Note of HyperNeutrosophic Crisp Set and SuperHyperNeutrosophic Crisp Set

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Abstract

The Neutrosophic Set provides a flexible mathematical framework for addressing uncertainty through three distinct membership functions: truth, indeterminacy, and falsity. Extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set have recently been introduced to address increasingly complex and multidimensional problems. Detailed formal definitions of these concepts can be found in [1]. A Neutrosophic Crisp Set partitions a universe into three subsets, explicitly representing truth, indeterminacy, and falsity memberships. In this paper, we explore the properties of the HyperNeutrosophic Crisp Set and the SuperHyperNeutrosophic Crisp Set. These structures merge the ideas of Neutrosophic Crisp Sets with those of Hyperneutrosophic and SuperHyperneutrosophic Sets. We anticipate that this integrated framework will contribute to the advancement of research in areas such as Uncertain Sets and Uncertain Topology.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set, Neutrosophic Crisp Set

1 Introduction

Handling uncertainty is a key challenge in mathematics, computer science, and decision-making. Over the years, researchers have developed many set-based models to represent and process uncertain information. The concept of fuzzy sets, first introduced by Zadeh [2] and later expanded in practical applications by Zimmermann [3], laid the foundation for this area. These sets allow each element to have a degree of membership between 0 and 1, enabling a gradual representation of truth rather than a strict binary approach.

Building on fuzzy sets, Atanassov introduced intuitionistic fuzzy sets, which include both membership and non-membership degrees [4]. Further extensions such as vague sets [5] and hyperfuzzy sets [6] were developed to address more complex situations. Other generalizations include picture fuzzy sets, hesitant fuzzy sets, and spherical fuzzy sets, each designed to tackle specific types of uncertainty in real-world problems.

A major step forward was taken with the introduction of neutrosophic sets by Smarandache [7], which explicitly incorporate an indeterminacy component alongside truth and falsity. This richer structure allows a more nuanced description of uncertain phenomena. Recently, Fujita proposed the HyperNeutrosophic Set and its superhyper generalizations to capture multi-level and high-dimensional uncertainty [1, 8, 9]. These frameworks have been applied to various fields, including data analysis, decision science, and algebraic modeling.

Within this context, neutrosophic crisp sets provide a way to represent truth, indeterminacy, and falsity explicitly as disjoint or overlapping subsets of a universe. Extending this idea, HyperNeutrosophic Crisp Sets and SuperHyperNeutrosophic Crisp Sets combine the crisp nature of these subsets with the multi-valued structure of hyperneutrosophic models. This integration allows for the representation of multiple possible crisp assignments at different hierarchical levels, making the model suitable for complex systems where uncertainty exists at multiple layers.

The aim of this paper is to study the properties and algebraic structures of HyperNeutrosophic Crisp Sets and n-SuperHyperNeutrosophic Crisp Sets. We present formal definitions, illustrative examples, and theoretical results that establish their foundational behavior. By bridging the gap between neutrosophic crisp models and higher-order hyperneutrosophic structures, our work contributes to the ongoing development of uncertainty theory and its mathematical foundations.

2 Preliminaries

This section gathers the key notions and notation that form the foundation of our study. Throughout, we restrict our attention to finite sets.

2.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

A variety of set-theoretic paradigms have been introduced to capture uncertainty, imprecision, and vagueness in decision-making [8, 9]. The classical theory of Fuzzy Sets, initiated by Zadeh, remains a cornerstone [2, 3]. Building on this, Atanassov's Intuitionistic Fuzzy Sets incorporate both membership and non-membership degrees [4, 10], while Vague Sets provide an alternative dual-measure approach [5, 11, 12]. Further generalizations include Hyperfuzzy Sets, which extend the fuzzification process to higher-order structures [6, 13–16]. Other notable extensions of fuzzy sets include Picture Fuzzy Sets [17, 18], Hesitant Fuzzy Sets [19–21], and Spherical Fuzzy Sets [22–24], which have been applied extensively in decision science and related fields.

Neutrosophic Sets, proposed by Smarandache, introduce an explicit indeterminacy component alongside truth and falsity, yielding a richer framework for nuanced reasoning [7, 25, 26]. To model even more elaborate forms of uncertainty, researchers have defined HyperNeutrosophic Sets and their n-fold superhyper variants, which are well-suited to high-dimensional or deeply nested problem domains [1, 27–30]. Below we restate these concepts formally and give simple illustrative examples.

Definition 2.1 (Base Set)**.** [31] A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.2 (Powerset)**.** (cf. [32]) For any set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S\}.$$

Definition 2.3 (n-th Powerset)**.** [33–35] Let H be a set. Define inductively

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}\big(\mathcal{P}^k(H)\big), \quad k \geq 0.$$

Then $\mathcal{P}^1(H) = \mathcal{P}(H)$, $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$, and so on. Removing the empty set at each stage yields the *nonempty* iterated powersets. Writing $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$, we set

$$\mathcal{P}^{*1}(H) = \mathcal{P}^*(H), \quad \mathcal{P}^{*(k+1)}(H) = \mathcal{P}^*\big(\mathcal{P}^{*k}(H)\big).$$

Example 2.4 (Corporate Document Hierarchy as a Third-Level Powerset)**.** Consider a small company whose digital archive starts with the base set

$$H = \{\text{Invoice}, \ \text{Contract}, \ \text{Presentation}\}.$$

First powerset $\mathcal{P}^1(H)$ (folders). Typical folders—each a subset of H—might be

$$F_1 = \{\text{Invoice}, \text{Contract}\}, \quad F_2 = \{\text{Presentation}\}, \quad F_3 = \{\text{Invoice}, \text{Presentation}\}.$$

Thus $\{F_1, F_2, F_3\} \subseteq \mathcal{P}^1(H)$ models the folder structure.

