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Single Valued Pythagorean Neutrosophic Implicative Ideals in KU-algebras

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Abstract. In this paper, we introduce the notion of Pythagorean neutrosophic implicative ideals in KU-algebras, provide several illustrative examples, and investigate some of their key properties. We also define the image and inverse image of such ideals in KU-algebras and establish the conditions under which these images preserve the structure of Pythagorean neutrosophic implicative ideals. Moreover, we explore the Cartesian product of Pythagorean neutrosophic implicative ideals within the Cartesian product of KU-algebras.

Keywords: Pythagorean neutrosophic implicative ideal, image (inverse image) of Pythagorean neutrosophic implicative ideals, Cartesian product of Pythagorean neutrosophic implicative ideals.

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1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [11–13], and since their inception, they have been extensively investigated by several researchers. One of the fundamental approaches to studying these algebras is through their ideals, since ideals provide a powerful tool for understanding the intrinsic properties and internal structure of algebraic systems. The notions of ideals in BCK-algebras, along with the concept of positive implicative ideals (also known as Iseki's implicative ideals), were first introduced by Iseki himself [11–13]. Later, the study was extended to include commutative ideals and implicative ideals in BCK-algebras, which were systematically introduced and investigated in [17, 19–24]. In a parallel line of research, Zadeh [36] introduced the concept of fuzzy sets, a groundbreaking idea that has since found applications across numerous mathematical domains, including group theory, functional analysis, probability theory, and topology. In 1991, Xi [35] pioneered the application of fuzzy set theory to BCK-algebras, introducing the notion of fuzzy subalgebras and fuzzy ideals of BCK-algebras with respect to the minimum operator. Following this development, Jun et al. further studied fuzzy ideals in BCK-algebras [10,14,15], and subsequently, a number of fuzzy structures in BCC-algebras were proposed and examined [2–9]. Building on these ideas, Prabpayak and Leerawat [30, 31] introduced a new algebraic structure, called the KU-algebra. They also developed the concept of homomorphisms of KU-algebras and explored several of their fundamental properties. Extending this work, Mostafa et al. [25, 26, 28, 33] introduced the notion of fuzzy KU-ideals in KU-algebras and investigated their essential properties, thereby broadening the framework of fuzzy algebraic systems. In addition, Meng et al. [22, 23] introduced the concept of implicative ideals and commutative ideals in BCI-algebras and studied their fundamental properties. Motivated by this, Mostafa et al. [28, 33] extended these notions to KU-algebras, where they defined implicative ideals and commutative ideals and investigated their structural attributes in detail. Thus, the study of BCK, BCI, BCC, and KU-algebras, particularly through their ideals and fuzzy extensions, has become a significant branch of modern algebra, blending classical algebraic structures with fuzzy logic and neutrosophic theories to model uncertainty in a wide range of applications.

In this paper, we introduce the notion of Pythagorean neutrosophic implicative ideals in KU-algebras and investigate several of their basic properties. Furthermore, we examine the conditions under which the image and pre-image of a Pythagorean neutrosophic implicative ideal, under a homomorphism of KU-algebras, remain Pythagorean neutrosophic implicative ideals. In addition, we establish the relationship between the product of Pythagorean neutrosophic implicative ideals and the product of fuzzy implicative ideals.

2. Preliminaries

Definition 2.1. [30,31] Algebra $(\Upsilon, *, 0)$ of type $(2, 0)$ is said to be a KU-algebra, if it satisfies the following conditions:

(KU_1) $(\imath * \jmath) * [(\jmath * \ell) * (\imath * \ell)] = 0$,

(KU_2) $\imath * 0 = 0$,

(KU_3) $0 * \imath = \imath$,

(KU_4) $\imath * \jmath = 0$ and $\jmath * \imath = 0$ implies $\imath = \jmath$,

(KU_5) $\imath * \imath = 0$, for all $\imath, \jmath, \ell \in \Upsilon$.

On a KU-algebra $(\Upsilon, *, 0)$ we can define a binary relation $\leq$ on Υ by putting

$$\imath \leq \jmath \Leftrightarrow \jmath * \imath = 0.$$

Thus a KU-algebra Υ satisfies the conditions:

(kU_1) $(\jmath * \ell) * (\imath * \ell) \leq (\imath * \jmath)$,

(kU_1) $0 \leq \imath$,

(kU_1) $\imath \leq \jmath, y \leq \imath$ implies $\imath = \jmath$,

(kU_1) $\jmath * \imath \leq \imath$.

Theorem 2.2. [26] In a KU-algebra Υ, the following axioms are satisfied:

For all $\imath, \jmath, \ell \in \Upsilon$

(1) $\imath \leq \jmath$ imply $\jmath * \ell \leq \imath * \ell$,

(2) $\imath * (\jmath * \ell) = \jmath * (\imath * \ell)$, for all $\imath, \jmath, \ell \in \Upsilon$,

(3) $((\jmath * \imath) * \imath) \leq \jmath$,

(4) $((\jmath * \imath) * \imath) * \imath)) = (\jmath * \imath)$.

Definition 2.3. [30, 31] Let I be a non empty subset of a KU-algebra Υ. Then, I is said to be an ideal of Υ, if

(I_0) $0 \in I$,

(II_0) $\forall\ \jmath, \ell \in \Upsilon$, if $(\jmath * \ell) \in I$ and $\jmath \in I$, imply $\ell \in I$.

Definition 2.4. [26] A non empty subset I of a KU-algebra Υ is said to be an KU-ideal of Υ if it satisfies:

(K_1) $0 \in I$,

(K_2) $\imath * (\jmath * \ell) \in I$ and $\jmath \in I$ imply $\imath * \imath \in I$ for all $\imath, \jmath$ and $\ell \in \Upsilon$.

Definition 2.5. [33] A KU-algebra Υ is said to be implicative if it satisfies the identity

$$\Upsilon = (\imath * \jmath) * \imath, \text{ for all } \imath, \jmath \in \Upsilon.$$

For the properties of KU-algebras, we refer the reader to [12–16].

Definition 2.6. [36] Let Υ be a non empty set, a fuzzy subset μ in Υ is a function

$$f : \Upsilon \to [0, 1].$$

Definition 2.7. [26] Let Υ be a KU-algebra, a fuzzy subset μ in Υ is called a fuzzy ideal of Υ if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(\imath)$, for all $\imath \in \Upsilon$,

(ii) $\forall \imath, \jmath \in \Upsilon, \mu(\jmath) \geq \min\{\mu(\imath * \jmath), \mu(\imath)\}$.

Definition 2.8. [27] Let Υ be a KU-algebra. A fuzzy set μ in Υ is called a fuzzy KU-ideal of Υ if it satisfies:

(FK_1) $\mu(0) \geq \mu(\imath)$, (FK_2) $\mu(\imath * \ell) \geq \min\{\mu(\imath * (\jmath * \ell)), \mu(\jmath)\}$, for all $\imath, \jmath$ and $\ell \in \Upsilon$.

Definition 2.9. [26] Let μ be a fuzzy set in a set Υ. For $t \in [0, 1]$, the set

$$\mu_t = \{\imath \in \Upsilon | \mu(\imath) \geq t\}$$

is called upper level cut (level subset) of μ.

