



A study on multiset and multi-real number system and its use to develop metric in the multiset context

Debjyoti Chatterjee^{1,*} and Shashi Bajaj Mukherjee²

¹Dept. of Mathematics, Jalpaiguri Government Engineering College, Jalpaiguri, 735102, West Bengal, India;
debjyoti.chatterjee@jgec.ac.in

²Dept. of Mathematics, Dr. B. C. Roy Engineering College, Durgapur, 713206, West Bengal, India;
shashiagarwala@gmail.com

*Correspondence: debjyoti.chatterjee@jgec.ac.in

Abstract. In this paper, we present an alternative development of multiset theory through the notions of the general multiset, the \mathbb{Q}^+ -multiset, and \mathbb{N} -multiset. We further propose the notion of a multi-real number system. Several properties of the general multiset, the \mathbb{Q}^+ -multiset, \mathbb{N} -multiset, and the multi-real number system are systematically explored. Also we use the multi-real number system to introduce the concept of *multi-metric space*, extending traditional metric space concepts such as distance, neighbourhood, open ball and open set to the context of multisets, and investigate its fundamental topological properties. This study offers a new perspective on multiset theory and provides a foundation for further research in algebraic and topological structures enriched by multiplicity.

Keywords: Multiset; General multiset; Multi-field; Multi-real number; \mathbb{Q}^+ -multiset; Multi-metric space; Multi-open set.

1. Introduction

A *multiset* (mset in short) is a collection of objects in which objects may occur more than once. The number of times an element occurs in a multiset is called the multiplicity of the element. The studies on multisets revolved around combinatorics in earlier times [1]. Modern research in this field on the structural development in multiset domain is relatively new. To obtain a structure of multisets, many researchers rediscovered the theory of multisets several times, although they use different names, e.g. bags, heaps, lists, bunch, and weighted set. Wayne D. Blizard proposed the first formal theory of multisets in [2–4] after an excellent literature survey in [2]. A classical introduction to the concept of multiset is [5] by D.E.

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Knuth. Many researchers like A. Syropoulos [6], D. Singh et al. [7], Yagar [8], Miyamoto [9], Hickman [10], K. P. Girish et al. [11–13] have studied the properties of multisets. Some authors also have generalized the notion of multisets to form fuzzy multisets [14], intuitionistic fuzzy multisets [15, 16], soft multisets [17, 18] etc. Various research works on multiset ordering [7, 17, 19], relations and functions in multiset context [9, 20], multiset topology [12, 13], multi group theory [21] etc. have been done recently by some researchers. However, in most of the cases researchers have considered multisets as just functions from sets into some subsets of the set of all real numbers. But some of them have considered the true multiset, existing by axiomatic theories of objects [22, 23]. However, in [12, 13] Girish presented a topological structure in multiset, which is actually the generalization of general topology on classical sets on multisets. But in [24], Ghareeb concluded that multiset topology is exactly a special case of general topology, also in [25], L. Wang and F. G. Shi establishes that an mset topology can be viewed as an L-topology.

In [29] A. B. Petrivsky, introduced the concept of theory of multiset metric spaces and also used it for clustering and sorting objects that are described with many quantitative and/or qualitative attributes and may exist in several copies with inconsistent and contradictory attributes. Also, in [31], he considers new classes of spaces of finite, bounded, measurable multisets with different metrics, pseudometrics, quasimetrics, symmetric, and some properties of these metrics. Also, discuss the possibilities to apply new types of metrics for estimating proximity of objects with many numerical and/or verbal attributes. He uses the introduced indexes of similarity and dissimilarity of objects represented as multisets in new methods of group multiple criteria decision making. In [30], A. M. Ibrahim et al. develop a perspective of multiset metric spaces parameterized in terms of multiplicities of objects occurring in multisets of a cardinality-bounded multiset universe. In [32], Ray-Ming Chen introduces a variety of metrics for comparing full graphs and subgraphs, based on minimal matching between multisets of positive real numbers representing multiple edges relative to their vertices. It also presents an implementation approach using adjacency matrices which enable practical computation. The proposed metrics are adaptable for various applications, including the comparison of graphs, trees, and fuzzy networks. In [33], K. Shrava studies the metrizable of multiset topological spaces by introducing a metric between two multi-points in a finite multiset and exploring key properties of the resulting metric space. Using this metric, the concept of metrizable is analyzed and Urysohn's lemma is examined in the context of multisets. In all the cases, the real number system is used to define the metric in the multiset context.

