Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

MADM Model Based on Trigonometric Aggregation Operators of Linguistic Neutrosophic Values and Its Application in Selecting the Crawling Robot Design Schemes

Yongbiao Luo , Hui Pan *

[1] School of Information and Mechatronic Engineering, Shaoxing University Yuanpei College, Shaoxing. Shaoxing China; yongbiaoluo8@gmail.com, panhui33@163.com

* Correspondence: panhui33@163.com; Tel.: (+86 13575585569)

Abstract: Linguistic neutrosophic multi-attribute decision making (MADM) has become one of critical research topics in decision theory. However, existing aggregation operations of linguistic neutrosophic values (LNVs) do not contain operational periodicity. As a result, they are difficult to handle MADM problems with the polytemporal/periodic needs in LNV scenarios. To fill this research gap, this paper intends to develop a MADM model based on the trigonometric aggregation operators (TAOs) of LNVs for MADM applications with polytemporal/periodic needs in LNV scenarios. To do it, we first introduce the linguistic trigonometric t-norms and t-conorms and the LNV trigonometric operation laws based on the trigonometry 1 (including tangent, cotangent, inverse cotangent, and arctangent functions) and the trigonometry 2 (including sine, cosine, arccosine, and arcsine functions). Second, we propose the LNV trigonometric weighted average ($LNVTWA_{T1}$ and $LNVTWA_{T2}$) operators and LNV trigonometric weighted geometric ($LNVTWG_{T1}$ and $LNVTWG_{T2}$) operators based on the trigonometry 1 and 2. Third, a MADM model is developed in terms of one of the proposed four TAOs to address MADM problems with polytemporal phases/periodicity in a LNV scenario. Lastly, the developed MADM model is used for a MADM application of crawling robot design schemes to verify its validity and rationality.

Keywords: Linguistic neutrosophic value; Linguistic trigonometric operation; Linguistic neutrosophic value trigonometric aggregation operator; Decision making; Crawling robot design scheme

1. Introduction

Linguistic multi-attribute decision making (MADM) has become one of the research hotspots because the linguistic assessment reflects the qualitative judgments and representation of decision makers in MADM applications. As a result, linguistic MADM issues have revealed their importance and necessity in decision theory and methodology. Consequently, they have received a great deal of attention from scholars since a linguistic variable was presented by Zadeh [1]. For example, linguistic decision-making (DM) techniques have been developed to perform linguistic DM problems [2-6]. Based on an extension of linguistic DM techniques, linguistic intuitionistic fuzzy DM techniques have been proposed to solve linguistic intuitionistic fuzzy DM problems [7-13]. However, the linguistic intuitionistic fuzzy value (LIFV) can only convey membership/truth and nonmembership/falsity linguistic values but cannot express true, false, and uncertain linguistic values in inconsistent and indeterminate linguistic DM scenarios. Then, linguistic neutrosophic values (LNVs) and their DM techniques [14] were proposed because their more general expression and decision frameworks can fill the research gap of linguistic intuitionistic fuzzy DM techniques. It is well known that LNV aggregation operators (AOs) play an important role in linguistic neutrosophic DM applications. Therefore, Fang and Ye [14] first presented the LNV weighted AOs to address linguistic neutrosophic group DM issues. Then, Fan et al. [15] introduced LNV Bonferroni mean AOs and their group DM approach. Liu and You [16] presented some LNV Hamy mean AOs and their group DM technique. Liang et al. [17] put forward the LNV Hamacher AOs and applied them in the assessment of mine land reclamation. Liu et al. [18] proposed the power Heronian AOs of LNVs for group DM. Fan et al. [19] introduced the LNV Einstein AOs and their DM application. Zhang et al. [20] developed the LNV Dombi AOs for the DM application of slope treatment schemes. Liu and You [21] developed the partitioned Maclaurin symmetric mean AOs for group DM. Zhang and Ye [22] proposed the single-valued neutrosophic value (SVNV) and LNV hybrid AOs for group DM. Luo et al. [23] presented the LNV Maclaurin symmetric mean AOs and used them for the performance evaluation of human resources. Li et al. [24] introduced the reliability allocation method using the LNV weighted Muirhead mean AO. Recently, Ye et al. [25, 26] proposed the trigonometric aggregation operators (TAOs) of SVNVs and single-valued neutrosophic credibility values and their DM techniques, but these TAOs cannot be used in linguistic neutrosophic DM scenarios.

Considering the emerging LNV operations or aggregation algorithms in the existing literature, none of them implies the operational properties of polytemporal phases/periodicity in LNV scenarios. In this case, their DM techniques are also difficult to perform the MADM problems with periodicity/polytemporal phases in LNV scenarios, which show the research gap. However, LNV is a more general linguistic framework that includes the linguistic value and IFLV. Therefore, in LNV scenarios, it is necessary to develop a new MADM technique to fill the existing research gap.

Motivated based on TAOs [25-27], this paper intends to propose a MADM model based on the TAOs of LNVs for addressing the MADM problem with polytemporal phases/periodicity in the LNV scenario. First, we propose the linguistic trigonometric operations (LTOs) of the linguistic trigonometric t-norms and t-conorms and the trigonometric operation laws (TOLs) of LNVs based on the trigonometry 1 (including tangent, cotangent, inverse cotangent, and arctangent functions) and the trigonometry 2 (including sine, cosine, arccosine, and arcsine functions). Next, we propose the LNV trigonometric weighted average (LNVTWAT1 and LNVTWAT2) operators and the LNV trigonometric weighted geometric (LNVTWGT1 and LNVTWGT2) operators in terms of TOLs of LNVs based on the trigonometry 1 and 2. Furthermore, a MADM model is developed based on one of the four proposed LNV TAOs. Finally, the developed MADM model is used for a DM application of crawling robot design schemes (CRDSs). Through the comparison of the developed model with the existing linguistic neutrosophic DM model, the validity and rationality of the linguistic neutrosophic DM application are verified.

Summerly, this original work mainly creates these new achievements below:

- The new TOLs of LNVs are presented based on the trigonometry 1 and 2 to provide the polytemporal/periodic benefits for the LNV operations.
- The $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators are proposed to provide the critical mathematical tools for MADM modeling in LNV scenarios.
- The MADM model using one of the proposed TAOs of LNVs can effectively help decision makers to select the best CRDS and meet the polytemporal/periodic DM requirements in LNV scenarios.

The remainder of the paper includes the following sections. Section 2 reviews the preliminaries of LNVs for the further study of this paper. Section 3 introduces the linguistic trigonometric t-norms and t-conorms and the TOLs of LNVs based on the trigonometry 1 and 2. Section 4 presents the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators based on the TOLs of LNVs and their properties. In Section 5, a MADM model is developed based on one of the four proposed TAOs of LNVs in a LNV scenario. Section 6 uses the developed MADM model for a DM application of CRDSs to verify its rationality and validity, and a comparison with the existing MADM model in the scenario of LNVs reflects the superiority of the developed model. Section 7 presents some conclusions and future research..

2. Preliminaries of LNVs

Set a linguistic term set (LTS) as $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ subject to odd cardinality b+1. Fang and Ye [14] first defined the LNV $\lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ on λ_L such that $\lambda_{td}, \lambda_{ud}, \lambda_{fd} \in \lambda_L$ and $td, ud, fd \in [0, b]$, where λ_{td}, λ_{ud}, and λ_{fd} are the true, uncertain, and false linguistic variables, respectively.

