), \dot{\boldsymbol{\gamma}}(\mathbb{x}) : \underset{..}{\mathrm{U}} \to [0,1]$ and Δ is the subset of $\underset{..}{\mathrm{U}}$. The sum of truth $(\breve{\boldsymbol{\delta}})$, indeterminacy $(\check{\boldsymbol{\tau}})$ and falsity $(\dot{\boldsymbol{\gamma}})$ is less than or equal to 3 and the radius $(\breve{\boldsymbol{\mu}})$ belongs to [0, 1].

Let $\{\langle \breve{\boldsymbol{\delta}}_{c,1}, \check{\boldsymbol{\tau}}_{c,1}, \dot{\boldsymbol{\gamma}}_{c,1}\rangle, \langle \breve{\boldsymbol{\delta}}_{c,2}, \check{\boldsymbol{\tau}}_{c,2}, \dot{\boldsymbol{\gamma}}_{c,2}\rangle, \dots, \langle \breve{\boldsymbol{\delta}}_{c,p_c}, \check{\boldsymbol{\tau}}_{c,p_c}, \dot{\boldsymbol{\gamma}}_{c,p_c}\rangle\}$ be the collection of NSs. The center of the sphere is $\langle \breve{\boldsymbol{\delta}}(\mathbb{x}_c), \check{\boldsymbol{\tau}}(\mathbb{x}_c), \dot{\boldsymbol{\gamma}}(\mathbb{x}_c)\rangle = \langle \frac{\sum_{d=1}^{p_c} \breve{\delta}_{c,d}}{p_c}, \frac{\sum_{d=1}^{p_c} \check{\tau}_{c,d}}{p_c}, \frac{\sum_{d=1}^{p_c} \dot{\gamma}_{c,d}}{p_c}\rangle$.

The radius is $\breve{\boldsymbol{\mu}}_c = \min\left\{\max_{1 \leq d \leq p_c} \sqrt{\left(\breve{\boldsymbol{\delta}}(\mathbb{x}_c) - \breve{\boldsymbol{\delta}}_{c,d}\right)^{\mathbb{2}} + \left(\check{\boldsymbol{\tau}}(\mathbb{x}_c) - \check{\boldsymbol{\tau}}_{c,d}\right)^{\mathbb{2}} + \left(\dot{\boldsymbol{\gamma}}(\mathbb{x}_c) - \dot{\boldsymbol{\gamma}}_{c,d}\right)^{\mathbb{2}}}, \mathbb{1}\right\}$.

Definition 2.2 [21] Let $\ddot{\mathrm{P}}(\underset{..}{\mathrm{U}})$ be the power set of $\underset{..}{\mathrm{U}}$ (universal set). The well defined attributes $a_1, a_2, \dots, a_n$ where $\mathbb{m} \geq 1$ corresponding to the sets $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_{\mathbb{m}}$ and $\ddot{\mathrm{P}}(\mathbb{A}_1), \ddot{\mathrm{P}}(\mathbb{A}_2), \dots, \ddot{\mathrm{P}}(\mathbb{A}_{\mathbb{m}})$ be the power set of the set $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_{\mathbb{m}}$. The pair $(F, \ddot{\mathbb{p}}(\mathbb{A}_1) \times \ddot{\mathbb{p}}(\mathbb{A}_2) \times \ldots \times \ddot{\mathbb{p}}(\mathbb{A}_{\mathbb{m}})$ is said to be NSHSSs over $\ddot{\mathrm{U}}$ then the mapping $F: \ddot{\mathbb{p}}(\mathbb{A}_1) \times \ddot{\mathbb{p}}(\mathbb{A}_2) \times \ldots \times \ddot{\mathbb{p}}(\mathbb{A}_{\mathbb{m}}) \rightarrow \ddot{\mathbb{p}}(\ddot{\mathrm{U}})$.

$\ddot{\mathbb{p}}(\mathbb{A}_1) \times \ddot{\mathbb{p}}(\mathbb{A}_2) \times \ldots \times \ddot{\mathbb{p}}(\mathbb{A}_{\mathbb{m}}) = \{\rho, \langle \mathbb{x}, \breve{\boldsymbol{\delta}}_\rho(\mathbb{x}), \check{\boldsymbol{\tau}}_\rho(\mathbb{x}), \dot{\boldsymbol{\gamma}}_\rho(\mathbb{x})\rangle : x \in \ddot{\mathrm{U}},\ \rho \in \ddot{\mathbb{p}}(\mathbb{A}_1) \times \ddot{\mathbb{p}}(\mathbb{A}_2) \times \ldots \times \ddot{\mathbb{p}}(\mathbb{A}_{\mathbb{m}})\}$ where $\breve{\boldsymbol{\delta}}_\rho(\mathbb{x}), \check{\boldsymbol{\tau}}_\rho(\mathbb{x}), \dot{\boldsymbol{\gamma}}_\rho(\mathbb{x})$ represents the truth, indeterminacy and falsity.

Definition 2.3[18] The NS A is defined as $\langle \breve{\boldsymbol{\delta}}, \check{\boldsymbol{\tau}}, \dot{\boldsymbol{\gamma}} \rangle$ then the score function is written as

$s_d(A) = \frac{2+\breve{\delta}-\check{\tau}-\dot{\gamma}}{3}; s_d(A) \in [0,1]$.

Definition 2.4 [7] Let $\mathbb{x}_1 = \langle \breve{\boldsymbol{\delta}}_1, \check{\boldsymbol{\tau}}_1, \dot{\boldsymbol{\gamma}}_1; \breve{\boldsymbol{\mu}}_1 \rangle$, $\mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_2, \dot{\boldsymbol{\gamma}}_2; \breve{\boldsymbol{\mu}}_2 \rangle$ are the two CSNSs over the universal set $\ddot{\mathrm{U}}$ and $0 \le \mathbb{X} \le 1$. The operators defined as follows:

1. $\mathbb{x}_1 \oplus \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1 + \breve{\boldsymbol{\delta}}_2 - \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2;\ \breve{\boldsymbol{\mu}}_1 + \breve{\boldsymbol{\mu}}_2 - \breve{\boldsymbol{\mu}}_1\breve{\boldsymbol{\mu}}_2 \rangle$.
2. $\mathbb{x}_1 \oplus \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1 + \breve{\boldsymbol{\delta}}_2 - \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2; \max(\breve{\boldsymbol{\mu}}_1, \breve{\boldsymbol{\mu}}_2) \rangle$.
3. $\mathbb{x}_1 \oplus \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1 + \breve{\boldsymbol{\delta}}_2 - \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2;\ \min(\breve{\boldsymbol{\mu}}_1, \breve{\boldsymbol{\mu}}_2) \rangle$.
4. $\oplus_{c=1}^{\mathbb{m}} = \langle \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \breve{\boldsymbol{\delta}}_c), \prod_{c=1}^{\mathbb{m}} \check{\boldsymbol{\tau}}_c, \prod_{c=1}^{\mathbb{m}} \dot{\boldsymbol{\gamma}}_c; \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \breve{\boldsymbol{\mu}}_c) \rangle$.
5. $\oplus_{c=1}^{\mathbb{m}} = \langle \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \breve{\boldsymbol{\delta}}_c), \prod_{c=1}^{\mathbb{m}} \check{\boldsymbol{\tau}}_c, \prod_{c=1}^{\mathbb{m}} \dot{\boldsymbol{\gamma}}_c; \max(\breve{\boldsymbol{\mu}}_c)(c = 1, 2, \ldots, \mathbb{m}) \rangle$.
6. $\oplus_{c=1}^{\mathbb{m}} = \langle \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \breve{\boldsymbol{\delta}}_c), \prod_{c=1}^{\mathbb{m}} \check{\boldsymbol{\tau}}_c, \prod_{c=1}^{\mathbb{m}} \dot{\boldsymbol{\gamma}}_c; \min(\breve{\boldsymbol{\mu}}_c)(c = 1, 2, \ldots, \mathbb{m}) \rangle$.
7. $\mathbb{x}_1 \otimes \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2,\ \check{\boldsymbol{\tau}}_1 + \check{\boldsymbol{\tau}}_2 - \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1 + \dot{\boldsymbol{\gamma}}_2 - \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2;\ \breve{\boldsymbol{\mu}}_1\breve{\boldsymbol{\mu}}_2 \rangle$.
8. $\mathbb{x}_1 \otimes \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_1 + \check{\boldsymbol{\tau}}_2 - \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1 + \dot{\boldsymbol{\gamma}}_2 - \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2; \max(\breve{\boldsymbol{\mu}}_1, \breve{\boldsymbol{\mu}}_2) \rangle$.
9. $\mathbb{x}_1 \otimes \mathbb{x}_2 = \langle \breve{\boldsymbol{\delta}}_1\breve{\boldsymbol{\delta}}_2, \check{\boldsymbol{\tau}}_1 + \check{\boldsymbol{\tau}}_2 - \check{\boldsymbol{\tau}}_1\check{\boldsymbol{\tau}}_2,\ \dot{\boldsymbol{\gamma}}_1 + \dot{\boldsymbol{\gamma}}_2 - \dot{\boldsymbol{\gamma}}_1\dot{\boldsymbol{\gamma}}_2; \min(\breve{\boldsymbol{\mu}}_1, \breve{\boldsymbol{\mu}}_2) \rangle$.
10. $\otimes_{c=1}^{\mathbb{m}} = \langle \prod_{c=1}^{\mathbb{m}} \breve{\boldsymbol{\delta}}_c, \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \check{\boldsymbol{\tau}}_c),\ \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \dot{\boldsymbol{\gamma}}_c);\ \prod_{c=1}^{\mathbb{m}} \breve{\boldsymbol{\mu}}_c \rangle$.
11. $\otimes_{c=1}^{\mathbb{m}} = \langle \prod_{c=1}^{\mathbb{m}} \breve{\boldsymbol{\delta}}_c, \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \check{\boldsymbol{\tau}}_c),\ \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \dot{\boldsymbol{\gamma}}_c);\ \max(\breve{\boldsymbol{\mu}}_c)\ (c = 1, 2, \ldots, \mathbb{m}) \rangle$.
12. $\otimes_{c=1}^{\mathbb{m}} = \langle \prod_{c=1}^{\mathbb{m}} \breve{\boldsymbol{\delta}}_c, \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \check{\boldsymbol{\tau}}_c),\ \mathbb{1} - \prod_{c=1}^{\mathbb{m}}(\mathbb{1} - \dot{\boldsymbol{\gamma}}_c);\ \min(\breve{\boldsymbol{\mu}}_c)(c = 1, 2, \ldots, \mathbb{m}) \rangle$.
13. $\varphi\mathbb{x} = \langle \mathbb{1} - (\mathbb{1} - \breve{\boldsymbol{\delta}})^\varphi, \check{\boldsymbol{\tau}}^\varphi,\ \dot{\boldsymbol{\gamma}}^\varphi; \mathbb{1} - (\mathbb{1} - \breve{\boldsymbol{\mu}})^\varphi \rangle$.
14. $\mathbb{x}^\varphi = \langle \breve{\boldsymbol{\delta}}^\varphi,\ \mathbb{1} - (\mathbb{1} - \check{\boldsymbol{\tau}})^\varphi,\ \mathbb{1} - (\mathbb{1} - \dot{\boldsymbol{\gamma}})^\varphi;\ \breve{\boldsymbol{\mu}}^\varphi \rangle$.