But to develop metric structure on multiset we start from the beginning. In [26–28] we develop the multi number system from the axiomatic point of view, and in this paper we propose an alternative treatment to deal with multiset.

The motivation of this study lies in extending the classical idea of metric space to the multiset setting, where a well-established metric structure is still lacking. Existing approaches to defining the metric space on multiset are based exclusively on the conventional real number system, restricting their scope and flexibility. To address this gap, we introduce the notion of a multi-real number system together with a restructured definition of multiset, and employ these foundations to develop a new notion of metric space in the multiset context. The main objective is to establish a richer and more natural metric framework for multiset, enabling deeper theoretical insights and broader applications.

The paper is organized as follows. Section 1 presents the introduction and methodology; Section 2 provides the basic notation of multiset theory; Section 3 develops the theory of the general multiset, where we introduce concepts such as multi-group, multi-distributive property, multi-ring, multi-integral domain, and multi-field. In Section 4, we propose the multi-real number system, which is shown to be both a complete distributive lattice and a multi-field. Section 5 introduces the \mathbb{Q}^+ -multiset theory, including the notions of \mathbb{Q}^+ -submultiset, \mathbb{Q}^+ -multiset union, \mathbb{Q}^+ -multiset intersection, and related operations, along with some important consequences of the theory. In Section 6, the multi-real number system is used to define a multi-metric space in the multiset setting, where we study the notions of open balls, open sets, and related topological properties. Section 7 presents a comparative analysis with neutrosophic-based methods. Section 8 provides the conclusion, and Section 9 discusses the limitations and directions for future research.

Throughout this paper, we denote \mathbb{N} as the set of all natural numbers, \mathbb{Q} as the set of all rational numbers, \mathbb{Q}^+ as the set of all positive rational numbers and \mathbb{R} as the set of all real numbers.

1.1. Methodology

We first formalize multisets via a general multiset model and a \mathbb{Q}^+ -multiset model, and introduce a multi-real number system to represent multiplicities. Also, we introduce \mathbb{N} -multiset model and the notion of \mathbb{N} -subm-elements. Finally, given two \mathbb{N} -subm-elements, we define their distance as a non-negative multi-real number by introducing multi-metric with non-negativity, identity, symmetry, and triangle inequality restrictions. Finally, we investigate

several fundamental topological properties of the multi-metric space. The following flowchart (FIGURE 1.) presents the core contributions of the paper.

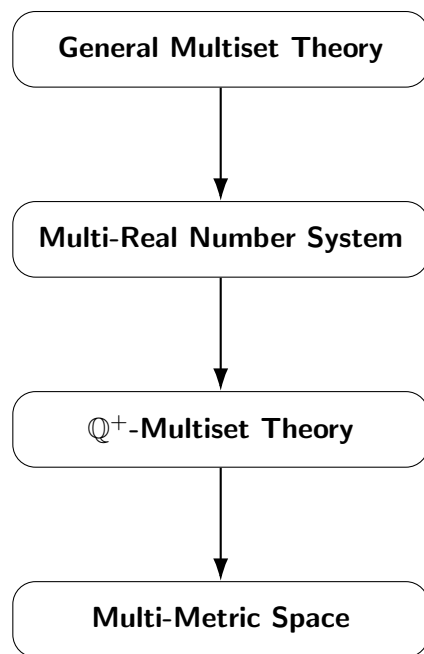


FIGURE 1. Core contributions of the paper

2. Multiset

The notion of **Multiset** (**mset** in short) was introduced by Yagar [8]. The basic definitions and notions of relations and functions in multiset context were introduced by Girish and John [12,13]. In [12] an mset M drawn from the set X is presented by a function $Count_M$ or C_M defined as $C_M : X \rightarrow \mathbb{N}$. Let M be an mset drawn from the set $X = \{x_1, x_2, \dots, x_n\}$ with x_i appearing k_i times in M , then it is denoted by $x_i \in^{k_i} M$. Clearly, a crisp set is a special case of an mset. The mset M drawn from the set X is then denoted by $\{k_1/x_1, k_2/x_2, \dots, k_n/x_n\}$. Also, $C_M(x)$ is the number of occurrences of the element x in the mset M . However, those elements that are not included in the mset M have zero count. Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{3/a, 2/b, 1/e\}$ is an mset drawn from X .