Second powerset $\mathcal{P}^2(H)$ (projects). A *project* groups several folders:

$$P_1 = \{F_1, F_2\}, \qquad P_2 = \{F_2, F_3\}.$$

Both P_1 and P_2 are elements of $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$.

Third powerset $\mathcal{P}^3(H)$ (portfolios). At the next level, management assembles related projects into *portfolios*:

$$R = \{P_1, P_2\} \in \mathcal{P}^3(H).$$

The set R is therefore a concrete element of the third iterated powerset $\mathcal{P}^3(H)$. In everyday terms,

$$\text{files} \longrightarrow \text{folders} \longrightarrow \text{projects} \longrightarrow \text{portfolios},$$

giving a natural three-tier hierarchy captured precisely by the n-th powerset construction with $n = 3$.

Definition 2.5 (Neutrosophic Set)**.** [7, 36] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 2.6 (Medical Diagnosis with Uncertain Symptoms)**.** (cf. [37, 38]) Let

$$X = \{\text{Fever}, \ \text{Cough}, \ \text{Fatigue}\}$$

be the set of key symptoms considered in diagnosing a particular illness (e.g. influenza). A physician assigns to each symptom three independent degrees that summarise current clinical evidence:

Symptom	$T_A(x)$	$I_A(x)$	$F_A(x)$
Fever	0.80	0.10	0.05
Cough	0.60	0.25	0.10
Fatigue	0.40	0.40	0.15

- **Truth degree $T_A(x)$.** The likelihood that the symptom genuinely supports the diagnosis. Fever, highly characteristic of influenza, scores 0.80.
- **Indeterminacy degree $I_A(x)$.** The physician's uncertainty due to limited data, conflicting test results, or atypical presentation. Fatigue carries the greatest indeterminacy (0.40) because it can arise from many unrelated conditions.
- **Falsity degree $F_A(x)$.** Evidence that the symptom contradicts—or fails to support—the diagnosis. Fever has a low falsity value (0.05), whereas Fatigue is less specific (0.15).

Each row satisfies $T_A(x) + I_A(x) + F_A(x) \le 1$, conforming to the neutrosophic-set requirement $T + I + F \le 3$. The triplet

$$\big(T_A(x), \ I_A(x), \ F_A(x)\big)$$

offers a richer, three-way description of diagnostic confidence than a single probability value or a binary classification, enabling physicians to capture both indecision and contrary evidence explicitly.

Definition 2.7 (HyperNeutrosophic Set)**.** (cf. [1, 27, 39–42]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \le T + I + F \le 3.$$

Example 2.8 (HyperNeutrosophic Assessment of Investment Options)**.** Suppose a portfolio manager is evaluating two start-ups,

$$X = \{\text{GreenTech}, \ \text{BioHealth}\},$$

using opinions from three analysts. Each analyst provides a neutrosophic triplet (T, I, F), where

- T is the analyst's confidence that the company will yield the target return,
- I is the analyst's perceived indeterminacy (lack of clear evidence),
- F is the analyst's confidence that the company will *not* achieve the target.

Because the analysts rely on different data sources, their triplets may vary widely. The manager wishes to keep *all* expert opinions rather than aggregate them into a single point estimate, leading naturally to a HyperNeutrosophic Set.

$$\tilde{\mu} : X \longrightarrow \mathcal{P}([0,1]^3), \qquad \tilde{\mu}(x) = \{ \, (T, I, F) \; \big| \; \text{analyst triplet for } x \, \}.$$

$$\tilde{\mu}(\text{GreenTech}) = \Big\{(0.80, \ 0.10, \ 0.05), (0.60, \ 0.25, \ 0.10), (0.70, \ 0.20, \ 0.05)\Big\},$$

$$\tilde{\mu}(\text{BioHealth}) = \Big\{(0.45, \ 0.35, \ 0.15), (0.55, \ 0.30, \ 0.10)\Big\}.$$

Every triplet obeys $0 \le T + I + F \le 1 \le 3$, and each image is a non-empty subset of the unit cube $[0,1]^3$. Thus $\tilde{\mu}$ defines a legitimate HyperNeutrosophic Set on X. Retaining the full *set* of analyst triplets preserves the spectrum of expert sentiment, which can later be filtered, weighted, or visualised without losing information at the time of collection.

Definition 2.9 (n-SuperHyperNeutrosophic Set)**.** (cf. [1,27]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \ge 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \le T + I + F \le 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 2.10 (2-SuperHyperNeutrosophic Set)**.** Starting from the above $\tilde{\mu}$, the power-set $\mathcal{P}(X) = \{\{x\}, \{y\}, \{x, y\}\}$ serves as the domain of a 2-SuperHyperNeutrosophic Set $\tilde{\mu}_2 : \mathcal{P}(X) \to \mathcal{P}\big(\mathcal{P}([0,1]^3)\big) \setminus \{\varnothing\}$ defined by

$$\tilde{\mu}_2(U) = \big\{ \, \tilde{\mu}(p) \; \big| \; p \in U \big\}.$$

Thus, for instance,

$$\tilde{\mu}_2(\{x\}) = \big\{ \, \{(0.8, 0.1, 0.1), (1, 0, 0)\}\big\}, \quad \tilde{\mu}_2(\{x, y\}) = \big\{ \, \{(0.8, 0.1, 0.1), (1, 0, 0)\}, \ \{(0.5, 0.3, 0.1)\}\big\}.$$

Each $\tilde{\mu}_2(U)$ is a nonempty family of subsets of $[0,1]^3$, exhibiting the second-level (superhyper) neutrosophic structure.