3. Proposed work

In this section, we introduce the new aggregation operator with weight through HM operator in geometric representation called CSNWGHM. The theorem 1 gives the proof of this operator using induction hypothesis method. Also, the fundamental properties idempotency, monotonicity and boundedness are verified one by one. The proposed operator is used to aggregate the CSNS values. The application of this operator is given the illustrative example section.

Definition 3.1 Let $\mathbb{x}_c = \langle \breve{\boldsymbol{\delta}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c}, \dot{\boldsymbol{\gamma}}_{n_c}; \breve{\boldsymbol{\mu}}_{n_c} \rangle (c = 1, 2, 3, \ldots, \mathbb{m})$ be a collection of CSNSs. For any $\bar{\bar{f}}, \bar{\bar{g}} > 0, \bar{\bar{f}} + \bar{\bar{g}} > 1, \mathbb{w} = (\mathbb{w}_1, \mathbb{w}_2, \ldots, \mathbb{w}_{\mathbb{m}})^T$ is the weight of the vector $\mathbb{x}_c$ (c =1, 2, …., $\mathbb{m}$) satisfies $\mathbb{w}_c \in [0,1]$ and $\sum_{c=1}^{\mathbb{m}} \mathbb{w}_c = 1$.

$$CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\mathbb{x}_1, \mathbb{x}_2, \ldots, \mathbb{x}_{\mathbb{m}}) = \frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}\left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\bar{\bar{f}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right) \oplus \bar{\bar{g}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}.$$

Theorem 3.1 Let $\mathbb{x}_c = \langle \breve{\boldsymbol{\delta}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c}, \dot{\boldsymbol{\gamma}}_{n_c}; \breve{\boldsymbol{\mu}}_{n_c} \rangle (c = 1, 2, 3, \ldots, \mathbb{m})$ be a collection of CSNSs. For any $\bar{\bar{f}}, \bar{\bar{g}}$ >0, $\bar{\bar{f}} + \bar{\bar{g}}$ >1, $\mathbb{w} = (\mathbb{w}_1, \mathbb{w}_2, \ldots, \mathbb{w}_{\mathbb{m}})^T$ is the weight of the vector $\mathbb{x}_c$ (c =1, 2, …, $\mathbb{m}$) satisfies $\mathbb{w}_c \in [0,1]$ and $\sum_{c=1}^{\mathbb{m}} \mathbb{w}_c = 1$. Then

$$CSNWGHM^{\dot{\bar{\bar{f}}},\dot{\bar{\bar{g}}}}(\mathbb{x}_1,\mathbb{x}_2,\ldots,\mathbb{x}_{\mathbb{m}})=\begin{cases} \mathbb{1}-\left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \mathbb{1}-\left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \end{cases} \tag{1}$$

Proof:

By the basic operations,

$$\mathbb{x}_c^{\mathbb{w}_c}=\left(\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c},\mathbb{1}-(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c})^{\mathbb{w}_c},\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_c})^{\mathbb{w}_c};\ \breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_d}\right)$$

$$\mathbb{x}_d^{\mathbb{w}_d}=\left(\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d},\mathbb{1}-(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d})^{\mathbb{w}_d},\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_d})^{\mathbb{w}_d};\ \breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)$$

$$\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)=\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}},\left(\mathbb{1}-(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c})^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}},\left(\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_c})^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}};\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\right)$$

$$\dot{\bar{\bar{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)=\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}},\left(\mathbb{1}-(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d})^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}},\left(\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_d})^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}};\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)$$

$$\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\dot{\bar{\bar{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)=\begin{pmatrix}\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}},\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}},\\ \left(\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_c})^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_d})^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}};\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\end{pmatrix}$$

First let us prove,

$$\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\dot{\bar{\bar{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)=\begin{cases}\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\dot{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\\ \prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\dot{\bar{\bar{g}}}}\right)\end{cases}$$

By using the mathematical induction on $\mathbb{m}$, put $\mathbb{m}=2$,

$$\prod_{\substack{c,d=1\\c\neq d}}^{2}\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\dot{\bar{\bar{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)=\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_1^{\mathbb{w}_1}\right)\oplus\dot{\bar{\bar{g}}}\left(\mathbb{x}_2^{\mathbb{w}_2}\right)\right)\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_2^{\mathbb{w}_2}\right)\oplus\dot{\bar{\bar{g}}}\left(\mathbb{x}_1^{\mathbb{w}_1}\right)\right)$$

$$2\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus 2\left(\mathbb{x}_d^{\mathbb{w}_d}\right)=\begin{cases}\prod_{\substack{c,d=1\\c\neq d}}^{2}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{2}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{2}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \prod_{\substack{c,d=1\\c\neq d}}^{2}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\end{cases}$$

Assume that, this result is true for $\mathbb{m}=k$,

$$\prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\bar{\bar{\dot{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)=\begin{cases}\prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\end{cases}$$

If $\mathbb{m}=k+1$,

$$\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\bar{\bar{\dot{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)=\prod_{\substack{c,d=1\\c\neq d}}^{k}\left(\bar{\bar{\dot{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)\oplus\left(\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\bar{\bar{\dot{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(\mathbb{x}_{k+1}^{\mathbb{w}_{k+1}}\right)\right)\right)$$

$$\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\bar{\bar{\dot{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)=\begin{cases}\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \prod_{\substack{c,d=1\\c\neq d}}^{k+1}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\end{cases}$$

The above equation holds for $\mathbb{m}=k+1$,

$$\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\bar{\bar{\dot{f}}}\left(x_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{\dot{g}}}\left(x_d^{\mathbb{w}_d}\right)\right)=\begin{cases}\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \mathbb{1}-\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\\ \prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\bar{\bar{\dot{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{\dot{g}}}}\right)\end{cases}$$

$$\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{g}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}$$

$$=\begin{cases}\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\\ \mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\\ \mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\\ \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\end{cases}$$

$$\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\bar{\bar{g}}}\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\dot{\bar{\bar{f}}}\left(\mathbb{x}_c^{\mathbb{w}_c}\right)\oplus\bar{\bar{g}}\left(\mathbb{x}_d^{\mathbb{w}_d}\right)\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}$$

$$=\begin{cases}\mathbb{1}-\left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\bar{\bar{g}}}}\\ \left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\check{\boldsymbol{\tau}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\bar{\bar{g}}}}\\ \left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_c}\right)^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\left(\mathbb{1}-\grave{\boldsymbol{\gamma}}_{\mathbb{x}_d}\right)^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\bar{\bar{g}}}}\\ \mathbb{1}-\left(\mathbb{1}-\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1}-\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_c}^{\mathbb{w}_c}\right)^{\dot{\bar{\bar{f}}}}\left(\mathbb{1}-\breve{\boldsymbol{\mu}}_{\mathbb{x}_d}^{\mathbb{w}_d}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\bar{\bar{g}}}}\end{cases}$$

Hence this result is proved by the induction hypothesis.

Theorem 3.2 (Idempotency) Let $\mathbb{x}_c=\langle\breve{\boldsymbol{\delta}}_c,\check{\boldsymbol{\tau}}_c,\grave{\boldsymbol{\gamma}}_c;\breve{\boldsymbol{\mu}}_c\rangle$ (c =1, 2, …, $\mathbb{m}$) then $\mathbb{x}_c=\mathbb{x}=\langle\breve{\boldsymbol{\delta}},\check{\boldsymbol{\tau}},\grave{\boldsymbol{\gamma}};\breve{\boldsymbol{\mu}}\rangle$ for all c, then $CSNWGHM^{\dot{\bar{\bar{p}}},\bar{\bar{q}}}(\mathbb{x}_1,\mathbb{x}_2,\dots,\mathbb{x}_{\mathbb{m}})=CSNWGHM^{\dot{\bar{\bar{p}}},\bar{\bar{q}}}(\mathbb{x},\mathbb{x},\dots,\mathbb{x})$.