Support that there are two LNVs $\lambda_{LNV(1)} = <\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}>$ and $\lambda_{LNV(2)} = <\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}>$ in λ_L and $e > 0$. Then their operation laws are introduced below [14]:

(i) $\lambda_{LNV(1)} \oplus \lambda_{LNV(2)} = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle \oplus \left\langle \lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)} \right\rangle = \left\langle \lambda_{td(1)+td(2)-\frac{td(1)td(2)}{b}}, \lambda_{\frac{ud(1)ud(2)}{b}}, \lambda_{\frac{fd(1)fd(2)}{b}} \right\rangle$;

(ii) $\lambda_{LNV(1)} \otimes \lambda_{LNV(2)} = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle \otimes \left\langle \lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)} \right\rangle = \left\langle \lambda_{\frac{td(1)td(2)}{b}}, \lambda_{ud(1)+ud(2)-\frac{ud(1)ud(2)}{b}}, \lambda_{fd(1)+fd(2)-\frac{fd(1)fd(2)}{b}} \right\rangle$;

(ii) $e \cdot \lambda_{LNV(1)} = e \cdot \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle = \left\langle \lambda_{b-b\left(1-\frac{td(1)}{b}\right)^e}, \lambda_{b\left(\frac{ud(1)}{b}\right)^e}, \lambda_{b\left(\frac{fd(1)}{b}\right)^e} \right\rangle$;

(iv) $\lambda_{LNV(1)}^e = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle^e = \left\langle \lambda_{b\left(\frac{td(1)}{b}\right)^e}, \lambda_{b-b\left(1-\frac{ud(1)}{b}\right)^e}, \lambda_{b-b\left(1-\frac{fd(1)}{b}\right)^e} \right\rangle$.

Regarding a series of LNVs $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ subject to their weights e_j (j = 1, 2, …, p) for $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$, the LNV weighted average (LNVWA) and LNV weighted geometric (LNVWG) operators [14] are introduced below:

$$LNVWA(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \ldots, \lambda_{LNV(p)}) = \sum_{j=1}^{p} e_j \lambda_{LNV(j)} = \left\langle \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{td(j)}{b}\right)^{e_j}}, \lambda_{b\prod_{j=1}^{p}\left(\frac{ud(j)}{b}\right)^{e_j}}, \lambda_{b\prod_{j=1}^{p}\left(\frac{fd(j)}{b}\right)^{e_j}} \right\rangle, \quad (1)$$

$$LNVWG(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \ldots, \lambda_{LNV(p)}) = \prod_{j=1}^{p} \lambda_{LNV(j)}^{e_j} = \left\langle \lambda_{b\prod_{j=1}^{p}\left(\frac{td(j)}{b}\right)^{e_j}}, \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{ud(k)}{b}\right)^{e_j}}, \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{fd(j)}{b}\right)^{e_j}} \right\rangle, \quad (2)$$

Then, Fang and Ye [14] introduced the score and accuracy equations of $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$:

$$E(\lambda_{LNV(j)}) = (2b + td(j) - ud(j) - fd(j))/3b \text{ for } E(\lambda_{LNV(j)}) \in [0, 1], \quad (3)$$

$$F(\lambda_{LNV(j)}) = (td(j) - fd(j))/b \text{ for } E(\lambda_{LNV(j)}) \in [-1, 1], \quad (4)$$

Regarding two LNVs $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ for j = 1, 2, their comparative rules [14] are introduced below:

(i) $\lambda_{LNV(1)} > \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) > E(\lambda_{LNV(2)})$;

(ii) $\lambda_{LNV(1)} > \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) = E(\lambda_{LNV(2)})$ and $F(\lambda_{LNV(1)}) > F(\lambda_{LNV(2)})$;

(iii) $\lambda_{LNV(1)} \cong \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) = E(\lambda_{LNV(2)})$ and $F(\lambda_{LNV(1)}) = F(\lambda_{LNV(2)})$.

3. LTOs and LNV TOLs

Based on the trigonometry 1 and the trigonometry 2 [25–27], this section introduces the linguistic trigonometric t-norms and t-conorms and the LNV TOLs.

First, we define the linguistic trigonometric t-norms and t-conorms based on the trigonometry 1 and 2.

Definition 1. Let two linguistic variables be λ_v, λ_μ in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$. Based on the trigonometry 1 and 2, the linguistic trigonometric t-norms $N_{R1}(\lambda_v, \lambda_\mu)$ and $N_{R2}(\lambda_v, \lambda_\mu)$ and the linguistic trigonometric t-conorms $N_{S1}(\lambda_v, \lambda_\mu)$ and $N_{S2}(\lambda_v, \lambda_\mu)$ between λ_v and λ_μ are defined as the following LTOs:

$$N_{R1}\left(\lambda_v, \lambda_\mu\right) = \lambda_{2b/\pi\cot^{-1}\left(\cot(n\pi+v\pi/2b)+\cot(n\pi+\mu\pi/2b)\right)} = \lambda_{2b/\pi\cot^{-1}\left(\cot(v\pi/2b)+\cot(\mu\pi/2b)\right)}, \quad (5)$$

$$N_{S1}(\lambda_v, \lambda_\mu) = \lambda_{2b/\pi\tan^{-1}\left(\tan(n\pi+v\pi/2b)+\tan(n\pi+\mu\pi/2b)\right)} = \lambda_{2b/\pi\tan^{-1}\left(\tan(v\pi/2b)+\tan(\mu\pi/2b)\right)}, \quad (6)$$

$$N_{R2}\left(\lambda_{v},\lambda_{\mu}\right)=\lambda_{2b/\pi\sin^{-1}\left(\sin(2n\pi+v\pi/2b)\sin(2n\pi+\mu\pi/2b)\right)}=\lambda_{2b/\pi\sin^{-1}\left(\sin(v\pi/2b)\sin(\mu\pi/2b)\right)}, \quad (7)$$

$$N_{S2}(\lambda_{v},\lambda_{\mu})=\lambda_{2b/\pi\cos^{-1}\left(\cos(2n\pi+v\pi/2b)\cos(2n\pi+\mu\pi/2b)\right)}=\lambda_{2b/\pi\cos^{-1}\left(\cos(v\pi/2b)\cos(\mu\pi/2b)\right)}, \quad (8)$$

It is clear that there are $n\pi$ (n = 1, 2, ..., p) periodicity in Eqs. (5) and (6) and $2n\pi$ (n = 1, 2, ..., p) periodicity in Eqs. (7) and (8). In terms of the LTOs of the trigonometry 1 and 2, we can define the TOLs of LNVs.