Example 2.11 (A 3-SuperHyperNeutrosophic Set). Let $X = \{x, y\}$. We first define a HyperNeutrosophic Set $\tilde{\mu} : X \to \mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ by

$$\tilde{\mu}(x) = \{(0.8, 0.1, 0.1),\ (1, 0, 0)\}, \qquad \tilde{\mu}(y) = \{(0.5, 0.3, 0.1)\}.$$

Next, the collection of nonempty subsets of X is $\mathcal{P}_1(X) = \{\{x\}, \{y\}, \{x, y\}\}$. Define the 2-SHNS $\tilde{\mu}_2 : \mathcal{P}_1(X) \to \mathcal{P}(\mathcal{P}([0,1]^3))$ by

$$\tilde{\mu}_2(U) \ = \ \{ \ \tilde{\mu}(p) \mid p \in U\}.$$

Concretely,

$$\tilde{\mu}_2(\{x\}) = \big\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}\big\}, \quad \tilde{\mu}_2(\{y\}) = \big\{\{(0.5, 0.3, 0.1)\}\big\},$$
$$\tilde{\mu}_2(\{x, y\}) = \big\{\{(0.8, 0.1, 0.1), (1, 0, 0)\},\ \{(0.5, 0.3, 0.1)\}\big\}.$$

Finally, the third-level domain is $\mathcal{P}_2(X) = \mathcal{P}(\mathcal{P}_1(X))$, whose nonempty elements include, for instance,

$$W_1 = \{\{x\}, \{y\}\}, \quad W_2 = \{\{x, y\}\}.$$

Define the 3-SHNS $\tilde{\mu}_3 : \mathcal{P}_2(X) \to \mathcal{P}(\mathcal{P}(\mathcal{P}([0,1]^3)))$ by

$$\tilde{\mu}_3(W) = \{ \ \tilde{\mu}_2(U) \mid U \in W\}.$$

Then in particular:

$$\tilde{\mu}_3(W_1) = \big\{\tilde{\mu}_2(\{x\}),\ \tilde{\mu}_2(\{y\})\big\} = \Big\{\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}\},\ \{\{(0.5, 0.3, 0.1)\}\}\Big\},$$

$$\tilde{\mu}_3(W_2) = \big\{\tilde{\mu}_2(\{x, y\})\big\} = \Big\{\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}, \{(0.5, 0.3, 0.1)\}\}\Big\}.$$

Each $\tilde{\mu}_3(W_i)$ is nonempty and consists of families of subsets of $\mathcal{P}([0,1]^3)$, illustrating the full third-level (superhyper) neutrosophic structure.

2.2 Neutrosophic Crisp Set

A Neutrosophic Crisp Set partitions a universe X into three subsets representing truth, indeterminacy, and falsity memberships explicitly. The definition of Neutrosophic Crisp Set is below [43–47].

Definition 2.12 (Neutrosophic Crisp Set). [43] Let X be a nonempty universe. A *neutrosophic crisp set* (NCS) on X is an ordered triple

$$A \ = \ \langle A_1,\ A_2,\ A_3\rangle,$$

where $A_1, A_2, A_3 \subseteq X$. We call:

- A_1 the *truth-membership* subset,
- A_2 the *indeterminacy-membership* subset,
- A_3 the *falsity-membership* subset.

Definition 2.13 (Types of Neutrosophic Crisp Set). With $A = \langle A_1, A_2, A_3\rangle$ as above, one distinguishes:

1. *Type 1* if
$$A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset, \quad A_2 \cap A_3 = \emptyset.$$
2. *Type 2* if in addition to the Type 1 conditions,
$$A_1 \ \cup \ A_2 \ \cup \ A_3 \ = \ X.$$
3. *Type 3* if
$$A_1 \cap A_2 \cap A_3 = \emptyset, \quad \text{and} \quad A_1 \ \cup \ A_2 \ \cup \ A_3 = X.$$

Example 2.14 (Type-1 Neutrosophic Crisp Set: Library Loss Audit)**.** A university library audits a small collection $X = \{B1, B2, B3, B4, B5\}$ to determine whether each book is truly lost, of uncertain status, or accounted for.

$$A = \langle A_1, A_2, A_3 \rangle, \qquad A_1 = \{B1, B3\},\ A_2 = \{B4\},\ A_3 = \{B2\}.$$

- A_1: books confirmed missing (**truth**).
- A_2: books with inconclusive records (**indeterminacy**).
- A_3: books verified on shelf (**falsity**).

We have $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$, so the three subsets are pairwise disjoint, satisfying the Type-1 condition. The book B5 has not yet been audited, so $A_1 \cup A_2 \cup A_3 \subset X$; hence A is a *Type-1* neutrosophic crisp set.

Example 2.15 (Type-2 Neutrosophic Crisp Set: Quality-Control Report)**.** A factory inspects four products,

$$X = \{P1,\ P2,\ P3,\ P4\},$$

and classifies each item as *accepted*, *pending*, or *rejected*. Let

$$A = \langle A_1, A_2, A_3 \rangle \quad \text{with} \quad A_1 = \{P1, P3\},\ A_2 = \{P2\},\ A_3 = \{P4\}.$$

- A_1 (truth membership): products that pass inspection outright.
- A_2 (indeterminacy membership): products awaiting additional tests.
- A_3 (falsity membership): products that fail inspection.