Definition 2.10. [29] A non empty subset μ of a KU-algebra Υ is called a fuzzy implicative ideal of Υ, if $\forall\ \imath, \jmath, \ell \in \Upsilon$,

(F_0) $\mu(0) \geq \mu(\imath)$,

(F_0) $\mu((\imath * \jmath) * \imath) \geq \min\{\mu(\ell * ((\imath * \jmath) * \imath)), \mu(\ell)\}$.

Definition 2.11. [35] Let f be a mapping from the set Υ to a set Y. If μ is a fuzzy subset of Υ, then the fuzzy subset B of Y defined by

$$f(\mu)(\jmath) = B(\jmath) = \begin{cases} \sup\limits_{\imath \in f^{-1}(\jmath)} \mu(\imath) & \text{if} \quad f^{-1}(\jmath) = \{\imath \in \Upsilon, f(\imath) = \jmath\} \neq \phi \\ 0 & , \text{ otherwise} \end{cases}$$

is said to be the image of μ under f. Similarity if β is a fuzzy subset of Y, then the fuzzy subset $\mu = \beta \circ f$ in Υ (i. e. the fuzzy subset defined by $\mu(\imath) = \beta(f(\imath)$, for all $\imath \in \Upsilon$ is called the primage of β under f.

Definition 2.12. [27] Let μ be a fuzzy set on a KU-algebra Υ, then μ is called a fuzzy KU-subalgebra of Υ if $\mu(\imath * \jmath) \geq \min\{\mu(\imath), \mu(\jmath)\}$ for all $\imath, \jmath \in \Upsilon$.

Lemma 2.13. [27] Let μ be a fuzzy ideal of KU-algebra Υ. if the inequality $\imath * \jmath \leq \ell$ hold in Υ, then $\mu(\jmath) \geq \min\{\mu(\imath), \mu(\ell)\}$.

Lemma 2.14. [27] If μ be a fuzzy ideal of KU-algebra Υ and if $\imath \leq \jmath$, then $\mu(\imath) \geq \mu(\jmath)$.

Definition 2.15. [32]. Let Υ be a non-empty set (Universe) A Pythagorean neutrosophic set (briefly, PNS) T and F as dependent neutrosophic components A on Υ is an object of the form $\mathcal{P} = \{\langle \imath, \mu_{\mathcal{P}}(\imath), \nu_{\mathcal{P}}(\imath), \lambda_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$,
where $\mu_{\mathcal{P}}(\imath)$, $\nu_{\mathcal{P}}(\imath)$, $\lambda_{\mathcal{P}}(\imath)$ are the truth, indeterminacy and false respectively such that $\mu, \nu, \lambda \in [0, 1]$. Here when μ and λ are dependent components, then for all Υ in Υ; (i) $\mu + \lambda \leq 1$, (ii) $0 \leq \mu^2 + \lambda^2 \leq 1$, (iii) $0 \leq \mu^2 + \nu^2 + \lambda^2 \leq 2$.

We define these basic operations on PNS which can be described as follows: Let Υ be a nonempty set (universe). A Pythagorean Neutrosophic set μ and λ as dependent neutrosophic components $\mathcal{P}$ and $\mathcal{Q}$ of the form $\mathcal{P} = \{\langle x, \mu_{\mathcal{P}}(\imath), \nu_{\mathcal{P}}(\imath), \lambda_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$ and $\mathcal{Q} = \{\langle x, \mu_{\mathcal{Q}}(\imath), \nu_{\mathcal{Q}}(\imath), \lambda_{\mathcal{Q}}(\imath)\rangle | \imath \in \Upsilon\}$. The complement of $\mathcal{P}$ is $\mathcal{P}^c = \{\langle x, \lambda_{\mathcal{P}}(\imath), 1 - \nu_{\mathcal{P}}(\imath), \mu_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$. The union and intersection of $\mathcal{P}$ and $\mathcal{Q}$ are

(i) $\mathcal{P} \cup \mathcal{Q} = \{max(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), min(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), min(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$;

(ii) $\mathcal{P} \cap \mathcal{Q} = \{min(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), max(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), max(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$.

3. Pythagorean neutrosophic implicative ideals

Definition 3.1. Let Υ be a KU-algebra. A PNS $\mathfrak{C} = \{\langle \imath, \mu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\imath)\rangle | \imath \in \Upsilon\}$ is called a PN single valued ideal ($PNSVI$) of Υ if it satisfies:

(i) $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

(ii) $\mu_{\mathfrak{C}}(\jmath) \geq \min\{\mu_{\mathfrak{C}}(\imath \times \jmath), \mu_{\mathfrak{C}}(\imath)\}$, $\nu_{\mathfrak{C}}(\jmath) \leq \max\{\nu_{\mathfrak{C}}(\imath \times \jmath), \nu_{\mathfrak{C}}(\imath)\}$, $\lambda_{\mathfrak{C}}(\jmath) \leq \max\{\lambda_{\mathfrak{C}}(\imath \times \jmath), \lambda_{\mathfrak{C}}(\imath)\}$, $\forall \imath, \jmath \in \Upsilon$.

Definition 3.2. Let Υ be a KU-algebra, a PNS $\mathfrak{C} = \{\langle \imath, \mu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\imath)\rangle | \imath \in \Upsilon\}$ in Υ is called a PN implicative ideal (resp. $PNImpI$) of Υ if it satisfies the following conditions:

(PN_0) $\forall \imath \in \Upsilon$, $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

(PN_1) $\forall \imath, \jmath, \ell \in \Upsilon$, $\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) \geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \mu_{\mathfrak{C}}(\ell)\}$,
$\nu_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \nu_{\mathfrak{C}}(\ell)\}$,
$\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$.

Definition 3.3. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be PNS in Υ. For $s, t, r \in [0,1]$, the set, $U(\mu_{\mathfrak{C}}, t) = \{\imath \in \Upsilon / \mu_{\mathfrak{C}}(\imath) \geq t\}$ is called upper level cut of $\mu_{\mathfrak{C}}$, the set $L(\nu_{\mathfrak{C}}, s) = \{\imath \in \Upsilon / \nu_{\mathfrak{C}}(\imath) \leq s$ is called lower level cut of $\nu_{\mathfrak{C}}$ and the set $L(\lambda_{\mathfrak{C}}, r) = \{\imath \in \Upsilon / \lambda_{\mathfrak{C}}(\imath) \leq r\}$ is called lower level cut of $\lambda_{\mathfrak{C}}$.

Example 3.4. Let $\Upsilon = \{a, b, c, d, e\}$ in which the operation $*$ is given by the table

$*$	a	b	c	d	e
a	a	b	c	d	e
b	a	a	b	d	e
c	a	a	a	d	e
d	a	a	a	a	e
e	a	a	a	a	a

Then $(\Upsilon, *, 0 = a)$ is a KU-algebra. Define a PNS $\mu : \Upsilon \to [0,1]$, $\nu : \Upsilon \to [0,1]$ and $\lambda : \Upsilon \to [0,1]$ by

$$\mu(a) = s_0, \mu(b) = \mu(c) = s_1, \mu(d) = s_2, \mu(e) = s_3$$
$$\nu(a) = t_0, \nu(b) = \nu(c) = t_1, \nu(d) = t_2, \nu(e) = t_3$$
$$\lambda(a) = u_0, \lambda(b) = \lambda(c) = u_1, \lambda(d) = u_2, \lambda(e) = u_3$$

where $s_0, s_1, s_2, s_3 \in [0,1]$ with $s_0 < s_1 < s_2 < s_3$, $t_0, t_1, t_2, t_3 \in [0,1]$ with $t_0 < t_1 < t_2 < t_3$ and $u_0, u_1, u_2, u_3 \in [0,1]$ with $u_0 > u_1 > u_2 > u_3$

Routine calculation gives that μ is a $\langle \Upsilon, \mu, \nu, \lambda \rangle$ is a $PNImpI$ of KU-algebra Υ.