Let P and Q be two multisets drawn from a set X , then the following are defined:

- (i) $P = Q$ if $C_P(x) = C_Q(x) \forall x \in X$.
- (ii) $P \subseteq Q$ if $C_P(x) \leq C_Q(x) \forall x \in X$, then we call P a **subset** of Q .
- (iii) $M = P \cup Q$ if $C_M(x) = \max\{C_P(x), C_Q(x)\} \forall x \in X$.
- (iv) $M = P \cap Q$ if $C_M(x) = \min\{C_P(x), C_Q(x)\} \forall x \in X$.
- (v) $M = P \oplus Q$ if $C_M(x) = C_P(x) + C_Q(x) \forall x \in X$.
- (vi) $M = Q \ominus P$ if $C_M(x) = \max\{C_Q(x) - C_P(x), 0\} \forall x \in X$.

Here, \cup , \cap , \oplus and \ominus represent mset union, mset intersection, mset addition, and mset subtraction, respectively.

Let M be an mset drawn from a set X , then the **support set** of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and is also called the root set. The **cardinality** of an mset M drawn from a set X is denoted by $card(M)$ or $|M|$ and is given by $|M| = \sum_{x \in X} C_M(x)$. The **mset space** $[X]^m$ is the set of all msets whose elements are in X so that no element in the mset occurs more than m times. The **mset space** $[X]^\infty$ is the set of all msets drawn from X such that there is no limit on the number of occurrences of an object in an mset.

Let $\{M_i : i \in \Omega\}$ be a collection of msets drawn from $[X]^m$, then the following operations are defined:

(i) $P = \bigcup_{i \in \Omega} M_i$ if $C_P(x) = \max_{i \in \Omega} C_{M_i}(x), x \in X$.

(ii) $P = \bigcap_{i \in \Omega} M_i$ if $C_P(x) = \min_{i \in \Omega} C_{M_i}(x), x \in X$.

Let X be a support set and $[X]^m$ be the mset space defined over X , then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that [12, 13] $C_{M^c}(x) = m - C_M(x) \forall x \in X$.

3. General Multiset

3.1. Definition

[27] Let X be a non-empty set. A **general multiset** (or **general mset**) M drawn from the set X is characterized by a relation ρ_M from the set X to the set \mathbb{R} (\mathbb{R} being the set of all real numbers). In other words, a general mset M drawn from the set X is a subset of $X \times \mathbb{R}$. If for some $x \in X$ and $r \in \mathbb{R} - \{0\}$, $(x, r) \in \rho_M$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$.

Let M and P be two general msets drawn from the crisp sets A and B , respectively. If for $a \in A \cap B$ and $r \in \mathbb{R} - \{0\}$, $A_a^r \in M$ and $B_a^r \in P$, then we shall consider $A_a^r = B_a^r$ and we shall also represent it by (a, r) if it is not necessary to mention from which set a is chosen.

Let X be a non-empty set. Let us denote the general mset drawn from X and characterized by the universal relation from the set X to the set \mathbb{R} as $\pi(X)$ and accordingly $\rho_{\pi(X)} = X \times \mathbb{R}$. Let us call $\pi(X)$ the **most general multiset** drawn from the set X .

[27] Let X be a non-empty set. A **\mathbb{R} -multiset** (or **\mathbb{R} -mset** in short) M drawn from X is characterized by a function $Count_M$ or $C_M : X \rightarrow \mathbb{R}$.

If for some $x \in X$ and $r \in \mathbb{R} - \{0\}$, $C_M(x) = r$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$. Also, we shall denote a \mathbb{R} -mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, \dots, X_{x_n}^{k_n}, \dots\}$ or as $\{(x_1, k_1), (x_2, k_2), \dots, (x_n, k_n), \dots\}$ where $C_M(x_i) = k_i$, $x_i \in X$ and $k_i \in \mathbb{R} - \{0\}$.

[27] Let X be a non-empty set. A **\mathbb{N} -multiset** (or **\mathbb{N} -mset** in short) M drawn from X is characterized by a function $Count_M$ or $C_M : X \rightarrow \mathbb{N} \cup \{0\}$.

If $C_M(x) = r$ for some $x \in X$ and $r \in \mathbb{N} - \{0\}$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$.

Clearly, general multiset is a generalization of the \mathbb{R} -multiset. Also, \mathbb{R} -multiset is a generalization of \mathbb{N} -multiset.

[27] We note that for all $i, j \in \mathbb{N}$, R_i^j , Z_i^j and N_i^j both are immediately identical. i.e., $R_i^j = Z_i^j = N_i^j, \forall i, j \in \mathbb{N}$.