Definition 2. Let $\lambda_{LNV(1)} = \langle\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}\rangle$ and $\lambda_{LNV(2)} = \langle\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}\rangle$ be two LNVs in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ and $e > 0$. Then, their TOLs of LNVs based on the trigonometry 1 are defined in the following:

(1) $$\lambda_{LNV(1)}\oplus_{T1}\lambda_{LNV(2)}=\left\langle\begin{matrix}\lambda_{2b/\pi\tan^{-1}\left(\tan(td(1)\pi/2b)+\tan(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(ud(1)\pi/2b)+\cot(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(fd(1)\pi/2b)+\cot(fd(2)\pi/2b)\right)}\end{matrix}\right\rangle;$$

(2) $$\lambda_{LNV(1)}\otimes_{T1}\lambda_{LNV(2)}=\left\langle\begin{matrix}\lambda_{2b/\pi\cot^{-1}\left(\cot(td(1)\pi/2b)+\cot(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\tan^{-1}\left(\tan(ud(1)\pi/2b)+\tan((ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\tan^{-1}\left(\tan(fd(1)\pi/2b)+\tan(fd(2)\pi/2b)\right)}\end{matrix}\right\rangle;$$

(3) $$e\lambda_{LNV(1)}=\left\langle\lambda_{2b/\pi\tan^{-1}\left(e\tan(td(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(ud(1)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e\cot(fd(1)\pi/2b)\right)}\right\rangle;$$

(4) $$(\lambda_{LNV(1)})^{e}=\left\langle\lambda_{2b/\pi\cot^{-1}\left(e\cot(td(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(ud(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(fd(1)\pi/2b)\right)}\right\rangle.$$

However, the above operation results are still LNVs.

Definition 3. Let $\lambda_{LNV(1)} = \langle\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}\rangle$ and $\lambda_{LNV(2)} = \langle\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}\rangle$ be two LNVs in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ and $e > 0$. Then, their TOLs based on the trigonometry 2 are defined in the following:

(1) $$\lambda_{LNV(1)}\oplus_{T2}\lambda_{LNV(2)}=\left\langle\begin{matrix}\lambda_{2b/\pi\cos^{-1}\left(\cos(td(1)\pi/2b)\cos(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\sin^{-1}\left(\sin(ud(1)\pi/2b)\sin(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\sin^{-1}\left(\sin(fd(1)\pi/2b)\sin(fd(2)\pi/2b)\right)}\end{matrix}\right\rangle;$$

(2) $$\lambda_{LNV(1)}\otimes_{T2}\lambda_{LNV(2)}=\left\langle\begin{matrix}\lambda_{2b/\pi\sin^{-1}\left(\sin(td(1)\pi/2b)\sin(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cos^{-1}\left(\cos(ud(1)\pi/2)\cos(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cos^{-1}\left(\cos(fd(1)\pi/2)\cos(fd(2)\pi/2b)\right)}\end{matrix}\right\rangle;$$

(3) $$e\lambda_{LNV(1)}=\left\langle\lambda_{2b/\pi\cos^{-1}\left(\cos(td(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\sin^{-1}\left(\sin(ud(2)\pi/2b)\right)^{e}},\lambda_{2b/\pi\sin^{-1}\left(\sin(fd(2)\pi/2b)\right)^{e}}\right\rangle$$

(4) $$(\lambda_{LNV(1)})^{e}=\left\langle\lambda_{2b/\pi\sin^{-1}\left(\sin(td(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\cos^{-1}\left(\cos(ud(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\cos^{-1}\left(\cos(fd(1)\pi/2b)\right)^{e}}\right\rangle,$$

Obviously, the above operation results are still LNVs.

4. TAOs of LNVs based on the trigonometry 1 and 2

According to TOLs of LNVs based on the trigonometry 1 and 2 in Definitions 2 and 3, this section presents four TAOs of LNVs and their characteristics.

4.1 LNVTWA$_{T1}$ and LNVTWA$_{T2}$ operators

This part proposes the LNVTWA$_{T1}$ and LNVTWA$_{T2}$ operators based on the TOLs of LNVs in Definitions 2 and 3.

Definition 4. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and $LNVTWA_{T1}$ and $LNVTWA_{T2}$: $\Omega^p \to \Omega$. Assume that e_j (j = 1, 2, …, p) is the weight of $\lambda_{LNV(j)}$ with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the $LNVTWA_{T1}$ and $LNVTWA_{T2}$ operators are defined, respectively, in the following:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = e_1\lambda_{LNV(1)} \oplus_{T1} e_2\lambda_{LNV(2)} \oplus_{T1} ... \oplus_{T1} e_p\lambda_{LNV(p)} = \sum_{j=1}^{p}{}_{T1} e_j\lambda_{LNV(j)}, \quad (9)$$

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = e_1\lambda_{LNV(1)} \oplus_{T2} e_2\lambda_{LNV(2)} \oplus_{T2} ... \oplus_{T2} e_p\lambda_{LNV(p)} = \sum_{j=1}^{p}{}_{T2} e_j\lambda_{LNV(j)}. \quad (10)$$

Theorem 1. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs. Assume that e_j (j = 1, 2, …, p) is the weight of $\lambda_{LNV(j)}$ with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the $LNVTWA_{T1}$ operator is LNV, which is gotten by

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p}{}_{T1} e_j\lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(fd(j)\pi/2b)\right)\right)} \right\rangle, \quad (11)$$

Proof: Mathematical induction is applied to the proof of Eq. (11).

(1) Set p = 2. The operational result corresponding to the TOLs (1) and (3) in Definition 2 is gotten below:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}) = e_1\lambda_{LNV(1)} \oplus_{T1} e_2\lambda_{LNV(2)}$$

$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \tan^{-1}\left(\tan\left(\pi/2b \times 2b/\pi \tan^{-1}(e_1 \tan(td(1)\pi/2b))\right) + \tan\left(\pi/2b \times 2b/\pi \tan^{-1}(e_2 \tan(td(2)\pi/2b))\right)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(\cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_1 \cot(ud(1)\pi/2b))\right) + \cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_2 \cot(ud(2)\pi/2b))\right)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(\cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_1 \cot(fd(1)\pi/2b))\right) + \cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_2 \cot(fd(2)\pi/2b))\right)\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \tan^{-1}(e_1 \tan(td(1)\pi/2b) + e_2 \tan(td(2)\pi/2b))}, \\ \lambda_{2b/\pi \cot^{-1}(e_1 \cot(ud(1)\pi/2b) + e_2 \cot(ud(2)\pi/2b))}, \\ \lambda_{2b/\pi \cot^{-1}(e_1 \cot(fd(1)\pi/2b) + e_2 \cot(fd(2)\pi/2b))} \end{array} \right\rangle, \quad (12)$$

$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{2} e_j \tan(te(j)\pi/2b)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{2} e_j \cot(ue(j)\pi/2b)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{2} e_j \cot(fe(j)\pi/2b)\right)} \right\rangle.$$

(2) Set $p = m$. Eq. (11) can keep the equation:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(m)}) = \sum_{j=1}^{m}{}_{T1} e_j\lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{m}\left(e_j \tan(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{m}\left(e_j \cot(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{m}\left(e_j \cot(fd(j)\pi/2b)\right)\right)} \right\rangle, \quad (13)$$

(3) Set p = m+1. In terms of the TOLs (1) and (3) in Definition 2 and Eqs. (12) and (13) , the operational result is given below:

$$LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(m)},\lambda_{LNV(m+1)})=\sum_{j=1}^{m+1}{}_{T1}e_j\lambda_{LNFV(j)}$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(ud(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle$$

$$\oplus_{T1}e_{m+1}\lambda_{LNV(m+1)}$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle$$

$$\oplus_{T1}\left\langle \lambda_{2b/\pi\tan^{-1}\left(e_{m+1}\tan(td(j)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e_{m+1}\cot(ud(j)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e_{m+1}\cot(fd(j)\pi/2b)\right)}\right\rangle$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\cot(ud(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle.$$

Regarding the results of (1)–(3), Eq. (11) can exist for any *p*.