Proof:

Since $\mathbb{x}_c=\langle\breve{\boldsymbol{\delta}}_c,\check{\boldsymbol{\tau}}_c,\grave{\boldsymbol{\gamma}}_c;\breve{\boldsymbol{\mu}}_c\rangle$ (c =1, 2, …, $\mathbb{m}$)

$$CSNWGHM^{\dot{\bar{\bar{f}}},\bar{\bar{g}}}(\mathbb{x}_1,\mathbb{x}_2,\dots,\mathbb{x}_{\mathbb{m}})=CSNWGHM^{\dot{\bar{\bar{f}}},\bar{\bar{g}}}(\mathbb{x},\mathbb{x},\dots,\mathbb{x})$$

$$= \begin{cases} \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}}^{\mathbb{w}} \right)^{\dot{\bar{\bar{f}}}} \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}}^{\mathbb{w}} \right)^{\dot{\bar{\bar{g}}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - (\mathbb{1} - (\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}})^{\mathbb{w}})^{\dot{\bar{\bar{f}}}} (\mathbb{1} - (\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}})^{\mathbb{w}})^{\dot{\bar{\bar{g}}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - (\mathbb{1} - (\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}})^{\mathbb{w}})^{\dot{\bar{\bar{f}}}} (\mathbb{1} - (\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}})^{\mathbb{w}})^{\dot{\bar{\bar{g}}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - (\mathbb{1} - \breve{\boldsymbol{\mu}}_{\mathbb{x}}^{\mathbb{w}})^{\dot{\bar{\bar{f}}}} (\mathbb{1} - \breve{\boldsymbol{\mu}}_{\mathbb{x}}^{\mathbb{w}})^{\dot{\bar{\bar{g}}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \end{cases}$$

$$= \begin{cases} \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}} \right)^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right)^{\frac{\mathbb{m}(\mathbb{m}+\mathbb{1})}{\mathbb{2}} \frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - (\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right) \right)^{\frac{\mathbb{m}(\mathbb{m}+\mathbb{1})}{\mathbb{2}} \frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - (\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right) \right)^{\frac{\mathbb{m}(\mathbb{m}+\mathbb{1})}{\mathbb{2}} \frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - (\mathbb{1} - \breve{\boldsymbol{\mu}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right)^{\frac{\mathbb{m}(\mathbb{m}+\mathbb{1})}{\mathbb{2}} \frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \end{cases}$$

$$= \begin{cases} \mathbb{1} - \left(\mathbb{1} - \prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}} \right)^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - (\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right) \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \left(\mathbb{1} - \prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - (\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right) \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \\ \mathbb{1} - \left(\mathbb{1} - \prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - (\mathbb{1} - \breve{\boldsymbol{\mu}}_{\mathbb{x}})^{(\mathbb{w}\dot{\bar{\bar{f}}}+\mathbb{w}\dot{\bar{\bar{g}}})} \right) \right)^{\frac{\mathbb{1}}{\dot{\bar{\bar{f}}}+\dot{\bar{\bar{g}}}}} \end{cases}$$

$$= \begin{cases} 1 - \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{x}\right)^{w(\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}})}\right)\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}} \\ \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \left(1 - \check{\boldsymbol{\tau}}_{x}\right)^{w(\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}})}\right)\right)\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}} \\ \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \left(1 - \grave{\boldsymbol{\gamma}}_{x}\right)^{w(\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}})}\right)\right)\right)^{\frac{1}{\bar{\bar{\dot{p}}}+\bar{\bar{\dot{q}}}}} \\ 1 - \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\mu}}_{x}\right)^{w(\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}})}\right)\right)^{\frac{1}{\bar{\bar{\dot{p}}}+\bar{\bar{\dot{q}}}}} \end{cases} = \langle \breve{\boldsymbol{\delta}}_{x}, \check{\boldsymbol{\tau}}_{x}, \grave{\boldsymbol{\gamma}}_{x}; \breve{\boldsymbol{\mu}}_{x} \rangle.$$

Theorem 3.3 (Monotonicity) Let $x_c = \langle \breve{\boldsymbol{\delta}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c}, \grave{\boldsymbol{\gamma}}_{n_c}; \breve{\boldsymbol{\mu}}_{n_c} \rangle$ and $\widetilde{x}_c = \langle \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}, \tilde{\check{\boldsymbol{\tau}}}_{n_c}, \widetilde{\boldsymbol{\gamma}}_{n_c}; \widetilde{\boldsymbol{\mu}}_{n_c} \rangle$ (c =1, 2, …, m) be the two collections of CSNSs. If $x_c \geq \widetilde{x}_c$ for all d, suppose $\breve{\boldsymbol{\delta}}_{n_c} \geq \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c} \leq \tilde{\check{\boldsymbol{\tau}}}_{n_c}, \grave{\boldsymbol{\gamma}}_{n_c} \leq \widetilde{\boldsymbol{\gamma}}_{n_c}$ and $\breve{\boldsymbol{\mu}}_{n_c} \geq \widetilde{\boldsymbol{\mu}}_{n_c}$ then $CSNWGHM^{\bar{\bar{\dot{f}}},\bar{\bar{\dot{g}}}}(x_1, x_2, \dots, x_m) = CSNWGHM^{\bar{\bar{\dot{f}}},\bar{\bar{\dot{g}}}}(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_m)$.

Proof:

(I). Since $\breve{\boldsymbol{\delta}}_{n_c} \geq \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}$ for all c, $\bar{\bar{\dot{f}}}, \bar{\bar{\dot{g}}} > 0$,

$$\breve{\boldsymbol{\delta}}_{n_c} \geq \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}$$

$$1 - \breve{\boldsymbol{\delta}}_{n_c} \leq 1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}$$

$$\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}} \leq \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}$$

$$\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}} \leq \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}$$

$$1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}} \geq 1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}$$

$$\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}} \geq \left(1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}$$

$$\prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}} \geq \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}$$

$$1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}} \leq 1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}$$

$$\left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}}$$

$$\leq \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}}$$

$$1 - \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \breve{\boldsymbol{\delta}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}}$$

$$\geq 1 - \left(1 - \prod_{\substack{c,d=1 \\ c\neq d}}^{m}\left(1 - \left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{f}}}}\left(1 - \widetilde{\breve{\boldsymbol{\delta}}}_{n_c}^{w_c}\right)^{\bar{\bar{\dot{g}}}}\right)^{\frac{2}{m(m+1)}}\right)^{\frac{1}{\bar{\bar{\dot{f}}}+\bar{\bar{\dot{g}}}}}.$$

Similarly, we can prove the radius.

(II). Since $\check{\boldsymbol{\tau}}_{n_c} \leq \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}$ for all c,

$$\check{\boldsymbol{\tau}}_{n_c} \leq \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}$$

$$\left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c} \geq \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}$$

$$\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}} \leq \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}$$

$$\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}} \leq \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}$$

$$\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}} \geq \mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}$$

$$\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right) \geq \prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)$$

$$\left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \geq \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}$$

$$\mathbb{1} - \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \leq \mathbb{1} - \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}$$

$$\left(\mathbb{1} - \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \leq \left(\mathbb{1} - \left(\prod_{\substack{c,d=1\\c\neq d}}^{\mathbb{m}}\left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{f}}}\left(\mathbb{1} - \left(\mathbb{1} - \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right)^{\mathbb{w}_c}\right)^{\bar{\bar{g}}}\right)\right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}}\right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}}$$

Similarly, we can prove falsity.

Theorem 3.4 (Boundedness) Let $\widetilde{\mathbb{x}}_c = \langle \tilde{\tilde{\boldsymbol{\delta}}}_{n_c}, \tilde{\tilde{\boldsymbol{\tau}}}_{n_c}, \widetilde{\boldsymbol{\gamma}}_{n_c}; \widetilde{\boldsymbol{\mu}}_{n_c}\rangle$ be the collection of CSNS and $\widetilde{\mathbb{x}}^- = \left(min\left(\tilde{\tilde{\boldsymbol{\delta}}}_{n_c}\right), max\left(\tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right), max\left(\widetilde{\boldsymbol{\gamma}}_{n_c}\right); min\left(\widetilde{\boldsymbol{\mu}}_{n_c}\right)\right)$ and $\widetilde{\mathbb{x}}^+ = \left(max\left(\tilde{\tilde{\boldsymbol{\delta}}}_{n_c}\right), min\left(\tilde{\tilde{\boldsymbol{\tau}}}_{n_c}\right), min\left(\widetilde{\boldsymbol{\gamma}}_{n_c}\right); max\left(\widetilde{\boldsymbol{\mu}}_{n_c}\right)\right)$ then $\widetilde{\mathbb{x}}^- \leq CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1, \widetilde{\mathbb{x}}_2, \ldots, \widetilde{\mathbb{x}}_{\mathbb{m}}) \leq \widetilde{\mathbb{x}}^+$.

Proof:

Since $\mathbb{x}_c^- \geq \widetilde{\mathbb{x}}^-$, then based on the above theorems,

$$CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1, \widetilde{\mathbb{x}}_2, \ldots, \widetilde{\mathbb{x}}_{\mathbb{m}}) \geq CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1^-, \widetilde{\mathbb{x}}_2^-, \ldots, \widetilde{\mathbb{x}}_{\mathbb{m}}^-) = \widetilde{\mathbb{x}}^-$$

$$CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1, \widetilde{\mathbb{x}}_2, \ldots, \widetilde{\mathbb{x}}_{\mathbb{m}}) \leq CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1^+, \widetilde{\mathbb{x}}_2^+, \ldots, \widetilde{\mathbb{x}}_{\mathbb{m}}^+) = \widetilde{\mathbb{x}}^+$$

Then $\widetilde{\mathbb{x}}^{-} \leq CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\widetilde{\mathbb{x}}_1, \widetilde{\mathbb{x}}_2, \dots, \widetilde{\mathbb{x}}_{\mathbb{m}}) \leq \widetilde{\mathbb{x}}^{+}$.