3.2. Example

Consider the set $X = \{a, b, c\}$. Consider the relation ρ_M from the set X to the set \mathbb{R} where $\rho_M = \{(a, \frac{1}{4}), (b, 3), (b, \sqrt{2})\}$. Then ρ_M represents a general mset M drawn from X which is given by $M = \{X_a^{\frac{1}{4}}, X_b^3, X_b^{\sqrt{2}}\}$ or $M = \{(a, \frac{1}{4}), (b, 3), (b, \sqrt{2})\}$.

Next, consider the function $C_P : X \rightarrow \mathbb{R}$ defined by $C_P(a) = \frac{1}{4}$, $C_P(b) = 3$ and $C_P(c) = 0$.

Then C_P represents a \mathbb{R} -mset P drawn from X which is given by $P = \{X_a^{\frac{1}{4}}, X_b^3\}$ or by $\{(a, \frac{1}{4}), (b, 3)\}$.

Finally, consider the function $C_Q : X \rightarrow \mathbb{N} \cup \{0\}$ defined by $C_Q(a) = 1$, $C_Q(b) = 3$ and $C_Q(c) = 0$. Then C_Q represents a \mathbb{N} -mset Q drawn from X which is given by $Q = \{X_a^1, X_b^3\}$ or by $\{(a, 1), (b, 3)\}$.

3.3. Definition

1. The **elementary union** of two general msets A and B is denoted by $A \cup B$ and is defined by $A \cup B = \{(\alpha, k) : (\alpha, k) \in A \text{ or } (\alpha, k) \in B\}$.

2. The **elementary intersection** of two general msets A and B is denoted by $A \cap B$ and is defined by $A \cap B = \{(\alpha, k) : (\alpha, k) \in A \text{ and } (\alpha, k) \in B\}$.

3. The **elementary complement** of the general msets A in B is denoted by $B - A$ and is defined by $B - A = \{(\alpha, k) : (\alpha, k) \in B \text{ and } (\alpha, k) \notin A\}$.

3.4. Definition

Let $(X, *)$ be a group. Let M be a general mset drawn from the set X . Consider the function $\otimes : M \times M \rightarrow \pi(X)$ defined as follows:

For $(a, r), (b, s) \in M$, $(a, r) \otimes (b, s) = (a + b, r * s)$.

Let us call \otimes as **m-composition** defined on M induced by the group $(X, *)$.

If M is closed under $*$, then immediately \otimes obeys the commutative and associative property on M . So, then (M, \otimes) is a commutative semigroup. We define M as a **general mset drawn from the group** $(X, *)$.

3.5. Definition

Let $(X, *)$ be a group. Let M be a general mset drawn from the set X . Let $\otimes : M \times M \rightarrow \pi(X)$ be the m-composition defined on M induced by the group $(X, *)$. Then the structure (M, \otimes) is said to be a **multi-group** if the following conditions are satisfied:

(i) There exists $(\theta, 1) \in M$, where θ is the zero element of the group $(X, *)$, (ii) For $a \in X$ and $r \in [R - \{0\}]$, $(a, r) \in M \implies (-a, \frac{1}{r}) \in M$.

3.6. Example

Consider the group $(X, +)$ where Z_4 , the set of all residue classes modulo 4 and $+$ is the addition modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in \mathbb{Z}\}$ from the set X to the set G . Let \oplus be the m-composition defined in M induced by the group $(X, +)$. Then (M, \oplus) forms a multi-group induced by the group $(X, +)$.

3.7. Definition

Let $(X, +, \cdot)$ be a ring. Let M be a general mset drawn from X . Consider two functions $\oplus : M \times M \rightarrow \pi(X)$ and $\odot : M \times M \rightarrow \pi(X)$ defined as follows:

For $(a, r), (b, s) \in M$, $(a, r) \oplus (b, s) = (a + b, rs)$ and $(a, r) \odot (b, s) = (ar, bs)$.

Let us call \oplus and \odot respectively as **m-addition** and **m-multiplication** defined on M induced by the ring $(X, +, \cdot)$.

If M is closed under \oplus and \odot , then immediately \oplus obey commutative and associative property on M . So, (M, \oplus) is a commutative semigroups. Also, then \odot obey commutative and associative property on M and accordingly (M, \odot) is a semigroup. We define M as a **general mset drawn from the ring** $(X, +, \cdot)$.

3.8. Definition

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1.