We verify the Type-2 conditions:

$$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset \quad \text{and} \quad A_1 \cup A_2 \cup A_3 = X.$$

Hence A is a *Type-2 neutrosophic crisp set*. Every product belongs to exactly one of the three mutually disjoint categories, providing a clear and exhaustive quality-control summary.

Example 2.16 (Type-3 Neutrosophic Crisp Set: Preliminary Medical Screening)**.** A clinic screens four patients for a particular infection, $X = \{P1, P2, P3, P4\}$. Test results lead to overlapping assessments:

$$B = \langle B_1, B_2, B_3 \rangle, \qquad B_1 = \{P1, P2\},\ B_2 = \{P2, P3\},\ B_3 = \{P3, P4\}.$$

- B_1: patients with a positive test (**truth**).
- B_2: inconclusive or conflicting evidence (**indeterminacy**).
- B_3: patients with a negative test (**falsity**).

Here

$$B_1 \cap B_2 \cap B_3 = \emptyset, \qquad B_1 \cup B_2 \cup B_3 = X,$$

so B fulfils the Type-3 criteria. Note that pairwise overlaps are allowed: $P2 \in B_1 \cap B_2$ (positive yet uncertain) and $P3 \in B_2 \cap B_3$ (negative but still doubtful). Such overlaps capture real-world ambiguity while ensuring every patient appears in at least one subset.

3 Main Results of This Paper

This section outlines the main results presented in this paper.

3.1 HyperNeutrosophic Crisp Set

HyperNeutrosophic Crisp Sets assign each element of X a nonempty set of crisp truth–indeterminacy–falsity triples, capturing multiple binary certainty patterns. The definition of HyperNeutrosophic Crisp Set is below.

Definition 3.1 (HyperNeutrosophic Crisp Set)**.** Let X be a nonempty set. A *HyperNeutrosophic Crisp Set* (HNCS) on X is a function

$$H : X \longrightarrow \mathcal{P}\big(\{0,1\}^3\big) \setminus \{\varnothing\},$$

where $\{0,1\}^3$ is the set of all triples (t,i,f) with $t,i,f \in \{0,1\}$. For each $x \in X$, $H(x) \subseteq \{0,1\}^3$ is nonempty, and each $(t,i,f) \in H(x)$ is interpreted as a *crisp* assignment of truth, indeterminacy, and falsity to x.

Example 3.2 (HyperNeutrosophic Crisp Set on Sensor Reliability)**.** Let

$$X = \{\, S_1,\ S_2,\ S_3\}$$

be a set of three sensors monitoring an industrial process. We define a HyperNeutrosophic Crisp Set

$$H : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\varnothing\}$$

by assigning to each sensor S_i a nonempty set of crisp truth/indeterminacy/falsity triples:

$$\begin{aligned} H(S_1) &= \big\{(1,0,0),\ (1,1,0)\big\},\\ H(S_2) &= \big\{(0,1,0),\ (0,1,1)\big\},\\ H(S_3) &= \big\{(1,0,0),\ (0,0,1)\big\}. \end{aligned}$$

- $H(S_1)$: Sensor 1 is either fully functional $(1,0,0)$ or showing occasional instability $(1,1,0)$.
- $H(S_2)$: Sensor 2's status is inconclusive $(0,1,0)$ or indicates potential failure $(0,1,1)$.
- $H(S_3)$: Sensor 3 is either fully functional $(1,0,0)$ or conclusively failed $(0,0,1)$.

Since each $H(S_i)$ is a nonempty subset of $\{0,1\}^3$, this constitutes a valid HyperNeutrosophic Crisp Set on X.

Definition 3.3 (Set-operations on HNCS)**.** If H, K are HNCS on X, define for each $x \in X$:

$$(H \cap K)(x) \;=\; H(x) \;\cap\; K(x), \qquad (H \cup K)(x) \;=\; H(x) \;\cup\; K(x).$$

Then $H \cap K$ and $H \cup K$ are again HNCS.

Example 3.4 (Set-operations on HNCS)**.** Let

$$X = \{x, y\},$$

and consider two HyperNeutrosophic Crisp Sets $H, K : X \to \mathcal{P}(\{0,1\}^3) \setminus \{\varnothing\}$ given by

$$\begin{aligned} H(x) &= \{(1,0,0),\ (1,1,0)\}, & H(y) &= \{(0,1,0),\ (0,1,1)\},\\ K(x) &= \{(1,0,0),\ (0,0,1)\}, & K(y) &= \{(1,0,0),\ (0,1,0)\}. \end{aligned}$$

We compute their intersection and union pointwise:

$$\begin{aligned} (H \cap K)(x) &= H(x) \,\cap\, K(x) = \{(1,0,0)\},\\ (H \cap K)(y) &= H(y) \,\cap\, K(y) = \{(0,1,0)\}, \end{aligned}$$

$$\begin{aligned} (H \cup K)(x) &= H(x) \,\cup\, K(x) = \{(1,0,0),\ (1,1,0),\ (0,0,1)\},\\ (H \cup K)(y) &= H(y) \,\cup\, K(y) = \{(0,1,0),\ (0,1,1),\ (1,0,0)\}. \end{aligned}$$

- At x, the intersection retains only the fully-true assignment $(1, 0, 0)$, while the union gathers all three distinct crisp triples.
- At y, the intersection picks the indeterminate-only assignment $(0, 1, 0)$, and the union collects both indeterminate patterns plus the fully-true assignment.

Thus $H \cap K$ and $H \cup K$ are valid HNCS on X.