Hence prove the boundedness.

Theorem 3.5 Let $\mathbb{x}_c = \langle \breve{\boldsymbol{\delta}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c}, \dot{\boldsymbol{\gamma}}_{n_c}; \breve{\boldsymbol{\mu}}_{n_c} \rangle (c = 1, 2, \dots, \mathbb{m})$ be a collection of CSNSs. For any $\bar{\bar{f}} + \bar{\bar{g}} > 0$, $\bar{\bar{f}} + \bar{\bar{g}} > \mathbb{1}$, $\mathbb{w} = (\mathbb{w}_1, \mathbb{w}_2, \dots, \mathbb{w}_{\mathbb{m}})^T$ is the weight of the vector $\mathbb{x}_c$ (c =1, 2, …, $\mathbb{m}$) satisfies $\mathbb{w}_c \in [0,1]$ and $\sum_{c=1}^{\mathbb{m}} \mathbb{w}_c = \mathbb{1}$. Then

$$CSNWGHM^{\bar{\bar{p}},\bar{\bar{q}}}(\mathbb{x}_1, \mathbb{x}_2, \dots, \mathbb{x}_{\mathbb{m}}) = \begin{cases} \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}_d} \right)^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \left(\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}_d} \right)^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \max \left(\breve{\boldsymbol{\mu}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \end{cases}$$

The proof is similar to the above theorem and verified the properties.

Theorem 3.6 Let $\mathbb{x}_c = \langle \breve{\boldsymbol{\delta}}_{n_c}, \check{\boldsymbol{\tau}}_{n_c}, \dot{\boldsymbol{\gamma}}_{n_c}; \breve{\boldsymbol{\mu}}_{n_c} \rangle (c = 1, 2, \dots, \mathbb{m})$ be a collection of CSNSs. For any $\bar{\bar{f}} + \bar{\bar{g}} > 0$, $\bar{\bar{f}} + \bar{\bar{g}} > \mathbb{1}$, $\mathbb{w} = (\mathbb{w}_1, \mathbb{w}_2, \dots, \mathbb{w}_{\mathbb{m}})^T$ is the weight of the vector $\mathbb{x}_c$ (c =1, 2, …, $\mathbb{m}$) satisfies $c \in [0,1]$ and $\sum_{c=1}^{\mathbb{m}} \mathbb{w}_c = \mathbb{1}$. Then

$$CSNWGHM^{\bar{\bar{f}},\bar{\bar{g}}}(\mathbb{x}_1, \mathbb{x}_2, \dots, \mathbb{x}_{\mathbb{m}}) = \begin{cases} \mathbb{1} - \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}_c}^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \breve{\boldsymbol{\delta}}_{\mathbb{x}_d}^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \left(\mathbb{1} - \check{\boldsymbol{\tau}}_{\mathbb{x}_d} \right)^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \left(\mathbb{1} - \left(\prod_{\substack{c,d=1 \\ c \neq d}}^{\mathbb{m}} \left(\mathbb{1} - \left(\mathbb{1} - \left(\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \right)^{\bar{\bar{f}}} \left(\mathbb{1} - \left(\mathbb{1} - \dot{\boldsymbol{\gamma}}_{\mathbb{x}_d} \right)^{\mathbb{w}_d} \right)^{\bar{\bar{g}}} \right) \right)^{\frac{\mathbb{2}}{\mathbb{m}(\mathbb{m}+\mathbb{1})}} \right)^{\frac{\mathbb{1}}{\bar{\bar{f}}+\bar{\bar{g}}}} \\ \min \left(\breve{\boldsymbol{\mu}}_{\mathbb{x}_c} \right)^{\mathbb{w}_c} \end{cases}$$

The proof similar to the theorem 3.5 and verified the fundamental properties.

4. Algorithm for CSNWGHM in MCDM problem using NSHSSs

Following the below steps to finding the results:

Step 1: Step 1: Let take the set of alternatives as $Z\{@z_1, @z_2, \dots, @z_m\ z_m\}$. The set of criteria as $Y\{Cy_1, Cy_2, \dots, Cz_{\mathbb{m}}\ y_n\}$ and the field experts namely $\{FEFE_1, FEFE_2, \dots, FEz_kFE_p\}$.

Step 2: Create the linguistic table to measure the experts opinion in truth, indeterminacy and falsity values.

Step 3: Find the weights of the field experts using the below equation:

$$FE_k = \frac{1-\sqrt{\left\{\left(1-\breve{\boldsymbol{\delta}}_k(x)\right)^2+(1-\check{\boldsymbol{\tau}}_k(x))^2+(1-\grave{\boldsymbol{\gamma}}_k(x))^2\right\}/3}}{\sum_{k=1}^{p}\left(1-\sqrt{\{(1-\boldsymbol{\delta}_k(x))^2+(1-\check{\boldsymbol{\tau}}_k(x))^2+(1-\grave{\boldsymbol{\gamma}}_k(x))^2\}/3}\right)} \quad (2)$$

and $\sum_{k=1}^{p} FE_k = 1$.

Step 4: Calculate the criteria weights using SWARA method. Firstly, field experts give the ratings for the criteria through the linguistic table.

Step 4.1: Aggregate the values using below operator (3) and calculate the score value (s_j) using definition.

$$AGO(y_{cd}^1, y_{cd}^2, \ldots, y_{cd}^n) = (1-\prod_{k=1}^{\mathbb{m}}(1-\boldsymbol{\delta}_k)^{FE_k}, \prod_{k=1}^{\mathbb{m}}(\check{\boldsymbol{\tau}}_k)^{FE_k}, \prod_{k=1}^{\mathbb{m}}(\grave{\boldsymbol{\gamma}}_k)^{FE_k}). \quad (3)$$

Step 4.2: Calculate the relative coefficient (l_j), weight (o_j) and normalize the weight value using below equations:

$$l_d = \begin{cases} 1, & d = 1 \\ s_d + 1, & d > 1 \end{cases} \text{ and } o_{\mathrm{j}} = \begin{cases} 1, & d = 1 \\ \frac{l_{d-1}}{l_d}, & \mathrm{j} > 1 \end{cases} \quad (4)$$

$$w_d = \frac{o_d}{\sum_{d=1}^{n} o_d}. \quad (5)$$

Step 5: Make the propositions using NSHSSs.

Step 6: Forming the decision table with the help of the field experts.

Step 7: Convert the NSs into CSNSs.

Step 8: Aggregate the values using equation (1) and calculate the final value using score value then ranking the alternatives.

5. Illustrative Example

The farmer wants to find the best fertilizer for guava and banana tree. He had a set of fertilizers as organic compost, NPK 20-20-20, urea + potash blend, vermicompost and bio - fertilizer. To the convenient of the systematic approach assume the fertilizer as alternatives. The characteristic or attributes of the fertilizers are nutrient content, soil health impact, water retention support, environmental friendliness and specific yield impact. These attributes are taken as the criteria and it is divided into sub - criteria.

Step 1:

Here considering the 5 alternatives, 6 criteria and their sub - criteria and 3 field experts to sharing their field knowledge for finding the best results.

Alternatives:

- @1: Organic compost
- @2: NPK 20-20-20
- @3: Urea + Potash blend
- @4: Vermicompost
- @5: Bio - fertilizer.

The alternative 1 is organic compost which represented in @1, the second alternative is NPK 20-20-20 which is represented in @2. Similarly, the alternative 3, 4 and 5 is represented in @3, @4 and @5 respectively. This symbol are using the throughout process for referring their fertilizer names and it reduces the typing reputation of the fertilizers.

Criteria & sub-criteria

Here we generate the common characteristic of the all fertilizer. Generally, take fertilizer we thought about the cost, yield, side effects and benefits. Some of these are used our problem which is taken as the criteria and sub - criteria. Each sub - criteria is commonly divided into high, medium and low which is very helpful to measure.

Table 3. Criteria & sub - criteria

Description	Criterion	Sub – criteria
Quality and balance of nutrients	Nutrient content (C1)	C11:Nitrogen content(N) High(C11H), Medium(C11M), Low(C11L)
		C12: Phosphorus content (P) High(C12H), Medium(C12M), Low(C12L)
		C13: Potassium content (K) High(C13H), Medium(C13M), Low(C13L)
Long-term benefits to soil	Soil health impact (C2)	C21: Organic matter support High(C21H), Medium(C21M), Low(C21L)
		C22: Microbial activity High(C22H), Medium(C22M), Low(C22L)
		C23: pH balance contribution High(C23H), Medium(C23M), Low(C23L)
Total expense of using the fertilizer	Cost-effectiveness (C3)	C31: Market price High(C31H), Medium(C31M), Low(C31L)
		C32: Required quantity per acre High(C32H), Medium(C32M), Low(C32L)
		C33: Application frequency High(C33H), Medium(C33M), Low(C33L)
Moisture holding support	Water retention support (C4)	C41: Water holding capacity High(C41H), Medium(C41M), Low(C41L)
		C42: Surface mulch effect High(C42H), Medium(C42M), Low(C42L)

Eco-safe characteristics	Environmental friendliness (C5)	C51: Runoff pollution High(C51H), Medium(C51M), Low(C51L)
		C52: Biodegradability High(C52H), Medium(C52M), Low(C52L)
Crop yield enhancement	Plant - specific yield Impact (C6)	C61: Yield in guava High(C61H), Medium(C61M), Low(C61L)
		C62: Yield in banana High(C62H), Medium(C62M), Low (C62L)
		C63: Yield in coconut High(C63H), Medium(C63M), Low(C63L)

Figure 2. Representation of the criteria

Step 2:

The formed the linguistic table is to scaling the quality of the products. This is divided into three parts truth, indeterminacy and falsity. Here it used the field experts to show their opinions through values. Here the measure much more effective represents the first preference because the truth value is one. The second is more effective which is slightly less than the much more effective and so on. The sum of the three terms (truth, indeterminacy, falsity) is less than or equal to 3 as per NS condition.