Lemma 3.5. *If $\mathfrak{C} = \{\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}}\}$ is a $PNImpI$ of KU- algebra Υ and if $\imath \leq \ell$, then $\mu_{\mathfrak{C}}(\imath) \geq \mu_{\mathfrak{C}}(\ell)$, $\nu_{\mathfrak{C}}(\imath) \leq \nu_{\mathfrak{C}}(\ell)$, $\lambda_{\mathfrak{C}}(\imath) \leq \lambda_{\mathfrak{C}}(\ell)$.*

Proof. If $\imath \leq \ell$, then $\ell * \imath = 0$, this together with $0 * \imath = \imath$, $\imath * \imath = \imath * 0 = 0$ and $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\ell)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\ell)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\ell)$. Put $\jmath = 0$ in (PN_1), we get

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * 0) * \imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * 0) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(0 * \imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * (0 * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * \imath), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(\ell)\} \\
&= \min\{\mu_{\mathfrak{C}}(0 * \ell)\} = \mu_{\mathfrak{C}}(\ell) \\
\Rightarrow \mu_{\mathfrak{C}}(\imath) &\geq \mu_{\mathfrak{C}}(\ell).
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * 0) * \imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * 0) * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(0 * \imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * (0 * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(\imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * \imath), \nu_{\mathfrak{C}}(\ell)\} \\
&= \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(\ell)\} \\
&= \max\{\nu_{\mathfrak{C}}(0 * \ell)\} \\
&= \nu_{\mathfrak{C}}(\ell)
\end{aligned}$$

$$\Rightarrow \nu_{\mathfrak{C}}(\imath) \leq \nu_{\mathfrak{C}}(\ell).$$

Similarly we can prove for $\lambda_{\mathfrak{C}}(\imath) \leq \lambda_{\mathfrak{C}}(\ell)$.

Lemma 3.6. *Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be a $PNImpI$ of KU-algebra Υ, if the inequality, $\ell * \imath \leq \jmath$ hold in Υ, then $\mu_{\mathfrak{C}}(\imath) \geq min\{\mu_{\mathfrak{C}}(\jmath), \mu_{\mathfrak{C}}(\ell)\}$, $\nu_{\mathfrak{C}}(\imath) \leq max\{\nu_{\mathfrak{C}}(\jmath), \nu_{\mathfrak{C}}(\ell)\}$ and $\lambda_{\mathfrak{C}}(\imath) \leq max\{\lambda_{\mathfrak{C}}(\jmath), \lambda_{\mathfrak{C}}(\ell)\}$.*

Proof. Assume that the inequality $\ell * \imath \leq \jmath$ holds in Υ, then $\mu_{\mathfrak{C}}(\ell * \imath) \geq \mu_{\mathfrak{C}}(\jmath)$, $\nu_{\mathfrak{C}}(\ell * \imath) \leq \nu_{\mathfrak{C}}(\jmath)$ and $\lambda_{\mathfrak{C}}(\ell * \imath) \leq \lambda_{\mathfrak{C}}(\jmath)$ (by Lemma 3.5). Put $\imath = \jmath$ in (PN_2), we have

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * \imath) * \imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}((0 * \imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * (0 * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * \imath), \mu_{\mathfrak{C}}(\ell)\} \\
&= min\{\mu_{\mathfrak{C}}(\jmath), \mu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * \imath) * \imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(0 * \imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * (0 * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(\imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * \imath), \nu_{\mathfrak{C}}(\ell)\} \\
&= max\{\nu_{\mathfrak{C}}(\jmath), \nu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

and

$$\lambda_{\mathfrak{C}}((\imath * \imath) * \imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$$
$$\lambda_{\mathfrak{C}}(0 * \imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * (0 * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$$
$$\lambda_{\mathfrak{C}}(\imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * \imath), \lambda_{\mathfrak{C}}(\ell)\}$$
$$= max\{\lambda_{\mathfrak{C}}(\jmath), \lambda_{\mathfrak{C}}(\ell)\}.$$

Proposition 3.7. The intersection of any collection of $PNImpI$'s of a KU-algebra Υ is also a $PNImpI$.

Proof. Let $A_1 = \{\mu_{A_i}, \nu_{A_i}, \lambda_{A_i}\}$ be a family of $PNImpI$'s of KU-algebra Υ, then for any $\imath, \jmath, \ell \in \Upsilon$.
$(\bigcap_i \mu_{A_i})(0) = \inf(\mu_{A_i}(0)) \geq \inf(\mu_{A_i}(\imath)) = (\bigcap_i \mu_{A_i})(\imath)$,
$(\bigcup_i \nu_{A_i})(0) = \sup(\nu_{A_i}(0)) \leq \sup(\nu_{A_i}(\imath)) = (\bigcup_i \nu_{A_i})(\imath)$
and $(\bigcup_i \lambda_{A_i})(0) = \sup(\lambda_{A_i}(0)) \leq \sup(\lambda_{A_i}(\imath)) = (\bigcup_i \lambda_{A_i})(\imath)$.

$$(\bigcap_i \mu_{A_i})((\imath * \jmath) * \imath) = \inf(\mu_{A_i}((\imath * \jmath) * \imath))$$
$$\geq \inf(\min\{\mu_{A_i}(\ell * ((\imath * \jmath) * \imath)), \mu_{A_i}(\ell)\})$$
$$= \min\{\inf(\mu_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \inf(\mu_{A_i}(\ell))\}$$
$$= \min\{(\bigcap_i \mu_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcap_i \mu_{A_i})(\ell)\}$$

$$(\bigcup_i \nu_{A_i})((\imath * \jmath) * \imath) = \sup(\nu_{A_i}((\imath * \jmath) * \imath))$$
$$\leq \sup(\max\{\nu_{A_i}(\ell * ((\imath * \jmath) * \imath)), \nu_{A_i}(\ell)\})$$
$$= \max\{\sup(\nu_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \sup(\nu_{A_i}(\ell))\}$$
$$= \max\{(\bigcup_i \nu_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcup_i \nu_{A_i})(\ell)\}$$

$$(\bigcup_i \lambda_{A_i})((\imath * \jmath) * \imath) = \sup(\lambda_{A_i}((\imath * \jmath) * \imath))$$
$$\leq \sup(\max\{\lambda_{A_i}(\ell * ((\imath * \jmath) * \imath)), \lambda_{A_i}(\ell)\})$$
$$= \max\{\sup(\lambda_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \sup(\lambda_{A_i}(\ell))\}$$
$$= \max\{(\bigcup_i \lambda_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcup_i \lambda_{A_i})(\ell)\}.$$

Definition 3.8. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}})$ be PNS in Υ for $s, t, r \in [0,1]$, the set, $\cup(\mu_{\mathfrak{C}}, t) = \{\imath \in \Upsilon / \mu_{\mathfrak{C}} \geq t\}$ is called upper level cut of $\mu_{\mathfrak{C}}$, the set $L(\nu, s) = \{\imath \in \Upsilon / \nu_{\mathfrak{C}} \geq s$ is called loweer level cut of $\nu_{\mathfrak{C}}$ and the set $L(\lambda_{\mathfrak{C}}, \nu) = \{\imath \in \Upsilon / \lambda_{\mathfrak{C}} \leq r\}$ is called lower level cut of $\lambda_{\mathfrak{C}}$

Theorem 3.9. A PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ is a $PNImpI$ of Υ iff, for every $t, s, r \in [0,1]$, $U(\mu_{\mathfrak{C}}, t)$, $L(\nu_{\mathfrak{C}}, s)$ and $L(\lambda_{\mathfrak{C}}, r)$ are either empty or an implicative ideals of Υ.