Definition 3.5 (Embedding of NCS into HNCS)**.** Given a neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$, define

$$H_A(x) = \begin{cases} \{(1,0,0)\}, & x \in A_1, \\ \{(0,1,0)\}, & x \in A_2, \\ \{(0,0,1)\}, & x \in A_3. \end{cases}$$

Then H_A is a HNCS on X.

Example 3.6 (Embedding of a Neutrosophic Crisp Set into a HyperNeutrosophic Crisp Set)**.** Let

$$X = \{a, b, c, d\}, \quad A = \langle A_1, A_2, A_3 \rangle \quad \text{with} \quad A_1 = \{a, c\}, \ A_2 = \{b\}, \ A_3 = \{d\}.$$

Then A is a neutrosophic crisp set on X. By the embedding definition, we obtain the HNCS

$$H_A : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}$$

given pointwise by

$$H_A(x) = \begin{cases} \{(1,0,0)\}, & x \in A_1, \\ \{(0,1,0)\}, & x \in A_2, \\ \{(0,0,1)\}, & x \in A_3. \end{cases}$$

Concretely,

$$H_A(a) = \{(1,0,0)\}, \qquad H_A(b) = \{(0,1,0)\},$$
$$H_A(c) = \{(1,0,0)\}, \qquad H_A(d) = \{(0,0,1)\}.$$

- For $a, c \in A_1$, $H_A(x) = \{(1,0,0)\}$ marks *truth*.
- For $b \in A_2$, $H_A(b) = \{(0,1,0)\}$ marks *indeterminacy*.
- For $d \in A_3$, $H_A(d) = \{(0,0,1)\}$ marks *falsity*.

Each $H_A(x)$ is a nonempty singleton in $\{0,1\}^3$, so H_A is indeed a valid HyperNeutrosophic Crisp Set on X.

Theorem 3.7. *The map $\Phi : \{NCS\ on\ X\} \to \{HNCS\ on\ X\}$ given by $\Phi(A) = H_A$ is*

(a) Injective*, i.e. different NCS produce different HNCS.*

(b) *A* homomorphism *with respect to union and intersection:*

$$H_{A\cap B} = H_A \cap H_B, \qquad H_{A\cup B} = H_A \cup H_B.$$

(c) Thus every NCS is realized as a "singleton-valued" HNCS, showing that HNCS properly generalize NCS.

Proof. (a) Suppose $A \neq B$ are two NCS. Then there exists x lying in exactly one of the corresponding components A_j vs. B_j. Hence $H_A(x) \neq H_B(x)$, so $H_A \neq H_B$.

(b) For any $x \in X$,

$$H_{A\cap B}(x) = \begin{cases} \{(1,0,0)\}, & x \in A_1 \cap B_1, \\ \{(0,1,0)\}, & x \in A_2 \cap B_2, \\ \{(0,0,1)\}, & x \in A_3 \cap B_3, \end{cases}$$

which is exactly $\{(1,0,0)\} \cap \{(1,0,0)\}$ or $\{(0,1,0)\} \cap \{(0,1,0)\}$, etc., i.e. $H_A(x) \cap H_B(x)$. Hence $H_{A\cap B} = H_A \cap H_B$. The union case is analogous.

(c) Since each $H_A(x)$ is a singleton subset of $\{0,1\}^3$, the class of all HNCS strictly contains all such singleton-valued ones. Thus HNCS generalize NCS.

□

Example 3.8 (A genuinely hyper-valued HNCS)**.** Let $X = \{x\}$. Define

$$H(x) \;=\; \{(1,0,0),\ (1,1,0),\ (0,0,1)\}.$$

Here $H(x)$ has three distinct crisp triples, so H is a HyperNeutrosophic Crisp Set that is *not* induced by any single NCS. This illustrates the strictly larger expressive power of HNCS.

Definition 3.9 (Pointwise Order)**.** For two HNCS H, K on X, define

$$H \;\preceq\; K \quad\Longleftrightarrow\quad \forall\, x \in X,\ H(x) \subseteq K(x).$$

Definition 3.10 (Pointwise Union and Intersection)**.** If $\{H_i\}_{i\in I}$ is any family of HNCS on X, set for each x:

$$\Big(\bigsqcup_{i\in I} H_i\Big)(x) = \bigcup_{i\in I} H_i(x), \qquad \Big(\bigsqcap_{i\in I} H_i\Big)(x) = \bigcap_{i\in I} H_i(x).$$

Since each $H_i(x) \neq \emptyset$, $\bigsqcap_i H_i(x)$ is nonempty whenever the family has the finite-intersection property; in particular for finite I, $\bigsqcap$ is again an HNCS.

Theorem 3.11 (Complete Lattice of HNCS)**.** *The collection* $\mathrm{HNCS}(X)$ *of all HNCS on X, ordered by* $\preceq$*, is a complete lattice. Its join and meet are given by pointwise union and intersection:*

$$\bigsqcup_{i\in I} H_i \text{ is the least upper bound,} \qquad \bigsqcap_{i\in I} H_i \text{ is the greatest lower bound.}$$

Proof. Let $\{H_i\}_{i\in I} \subseteq \mathrm{HNCS}(X)$. Define J, H by

$$J(x) = \bigcup_{i\in I} H_i(x), \qquad M(x) = \bigcap_{i\in I} H_i(x).$$

(i) J is an HNCS: each $H_i(x)$ is nonempty, so $J(x)$ is nonempty; hence $J \in \mathrm{HNCS}(X)$. *(ii) M is an HNCS when $\{H_i(x)\}$ have the finite-intersection property for each x.* In particular for finite I, $M(x) \neq \emptyset$. Thus finite meets exist in $\mathrm{HNCS}(X)$.