Table 3. Linguistic table

Measure	Value	Notation
Much more effective(MME)	$\langle 1.000, 0.000, 0.000\rangle$	ɣɤ_1
More effective(ME)	$\langle 0.940, 0.010, 0.050\rangle$	ɣɤ_2
Above moderate effective(AME)	$\langle 0.870, 0.030, 0.100\rangle$	ɣɤ_3
Moderate effective(MOE)	$\langle 0.760, 0.055, 0.185\rangle$	ɣɤ_4
Below moderate effective(BME)	$\langle 0.620, 0.080, 0.300\rangle$	ɣɤ_5
Less effective(LE)	$\langle 0.390, 0.090, 0.520\rangle$	ɣɤ_6

Step 3:

Here we take three field experts to finding the best and worst alternatives. The three field experts namely FE_1, FE_2 and FE_3. Now calculate the weight of the field experts based on their knowledge and experience in this particular field. The total weight of three field experts is must be equal to 1.

Table 4. Field expert weights

Experts	Measure	Weight
FE_1	ɣɤ_1	0.267
FE_2	ɣɤ_2	0.301
FE_3	ɣɤ_1	0.432

Using the equation (2),

Weight of $FE_1 = \frac{1-\sqrt{((1-1)^2+(1-0)^2+(1-0)^2)/3}}{\sqrt{((1-1)^2+(1-0)^2+(1-0)^2)/3}+\sqrt{((1-0.940)^2+(1-0.010)^2+(1-0.050)^2)/3}+\sqrt{((1-1)^2+(1-0)^2+(1-0)^2)/3}}$

$= \frac{0.1835}{0.1835+0.2071+0.2974} = 0.267.$

Similarly, calculate other two field expert weights.

Step 4.1:

Gathering the field expert opinion of the each criteria using the linguistic table to aggregate the values and calculate score value by using the definition.

Table 5. Criteria aggregated & score value

Criteria	FE's opinion	Aggregated value	s_j
C1	$\langle \text{ɣɤ}_1, \text{ɣɤ}_2, \text{ɣɤ}_1\rangle$	$\langle 1.000, 0.000, 0.000\rangle$	1.000
C2	$\langle \text{ɣɤ}_2, \text{ɣɤ}_4, \text{ɣɤ}_1\rangle$	$\langle 1.000, 0.000, 0.000\rangle$	1.000
C3	$\langle \text{ɣɤ}_6, \text{ɣɤ}_5, \text{ɣɤ}_4\rangle$	$\langle 0.827, 0.039, 0.134\rangle$	0.765
C4	$\langle \text{ɣɤ}_3, \text{ɣɤ}_2, \text{ɣɤ}_6\rangle$	$\langle 0.647, 0.070, 0.282\rangle$	0.866
C5	$\langle \text{ɣɤ}_4, \text{ɣɤ}_3, \text{ɣɤ}_4\rangle$	$\langle 0.800, 0.046, 0.154\rangle$	0.867

C6	$\langle ɣɤ_5, ɣɤ_5, ɣɤ_3 \rangle$	⟨0.799,0.035,0.166⟩	0.885

To get the aggregate value using the equation (3),

$$\breve{\boldsymbol{\delta}}_1 = 1 - \left(\left((1-1)^\wedge 0.267\right) * \left((1-0.94)^\wedge 0.301\right) * \left((1-1)^\wedge 0.432\right)\right) = 1.000,$$

$$\check{\boldsymbol{\tau}}_1 = (0.000^\wedge 0.267) * (0.010^\wedge 0.301) * (0.000^\wedge 432) = 0.000,$$

$$\dot{\boldsymbol{\gamma}}_1 = (0.000^\wedge 0.267) * (0.050^\wedge 0.301) * (0.000^\wedge 432) = 0.000.$$

Then C1= ⟨1.000,0.000,0.000⟩ and $s_1 = \frac{2+1.000-0.000-0.000}{3} = 1.000.$

Similarly, get other criteria values.

Step 4.2:

Apply the equation (4) and (5) to calculate the criteria weights.

Table 6. SWARA method

Criteria	l_j	o_j	w_j
C1	1.000	1.000	0.325
C2	1.000	1.000	0.325
C6	1.885	0.531	0.172
C3	1.765	0.301	0.098
C5	1.867	0.161	0.052
C4	1.866	0.086	0.028

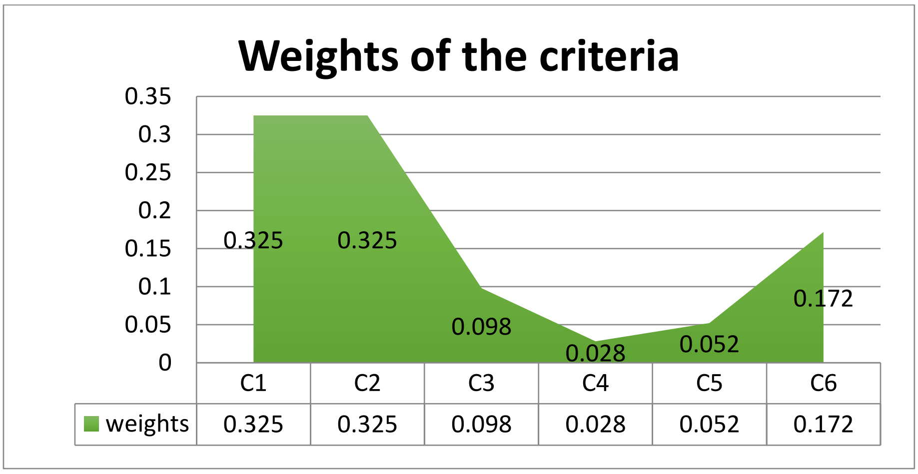

Figure 3. Weights

Step 5:

To make the propositions using the superhypersoft sets which is mapping from Cartesian product of the power sets to power set of the universal set.

The universal is set of fertilizers Ṳ ={@1, @2, @3, @4, @5} and $\ddot{p}$(Ṳ) = $\{\emptyset, @1, @2, @3, @4, @5, \{@1, @2\}, \{@1, @3\}, \{@1, @4\}, \{@1, @5\}, \{@2, @3\},$

$\{@2, @4\}, \{@2, @5\}, \{@3, @4\}, \{@3, @5\}, \{@4, @5\}, \{@1, @2, @3\}, \text{etc.}\}.$

Here we take {@1, @2, @3, @4, @5} as alternatives.

The power set of C1 is 512 elements. Then $\ddot{p}$(C1) = {∅, C11H, C11M, C11L, {C11H, C11M}, {C11H, C11L}, {C11M, C11L}, {C11H, C11M, C11L}, C12H, C12M, C12L, {C12H, C12M}, {C12H, C12L}, {C12M, C12L}, {C12H, C12M, C12L}, C13H, C13M, C13L, etc.}.

The power set of C2 is 512 elements. Then $\ddot{p}$(C2) = {∅, C21H, C21M, C21L, {C21H, C21M}, {C21H, C21L}, {C21M, C21L}, {C21H, C21M, C21L}, C22H, C22M, C22L, {C22H, C22M}, {C22H, C22L}, {C22M, C22L}, {C22H, C22M, C22L}, C23H, C23M, C23L, etc.}.

The power set of C3 is 512 elements. Then $\ddot{p}$(C3) = {∅, C31H, C31M, C31L, {C31H, C31M}, {C31H, C31L}, {C31M, C31L}, C32H, C32M, C32L, {C32H, C32M}, {C32H, C32L}, {C32M, C32L}, {C32H, C32M, C32L}, C33H, C33M, C33L, etc.}.

The power set of C4 is 64 elements. Then $\ddot{p}$(C4) = {∅, C41H, C41M, C41L, {C41H, C41M}, {C41H, C41L}, {C41M, C41L}, {C41H, C41M, C41L}, C42H, C42M, C42L, {C42H, C42M}, {C42H, C42L}, {C42M, C42L}, {C42H, C42M, C42L}, etc.}.

The power set of C5 is 64 elements. Then $\ddot{p}$(C5) = {∅, C51H, C51M, C52L, {C51H, C51M}, {C51H, C51L}, {C51M, C51L}, {C51H, C51M, C51L}, C52H, C52M, C52L, {C52H, C52M}, {C52H, C52L}, etc.}.

The power set of C6 is 512 elements. Then $\ddot{p}$(C6) = {∅, C61H, C61M, C61L, {C61H, C61M}, {C61H, C61L}, {C61M, C61L}, C62H, C62M, C62L, {C62H, C62M}, {C62H, C62L}, {C62M, C62L}, {C62H, C62M, C62L}, C63H, C63M, C63L, etc.}.

The total Cartesian product is $512 \times 512 \times 512 \times 64 \times 64 \times 512$ elements. The NSHSS is mapping: $\ddot{p}(C1) \times \ddot{p}(C2) \times \ddot{p}(C3) \times \ddot{p}(C4) \times \ddot{p}(C5) \times \ddot{p}(C6) \rightarrow \ddot{p}(\text{Ṳ})$.

$$\ddot{p}(C1) \times \ddot{p}(C2) \times \ddot{p}(C3) \times \ddot{p}(C4) \times \ddot{p}(C5) \times \ddot{p}(C6) =$$

$$\left\{\begin{array}{c}
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C11H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C11M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C11L),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C12H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C12M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C12L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C13H),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C13M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C13L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C21H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C21M),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C21L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C22H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C22M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C22L),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C23H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C23M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C23L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C31H),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C31M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C31L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C32H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C32M),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C32L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C33H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C33M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C33L),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C41H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C41M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C41L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C42H),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C42M), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C42L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C51H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C52M),\\
(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C52L), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C61H), (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,C62M), etc.
\end{array}\right\}$$

The total proposition is 2, 36, 196 from the Cartesian products. Here we take the two suitable propositions for guava and banana tree.