Thus, this proof is ended.

Theorem 2. The $LNVTWA_{T1}$ operator of Eq. (11) implies the characteristics:

(1)Idempotency: Set $\lambda_{LNV(j)}$ = <$\lambda_{td(j)}$, $\lambda_{ud(j)}$, $\lambda_{fd(j)}$> (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)}=\lambda_{LNV}$ = <λ_{td}, λ_{ud}, λ_{fd}> (j = 1, 2, …, p), then $LNVTWA_{T1}\left(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)}\right)=\lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)}$ = <$\lambda_{td(j)}$, $\lambda_{ud(j)}$, $\lambda_{fd(j)}$> (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min}=\left\langle \min_j\left(\lambda_{td(j)}\right),\max_j\left(\lambda_{ud(j)}\right),\max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max}=\left\langle \max_j\left(\lambda_{td(j)}\right),\min_j\left(\lambda_{ud(j)}\right),\min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, $\lambda_{LNV\min}\le LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})\le\lambda_{LNV\max}$ exists.

(3)Monotonicity: Set $\lambda_{LNV(j)}$ = <$\lambda_{td(j)}$, $\lambda_{ud(j)}$, $\lambda_{fd(j)}$> and $\lambda^*_{LNV(j)}=\left\langle \lambda^*_{td(j)},\lambda^*_{ud(j)},\lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)}\le\lambda^*_{LNV(j)}$, then there is $LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})\le LNVTWA_{T1}(\lambda^*_{LNV(1)},\lambda^*_{LNV(2)},...,\lambda^*_{LNV(p)})$.

Proof:

(1) Since $\lambda_{LNV(j)}=\lambda_{LNV}$ = <λ_{td}, λ_{ud}, λ_{fd}> (j = 1, 2, …, p), the aggregated result of Eq. (11) is given below:

$$LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})=\sum_{j=1}^{p}{}_{T1}e_j\lambda_{LNV}$$

$$=\left\langle \begin{matrix}\lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{p}e_j\tan(td\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}e_j\cot(ud\cdot\pi/2b)\right)}\\ \lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}e_j\cot(fd\cdot\pi/2b)\right)}\end{matrix}\right\rangle=\left\langle \begin{matrix}\lambda_{2b/\pi\tan^{-1}\left(\tan(td\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(ud\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(fd\cdot\pi/2b)\right)}\end{matrix}\right\rangle=\left\langle \lambda_{td},\lambda_{ud},\lambda_{fd}\right\rangle=\lambda_{LNV}.$$

(2)Since $\lambda_{LNV\min}$ and $\lambda_{LNV\max}$ are the minimum and maximum LNVs, $\lambda_{LNV\min} \le \lambda_{LNV(j)} \le \lambda_{LNV\max}$ exists. Consequently, $\sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV\max}$ also exists. In terms of the characteristic (1) and the characteristics of the trigonometric functions, $\lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \lambda_{LNV\max}$ can exist, i.e., $\lambda_{LNV\min} \le LNVTWA_{T1}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) \le \lambda_{LNV\max}$.

(3)Since $\lambda_{LNV(j)} \le \lambda_{LNV(j)}^{*}$, there is $\sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)}^{*}$, i.e., $LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T1}(\lambda_{LNV(1)}^{*}, \lambda_{LNV(2)}^{*}, ..., \lambda_{LNV(p)}^{*})$.

Therefore, these characteristics are true.

Theorem 3. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) (j = 1, 2, …, p) be p LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (j = 1, 2, …, p) with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the $LNVTWA_{T2}$ operator is LNV, which is given by

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(td(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(ud(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(fd(j)\pi/2b)\right)^{e_j}\right)} \right\rangle, \quad (14)$$

Proof: Mathematical induction is applied to the proof of Eq. (14).

(1) Set p = 2. Using the TOLs (1) and (3) in Definition 3, we give the result:

$$LNVTWA_{T2}(\lambda_{IFV(1)}, \lambda_{IFV(2)}) = e_1\lambda_{LNV(1)} \oplus_{T2} e_2\lambda_{LNV(2)} = \sum_{j=1}^{2}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi\cos^{-1}\left(\cos\left(\pi/2b\times 2b/\pi\cos^{-1}\left(\cos(td(1)\pi/2b)\right)^{e_1}\right)\cos\left(\pi/2b\times 2b/\pi\cos^{-1}\left(\cos(td(2)\pi/2b)\right)^{e_2}\right)\right)}, \\ \lambda_{2b/\pi\sin^{-1}\left(\sin\left(\pi/2b\times 2b/\pi\sin^{-1}\left(\sin(ud(1)\pi/2b)\right)^{e_1}\right)\sin\left(\pi/2b\times 2b/\pi\sin^{-1}\left(\sin(ud(2)\pi/2)\right)^{e_2}\right)\right)}, \\ \lambda_{2b/\pi\sin^{-1}\left(\sin\left(\pi/2b\times 2b/\pi\sin^{-1}\left(\sin(fd(1)\pi/2b)\right)^{e_1}\right)\sin\left(\pi/2b\times 2b/\pi\sin^{-1}\left(\sin(fd(2)\pi/2)\right)^{e_2}\right)\right)} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi\cos^{-1}\left(\left(\cos(td(1)\pi/2b)\right)^{e_1}\left(\cos(td(2)\pi/2b)\right)^{e_2}\right)}, \\ \lambda_{2b/\pi\sin^{-1}\left(\left(\sin(ud(1)\pi/2b)\right)^{e_1}\left(\sin(ud(2)\pi/2b)\right)^{e_2}\right)}, \\ \lambda_{2b/\pi\sin^{-1}\left(\left(\sin(fd(1)\pi/2b)\right)^{e_1}\left(\sin(fd(2)\pi/2b)\right)^{e_2}\right)} \end{array} \right\rangle \quad (15)$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{2}\left(\cos(td(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{2}\left(\sin(ud(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{2}\left(\sin(fd(j)\pi/2b)\right)^{e_j}\right)} \right\rangle.$$

(2)Set $p = m$. Eq. (14) can keep the following equation:

$$LNVTWA_{T2}(\lambda_{IFV(1)}, \lambda_{IFV(2)}, ..., \lambda_{IFV(m)}) = \sum_{j=1}^{m}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{m}\left(\cos(td(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}\left(\sin(ud(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}\left(\sin(fd(j)\pi/2b)\right)^{e_j}\right)} \right\rangle, \quad (16)$$

(3)Set $p = m+1$. In terms of the TOLs (1) and (3) in Definition 3 and Eqs. (15) and (16), we can give the result:

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(m+1)}) = \sum_{j=1}^{m+1} {}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{m}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle$$

$$\oplus_{T2} e_{m+1}\lambda_{LNV(m+1)}$$

$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{m+1}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m+1}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m+1}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle.$$

In terms of the results of (1)–(3), Eq. (14) can exist for any p.

Consequently, this proof is ended. □

Theorem 4. The LNVTWA$_{T2}$ operator of Eq. (14) implies the characteristics below.