Proof. Assume that $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is a $PNImpI$ of Υ, by (PN_1), we have $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$ for all $\imath \in \Upsilon$, therefore $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath) \geq t$ for $\imath \in U(\mu_{\mathfrak{C}}, t)$ and so $0 \in U(\mu_{\mathfrak{C}}, t)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath) \leq s$ for $\imath \in L(\nu_{\mathfrak{C}}, s)$ and so $0 \in L(\nu_{\mathfrak{C}}, s)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath) \leq r$ for $\imath \in L(\lambda_{\mathfrak{C}}, r)$ and so $0 \in L(\lambda_{\mathfrak{C}}, r)$. Let $\ell * ((\imath * \jmath) * \imath) \in U(\mu_{\mathfrak{C}}, t)$ and $\ell \in U(\mu_{\mathfrak{C}}, t)$, then $\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \geq t$ and $\mu_{\mathfrak{C}}(\ell) \geq t$, since $\mu_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) \geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \geq t$ and therefore $(\imath * \jmath) * \imath \in U(\mu_{\mathfrak{C}}, t)$. Hence $U(\mu_{\mathfrak{C}}, t)$ is an KU-ideal of Υ. Let $\ell * ((\imath * \jmath) * \imath) \in L(\nu_{\mathfrak{C}}, s)$ and $\ell \in L(\nu_{\mathfrak{C}}, s)$, then $\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \leq s$ and $\nu_{\mathfrak{C}}(\ell) \leq s$, since $\nu_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\nu_{\mathfrak{C}}((\imath*\jmath)*\imath) \leq \max\{\nu_{\mathfrak{C}}(\ell*((\imath*\jmath)*\imath)), \nu_{\mathfrak{C}}(\ell)\} \leq s$ and therefore $(\imath * \jmath) * \imath \in L(\nu_{\mathfrak{C}}, s)$. Hence $\nu_{\mathfrak{C}}$ is an KU-ideal of Υ.

Let $\ell * ((\imath * \jmath) * \imath) \in L(\lambda_{\mathfrak{C}}, r)$ and $\ell \in L(\lambda_{\mathfrak{C}}, r)$, then $\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \leq r$ and $\lambda_{\mathfrak{C}}(\ell) \leq r$, since $\lambda_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\} \leq r$ and therefore $(\imath * \jmath) * \imath \in L(\lambda_{\mathfrak{C}}, r)$. Hence $\lambda_{\mathfrak{C}}$ is an KU-ideal of Υ.

Conversely, we only need to show that (PN_1) $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$ and (PN_2) $\mu_{\mathfrak{C}}((\imath*\jmath)*\imath) \geq \min\{\mu_{\mathfrak{C}}(\ell*((\imath*\jmath)*\imath)), \mu_{\mathfrak{C}}(\ell)\}$, $\nu_{\mathfrak{C}}((\imath*\jmath)*\imath) \leq \max\{\nu_{\mathfrak{C}}(\ell*((\imath*\jmath)*\imath)), \nu_{\mathfrak{C}}(\ell)\}$, $\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$ are true. If PN_1 is false then there exist $\imath' \in \Upsilon$ such that $\mu_{\mathfrak{C}}(0) < \mu_{\mathfrak{C}}(\imath')$, $\nu_{\mathfrak{C}}(0) > \nu_{\mathfrak{C}}(\imath')$, $\lambda_{\mathfrak{C}}(0) > \lambda_{\mathfrak{C}}(\imath')$. If we take $t' = (\mu_{\mathfrak{C}}(\imath') + \mu_{\mathfrak{C}}(0))/2$, $s' = (\nu_{\mathfrak{C}}(\imath') + \nu_{\mathfrak{C}}(0))/2$, $r' = (\lambda_{\mathfrak{C}}(\imath') + \lambda_{\mathfrak{C}}(0))/2$, then $\mu_{\mathfrak{C}}(0) < t'$, $\nu_{\mathfrak{C}}(0) > s'$, $\lambda_{\mathfrak{C}}(0) > r'$ and $0 \leq t' < \mu_{\mathfrak{C}}(\imath') \leq 1$, $1 \geq s' > \nu_{\mathfrak{C}}(\imath') \geq 0$, $1 \geq r' > \lambda_{\mathfrak{C}}(\imath') \geq 0$ thus $\imath' \in U(\mu_{\mathfrak{C}}, t')$, $\imath' \in L(\nu_{\mathfrak{C}}, s')$, $\imath' \in L(\lambda_{\mathfrak{C}}, r')$ and $U(\mu_{\mathfrak{C}}, t') \neq \phi$, $L(\nu_{\mathfrak{C}}, s') \neq \phi$, $L(\lambda_{\mathfrak{C}}, r') \neq \phi$. As $\mu_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in U(\mu_{\mathfrak{C}}, t')$ and so $\mu_{\mathfrak{C}}(0) \geq t'$. This is a contradiction. As $\nu_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in L(\nu_{\mathfrak{C}}, s')$ and so $\nu_{\mathfrak{C}}(0) \leq s'$. This is a contradiction. As $\lambda_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in L(\lambda_{\mathfrak{C}}, r')$ and so $\lambda_{\mathfrak{C}}(0) \leq r'$. This is a contradiction. Now, assume (PN_2) is not true, then there exist $\imath'$, $\jmath'$ and ℓ' such that, $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < \min\{\mu_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \mu_{\mathfrak{C}}(\ell')\}$, $\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') > \max\{\nu_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \nu_{\mathfrak{C}}(\ell')\}$, $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') > \max\{\lambda_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \lambda_{\mathfrak{C}}(\ell')\}$. Putting $t' = \{\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \min\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \mu_{\mathfrak{C}}(\ell')\}\}/2$ then $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < t'$ and $0 \leq t' < \min\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \mu_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence $\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath') > t'$ and $\mu_{\mathfrak{C}}(\ell') > t' \Rightarrow (\imath' * \jmath') * \imath' \in U(\mu_{\mathfrak{C}}, t')$ and $\ell' \in U(\mu_{\mathfrak{C}}, t')$. Since $\mu_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in U(\mu_{\mathfrak{C}}, t')$ and that $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') > t'$ this is also a contradiction. Hence $U(\mu_{\mathfrak{C}}, t')$ is a $PNImpI$ of Υ. Putting $s' = \{\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \max\{\nu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \nu_{\mathfrak{C}}(\ell')\}\}/2$ then $\nu_{\mathfrak{C}}((\imath'*\jmath')*\imath') > s'$ and $0 \leq s' \leq \max\{\nu_{\mathfrak{C}}(\ell'*(\imath'*\jmath')*\imath'), \nu_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence $\{\nu_{\mathfrak{C}}(\ell'*(\imath'*\jmath')*\imath') < s'$ and $\nu_{\mathfrak{C}}(\ell') < s' \Rightarrow (\imath' * \jmath') * \imath' \in L(\nu_{\mathfrak{C}}, s')$ and $\ell' \in L(\nu_{\mathfrak{C}}, s')$. Since $\nu_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in L(\nu_{\mathfrak{C}}, s')$ and that $\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < s'$ this is also a contradiction. Hence $L(\nu_{\mathfrak{C}}, s')$ is a $PNImpI$ of Υ. Putting $r' = \{\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \max\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \lambda_{\mathfrak{C}}(\ell')\}\}/2$ then $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') > r'$ and $0 \leq r' \leq \max\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \lambda_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence

$\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath') < r'$ and $\lambda_{\mathfrak{C}}(\ell') < r' \Rightarrow (\imath' * \jmath') * \imath' \in L(\lambda_{\mathfrak{C}}, r')$ and $\ell' \in L(\lambda_{\mathfrak{C}}, r')$. Since $\lambda_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in L(\lambda_{\mathfrak{C}}, r')$ and that $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') < r'$ this is also a contradiction. Hence $L(\lambda_{\mathfrak{C}}, s')$ is a $PNImpI$ of Υ.

Corollary 3.10. If a PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ is a $PNImpI$ then for every $t \in I_m(\mu_{\mathfrak{C}})$, $s \in I_m(\nu_{\mathfrak{C}})$, $r \in I_m(\lambda_{\mathfrak{C}})$, $U(\mu_{\mathfrak{C}}, t)$, $L(\nu_{\mathfrak{C}}, s)$, $L(\lambda_{\mathfrak{C}}, r)$ is an $ImpI$ of Υ.

Theorem 3.11. An onto homomorphic preimage of a $PNImpI$ is also a $PNImpI$.

Proof. Let $\zeta : \Upsilon \to Y$ be an into homomorphic of KU-algebras, $B = (\jmath, \mu_B, \nu_B, \lambda_B)$ is a $PNImpI$ of Y and $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ the preimage of B under ζ, then

$$\mu_B(\zeta(\imath)) = \mu_{\mathfrak{C}}(\imath),\ \forall \imath \in \Upsilon$$
$$\nu_B(\zeta(\imath)) = \nu_{\mathfrak{C}}(\imath),\ \forall \imath \in \Upsilon$$
$$\lambda_B(\zeta(\imath)) = \lambda_{\mathfrak{C}}(\imath), \forall \imath \in \Upsilon.$$

Let $\imath \in \Upsilon$ then, $\mu_{\mathfrak{C}}(0) = \mu_B(\zeta(0)) \geq \mu_B(\zeta(\imath)) = \mu_{\mathfrak{C}}(\imath)$,
$\nu_{\mathfrak{C}}(0) = \nu_B(\zeta(0)) \leq \nu_B(\zeta(\imath)) = \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) = \lambda_B(\zeta(0)) \leq \lambda_B(\zeta(\imath)) = \lambda_{\mathfrak{C}}(\imath)$. Now, let $\imath, \jmath, \ell \in \Upsilon$ then

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \mu_B(\zeta((\imath * \jmath) * \imath)) \\
&= \mu_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\
&\geq \min\{\mu_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \mu_B(\zeta(\ell))\} \\
&= \min\{\mu_B(\zeta(\ell * ((\jmath * \imath) * \imath)), \mu_B(\zeta(\ell))\} \\
&= \min\{\mu_{\mathfrak{C}}(\ell * ((\jmath * \imath) * \imath)), \mu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \nu_B(\zeta((\imath * \jmath) * \imath)) \\
&= \nu_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\
&\leq \max\{\nu_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \nu_B(\zeta(\ell))\} \\
&= \max\{\nu_B(\zeta(\ell * ((\imath * \jmath) * \imath)), \nu_B(\zeta(\ell))\} \\
&= \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \nu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \lambda_B(\zeta((\imath * \jmath) * \imath)) \\ &= \lambda_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\ &\leq \max\{\lambda_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \lambda_B(\zeta(\ell))\} \\ &= \max\{\lambda_B(\zeta(\ell * ((\imath * \jmath) * \imath)), \lambda_B(\zeta(\ell))\} \\ &= \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}\end{aligned}$$

Hence the proof.

Definition 3.12. A PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of Υ has sup property if for any subset $T = (\Upsilon, \mu_T, \nu_T, \lambda_T)$ of Υ there exist $t_0, s_0, r_0 \in T$ such that

$$\begin{aligned}\mu_{\mathfrak{C}}(t_0) &= \sup_{t \in T} \mu_{\mathfrak{C}}(t) \\ \nu_{\mathfrak{C}}(s_0) &= \inf_{s \in T} \nu_{\mathfrak{C}}(s) \\ \lambda_{\mathfrak{C}}(r_0) &= \inf_{r \in T} \lambda_{\mathfrak{C}}(r).\end{aligned}$$

Theorem 3.13. Let $\zeta : \Upsilon \to Y$ be a homomorphism between KU-algebras Υ and Y. For every $PNImpI$ $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ in Υ, $B = (\zeta(\mu_{\mathfrak{C}}), \zeta(\nu_{\mathfrak{C}}), \zeta(\lambda_{\mathfrak{C}}))$ is a $PNImpI$ of Y.

Poof. By definition $\mu_B(\jmath') = \zeta(\mu_{\mathfrak{C}})(\jmath') = \sup_{\imath \in f^{-1}(\jmath')} \mu_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\sup \phi = 0$.
$\nu_B(\jmath') = \zeta(\nu_{\mathfrak{C}})(\jmath') = \inf_{\imath \in \zeta^{-1}(\jmath')} \nu_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\inf \phi = 0$.
$\lambda_B(\jmath') = \zeta(\lambda_{\mathfrak{C}})(\jmath') = \inf_{\imath \in \zeta^{-1}(\jmath')} \lambda_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\inf \phi = 0$.
We have to prove that
$\mu_B((\imath' * \jmath') * \imath') \geq \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}$
$\nu_B((\imath' * \jmath') * \imath') \leq \max\{\nu_B(\ell' * (\imath' * \jmath') * \imath'), \nu_B(\ell')\}$
$\lambda_B((\imath' * \jmath') * \imath') \leq \max\{\lambda_B(\ell' * (\imath' * \jmath') * \imath'), \lambda_B(\ell')\}$ $\forall \imath', \jmath', \ell' \in Y$.