P1: [C11H, C21H, C31M, C41M, C51L, C61H].

P2: [C13H, C23M, C31H, C41L, C52M, C62M].

Step 6:

Take the propositions one by one and applying the described steps in algorithm section. First take proposition 1 which is showed below:

Proposition 1:

The below table gives the linguistic notation based on the field experts opinion.

Table 7. FE's opinion for proposition 1

Alternatives	@1	@2	@3	@4	@5
C11H	⟨ɣɤ$_1$, ɣɤ$_2$, ɣɤ$_1$⟩	⟨ɣɤ$_2$, ɣɤ$_1$, ɣɤ$_1$⟩	⟨ɣɤ$_1$, ɣɤ$_4$, ɣɤ$_2$⟩	⟨ɣɤ$_1$, ɣɤ$_1$, ɣɤ$_2$⟩	⟨ɣɤ$_2$, ɣɤ$_3$, ɣɤ$_3$⟩
C21H	⟨ɣɤ$_3$, ɣɤ$_1$, ɣɤ$_2$⟩	⟨ɣɤ$_4$, ɣɤ$_5$, ɣɤ$_4$⟩	⟨ɣɤ$_3$, ɣɤ$_4$, ɣɤ$_4$⟩	⟨ɣɤ$_4$, ɣɤ$_3$, ɣɤ$_4$⟩	⟨ɣɤ$_1$, ɣɤ$_2$, ɣɤ$_1$⟩
C31M	⟨ɣɤ$_4$, ɣɤ$_4$, ɣɤ$_3$⟩	⟨ɣɤ$_2$, ɣɤ$_3$, ɣɤ$_3$⟩	⟨ɣɤ$_1$, ɣɤ$_1$, ɣɤ$_2$⟩	⟨ɣɤ$_1$, ɣɤ$_5$, ɣɤ$_2$⟩	⟨ɣɤ$_3$, ɣɤ$_2$, ɣɤ$_3$⟩
C41M	⟨ɣɤ$_1$, ɣɤ$_2$, ɣɤ$_1$⟩	⟨ɣɤ$_2$, ɣɤ$_1$, ɣɤ$_1$⟩	⟨ɣɤ$_1$, ɣɤ$_2$, ɣɤ$_4$⟩	⟨ɣɤ$_1$, ɣɤ$_1$, ɣɤ$_2$⟩	⟨ɣɤ$_2$, ɣɤ$_3$, ɣɤ$_3$⟩
C51L	⟨ɣɤ$_1$, ɣɤ$_2$, ɣɤ$_2$⟩	⟨ɣɤ$_1$, ɣɤ$_3$, ɣɤ$_3$⟩	⟨ɣɤ$_3$, ɣɤ$_3$, ɣɤ$_2$⟩	⟨ɣɤ$_2$, ɣɤ$_2$, ɣɤ$_1$⟩	⟨ɣɤ$_2$, ɣɤ$_3$, ɣɤ$_3$⟩
C61H	⟨ɣɤ$_2$, ɣɤ$_3$, ɣɤ$_3$⟩	⟨ɣɤ$_4$, ɣɤ$_5$, ɣɤ$_5$⟩	⟨ɣɤ$_4$, ɣɤ$_5$, ɣɤ$_4$⟩	⟨ɣɤ$_4$, ɣɤ$_5$, ɣɤ$_4$⟩	⟨ɣɤ$_1$, ɣɤ$_1$, ɣɤ$_2$⟩

Step 7:

To convert neutrosophic values into CSNS value using the definition which is shown the preliminaries part. Here given the CSNS representation based on the experts opinion which is tabulated below:

Table 8. CSNSs for proposition 1

C/A	C11H	C21H	C31H	C41M	C51L	C61H
@1	⟨0.980,0.003,0.017; 0.052⟩	⟨0.937,0.013,0.050; 0.085⟩	⟨0.797,0.047,0.157; 0.094⟩	⟨0.980,0.003,0.017; 0.052⟩	⟨0.960,0.007,0.033; 0.052⟩	⟨0.893,0.023,0.083; 0.059⟩
@2	⟨0.980,0.003,0.017; 0.052⟩	⟨0.713,0.063,0.223; 0.122⟩	⟨0.893,0.023,0.083; 0.059⟩	⟨0.980,0.003,0.017; 0.052⟩	⟨0.913,0.020,0.067; 0.111⟩	⟨0.667,0.072,0.262; 0.122⟩
@3	⟨0.900,0.022,0.078; 0.179⟩	⟨0.797,0.047,0.157; 0.094⟩	⟨0.980,0.003,0.017; 0.052⟩	⟨0.900,0.022,0.078; 0.179⟩	⟨0.893,0.023,0.083; 0.059⟩	⟨0.713,0.063,0.223; 0.122⟩
@4	⟨0.980,0.003,0.017; 0.052⟩	⟨0.797,0.047,0.157; 0.094⟩	⟨0.853,0.030,0.117; 0.301⟩	⟨0.980,0.003,0.017; 0.052⟩	⟨0.960,0.007,0.033; 0.052⟩	⟨0.713,0.063,0.223; 0.122⟩
@5	⟨0.893,0.023,0.083; 0.059⟩	⟨0.980,0.003,0.017; 0.052⟩	⟨0.893,0.023,0.083; 0.059⟩	⟨0.893,0.023,0.083; 0.059⟩	⟨0.893,0.023,0.083; 0.059⟩	⟨0.980,0.003,0.017; 0.052⟩

Using the definition (2.1),

Calculate the center and radius for alternative 1 (@1) in criteria 1(C11H) by taking Table 8.

$$\text{Center} = \frac{1.000+0.940+1.000}{3}, \frac{0.000+0.010+0.000}{3}, \frac{0.000+0.050+0.000}{3} = (0.980, 0.003, 0.017).$$

$$\text{Radius} = max\begin{pmatrix} \sqrt{(0.980-1.000)^2+(0.003-0.000)^2+(0.017-0.000)^2} \\ \sqrt{(0.980-0.940)^2+(0.003-0.010)^2+(0.017-0.050)^2} \\ \sqrt{(0.980-1.000)^2+(0.003-0.000)^2+(0.017-0.000)^2} \end{pmatrix} = 0.052.$$

Proceeding the similarly to get other values.

Step 8:

Using the theorem 3.1,

Let $\bar{\bar{f}}, \bar{\bar{g}} = 1$ in equation (1) then getting the aggregated values for each alternative.

Table 9. Final value for proposition 1

Alternative	$CSNWGHM^{1,1}$	Score value	Ranking
@1	$\langle 0.988, 0.003, 0.010; 0.649 \rangle$	0.351	2
@2	$\langle 0.964, 0.007, 0.027; 0.674 \rangle$	0.326	3
@3	$\langle 0.969, 0.006, 0.023; 0.703 \rangle$	0.297	5
@4	$\langle 0.972, 0.006, 0.021; 0.679 \rangle$	0.321	4
@5	$\langle 0.988, 0.002, 0.009; 0.632 \rangle$	0.368	1

Use the Table 8 values to obtain the $CSNWGHM^{1,1}$ value.

$$\breve{\delta} = 1 - \left\{ 1 - \left\{ \begin{array}{l} \left(1-\left((1-0.980^{0.325})^1 \times (1-0.980^{0.325})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.325})^1 \times (1-0.937^{0.325})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.325})^1 \times (1-0.797^{0.098})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.325})^1 \times (1-0.980^{0.028})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.325})^1 \times (1-0.960^{0.052})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.325})^1 \times (1-0.893^{0.172})^1\right)\right) \times \\ \left(1-\left((1-0.937^{0.325})^1 \times (1-0.937^{0.325})^1\right)\right) \times \\ \left(1-\left((1-0.937^{0.325})^1 \times (1-0.797^{0.098})^1\right)\right) \times \\ \left(1-\left((1-0.937^{0.325})^1 \times (1-0.980^{0.028})^1\right)\right) \times \\ \left(1-\left((1-0.937^{0.325})^1 \times (1-0.960^{0.052})^1\right)\right) \times \\ \left(1-\left((1-0.937^{0.325})^1 \times (1-0.893^{0.172})^1\right)\right) \times \\ \left(1-\left((1-0.797^{0.098})^1 \times (1-0.797^{0.098})^1\right)\right) \times \\ \left(1-\left((1-0.797^{0.098})^1 \times (1-0.980^{0.028})^1\right)\right) \times \\ \left(1-\left((1-0.797^{0.098})^1 \times (1-0.960^{0.052})^1\right)\right) \times \\ \left(1-\left((1-0.797^{0.098})^1 \times (1-0.893^{0.172})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.028})^1 \times (1-0.980^{0.028})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.028})^1 \times (1-0.960^{0.052})^1\right)\right) \times \\ \left(1-\left((1-0.980^{0.028})^1 \times (1-0.893^{0.172})^1\right)\right) \times \\ \left(1-\left((1-0.960^{0.052})^1 \times (1-0.960^{0.052})^1\right)\right) \times \\ \left(1-\left((1-0.960^{0.052})^1 \times (1-0.893^{0.172})^1\right)\right) \times \\ \left(1-\left((1-0.893^{0.172})^1 \times (1-0.893^{0.172})^1\right)\right) \end{array} \right\}^{\frac{2}{42}} \right\}^{\frac{1}{2}}$$

$$= 1 - \{0.000153\}^{\frac{1}{2}} = 0.988.$$

Similarly, we can find radius value.

$$\breve{\tau} = \left\{ \mathbb{1} - \left\{ \begin{array}{l} \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.013)^{0.325})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.047)^{0.098})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.003)^{0.028})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.007)^{0.052})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \times \\ \left(\mathbb{1} - \left((\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1} \times (\mathbb{1} - (\mathbb{1} - 0.023)^{0.172})^{1}\right)\right) \end{array} \right\}^{\frac{\mathbb{2}}{42}} \right\}^{\frac{\mathbb{1}}{2}}$$

$$= \left\{\mathbb{1} - \{0.99986\}^{\frac{\mathbb{2}}{42}}\right\}^{\frac{\mathbb{1}}{2}} = 0.003.$$

Similarly, we can find falsity value.