Then for all $(x, p), (y, q), (z, r) \in M$, $(1, p) \odot [(x, p) \odot ((y, q) \oplus (z, r))] = [(x, p) \odot (y, q)] \oplus [(x, p) \odot (z, r)]$.

Let us define the above property to be the **multi-distributive** property of \odot over \oplus on M .

3.9. Definition

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1. Let \oplus and \odot are m-addition and m-multiplication, respectively defined on M induced by the ring $(X, +, \cdot)$. If the structure (M, \oplus, \odot) satisfies the following:

(1) (M, \oplus) is an abelian group.

(2) (M, \odot) is a semigroup and

(3) \odot is multi-distributive over \oplus ,

then we define (M, \oplus, \odot) to be a **multi-ring** induced by the ring $(X, +, \cdot)$ with unity 1.

3.10. Proposition

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1. Let \oplus and \odot are m-addition and m-multiplication, respectively defined on M induced by the ring $(X, +, \cdot)$ with unity 1. Then (M, \oplus, \odot) will be a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1 if and only if the following conditions are satisfied:

(1) $\exists (\theta, 1) \in M$, θ being the zero element in the ring $(X, +, \cdot)$.

(2) For $a \in X$ and $r \in [\mathbb{R} - \{0\}]$, $(a, r) \in M \Rightarrow (-a, \frac{1}{r}) \in M$.

3.11. Remark

If (M, \oplus, \odot) is a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1, then (M, \oplus, \odot) is immediately a **commutative multi-ring**.

3.12. Example

Let us consider the ring $(X, +, \cdot)$ with unity $\bar{1}$ where $X = Z_4$, the set of all residue classes modulo 4, also $+$ and \cdot are respectively addition and multiplication modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in \mathbb{Z}\}$ from the set X to the set G . Then for all $\bar{a} \in X$ and for all $r \in G$, $(\bar{a}, r) \in M$. Let \oplus and \odot be m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$. Then (M, \oplus, \odot) forms a commutative multi-ring induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$.

3.13. Definition

Let (M, \oplus, \odot) be the multi-ring induced by the ring $(X, +, \cdot)$ with unity 1. Let θ be the zero element in $(X, +, \cdot)$. Then $(\theta, 1)$ must be the zero element in (M, \oplus, \odot) . Let us also define any element in M of the form (θ, r) for some $r \in \mathbb{R} - \{0\}$ to be the **multi-zero** elements of M (otherwise non-multi-zero elements) such that the m-multiplication of any element of the multi-ring with a multi-zero element of the same is again a multi-zero of the multi-ring. Clearly, the zero element in a multi-ring is a multi-zero element. Multi-zero elements which are not zero-element are called **special multi-zero** elements.

3.14. *Definition*

Let (M, \oplus, \odot) be the multi-ring induced by the ring $(X, +, \cdot)$ with unity 1. A non-zero element (a, p) in (M, \oplus, \odot) is said to be a divisor of zero if there exists a non-zero element (b, q) in (M, \oplus, \odot) such that $(a, p) \odot (b, q) = (\theta, 1)$ or a non-zero element (c, r) in (M, \oplus, \odot) such that $(c, r) \odot (a, p) = (\theta, 1)$, θ being the zero element of the ring $(X, +, \cdot)$. In the first case, (a, p) is said to be a left divisor of zero and in the second case, (a, p) is said to be a right divisor of zero. If, however, (M, \oplus, \odot) is a multi-ring, \odot immediately obey commutative property, and so every left divisor of zero is also a right divisor of zero. Thus, there is no distinction between left and right divisors of zero in a multi-ring. Also, every non-zero multi-zero element of a multi-ring are divisors of zero.

3.15. *Definition*

A multi-ring is said to have no non-multi-zero divisors of zero if all of its divisors of zero are special multi-zero elements of the ring.

3.16. *Remark*

Let, (M, \oplus, \odot) be a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1 and with divisors of zero. Let θ be the zero element of the ring $(X, +, \cdot)$. As $(X, +, \cdot)$ is a ring with divisors of zero, so \exists two non-zero elements a and b in the ring $(X, +, \cdot)$ such that $a \cdot b = \theta$.

Now, for some $r, s \in R - \{0\}$, let $(a, r), (b, s) \in M$.

Then $(a, r) \odot (b, s) = (ab, rs) = (\theta, rs) \in M$ (since M is closed under \odot). Again, (a, r) and (b, s) both are non-multi-zero elements of (M, \oplus, \odot) . Also, (a, r) and (b, s) are divisors of zero in the multi-ring (M, \oplus, \odot) . So, (a, r) and (b, s) are non-multi-zero divisors of zero in the multi-ring (M, \oplus, \odot) .