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, ..., p), then $LNVTWA_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, $\lambda_{LNV\min} \le LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$ exists.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. $LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$ exists when $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$.

Proof:

(1) For $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, ..., p), the aggregated result of Eq. (14) is given below:

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p} {}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{g}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle$$

$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\cos(td(j)\pi/2b)\right)^{\sum_{j=1}^{p} e_j}}, \lambda_{2b/\pi \sin^{-1}\left(\sin(ud(j)\pi/2b)\right)^{\sum_{j=1}^{p} e_j}}, \lambda_{2b/\pi \sin^{-1}\left(\sin(fd(j)\pi/2b)\right)^{\sum_{j=1}^{p} e_j}} \right\rangle$$

$$= \left\langle \lambda_{td}, \lambda_{ud}, \lambda_{fd} \right\rangle = \lambda_{LNV}.$$

(2) Since λ_{LNVmin} and λ_{LNVmax} are the minimum and maximum LNVs, $\lambda_{LNVmin} \le \lambda_{LNV(j)} \le \lambda_{LNVmax}$ exists. Consequently, $\sum_{j=1}^{p} {}_{T2} e_j \lambda_{LNV\min} \le \sum_{j=1}^{p} {}_{T2} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p} {}_{T2} e_j \lambda_{LNV\max}$ also exists. In terms of the characteristic (1) and the characteristics of the trigonometric functions, there is $\lambda_{LNV\min} \le \sum_{j=1}^{p} {}_{T2} e_j \lambda_{LNV(j)} \le \lambda_{LNV\max}$, i.e., $\lambda_{LNV\min} \le LNVTWA_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) \le \lambda_{LNV\max}$.

(3) For $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, there is $\sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T2} e_j \lambda^*_{LNV(j)}$, i.e., $LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Therefore, the characteristics of (1)-(3) are true. □

Example 1. Let three LNVs be $\lambda_{LNV(1)}$ = <λ_7, λ_2, λ_3>, $\lambda_{LNV(2)}$ = <λ_6, λ_1, λ_2>, and $\lambda_{LNV(3)}$ = <λ_5, λ_4, λ_5> in the LTS λ_L = {λ_0, λ_1, …, λ_8} subject to their weight vector ***e*** = (0.3, 0.4, 0.3). Using Eqs. (11) and (14), their aggregated values are calculated below:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \lambda_{LNV(3)}) = \sum_{j=1}^{3}{}_{T1} e_j \lambda_{LNV(j)}$$

$$= \left\langle \begin{array}{l} \lambda_{2\times 8/\pi \tan^{-1}(0.3\times\tan(7\pi/(2\times 8))+0.4\times\tan(6\pi/(2\times 8))+0.3\times\tan(5\pi/(2\times 8)))}, \\ \lambda_{2\times 8/\pi \cot^{-1}(0.3\times\cot(2\pi/(2\times 8))+0.4\times\cot(\pi/(2\times 8))+0.3\times\cot(4\pi/(2\times 8)))}, \\ \lambda_{2\times 8/\pi \cot^{-1}(0.3\times\cot(3\pi/(2\times 8))+0.4\times\cot(2\pi/(2\times 8))+0.3\times\cot(5\pi/(2\times 8)))} \end{array} \right\rangle = \langle \lambda_{6.3211}, \lambda_{1.6209}, \lambda_{2.8234} \rangle,$$

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \lambda_{LNV(3)}) = \sum_{j=1}^{3}{}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \begin{array}{l} \lambda_{2\times 8/\pi \cos^{-1}\left((\cos(7\pi/(2\times 8)))^{0.3}\times(\cos(6\pi/2))^{0.4}\times(\cos(5\pi/(2\times 8)))^{0.3}\right)}, \\ \lambda_{2\times 8/\pi \sin^{-1}\left((\sin(2\pi/(2\times 8)))^{0.3}\times(\sin(\pi/(2\times 8)))^{0.4}\times(\sin(4\pi/(2\times 8)))^{0.3}\right)}, \\ \lambda_{2\times 8/\pi \sin^{-1}\left((\sin(3\pi/(2\times 8)))^{0.3}\times(\sin(2\pi/(2\times 8)))^{0.4}\times(\sin(5\pi/(2\times 8)))^{0.3}\right)} \end{array} \right\rangle = \langle \lambda_{6.1808}, \lambda_{1.8287}, \lambda_{2.9061} \rangle.$$

4.2 LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators

This part proposes the LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators in terms of the TOLs of LNVs in Definitions 2 and 3.

Definition 5. Let $\lambda_{LNV(j)}$ = <$\lambda_{td(j)}$, $\lambda_{ud(j)}$, $\lambda_{fd(j)}$> (*j* = 1, 2, …, *p*) be *p* LNVs and LNVTWG$_{T1}$ and LNVTWG$_{T2}$: $\Omega^p \to \Omega$. Assume that e_j (*j* = 1, 2, …, *p*) is the weight of $\lambda_{LNV(j)}$ subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators are defined below:

$$LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = (\lambda_{LNV(1)})^{e_1} \otimes_{T1} (\lambda_{LNV(2)})^{e_2} \otimes_{T1} ... \otimes_{T1} (\lambda_{LNV(p)})^{e_p} = \prod_{j=1}^{p}{}_{T1} (\lambda_{LNV(j)})^{e_j}, \tag{17}$$

$$LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = (\lambda_{LNV(1)})^{e_1} \otimes_{T2} (\lambda_{LNV(2)})^{e_2} \otimes_{T2} ... \otimes_{T2} (\lambda_{LNV(p)})^{e_p} = \prod_{j=1}^{p}{}_{T2} (\lambda_{LNV(j)})^{e_j}, \tag{18}$$

Theorem 5. Let $\lambda_{LNV(j)}$ = <$\lambda_{td(j)}$, $\lambda_{ud(j)}$, $\lambda_{fd(j)}$> (*j* = 1, 2, …, *p*) be *p* LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (*j* = 1, 2, …, *p*) subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWG$_{T1}$ operator is LNV, which is gotten by

$$LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \prod_{j=1}^{p}{}_{T1} (\lambda_{LNV(j)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(fd(j)\pi/2b)\right)\right)} \right\rangle, \tag{19}$$

Proof: By a similar proof of Theorem 1, Theorem 5 can be verified (omitted here).

Theorem 6. The LNVTWG$_{T1}$ operator of Eq. (19) contains the characteristics below:

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), then $LNVTWG_{T1}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, there is this inequation $\lambda_{LNV\min} \le LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, then there is the inequation $LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWG_{T1}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Proof. In view of a similar proof of Theorem 2, Theorem 6 can be verified (omitted here).

Theorem 7. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) (j = 1, 2, …, p) be p LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (j = 1, 2, …, p) subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWG$_{T2}$ operator is LNV, which is gotten by

$$LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \prod_{j=1}^{p} T2(\lambda_{LNV(j)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}(\sin(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}(\cos(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}(\cos(fd(j)\pi/2b))^{e_j}\right)} \right\rangle, \quad (20)$$

Proof: According to a similar verification of Theorem 3, Theorem 7 can be verified (omitted here).