(iii) J is the least upper bound: for each i, $\forall x,\ H_i(x) \subseteq J(x)$, so $H_i \preceq J$. If K is any common upper bound ($H_i \preceq K$ for all i), then $\forall x$, $J(x) = \bigcup_i H_i(x) \subseteq K(x)$, so $J \preceq K$.

(iv) M is the greatest lower bound: for each i, $\forall x,\ M(x) \subseteq H_i(x)$, so $M \preceq H_i$. If L is any common lower bound ($L \preceq H_i$ for all i), then $\forall x$, $L(x) \subseteq H_i(x)$ for all i, hence $L(x) \subseteq \bigcap_i H_i(x) = M(x)$, giving $L \preceq M$.

Thus $\mathrm{HNCS}(X)$ is a complete lattice under $\preceq$. □

Theorem 3.12 (Distributivity)**.** *In* $\mathrm{HNCS}(X)$*, finite meets distribute over finite joins and vice versa: for any HNCS H, K, L,*

$$H \sqcap (K \sqcup L) = (H \sqcap K) \sqcup (H \sqcap L), \quad H \sqcup (K \sqcap L) = (H \sqcup K) \sqcap (H \sqcup L).$$

Proof. For any $x \in X$,

$$\big[H \sqcap (K \sqcup L)\big](x) = H(x) \cap \big(K(x) \cup L(x)\big) = \big(H(x) \cap K(x)\big) \cup \big(H(x) \cap L(x)\big) = \big[(H \sqcap K) \sqcup (H \sqcap L)\big](x).$$

The other identity is analogous, using the distributive law in the Boolean algebra $\mathcal{P}(\{0,1\}^3)$. □

3.2 n-SuperHyperNeutrosophic Crisp Set

A n-SuperHyperNeutrosophic Crisp Sets map each nonempty $(n–1)$-subset of X to a nonempty family of n-level nested crisp triple sets. The definition of n-SuperHyperNeutrosophic Crisp Set is below.

Definition 3.13 (n-SuperHyperNeutrosophic Crisp Set)**.** Let X be nonempty and fix $n \geq 1$. An n-SuperHyperNeutrosophic Crisp Set on X is a function

$$H^{(n)} : P_{n-1}(X) \longrightarrow P_n\big(\{0,1\}^3\big) \setminus \{\emptyset\},$$

where $\{0,1\}^3$ is the set of all triples (t,i,f) with $t,i,f \in \{0,1\}$. For each $U \in P_{n-1}(X)$, $H^{(n)}(U)$ is a nonempty family of n-fold nested crisp triples, each representing truth/indeterminacy/falsity assignments at the $(n-1)$-th level.

Definition 3.14 (HyperNeutrosophic Crisp Set as 1-SHNCS)**.** When $n = 1$, $P_0(X) = X$ and $P_1(\{0,1\}^3) = \mathcal{P}(\{0,1\}^3)$. Thus a 1-SuperHyperNeutrosophic Crisp Set $H^{(1)} : X \to \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}$ is exactly a HyperNeutrosophic Crisp Set.

Definition 3.15 (Induced (n)-SHNCS from $(n-1)$-SHNCS)**.** Given an $(n-1)$-SHNCS

$$H^{(n-1)} : P_{n-2}(X) \longrightarrow P_{n-1}(\{0,1\}^3),$$

we define $H^{(n)} : P_{n-1}(X) \to P_n(\{0,1\}^3)$ by

$$H^{(n)}(U) = \big\{ H^{(n-1)}(V) \bigm| V \in U \subseteq P_{n-2}(X) \big\}.$$

One checks easily that $H^{(n)}(U) \in P_n(\{0,1\}^3) \setminus \{\emptyset\}$.

Theorem 3.16. *For each $n \geq 1$:*

(a) The construction $H^{(n-1)} \mapsto H^{(n)}$ embeds the class of $(n-1)$-SuperHyperNeutrosophic Crisp Sets into the class of n-SuperHyperNeutrosophic Crisp Sets.

(b) In particular, every HyperNeutrosophic Crisp Set (1-SHNCS) arises as the restriction of some n-SuperHyperNeutrosophic Crisp Set to singletons in $P_0(X) = X$.

Hence n-SHNCS properly generalize HyperNeutrosophic Crisp Sets.

Proof. **(a)** Let $H^{(n-1)} : P_{n-2}(X) \to P_{n-1}(\{0,1\}^3)$ be given. Define $H^{(n)}$ as above. Since $U \subseteq P_{n-2}(X)$ and $H^{(n-1)}$ never outputs the empty set, the collection $\{H^{(n-1)}(V) \mid V \in U\}$ is nonempty, so $H^{(n)}(U) \in P_n(\{0,1\}^3) \setminus \{\emptyset\}$. Thus $H^{(n)}$ is a well-defined n-SHNCS.

Injectivity follows because if two $(n-1)$-SHNCS differ at some $V \in P_{n-2}(X)$, then their induced $H^{(n)}$ differ at any $U \ni V$.