Let $\zeta : \Upsilon \to Y$ be an onto homomorphism of KU-algebras, $\mathfrak{C}$ a $PNImpI$ of Υ with sup property and B the image of $\mathfrak{C}$ under ζ, since $\mathfrak{C}$ is a $PNImpI$ of Υ, we have

$$\begin{aligned}\mu_{\mathfrak{C}}(0) &\geq \mu_{\mathfrak{C}}(\imath) \\ \nu_{\mathfrak{C}}(0) &\leq \nu_{\mathfrak{C}}(\imath) \\ \lambda_{\mathfrak{C}}(0) &\leq \lambda_{\mathfrak{C}}(\imath), \ \forall \imath \in \Upsilon.\end{aligned}$$

Note that $0 \in \zeta'(0)$, where $0, 0'$ are the zero of Υ and Y respectively.
Thus, $\mu_B(0') = \sup_{t \in \zeta^{-1}(0')} \mu_{\mathfrak{C}}(t) = \mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$,
$\nu_B(0') = \inf_{t \in \zeta^{-1}(0')} \nu_{\mathfrak{C}}(t) = \nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$,
$\lambda_B(0') = \inf_{t \in \zeta^{-1}(0')} \lambda_{\mathfrak{C}}(t) = \lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

for all $\imath \in \Upsilon$, which implies that

$\mu_B(0^{'}) \geq \sup\limits_{t\in\zeta^{-1}(\imath')} \mu_{\mathfrak{C}}(t) = \mu_B(\imath')$,

$\nu_B(0^{'}) \leq \inf\limits_{t\in\zeta^{-1}(\imath')} \nu_{\mathfrak{C}}(t) = \nu_B(\imath')$,

$\lambda_B(0^{'}) \leq \inf\limits_{t\in\zeta^{-1}(\imath')} \lambda_{\mathfrak{C}}(t) = \lambda_B(\imath')$, $\forall \imath^{'} \in Y$. For any $\imath'$, $\jmath'$, $\ell' \in Y$, let $\imath_0 \in \zeta^{-1}(\imath')$, $\jmath_0 \in \zeta^{-1}(\jmath')$, $\ell_0 \in \zeta^{-1}(\ell')$ be such that

$$\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \sup_{t\in\zeta^{-1}(\ell_0*((\imath_0*\jmath_0)*\imath_0))} \mu_{\mathfrak{C}}(t),$$

$$\mu_{\mathfrak{C}}(\ell_0) = \sup_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}
\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) &= \mu_B(\zeta(\ell_0 * (\imath_0 * \jmath_0) * \imath_0)) \\
&= \mu_B(\ell' * ((\imath' * \jmath') * \imath')) \\
&= \sup_{(\ell_0*((\imath_0*\jmath_0)*\imath_0))\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0) \\
&= \sup_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t).
\end{aligned}$$

Then

$$\begin{aligned}
\mu_B((\imath' * \jmath') * \imath') &= \sup_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t) \\
&= \mu_{\mathfrak{C}}((\imath_0 * \jmath_0) * \imath_0) \\
&\geq \min\{\mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0), \mu_{\mathfrak{C}}(\ell_0)\} \\
&= \min\{\sup_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t), \sup_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\} \\
&= \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.
\end{aligned}$$

$$\nu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \inf_{t\in\zeta^{-1}(\ell^{'}*(\imath^{'}*\jmath^{'})*\imath^{'})} \nu_{\mathfrak{C}}(t)$$

$$\nu_{\mathfrak{C}}(\ell_0) = \inf_{t\in\zeta^{-1}(\ell^{'})} \nu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}
\nu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0) &= \nu_B(\zeta(\ell_0 * (\imath_0 * \jmath_0) * \imath_0)) \\
&= \nu_B(\ell' * (\imath' * \jmath') * \imath') \\
&= \inf_{\ell_0*((\imath_0*\jmath_0)*\imath_0)\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) \\
&= \inf_{t\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(t).
\end{aligned}$$

Then

$$\begin{aligned}\mu_B((\imath' * \jmath') * \imath') &= \inf_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t)\\ &= \mu_{\mathfrak{C}}((\imath_0 * \jmath_0) * \imath_0)\\ &\leq \max\{\mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0), \mu_{\mathfrak{C}}(\ell_0)\}\\ &= \min\{\inf_{t\in\zeta^{-1}(z'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(t), \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\}\\ &= \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.\end{aligned}$$

$$\lambda_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \lambda_{\mathfrak{C}}(t)$$

$$\lambda_{\mathfrak{C}}(\ell_0) = \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) &= \mu_B(\zeta(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)))\\ &= \lambda_B(\ell' * ((\imath' * \jmath') * \imath'))\\ &= \inf_{(\ell_0*((\imath_0*\jmath_0)*\imath_0))\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0))\\ &= \inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t).\end{aligned}$$

Then

$$\begin{aligned}\mu_B((\imath' * \jmath') * \imath') &= \inf_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t)\\ &\geq \max\{\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)), \mu_{\mathfrak{C}}(\ell_0)\}\\ &= \max\{\inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t), \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\}\\ &= \max\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.\end{aligned}$$

Hence $B = (\jmath, \mu_B, \nu_B, \lambda_B)$ is a $PNImpI$ of Y

4. Cartesian product of Pythagorean neutrosophic implicative ideal

Definition 4.1. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be PNS of a set Υ, the Cartesian product of $\mu_{\mathfrak{C}} \times \mu_B$, $\nu_{\mathfrak{C}} \times \nu_B$ and $\lambda_{\mathfrak{C}} \times \lambda_B$ is defined by

$$(\mu_{\mathfrak{C}} \times \mu_B)(\imath, \jmath) = \min\{\mu_{\mathfrak{C}}(\imath), \mu_B(\jmath)\}$$

$$(\nu_{\mathfrak{C}} \times \nu_B)(\imath, \jmath) = \max\{\nu_{\mathfrak{C}}(\imath), \nu_B(\jmath)\}$$

$$(\lambda_{\mathfrak{C}} \times \lambda_B)(\imath, \jmath) = \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_B(\jmath)\}\ \forall \imath, \jmath \in \Upsilon.$$

Definition 4.2. If $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is a PNS of a set Υ, the strongest PN relation on Υ, that is, a PN relation on $\mathfrak{C}$ is $\mu^f_{\mathfrak{C}}$ given by

$$\mu^f_{\mathfrak{C}}(\imath, \jmath) = \min\{\mu^f_{\mathfrak{C}}(\imath), \mu^f_{\mathfrak{C}}(\jmath)\}$$
$$\nu^f_{\mathfrak{C}}(\imath, \jmath) = \max\{\nu^f_{\mathfrak{C}}(\imath), \nu^f_{\mathfrak{C}}(\jmath)\}$$
$$\lambda^f_{\mathfrak{C}}(\imath, \jmath) = \max\{\lambda^f_{\mathfrak{C}}(\imath), \lambda^f_{\mathfrak{C}}(\jmath)\} \ \forall \imath, \jmath \in \Upsilon.$$

Proposition 4.3. For a given PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ, let $(\mu^f_{\mathfrak{C}}, \nu^f_{\mathfrak{C}}, \lambda^f_{\mathfrak{C}})$ is a $PNImpI$ of $\Upsilon \times \Upsilon$, then

$$\mu^f_{\mathfrak{C}}(0) \geq \mu^f_{\mathfrak{C}}(\imath),$$
$$\nu^f_{\mathfrak{C}}(0) \leq \nu^f_{\mathfrak{C}}(\imath),$$
$$\lambda^f_{\mathfrak{C}}(0) \leq \lambda^f_{\mathfrak{C}}(\imath) \ \forall \imath \in \Upsilon.$$