Theorem 8. The LNVTWG$_{T2}$ operator of Eq. (20) contains the characteristics below:

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), then $LNVTWG_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, there is this inequation $\lambda_{LNV\min} \le LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, then there is the inequation $LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWG_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Proof: According to a similar verification of Theorem 4, Theorem 8 can be verified (omitted here).

Example 2. Let three LNVs be $\lambda_{LNVV(1)} = <\lambda_7, \lambda_1, \lambda_3>$, $\lambda_{LNVV(2)} = <\lambda_6, \lambda_3, \lambda_3>$, and $\lambda_{LNVV(3)} = <\lambda_5, \lambda_2, \lambda_2>$ in the LTS $\lambda_L = \{\lambda_0, \lambda_1, …, \lambda_8\}$ subject to the weight vector $\boldsymbol{e}$ = (0.3, 0.3, 0.4). Using Eqs. (19) and (20), their aggregated values are given below:

$$LNVTWG_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},\lambda_{LNV(3)})=\prod_{j=1}^{3}{}_{T1}(\lambda_{LNV(j)})^{e_j}$$

$$=\left\langle\begin{array}{l}\lambda_{2\times 8/\pi\cot^{-1}(0.3\times\cot(7\pi/(2\times 8))+0.3\times\cot(6\pi/(2\times 8))+0.4\times\cot(5\pi/(2\times 8)))},\\ \lambda_{2\times 8/\pi\tan^{-1}(0.3\times\tan(\pi/(2\times 8))+0.3\times\tan(3\pi/(2\times 8))+0.4\times\tan(2\pi/(2\times 8)))},\\ \lambda_{2\times 8/\pi\tan^{-1}(0.3\times\tan(3\pi/(2\times 8))+0.3\times\tan(3\pi/(2\times 8))+0.4\times\tan(2\pi/(2\times 8)))}\end{array}\right\rangle=\langle\lambda_{5.8413},\lambda_{2.0502},\lambda_{2.6254}\rangle,$$

$$LNVTWG_{T2}(\lambda_{IFV(1)},\lambda_{IFV(2)},\lambda_{IFV(3)})=\prod_{j=1}^{3}{}_{T2}(\lambda_{LNV(j)})^{e_j}$$

$$=\left\langle\begin{array}{l}\lambda_{2\times 8/\pi\sin^{-1}\left((\sin(7\pi/(2\times 8)))^{0.3}\times(\sin(6\pi/(2\times 8)))^{0.3}\times(\sin(5\pi/(2\times 8)))^{0.4}\right)},\\ \lambda_{2\times 8/\pi\cos^{-1}\left((\cos(\pi/(2\times 8)))^{0.3}\times(\cos(3\pi/(2\times 8)))^{0.3}\times(\cos(2\pi/(2\times 8)))^{0.4}\right)},\\ \lambda_{2\times 8/\pi\cos^{-1}\left((\cos(3\pi/(2\times 8)))^{0.3}\times(\cos(3\pi/(2\times 8)))^{0.3}\times(\cos(2\pi/(2\times 8)))^{0.4}\right)}\end{array}\right\rangle=\langle\lambda_{5.7237},\lambda_{2.161},\lambda_{2.654}\rangle$$

5. MADM model using one of the four proposed LNV TAOs

This section develops a MADM model using one of the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators to handle MADM issues in the scenario of LNVs.

In a MADM problem, there are a set of s alternatives $Ka = \{Ka_1, Ka_2, \ldots, Ka_s\}$ and a set of p attributes $Ra = \{Ra_1, Ra_2, \ldots, Ra_p\}$. In the assessment process, the alternatives must meet the attribute requirements, then their assessment values can be assigned by the LNVs obtained from the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ with odd cardinality b+1. All the assessed LNVs are formed as their decision matrix $M_L = (\lambda_{LNV(kj)})_{s\times p}$, where $\lambda_{LNV(kj)} = <\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)}>$ for $\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)} \in \lambda_L$ ($j = 1, 2, \ldots, p$; $k = 1, 2, \ldots, s$) are LNVs to express the decision makers' true, false and uncertain linguistic values corresponding to their satisfactory degrees of each alternative Ka_k on the attributes Ra_j. The weight vector of the attributes is given by $\boldsymbol{e} = (e_1, e_2, \ldots, e_p)$ subject to $0 \le e_j \le 1$ and $\sum_{j=1}^{p} e_j = 1$. Regarding the MADM problem in the scenario of LNVs, the decision algorithm of the MADM model is indicated below.

Step 1: Get the aggregated value $\lambda_{LNV(k)}$ for each Ka_k ($k = 1, 2, \ldots, s$) by one of the following $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators:

$$\lambda_{LNV(k)}=LNVTWA_{T1}(\lambda_{LNV(k1)},\lambda_{LNV(k2)},\ldots,\lambda_{LNV(kp)})=\sum_{j=1}^{p}{}_{T1}e_j\lambda_{LNV(kj)}$$

$$=\left\langle\lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{p}\left(e_j\tan(td(kj)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}\left(e_j\cot(ud(kj)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}\left(e_j\cot(fd(kj)\pi/2b)\right)\right)}\right\rangle,\quad(21)$$

$$\lambda_{LNV(k)}=LNVTWA_{T2}(\lambda_{LNV(k1)},\lambda_{LNV(k2)},\ldots,\lambda_{LNV(kp)})=\sum_{j=1}^{p}{}_{T2}e_j\lambda_{LNV(kj)}$$

$$=\left\langle\lambda_{2b/\pi\cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(td(kj)\pi/2b)\right)^{e_j}\right)},\lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(ud(kj)\pi/2b)\right)^{e_j}\right)},\lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(fd(kj)\pi/2b)\right)^{e_j}\right)}\right\rangle,\quad(22)$$

$$\lambda_{LNV(k)} = LNVTWG_{T1}(\lambda_{LNV(k1)}, \lambda_{LNV(k2)}, ..., \lambda_{LNV(kp)}) = \prod_{j=1}^{p} {}_{T1}(\lambda_{LNV(kj)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(td(kj)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(ud(kj)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(fd(kj)\pi/2b)\right)\right)} \right\rangle, \quad (23)$$

$$\lambda_{LNV(k)} = LNVTWG_{T2}(\lambda_{LNV(k1)}, \lambda_{LNV(k2)}, ..., \lambda_{LNV(kp)}) = \prod_{j=1}^{p} {}_{T2}(\lambda_{LNV(kj)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(td(kj)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(ud(kj)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(fd(kj)\pi/2b)\right)^{e_j}\right)} \right\rangle. \quad (24)$$

Step 2: Calculate the score (accuracy) values of $E(\lambda_{LNV(k)})$ ($F(\lambda_{LNV(k)})$) (k = 1, 2, …, s) by Eq. (3) (Eq. (4) for necessity).

Step 3: Rank alternatives based on the descending order of the score values (accuracy values) and decide the best one.

Step 4: End.

6. Practical application of the proposed MADM model

6.1 Selection of CRDSs

This part applies the proposed MADM model in the choice problem of CRDSs for a manufacturing company in China to show the validity of the developed MADM model in the scenario of LNVs.