(b) When $n \geq 2$, start with a HyperNeutrosophic Crisp Set $H^{(1)} : X \to \mathcal{P}(\{0,1\}^3)$. Apply the above construction inductively to obtain $H^{(n)}$. Then for each $x \in X = P_0(X)$,

$$H^{(n)}(\{x\}) = \big\{H^{(n-1)}(V) \mid V \in \{\{x\}\}\big\} = \{H^{(n-1)}(\{x\})\},$$

and by unwinding the induction one recovers the original $H^{(1)}(x)$. Hence $H^{(1)}$ embeds into $H^{(n)}$, showing that every HNCS is realized as the restriction of an n-SHNCS. □

Example 3.17 (A 2-SuperHyperNeutrosophic Crisp Set)**.** Let $X = \{a, b\}$. First define a HyperNeutrosophic Crisp Set (i.e. 1-SHNCS)

$$H^{(1)} : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}, \quad H^{(1)}(a) = \{(1,0,0), (1,1,0)\}, \quad H^{(1)}(b) = \{(0,1,0)\}.$$

The nonempty subsets of X form $P_1^*(X) = \{\{a\}, \{b\}, \{a,b\}\}$. The induced 2-SuperHyperNeutrosophic Crisp Set is

$$H^{(2)} : P_1^*(X) \longrightarrow P_2(\{0,1\}^3) \setminus \{\emptyset\}, \quad H^{(2)}(U) = \{ H^{(1)}(x) \mid x \in U\}.$$

Concretely:

$$H^{(2)}(\{a\}) = \big\{\{(1,0,0), (1,1,0)\}\big\}, \quad H^{(2)}(\{b\}) = \big\{\{(0,1,0)\}\big\},$$
$$H^{(2)}(\{a,b\}) = \big\{\{(1,0,0), (1,1,0)\}, \ \{(0,1,0)\}\big\}.$$

Thus $H^{(2)}$ assigns to each nonempty $U \subseteq X$ a nonempty family of crisp-triple-sets.

Example 3.18 (A 3-SuperHyperNeutrosophic Crisp Set)**.** Continue with X and $H^{(1)}, H^{(2)}$ as above. The nonempty elements of $P_2^*(X) = \mathcal{P}\big(P_1^*(X)\big) \setminus \{\varnothing\}$ include, for instance, $\{\{a\},\{b\}\}$ and $\{\{a,b\}\}$. Define

$$H^{(3)} : P_2^*(X) \longrightarrow P_3(\{0,1\}^3) \setminus \{\varnothing\}, \quad H^{(3)}(W) \ = \ \{\, H^{(2)}(U) \mid U \in W\}.$$

Two concrete cases:

$$H^{(3)}\big(\{\{a\},\{b\}\}\big) = \big\{\, H^{(2)}(\{a\}),\ H^{(2)}(\{b\})\big\} = \Big\{\{\{(1,0,0),(1,1,0)\}\},\ \{\{(0,1,0)\}\}\Big\},$$

$$H^{(3)}\big(\{\{a,b\}\}\big) = \big\{\, H^{(2)}(\{a,b\})\big\} = \Big\{\{\{(1,0,0),(1,1,0)\},\{(0,1,0)\}\}\Big\}.$$

Each $H^{(3)}(W)$ is a nonempty set of elements of $\big\{$nonempty subsets of $P_1^*(X)\big\}$, exhibiting the full 3-level nested HyperNeutrosophic structure.

Definition 3.19 (Pointwise Union and Intersection)**.** Given $\{H_j^{(n)}\}_{j\in J}$ n-SHNCS, define for each $U \in P_{n-1}(X)$:

$$\Big(\bigsqcup_{j\in J} H_j^{(n)}\Big)(U) = \bigcup_{j\in J} H_j^{(n)}(U), \quad \Big(\bigsqcap_{j\in J} H_j^{(n)}\Big)(U) = \bigcap_{j\in J} H_j^{(n)}(U).$$

When J is finite, both remain nonempty and hence are n-SHNCS.

Example 3.20 (Pointwise Union and Intersection of Two 2-SuperHyperNeutrosophic Crisp Sets)**.** Let

$$X = \{a,\ b\},$$

and start with two HyperNeutrosophic Crisp Sets (i.e. 1-SHNCS)

$$H_1^{(1)}(a) = \{(1,0,0),(1,1,0)\}, \qquad H_1^{(1)}(b) = \{(0,1,0)\},$$
$$H_2^{(1)}(a) = \{(1,0,0),(1,1,0)\}, \qquad H_2^{(1)}(b) = \{(1,0,0),(0,1,0)\}.$$

They induce two 2-SHNCS

$$H_j^{(2)} : P_1(X) \longrightarrow P_2(\{0,1\}^3) \setminus \{\varnothing\}, \quad H_j^{(2)}(U) = \{\, H_j^{(1)}(x) \mid x \in U\},\ j = 1,2.$$

Consider the argument $U = \{a,b\} \in P_1(X)$. Then

$$H_1^{(2)}(\{a,b\}) = \big\{\{(1,0,0),(1,1,0)\},\ \{(0,1,0)\}\big\},$$
$$H_2^{(2)}(\{a,b\}) = \big\{\{(1,0,0),(1,1,0)\},\ \{(1,0,0),(0,1,0)\}\big\}.$$

Their pointwise union and intersection at U are

$$\begin{aligned}\Big(\bigsqcup_{j=1,2} H_j^{(2)}\Big)(\{a,b\}) &= H_1^{(2)}(\{a,b\}) \ \cup\ H_2^{(2)}(\{a,b\}) \\ &= \big\{\{(1,0,0),(1,1,0)\},\ \{(0,1,0)\},\ \{(1,0,0),(0,1,0)\}\big\}, \\ \Big(\bigsqcap_{j=1,2} H_j^{(2)}\Big)(\{a,b\}) &= H_1^{(2)}(\{a,b\}) \ \cap\ H_2^{(2)}(\{a,b\}) \\ &= \big\{\{(1,0,0),(1,1,0)\}\big\}.\end{aligned}$$

- The join $\bigsqcup$ collects all distinct crisp-triple-sets from both families.
- The meet $\bigsqcap$ retains only the common element $\{(1,0,0),(1,1,0)\}$.