Proof. Since, $\mu^f_{\mathfrak{C}},\ \nu^f_{\mathfrak{C}},\ \lambda^f_{\mathfrak{C}}$ are $PNImpI$ of $\Upsilon \times \Upsilon$, it follows from (PN_1) that

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(\Upsilon, x) &= \min\{\mu^f_{\mathfrak{C}}(\imath), \mu^f_{\mathfrak{C}}(\imath)\} \leq (0,0) \\
&= \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(0)\} \\
\nu^f_{\mathfrak{C}}(\Upsilon, x) &= \max\{\nu^f_{\mathfrak{C}}(\imath), \nu^f_{\mathfrak{C}}(\imath)\} = (0,0) \\
&\leq \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(0)\} \\
\lambda^f_{\mathfrak{C}}(\Upsilon, x) &= \max\{\lambda^f_{\mathfrak{C}}(\imath), \lambda^f_{\mathfrak{C}}(\imath)\} = (0,0) \\
&\leq \max\{\lambda_{\mathfrak{C}}(0), \lambda_{\mathfrak{C}}(0)\},\ \forall \imath \in \Upsilon,
\end{aligned}$$

where $(0,0) \in \Upsilon \times \Upsilon$, then

$$\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$$
$$\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$$
$$\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath).$$

Remark 4.4. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be KU-algebras, we define $*$ on $\mathfrak{C} \times \mathfrak{C}$ for every $(\imath, \jmath),\ (u, v) \in \mathfrak{C} \times B$, $(\imath, \jmath) * (u, v) = (\imath * u, \jmath * v)$, then clearly $(\imath * \jmath, *, (0,0))$ is a KU-algebra.

Theorem 4.5. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be a $PNImpI$'s of KU-algebra X, $\Upsilon \times B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

Proof. For any $(\imath, \jmath) \in \Upsilon \times \Upsilon$, we have,

$$\begin{aligned}
(\mu_{\mathfrak{C}} \times \mu_B)(0,0) &= \min\{\mu_{\mathfrak{C}}(0), \mu_B(0)\} \\
&\geq \min\{\mu_{\mathfrak{C}}(\imath), \mu_B(\imath)\} \\
&= (\mu_{\mathfrak{C}} \times \mu_B)(\Upsilon, x) \\
(\nu_{\mathfrak{C}} \times \nu_B)(0,0) &= \max\{\nu_{\mathfrak{C}}(0), \nu_B(0)\} \\
&\leq \max\{\nu_{\mathfrak{C}}(\imath), \nu_B(\imath)\} \\
&= (\nu_{\mathfrak{C}} \times \nu_B)(\Upsilon, x) \\
(\lambda_{\mathfrak{C}} \times \lambda_B)(0,0) &= \max\{\lambda_{\mathfrak{C}}(0), \lambda_B(0)\} \\
&\leq \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_B(\imath)\} \\
&= (\lambda_{\mathfrak{C}} \times \lambda_B)(\Upsilon, x).
\end{aligned}$$

Now, let $(\imath_1, \imath_2),\ (\jmath_1, \jmath_2),\ (\ell_1, \ell_2) \in \Upsilon \times \Upsilon$, then,

$$\begin{aligned}
&(\mu_{\mathfrak{C}} \times \mu_B)(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\
&= \min\{\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \mu_B((\imath_2 * \jmath_2) * \imath_2)\} \\
&\geq \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}(\ell_1)\}, \min\{\mu_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2), \mu_B(\ell_2)\}\} \\
&= \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2)\}, \min\{\mu_{\mathfrak{C}}(\ell_1), \mu_B(\ell_2)\}\} \\
&= \min\{(\mu_{\mathfrak{C}} \times \mu_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * (\imath_2 * \jmath_2) * \imath_2), (\mu_{\mathfrak{C}} \times \mu_B)(\ell_1, \ell_2)\}.
\end{aligned}$$

Hence, $\mu_{\mathfrak{C}} \times \mu_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

$$\begin{aligned}
&(\nu_{\mathfrak{C}} \times \nu_B)(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\
&= \max\{\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \nu_B((\imath_2 * \jmath_2) * \imath_2)\} \\
&\leq \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \nu_{\mathfrak{C}}(\ell_1)\}, \max\{\nu_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu_B(\ell_2)\}\} \\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \nu_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)\}, \max\{\nu_{\mathfrak{C}}(\ell_1), \nu_B(\ell_2)\}\} \\
&= \max\{(\nu_{\mathfrak{C}} \times \nu_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * (\imath_2 * \jmath_2) * \imath_2), (\nu_{\mathfrak{C}} \times \nu_B)(\ell_1, \ell_2)\}.
\end{aligned}$$

Hence, $\nu_{\mathfrak{C}} \times \nu_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

$$\begin{aligned}
&(\lambda_{\mathfrak{C}} \times \lambda_B)((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2) \\
&= \max\{\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \lambda_B((\imath_2 * \jmath_2) * \imath_2)\} \\
&\leq \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \lambda_{\mathfrak{C}}(\ell_1)\}, \max\{\lambda_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2), \lambda_B(\ell_2)\}\} \\
&= \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2))\}, \max\{\lambda_{\mathfrak{C}}(\ell_1), \lambda_B(\ell_2)\}\} \\
&= \max\{(\lambda_{\mathfrak{C}} \times \lambda_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), (\lambda_{\mathfrak{C}} \times \lambda_B)(\ell_1 \times \ell_2)\}.
\end{aligned}$$

Hence, $\lambda_{\mathfrak{C}} \times \lambda_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

Theorem 4.6. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be a PNS of KU-algebra Υ, such that $\mathfrak{C} \times B$ is $PNImpI$ of $\Upsilon \times \Upsilon$, then

(i) either $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ or $\mu_B(\imath) \leq \mu_B(0)$, $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ or $\nu_B(\imath) \geq \nu_B(0)$, $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ or $\lambda_B(\imath) \geq \lambda_B(0)$, $\forall \imath \in \Upsilon$.

(ii) if $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\mu_{\mathfrak{C}}(\imath) \leq \mu_B(0)$ or $\mu_B(\imath) \leq \mu_B(0)$,
if $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\nu_{\mathfrak{C}}(\imath) \geq \nu_B(0)$ or $\nu_B(\imath) \geq \nu_B(0)$,
if $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_B(0)$ or $\lambda_B(\imath) \geq \lambda_B(0)$.

(iii) if $\mu_B(\imath) \leq \mu_B(0)$ $\forall \imath \in \Upsilon$, then either $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ or $\mu_B(\imath) \leq \mu_{\mathfrak{C}}(0)$,
if $\nu_B \geq \nu_B(0)$ $\forall \imath \in \Upsilon$, then either $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ or $\nu_B(\imath) \geq \nu_{\mathfrak{C}}(0)$,
$\lambda_B(\imath) \geq \lambda_B(0)$ $\forall \imath \in \Upsilon$, then either $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ or $\lambda_B(\imath) \geq \lambda_{\mathfrak{C}}(0)$.

(iv) either $\mathfrak{C}$ or B is $PNImpI$ of Υ.

Proof. The proof is similar to previous Theorem 4.5.