Crawling robots have been used in a wide range of fields due to their flexibility and adaptability. For search and rescue missions into hazardous areas after earthquakes, floods, and other disasters, the technical department of a manufacturing company in China presents the four potential design schemes of crawling robots: the track-driven crawling robot (Ka_1), the wheeled crawling robot (Ka_2), the crawler robot (Ka_3), and the wheel-track crawler robot (Ka_4), then they are expressed as a set of the four alternatives Ka = {Ka_1, Ka_2, Ka_3, Ka_4} for the choice of decision makers. In the assessment process, the four design schemes must meet four key requirements (attributes): technical conditions (Ra_1), manufacturing cost (Ra_2), flexible and fast movement ability (Ra_3), and adaptability to complex environments (Ra_4), then their weigh vector is given by $\boldsymbol{e}$ = (0.3, 0.3, 0.2, 0.2) as the known attribute weights. In this MADM issue, decision makers/engineers are invited to assess the satisfaction of the four CRDSs corresponding to the four attributes. Subsequently, the assessment values of their satisfaction are presented by the true, false, and uncertain linguistic values obtained from the given LTS λ_L = {λ_0(Extremely low), λ_1(Very low), λ_2(Low), λ_3(Slightly high), λ_4(Medium), λ_5(Slightly high), λ_6(High), λ_7(Very high), λ_8(Extremely high} with b = 8. Thus, all the given LNVs $\lambda_{LNV(kj)} = <\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)}>$ for $\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)} \in \lambda_L$ (k, j = 1, 2, 3, 4) can be created as their assessment matrix:

$$M_L = \begin{bmatrix} <\lambda_6,\lambda_1,\lambda_2> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_5,\lambda_2,\lambda_2> & <\lambda_6,\lambda_3,\lambda_2> \\ <\lambda_7,\lambda_2,\lambda_3> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_6,\lambda_2,\lambda_4> \\ <\lambda_6,\lambda_2,\lambda_2> & <\lambda_6,\lambda_2,\lambda_4> & <\lambda_6,\lambda_2,\lambda_1> & <\lambda_6,\lambda_2,\lambda_3> \\ <\lambda_7,\lambda_3,\lambda_2> & <\lambda_7,\lambda_2,\lambda_3> & <\lambda_7,\lambda_2,\lambda_2> & <\lambda_5,\lambda_1,\lambda_2> \end{bmatrix}.$$

First, using one of Eqs. (21)–(24), we get the aggregated values of $\lambda_{LNV(k)}$ for Ka_k (k = 1, 2, …, s):

(1) The aggregated values of the LNVTWA$_{T1}$ operator are $\lambda_{LNV(1)} = <\lambda_{6.3688}, \lambda_{1.7826}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.8875}, \lambda_{2.4076}, \lambda_{2.5168}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.0655}>$, and $\lambda_{LNV(4)} = <\lambda_{6.8418}, \lambda_{1.8304}, \lambda_{2.2273}>$;

(2) The aggregated values of the LNVTWA$_{T2}$ operator are $\lambda_{LNV(1)} = <\lambda_{6.2502}, \lambda_{1.9671}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.8534}, \lambda_{2.4407}, \lambda_{2.5724}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.2819}>$, and $\lambda_{LNV(4)} = <\lambda_{6.763}, \lambda_{1.9523}, \lambda_{2.2523}>$;

(3) The aggregated values of the LNVTWG$_{T1}$ operator are $\lambda_{LNV(1)}$ = <$\lambda_{6.0603}$, $\lambda_{2.2651}$, λ_2>, $\lambda_{LNV(2)}$ = <$\lambda_{6.7909}$, $\lambda_{2.5264}$, $\lambda_{2.7806}$>, $\lambda_{LNV(3)}$ = <λ_6, λ_2, $\lambda_{2.7436}$>, and $\lambda_{LNV(4)}$ = <$\lambda_{6.5495}$, $\lambda_{2.1423}$, $\lambda_{2.3221}$>;

(4) The aggregated values of the LNVTWG$_{T2}$ operator are $\lambda_{LNV(1)}$ = <$\lambda_{5.9617}$, $\lambda_{2.3853}$, λ_2>, $\lambda_{LNV(2)}$ = <$\lambda_{6.7313}$, $\lambda_{2.5584}$, $\lambda_{2.84}$>, $\lambda_{LNV(3)}$ = <λ_6, λ_2, $\lambda_{2.8744}$>, and $\lambda_{LNV(4)}$ = <$\lambda_{6.3651}$, $\lambda_{2.2272}$, $\lambda_{2.3534}$>.

Then using Eq. (3) (Eq. (4)), the score (accuracy) values of $E(\lambda_{LNV(k)})$ (k = 1, 2, 3, 4) and the ranking results of the four CRDSs are shown in Table 1.

Table 1. Decision results of the developed MADM model and the existing MADM model

AO	Score value	Ranking	Best CRDS
LNVTWA$_{T1}$ operator	0.7744, 0.7485, 0.7473, 0.7827	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVTWA$_{T2}$ operator	0.7618, 0.7433, 0.7383, 0.7733	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVTWG$_{T1}$ operator	0.7415, 0.7285, 0.7190, 0.7535	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVTWG$_{T2}$ operator	0.7324, 0.7222, 0.7136, 0.7410	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVWA operator [14]	0.7604, 0.7420, 0.7365, 0.7721	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVWG operator [14]	0.7415, 0.7293, 0.7201, 0.7535	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4

In terms of the ranking results in Table 1, there is the same ranking order based on the four proposed TAOs of LNVs, which shows the ranking validity and robustness of the developed MADM model using one of Eqs. (21)–(24) in the scenario of LNVs.

6.2 Comparison with the existing MADM model using the LNVWA and LNVWG operators

To indicate the validity of the developed MADM model using one of the four presented TAOs, it is compared with the existing MADM model using the LNVWA and LNVWG operators [14] by the selection example of the four CRDSs in the scenario of LNVs.

Applying Eqs. (1)–(3) based on the existing DM model [14], we get the aggregated values of LNVs and the score values of $E(\lambda_{LNV(k)})$ (k = 1, 2, 3, 4) and the ranking results of the four CRDSs are also shown in Table 1.

In terms of all the ranking results in Table 1, there is the same ranking order regarding the existing LNVWA and LNVWG operators and the proposed four TAOs of LNVs, which shows the ranking validity and rationality of the developed MADM model using one of Eqs. (21)–(24) in the scenario of LNVs. Then, the developed MADM model using the proposed TAOs of LNVs contains periodic property and can perform MADM problems with periodicity/polytemporal phases, while the existing MADM model using the LNVWA and LNVWG operators [14] cannot contain the periodic properties so as to difficultly perform DM problems with periodic/ploytemporal needs. It is clear that in the LNV scenario the developed MADM model significantly outperforms the existing MADM model in the periodic/polytemporal DM capability.

7. Conclusion

This paper first defined the linguistic trigonometric t-norms and t-conorms based on the trigonometry 1 and 2 and the TOLs of LNVs, which contained periodic operational properties. Then, the LNVTWA$_{T1}$, LNVTWA$_{T2}$, LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators were proposed to provide appropriate mathematical tools for solving MADM issues with periodicity/ploytemporal phases in the scenario of LNVs. Meanwhile, the TAOs of LNVs can compensate for the defects of existing AOs that lack periodic operations of LNVs. Furthermore, the developed MADM model using one of the LNVTWA$_{T1}$, LNVTWA$_{T2}$, LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators can effectively address MADM problems with periodicity/ploytemporal phases in LNV scenarios. Finally, the developed MADM model was used for the MADM application in the choice problem of CRDSs and reflected its validity and robustness of the ranking results. By comparison with the existing MADM model in the LNV scenario, the developed MADM model verified the validity and rationality of the decision results and significantly outperformed the existing MADM model.