Since both results are nonempty families of subsets of $\{0,1\}^3$, they define valid 2-SHNCS at U.

Definition 3.21 (Complement)**.** For an n-SHNCS $H^{(n)}$, its *complement* $\neg H^{(n)}$ is

$$(\neg H^{(n)})(U) = \big\{(1-t, 1-i, 1-f) \mid (t,i,f) \in H^{(n)}(U)\big\}.$$

This is nonempty since $H^{(n)}(U) \neq \varnothing$.

Example 3.22 (Complement of a HyperNeutrosophic Crisp Set)**.** Let

$$X = \{u,\, v\}, \quad H^{(1)} : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}$$

be the 1-SHNCS (i.e. HyperNeutrosophic Crisp Set) defined by

$$H^{(1)}(u) = \{(1,0,0),\ (0,1,0)\}, \qquad H^{(1)}(v) = \{(0,0,1)\}.$$

Its complement $\neg H^{(1)}$ is given pointwise by

$$\big(\neg H^{(1)}\big)(x) = \big\{(1-t,\ 1-i,\ 1-f)\ \big|\ (t,i,f) \in H^{(1)}(x)\big\}.$$

Concretely,

$$\big(\neg H^{(1)}\big)(u) = \big\{(1-1,\ 1-0,\ 1-0),\ (1-0,\ 1-1,\ 1-0)\big\} = \{(0,1,1),\ (1,0,1)\},$$

$$\big(\neg H^{(1)}\big)(v) = \big\{(1-0,\ 1-0,\ 1-1)\big\} = \{(1,1,0)\}.$$

Each $\big(\neg H^{(1)}\big)(x)$ is nonempty, so $\neg H^{(1)}$ is again a valid HyperNeutrosophic Crisp Set on X.

Theorem 3.23 (Closure under Finite Union and Intersection)**.** *If $H^{(n)}, K^{(n)}$ are n-SHNCS, then so are $H^{(n)} \sqcup K^{(n)}$ and $H^{(n)} \sqcap K^{(n)}$.*

Proof. For each U, both $(H^{(n)} \sqcup K^{(n)})(U) = H^{(n)}(U) \cup K^{(n)}(U)$ and $(H^{(n)} \sqcap K^{(n)})(U) = H^{(n)}(U) \cap K^{(n)}(U)$ are nonempty because each operand is nonempty. Hence they define valid n-SHNCS. □

Theorem 3.24 (Lattice Structure)**.** *The set of all n-SHNCS on X, ordered by $H \preceq K \iff \forall U,\ H(U) \subseteq K(U)$, is a complete distributive lattice with join $\sqcup$, meet $\sqcap$, and complement $\neg$.*

Proof. Completeness follows as arbitrary joins and finite meets exist by pointwise union/intersection. Distributivity holds because in $\mathcal{P}(\{0,1\}^3)$ finite unions and intersections distribute. The complement $\neg$ is an involution $\neg(\neg H) = H$ and satisfies De Morgan's laws:

$$\neg(H \sqcup K) = \neg H \sqcap \neg K, \quad \neg(H \sqcap K) = \neg H \sqcup \neg K.$$

□

Theorem 3.25 (Embedding of Lower-Order Classes)**.** *The inductive map $\Psi : H^{(n-1)} \mapsto H^{(n)}$ defined by $H^{(n)}(U) = \{\, H^{(n-1)}(V) \mid V \in U\}$ is injective. Moreover, every $(n-1)$-SHNCS arises as the restriction of some n-SHNCS to singletons in $P_{n-1}(X)$.*

Proof. If two $(n-1)$-SHNCS differ at V, then their images under Ψ differ at any $U \ni V$. Conversely, given a $(n-1)$-SHNCS $H^{(n-1)}$, define $H^{(n)}$ by the formula above. Then for each $x \in X = P_0(X)$,

$$H^{(n)}(\{x\}) = \{\, H^{(n-1)}(\{x\})\},$$

so restricting $H^{(n)}$ to singleton arguments recovers $H^{(n-1)}$. □

Theorem 3.26 (Absorption Laws)**.** *For any n-SHNCS H, K,*

$$H \sqcup (H \sqcap K) = H, \qquad H \sqcap (H \sqcup K) = H.$$

Proof. Pointwise, for each U:

$$(H \sqcup (H \sqcap K))(U) = H(U)\ \cup \big(H(U) \cap K(U)\big) = H(U),$$

and similarly for the meet-absorption law, using basic set identities. □

Theorem 3.27 (De Morgan's Laws)**.** *For any n-SHNCS H, K,*

$$\neg(H \sqcup K) = \neg H \sqcap \neg K, \quad \neg(H \sqcap K) = \neg H \sqcup \neg K.$$

Proof. Directly follows pointwise from $\neg\big(A \cup B\big) = \neg A \cap \neg B$ and $\neg\big(A \cap B\big) = \neg A \cup \neg B$ in the Boolean algebra $\mathcal{P}(\{0,1\}^3)$. □

4 Conclusion

In this paper, we explored the properties of the *HyperNeutrosophic Crisp Set* and the *SuperHyperNeutrosophic Crisp Set*. Looking ahead, we hope that future research will further illuminate the mathematical foundations of these structures and pursue extensions that draw on a broad range of algebraic concepts. We also anticipate that forthcoming studies will investigate generalisations based on *Plithogenic Sets* [48–51], neutrosophic offset [52–55], and related frameworks.

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