Ranking: @5 > @1 > @2 > @4 > @3 it shows the alternative 5 bio - fertilizer is the best for guava tree among other fertilizers.

Now, use the theorem 3.5 and follows the similar steps above and the final ranking is,

Ranking: @5 > @1 > @2 > @3 > @4.

Now, use the theorem 3.6 and follows the similar steps above and the final ranking is,

Ranking: @5 > @1 > @4 > @2 > @3.

Obtain the three results, the alternative 5 bio - fertilizer is the best for guava tree among other fertilizers.

Proposition 2:

Taking the proposition 2 and applying the step 6 to step 8. Using the same criteria weight same as proposition 1 and using the aggregation operator. Next, apply the score function and getting the final values. Ranking the alternatives highest values is the best alternative and lowest value is the worst alternative.

Table 10. FE's opinion for proposition 2

Alternatives	@1	@2	@3	@4	@5
C13H	$\langle ɣɤ_2, ɣɤ_1, ɣɤ_4 \rangle$	$\langle ɣɤ_3, ɣɤ_4, ɣɤ_1 \rangle$	$\langle ɣɤ_3, ɣɤ_2, ɣɤ_5 \rangle$	$\langle ɣɤ_2, ɣɤ_1, ɣɤ_6 \rangle$	$\langle ɣɤ_2, ɣɤ_3, ɣɤ_4 \rangle$
C23M	$\langle ɣɤ_4, ɣɤ_5, ɣɤ_3 \rangle$	$\langle ɣɤ_2, ɣɤ_6, ɣɤ_3 \rangle$	$\langle ɣɤ_1, ɣɤ_2, ɣɤ_4 \rangle$	$\langle ɣɤ_1, ɣɤ_2, ɣɤ_6 \rangle$	$\langle ɣɤ_3, ɣɤ_4, ɣɤ_5 \rangle$
C31H	$\langle ɣɤ_1, ɣɤ_2, ɣɤ_4 \rangle$	$\langle ɣɤ_2, ɣɤ_1, ɣɤ_2 \rangle$	$\langle ɣɤ_1, ɣɤ_4, ɣɤ_5 \rangle$	$\langle ɣɤ_2, ɣɤ_1, ɣɤ_6 \rangle$	$\langle ɣɤ_4, ɣɤ_3, ɣɤ_5 \rangle$
C41L	$\langle ɣɤ_2, ɣɤ_1, ɣɤ_4 \rangle$	$\langle ɣɤ_2, ɣɤ_3, ɣɤ_6 \rangle$	$\langle ɣɤ_3, ɣɤ_2, ɣɤ_5 \rangle$	$\langle ɣɤ_2, ɣɤ_4, ɣɤ_3 \rangle$	$\langle ɣɤ_5, ɣɤ_6, ɣɤ_4 \rangle$
C52M	$\langle ɣɤ_1, ɣɤ_2, ɣɤ_3 \rangle$	$\langle ɣɤ_3, ɣɤ_4, ɣɤ_5 \rangle$	$\langle ɣɤ_3, ɣɤ_2, ɣɤ_4 \rangle$	$\langle ɣɤ_1, ɣɤ_2, ɣɤ_3 \rangle$	$\langle ɣɤ_3, ɣɤ_4, ɣɤ_6 \rangle$
C62M	$\langle ɣɤ_4, ɣɤ_6, ɣɤ_5 \rangle$	$\langle ɣɤ_4, ɣɤ_5, ɣɤ_6 \rangle$	$\langle ɣɤ_3, ɣɤ_4, ɣɤ_5 \rangle$	$\langle ɣɤ_5, ɣɤ_2, ɣɤ_6 \rangle$	$\langle ɣɤ_5, ɣɤ_4, ɣɤ_6 \rangle$

Table 11. CSNSs for proposition 2

C/A	C11H	C21H	C31H	C41M	C51L	C61H
@1	$\langle 0.900,0.022,0.078; 0.179 \rangle$	$\langle 0.750,0.055,0.195; 0.169 \rangle$	$\langle 0.900,0.022,0.078; 0.179 \rangle$	$\langle 0.900,0.022,0.078; 0.179 \rangle$	$\langle 0.937,0.013,0.050; 0.085 \rangle$	$\langle 0.590,0.075,0.335; 0.273 \rangle$
@2	$\langle 0.877,0.028,0.095; 0.158 \rangle$	$\langle 0.733,0.043,0.223; 0.456 \rangle$	$\langle 0.960,0.007,0.033; 0.052 \rangle$	$\langle 0.733,0.043,0.223; 0.456 \rangle$	$\langle 0.750,0.055,0.195; 0.169 \rangle$	$\langle 0.590,0.075,0.335; 0.273 \rangle$
@3	$\langle 0.810,0.040,0.150; 0.245 \rangle$	$\langle 0.900,0.022,0.078; 0.179 \rangle$	$\langle 0.793,0.045,0.162; 0.266 \rangle$	$\langle 0.810,0.040,0.150; 0.245 \rangle$	$\langle 0.857,0.032,0.112; 0.124 \rangle$	$\langle 0.750,0.055,0.195; 0.169 \rangle$
@4	$\langle 0.777,0.033,0.190; 0.511 \rangle$	$\langle 0.777,0.033,0.190; 0.511 \rangle$	$\langle 0.777,0.033,0.190; 0.511 \rangle$	$\langle 0.857,0.032,0.112; 0.124 \rangle$	$\langle 0.937,0.013,0.050; 0.085 \rangle$	$\langle 0.650,0.060,0.290; 0.380 \rangle$
@5	$\langle 0.857,0.032,0.112; 0.124 \rangle$	$\langle 0.750,0.055,0.195; 0.169 \rangle$	$\langle 0.750,0.055,0.195; 0.169 \rangle$	$\langle 0.590,0.075,0.335; 0.273 \rangle$	$\langle 0.673,0.058,0.268; 0.380 \rangle$	$\langle 0.590,0.075,0.335; 0.273 \rangle$

Table 12. Final value for proposition 2

Calculating the aggregated value using the theorem 3.1,

Alternative	$CSNWGHM^{1,1}$	Score value	Ranking
@1	$\langle 0.960,0.007,0.031; 0.757 \rangle$	0.791	2
@2	$\langle 0.956,0.007,0.035; 0.784 \rangle$	0.783	4
@3	$\langle 0.968,0.006,0.024; 0.770 \rangle$	0.792	1
@4	$\langle 0.955,0.006,0.038; 0.867 \rangle$	0.761	5
@5	$\langle 0.951,0.009,0.038; 0.755 \rangle$	0.787	3

Ranking: @3 > @1 > @5 > @2 > @4 it gives the alternative 3 urea + potash blend is best fertilizer for banana tree comparatively other four fertilizers.

Now using the theorem 3.5, the final ranking is

Ranking: @3 > @1 > @4 > @5 > @2.

Using the theorem 3.6, the final ranking is,

Ranking: @5 > @3 > @2 > @1 > @4. Among the three results majority of two results says, the alternative 3 urea + potash blend is suitable for banana tree.

5.1 Sensitive analysis

This part compares the final results if changing the parameter values then it affect the results are not. Here we changing the $\bar{\bar{f}}$, $\bar{\bar{g}}$ values but it did not affect the results except 0, 0.25 and 0, 0.5. These two are only changing the last two alternatives which means the very small change only occur. The below table show the effectiveness and stability of the proposed method. In proposition 1, the alternative 5 bio - fertilizer is the suitable fertilizer for guava tree. In proposition 2 the alternative 3 urea + potash blend is most effectiveness one for banana tree. This study shows the strong correlation of the proposed framework.