Theorem 4.7. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be a PNS of KU-algebra Υ and let $\mu^f_{\mathfrak{C}}$, $\nu^f_{\mathfrak{C}}$, $\lambda^f_{\mathfrak{C}}$ be the strongest PN relation on Υ, then $\mathfrak{C}$ is a $PNImpI$ of Υ iff $\mu^f_{\mathfrak{C}}$, $\nu^f_{\mathfrak{C}}$, $\lambda^f_{\mathfrak{C}}$ are $PNImpI$ of $\Upsilon \times \Upsilon$.

Proof. Assume that $(\mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is $PNImpI$ on Υ, we note from (PNI) that

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(0,0) &= \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(0)\} \\
&\geq \min\{\mu_{\mathfrak{C}}(\imath), \mu_{\mathfrak{C}}(\jmath)\} \\
&\geq \mu^f_{\mathfrak{C}}(\imath, \jmath). \\
\nu^f_{\mathfrak{C}}(0,0) &= \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(0)\} \\
&\leq \max\{\nu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\jmath)\} \\
&\leq \nu^f_{\mathfrak{C}}(\imath, \jmath). \\
\lambda^f_{\mathfrak{C}}(0,0) &= \max\{\lambda_{\mathfrak{C}}(0), \lambda_{\mathfrak{C}}(0)\} \\
&\leq \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\jmath)\} \\
&\leq \lambda^f_{\mathfrak{C}}(\imath, \jmath), \ \forall (\imath, \jmath) \in \Upsilon \times \Upsilon.
\end{aligned}$$

Now, for any $(\imath_1, \imath_2), (\jmath_1, \jmath_2), (\ell_1, \ell_2) \in \Upsilon \times \Upsilon$, we have from (PN_2)

$$
\begin{aligned}
&\mu^f_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2)\\
&= \min\{\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\}\\
&\geq \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \mu_{\mathfrak{C}}(\ell_1)\}, \min\{\mu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \mu_{\mathfrak{C}}(\ell_2)\}\}\\
&= \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \mu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2))\}, \min\{\mu_{\mathfrak{C}}(\ell_1), \mu_{\mathfrak{C}}(\ell_2)\}\}\\
&= \min\{(\mu_{\mathfrak{C}} \times \mu_{\mathfrak{C}})(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), (\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), (\mu_{\mathfrak{C}} \times \mu_{\mathfrak{C}})(\ell_1, \ell_2)\}\\
&\nu^f_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2)\\
&= \max\{\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\}\\
&\leq \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \nu_{\mathfrak{C}}(\ell_1)\}, \max\{\nu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu_{\mathfrak{C}}(\ell_2)\}\}\\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \nu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \max\{\nu_{\mathfrak{C}}(\ell_1), \nu_{\mathfrak{C}}(\ell_2)\}\}\\
&= \max\{(\nu_{\mathfrak{C}} \times \nu_{\mathfrak{C}})(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), (\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), (\nu_{\mathfrak{C}} \times \nu_{\mathfrak{C}})(\ell_1, \ell_2)\}
\end{aligned}
$$

Similarly, $\lambda^f_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2) \leq \max\{(\lambda_{\mathfrak{C}} \times \lambda_{\mathfrak{C}})(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), (\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), (\lambda_{\mathfrak{C}} \times \lambda_{\mathfrak{C}})(\ell_1, \ell_2)\}$ Hence $(\mu^f_{\mathfrak{C}}, \nu^f_{\mathfrak{C}}, \lambda^f_{\mathfrak{C}})$ is $PNImpI$ of $\Upsilon \times \Upsilon$.

Conversely, $\forall (\imath, \jmath) \in \Upsilon \times \Upsilon$, we have

$\min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(0)\} = \mu^f_{\mathfrak{C}}(\imath, \jmath) = \min\{\mu_{\mathfrak{C}}(\imath), \mu_{\mathfrak{C}}(\jmath)\}$.

It follows that $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\forall \imath \in \Upsilon$

$\max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(0)\} = \nu^f_{\mathfrak{C}}(\imath, \jmath) = \max\{\nu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\jmath)\}$.

It follows that $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\forall \imath \in \Upsilon$

$\max\{\lambda_{\mathfrak{C}}(0), \lambda_{\mathfrak{C}}(0)\} = \lambda^f_{\mathfrak{C}}(\imath, \jmath) = \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\jmath)\}$.

It follows that $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$ $\forall \imath \in \Upsilon$ which proves (PN_1) Now, let $(\imath_1, \imath_2),\ (\jmath_1, \jmath_2),\ (\ell_1, \ell_2) \in \Upsilon \times \Upsilon$, then

$$
\begin{aligned}
&\min\{\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\}\\
&= \mu^f_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2)\\
&\geq \min\{\mu^f_{\mathfrak{C}}((\ell_1, \ell_2) * ((\imath_1, \imath_2) * (\jmath_1, \jmath_2)) * (\imath_1, \imath_2)), \mu^f_{\mathfrak{C}}(\ell_1, \ell_2)\}\\
&= \min\{\mu^f_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * (\imath_2 * \jmath_2) * \imath_2), \mu^f_{\mathfrak{C}}(\ell_1, \ell_2)\}\\
&= \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \mu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2))\}, \min\{\mu_{\mathfrak{C}}(\ell_1), \mu_{\mathfrak{C}}(\ell_2)\}\}\\
&= \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \mu_{\mathfrak{C}}(\ell_1)\}, min\{\mu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \mu_{\mathfrak{C}}(\ell_2)\}\}.
\end{aligned}
$$

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \geq \min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}(\ell_1)\}$

$$\begin{aligned}
&\max\{\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\} \\
&= \nu^f_{\mathfrak{C}}(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\
&\leq \max\{\nu^f_{\mathfrak{C}}((\ell_1, \ell_2) * ((\imath_1 * \imath_2) * (\jmath_1 * \jmath_2)) * (\imath_1 * \imath_2)), \nu^f_{\mathfrak{C}}(\ell_1, \ell_2)\} \\
&= \max\{\nu^f_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), (\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu^f_{\mathfrak{C}}(\ell_1, \ell_2)\} \\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_2 * (\imath_2 * \jmath_2) * \imath_2)\}, \max\{\nu_{\mathfrak{C}}(\ell_1), \nu_{\mathfrak{C}}(\ell_2)\}\} \\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_1)\}, max\{\nu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu_{\mathfrak{C}}(\ell_2)\}\}.
\end{aligned}$$

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \leq \max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_1)\}$. Similarly, $\max\{\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \lambda_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\} \leq \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_{\mathfrak{C}}(\ell_1)\}, \max\{\lambda_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \lambda_{\mathfrak{C}}(\ell_2)\}$.

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \leq \max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_{\mathfrak{C}}(\ell_1)\}$.

5. Conclusions

We have investigated Pythagorean neutrosophic $ImpI$s in KU-algebras and discussed several related results. In particular, we defined the image and pre-image of Pythagorean neutrosophic $ImpI$s under homomorphisms of KU-algebras and studied the conditions under which these images remain Pythagorean neutrosophic $ImpI$s. Moreover, we established the product of Pythagorean neutrosophic $ImpI$s as a product Pythagorean neutrosophic $ImpI$. As a direction for future work, we aim to explore the foldedness of other classes of Pythagorean neutrosophic ideals with special properties, such as bipolar intuitionistic (interval-valued) fuzzy n-fold $ImpI$s in certain algebraic structures.

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