Generally, the MADM model using one of the LNVTWA$_{T1}$, LNVTWA$_{T2}$, LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators proposed in this study can not only extend the existing linguistic neutrosophic DM models, but also handle linguistic or linguistic intuitionistic fuzzy DM problems by a special case of the proposed MADM model. However, this study developed the TAOs of LNVs and their DM method for the first time. Therefore, we need to provide more research work in the future. To do so, it is necessary to further develop some new LNV TAOs, including LNV trigonometric Einstein, Hamacher, Bonferroni, Maclaurin, and Heronian AOs, etc., and their applications in computer science, medical diagnosis, civil engineering management in LNV scenarios.

Data Availability

All data generated or analyzed during this study are included in this article.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animalsperformed by any of the authors.

Informed Consent All authors agreed with the content of the manuscript and the accepted submission and agree to be accountable for all aspects of the work.

References

References must be numbered in order of appearance in the text (including citations in tables and legends) and listed individually at the end of the manuscript. We recommend preparing the references with a bibliography software package, such as EndNote, Reference Manager or Zotero to avoid typing mistakes and duplicated references. Include the digital object identifier (DOI) for all references where available.

Citations and References in Supplementary files are permitted provided that they also appear in the reference list here.

In the text, reference numbers should be placed in square brackets [ ], and placed before the punctuation; for example [1], [1–3] or [1,3]. For embedded citations in the text with pagination, use both parentheses and brackets to indicate the reference number and page numbers; for example [5] (p. 10), or [6] (pp. 101–105).

1. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning Part I. Information Sciences 1975, 8(3),pp.199–249.
2. Herrera F, Herrera-Viedma E, Verdegay L. A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets and Systems* **1996,** 79(1),pp.73–87.
3. Herrera F, Herrera-Viedma E. Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems* **2000,** 115(1),pp.67–82.
4. Xu ZS. A note on linguistic hybrid arithmetic averaging operator in multiple attribute group decision making with linguistic information. *Group Decision and Negotiation* **2006 ,** 15(6),pp.593–604.
5. Martínez L, Ruan D, Herrera F, Herrera-Viedma E, Wang PP. Linguistic decision making: Tools and applications. *Information Sciences* **2009 ,** 179(14),pp. 2297–2298.
6. Zhang Z, Chen SM, Wang C. Group decision making based on multiplicative consistency and consensus of fuzzy linguistic preference relations. *Information Sciences* **2020,** 509,pp. 71–86.
7. Chen ZC, Liu PH, Pei Z. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *International Journal of Computational Intelligence Systems* **2015,** 8(4),pp. 747–760.
8. Yager RR.Multi-criteria decision making with ordinal/linguistic intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems* **2016,** 24,pp. 590–599.
9. Liu P, Liu X, Ma G, Liang Z, Wang C, Alsaadi FE. A multi-attribute group decision-making method based on linguistic intuitionistic fuzzy numbers and Dempster–Shafer evidence theory. *International Journal of Information Technology & Decision Making* **2020,** 19(02),pp.499–524.
10. Kumar K, Chen SM. Multiple attribute group decision making based on advanced linguistic intuitionistic fuzzy weighted averaging aggregation operator of linguistic intuitionistic fuzzy numbers. *Information Sciences* **2022,** 587,pp. 813–824.
11. Kumar K, Chen SM. Group decision making based on linguistic intuitionistic fuzzy Yager weighted arithmetic aggregation operator of linguistic intuitionistic fuzzy numbers. *Information Sciences* **2023 ,** 647: 119228.

12. Li T, Zhang L, Zhang Z. The consistency and consensus analysis for group decision-making with incomplete linguistic interval-valued intuitionistic fuzzy preference relations. *Applied Intelligence* **2023,** 53,pp. 23500–23521.
13. Li J, Chen M. Advanced linguistic intuitionistic fuzzy Maclaurin symmetric means for MAGDM. *Applied Intelligence* 2024 , 54,pp.3680–3722.
14. Fang Z, Ye J. Multiple attribute group decision-making method based on linguistic neutrosophic numbers. Symmetry **2017,** 9(7),pp.111.
15. Fan C, Ye J, Hu K, Fan E. Bonferroni mean operators of linguistic neutrosophic numbers and their multiple attribute group decision-making methods. *Information* **2017 ,** 8(3),pp. 107.
16. Liu P, You X. Some linguistic neutrosophic Hamy mean operators and their application to multi-attribute group decision making. *PloS One* **2018,** 13(3): e0193027.
17. Liang W, Zhao G, Luo S. Linguistic neutrosophic Hamacher aggregation operators and the application in evaluating land reclamation schemes for mines. *PloS One* **2018,** 13(11): e0206178.
18. Liu P, Mahmood T, Khan Q. Group decision making based on power Heronian aggregation operators under linguistic neutrosophic environment. *International Journal of Fuzzy Systems* **2018,** 20,pp. 970–985.
19. Fan C, Feng S, Hu K. Linguistic neutrosophic numbers Einstein operator and its application in decision making. *Mathematics* **2019,** 7(5): 389.
20. Liu P, You X. Linguistic neutrosophic partitioned Maclaurin symmetric mean operators based on clustering algorithm and their application to multi-criteria group decision-making. *Artificial Intelligence Review* **2020,** 53(3),pp.2131–2170.
21. Zhang Y, Yong R, Ye J, Zhong Z, Zheng S. Dombi aggregation operators of linguistic neutrosophic numbers for multiple attribute group decision-making problems in landslide treatment schemes. *Neutrosophic Sets and Systems* **2022,** 51: 724–743.
22. Zhang S, Ye J. Multiple attribute group decision-making models using single-valued neutrosophic and linguistic neutrosophic hybrid element aggregation algorithms. *Journal of Mathematics* **2022,** Article ID 1021280. https://doi.org/10.1155/2022/1021280.
23. Luo SZ, Xing LN, Ren T. Performance evaluation of human resources based on linguistic neutrosophic Maclaurin symmetric mean operators. *Cognitive Computation* **2022,** 14(2),pp. 547–562.
24. Li G, Zhong Y, Chen C, Jin T, Liu Y. Reliability allocation method based on linguistic neutrosophic numbers weight Muirhead mean operator. *Expert Systems with Applications* **2022,** 193: 116504.
25. Ye J, Du S, Yong R. Multi-criteria decision-making model using trigonometric aggregation operators of single-valued neutrosophic credibility numbers. *Information Sciences* **2023,** 644: 118968.
26. Ye J, Du S, Yong R. MCDM technique using single-valued neutrosophic trigonometric weighted aggregation operators. Journal of Management Analytics 2024, 11(1),pp. 45–61.
27. Ye J. Group decision-making strategy based on aggregation operators of linguistic confidence interval neutrosophic numbers in a linguistic neutrosophic multivalued scenario. *Engineering Applications of Artificial Intelligence* **2025,** 141, 109823.

Received: July 2, 2025. Accepted: Sep 10, 2025