Table 13. Different parameter values

Operator	Proposition	Ranking	Best & worst
$CSNWGHM^{2,1}$	Proposition 1	@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{1,2}$		@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{0,0.25}$		@5 > @1 > @2 > @3 > @4	(@5,@4)
$CSNWGHM^{0,0.5}$		@5 > @1 > @2 > @3 > @4	(@5,@4)
$CSNWGHM^{0,0.75}$		@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{0.25,0}$		@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{0.5,0}$		@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{0.75,0}$		@5 > @1 > @2 > @4 > @3	(@5,@3)
$CSNWGHM^{2,1}$	Proposition 2	@3 > @1 > @5 > @2 > @4	(@3,@4)
$CSNWGHM^{1,2}$		@3 > @5 > @1 > @2 > @4	(@3,@4)
$CSNWGHM^{0,0.25}$		@3 > @1 > @2 > @5 > @4	(@3,@4)
$CSNWGHM^{0,0.5}$		@3 > @1 > @2 > @5 > @4	(@3,@4)
$CSNWGHM^{0,0.75}$		@3 > @1 > @2 > @5 > @4	(@3,@4)
$CSNWGHM^{0.25,0}$		@1 > @5 > @3 > @2 > @4	(@1,@4)
$CSNWGHM^{0.5,0}$		@1 > @5 > @3 > @2 > @4	(@1,@4)
$CSNWGHM^{0.75,0}$		@1 > @5 > @3 > @2 > @4	(@1,@4)

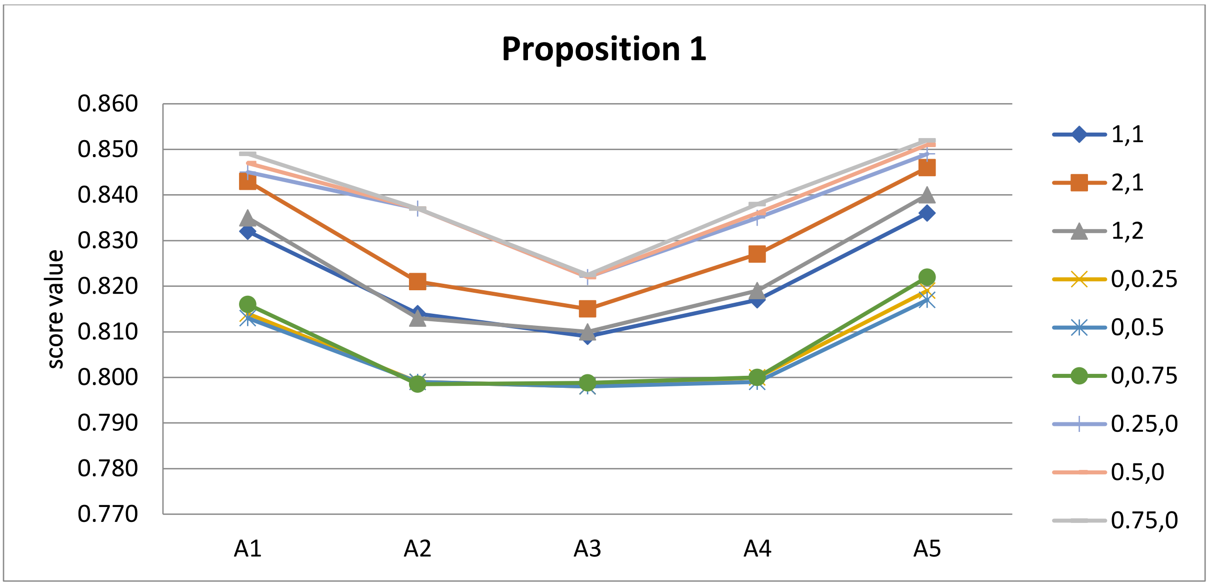

Figure 4. Sensitive analysis for proposition 1

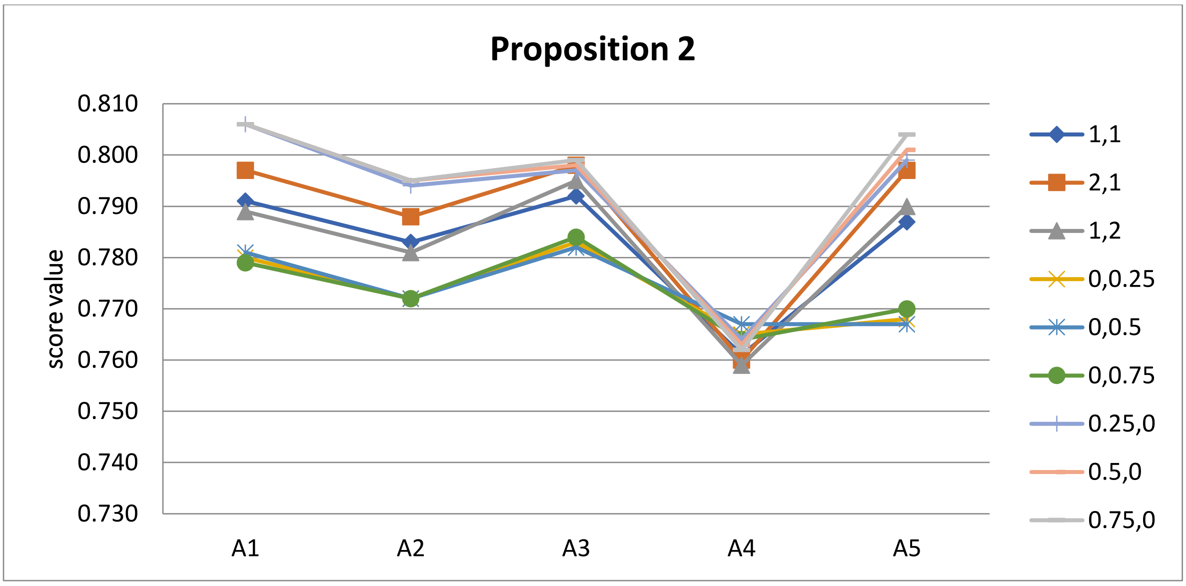

Figure 5. Sensitive analysis for proposition 2

Now, compare this proposed method to other aggregation operators. The below table shows the final ranking of the each methods.

Table 14. Comparison with other operators

Propositions	Operator	Ranking
Proposition 1	Gomathi et al. [9] (Arithmetic)	@5 > @1 > @4 > @2 > @3
	Gomathi et al. [9] (Geometric)	@5 > @1 > @4 > @2 > @3
	Biswas et al. [4]	@5 > @1 > @4 > @2 > @3
	Kamal et al. [11]	@5 > @1 > @4 > @2 > @3

	Proposed method	@5 > @1 > @4 > @2 > @3
Proposition 2	Gomathi et al. [9] (Arithmetic)	@3 > @1 > @5 > @2 > @4
	Gomathi et al. [9] (Geometric)	@3 > @1 > @2 > @5 > @4
	Biswas et al. [4]	@3 > @1 > @5 > @2 > @4
	Kamal et al. [11]	@3 > @1 > @5 > @2 > @4
	Proposed method	@3 > @1 > @5 > @2 > @4

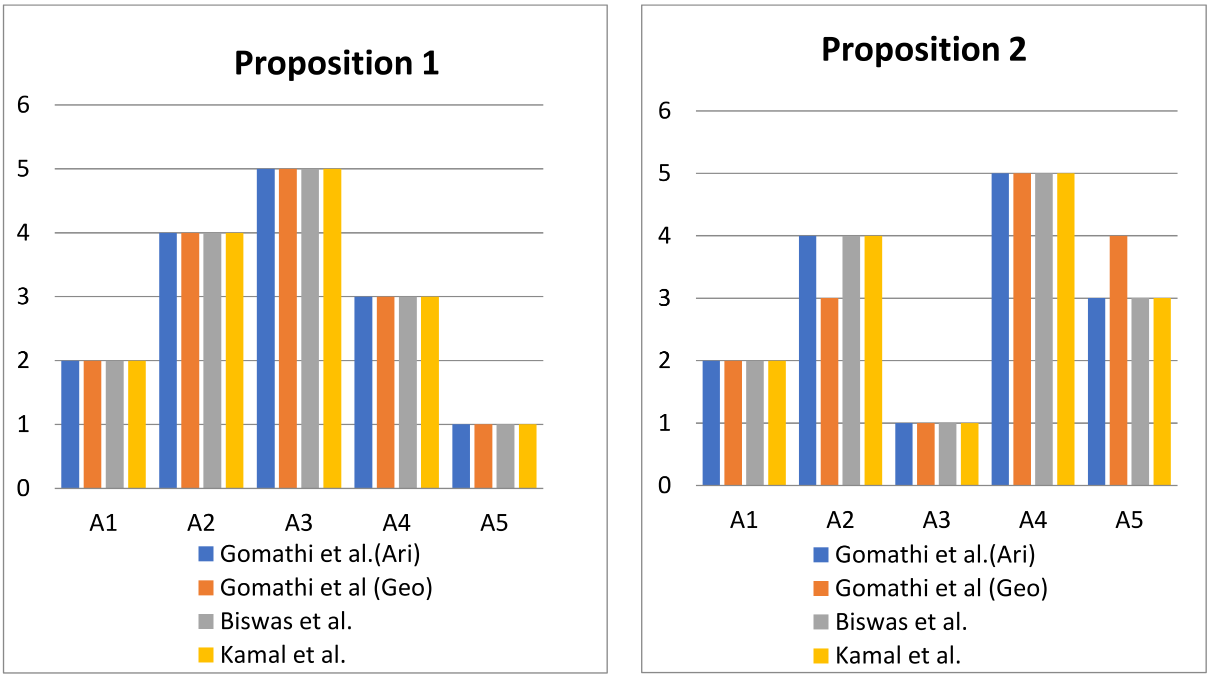

Figure 6. Comparison graph for both propositions

The above table shows the stability and effectiveness of the proposed method. All the ranking gives the same best and worst alternatives. In this figure, the lowest graph gives the high impact factor. As per results, bio – fertilizer (alternative 5) is best for proposition 1 and urea + potash blend (alternative 3) is best for proposition 2.

6. Comparison

This study introduced the new aggregation operator namely CSNWGHM using CSNSs. The literature review gives the existing studies using HM operator. From that there is no study using HM in CSNSs. Next, proposed the aggregation operator and verifying the properties as idempotency, monotonicity and boundedness. To make the algorithm for CSNWGHM in MCDM problem and the SWARA method is used to finding criteria weights. Here, the proposed method is used to finding the best fertilizer for guava and banana trees which helps to yielding the maximum profit and lead to use the appropriate fertilizer. In this problem, we use 5 alternatives and 6 criteria under 3 field expert opinions. Make the propositions of criteria and sub - criteria through the NSHSSs. Applied the proposed operator and score value to get the right fertilizer. From the application of this method alternative 5 bio - fertilizer gives high effective for guava tree and alternative 3 urea + potash blend is suitable for banana tree. This method is applicable for plants, industries, etc. The sensitive analysis and comparative analysis gives the strong correlation and effectiveness of this approach.

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