3.17. *Example*

Consider the multi-ring (M, \oplus, \odot) induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$ as mentioned in Example 11 where $X = Z_4$. Then, for $(\bar{2}, 2), (\bar{2}, \frac{1}{2}) \in M$, $X(\bar{2}, 2) \odot (\bar{2}, \frac{1}{2}) = (\bar{0}, 1)$ which is the zero element of the multi-ring (M, \oplus, \odot) induced by the ring $(X, +, \cdot)$. Also, $(\bar{2}, 2)$ and $(\bar{2}, \frac{1}{2})$ are the non-multi-zero elements of the multi-ring (M, \oplus, \odot) induced by the ring $(X, +, \cdot)$. So, the multi-ring (M, \oplus, \odot) induced by the ring $(X, +, \cdot)$ contains non-multi-zero divisors of zero.

3.18. *Definition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1 (or an integral domain $(X, +, \cdot)$). Let \oplus and \odot are m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity 1 (or the integral domain $(X, +, \cdot)$). If the structure

(M, \oplus, \odot) satisfies the following:

- (1) (M, \oplus) is a commutative group.
- (2) (M, \odot) is a commutative monoid
- (3) \odot is multi-distributive over \oplus and
- (4) M has no non-multi-zero divisors of zero,

then we define (M, \oplus, \odot) to be a **multi-integral domain** induced by the ring (or the integral domain) $(X, +, \cdot)$ with unity 1.

It is worth noting that if M is a general mset drawn from an integral domain $(X, +, \cdot)$ which is closed under \oplus and \odot , then immediately (M, \oplus, \odot) has no non-multi-zero divisors of zero.

3.19. Example

The multi-ring (M, \oplus, \odot) induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$ as mentioned in Example 11 and Example 16 where $X = Z_4$ is not a multi-integral domain.

3.20. Definition

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity (or a field $(X, +, \cdot)$). Let \oplus and \odot are m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity (or the field $(X, +, \cdot)$). If the structure (M, \oplus, \odot) satisfies the following:

- (1) (M, \oplus) is a commutative group.
- (2) (M, \odot) is a commutative monoid
- (3) Every non-multi-zero element of M has its inverse in M with respect to \odot .
- (4) \odot is multi-distributive over \oplus

then we define (M, \oplus, \odot) to be a **multi-field** induced by the ring (or the field) $(X, +, \cdot)$ with unity.

3.21. Example

Consider the field $(X, +, \cdot)$ where $X = Z_3$, the set of all residue classes modulo 3, also $+$ and \cdot respectively are addition and multiplication modulo 3. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in Z\}$ between X and G . Then for all $a \in X$ and for all $r \in G$, $(a, r) \in M$. Let \oplus and \odot be m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$. Then (M, \oplus, \odot) forms a multi-field induced by the field $(X, +, \cdot)$.

4. The multi-real number system

4.1. Definition

Let us consider the general mset $m(\mathbb{R})$ drawn from the field $(\mathbb{R}, +, \cdot)$, \mathbb{R} being the set of all real numbers, characterized by the universal relation $\rho_{m(\mathbb{R})} = \mathbb{R} \times \mathbb{Q}^+$ from the set \mathbb{R} to the set \mathbb{Q}^+ i.e. $(p, q) \in m(\mathbb{R})$ if and only if $p \in \mathbb{R}$ and $q \in \mathbb{Q}^+$.

Let us define two m-compositions \oplus and \odot on $m(\mathbb{R})$ as follows:

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \oplus (r, s) = (p + r, qs)$ and $(p, q) \odot (r, s) = (pr, qs)$.

Also, define $<$ on $m(\mathbb{R})$ as follows: For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) < (r, s)$ if and only if there exists $(a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ (\mathbb{R}^+ is the set of all positive real numbers) and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$.

For $(p, q), (r, s) \in m(\mathbb{R})$, we define $(p, q) = (r, s)$ if and only if $p = r$ and $q = s$.

Also, for $(p, q), (r, s) \in m(\mathbb{R})$, we define $(p, q) \leq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) = (r, s)$.

Then every element of $m(\mathbb{R})$ is defined as a **multi-real number**.

4.2. Remark

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \leq (r, s)$ if and only if there exist $(a, b) \in m(\mathbb{R})$ with $a \geq 0$ and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$.

4.3. Definition

(i) Define $m^+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{N}\}$. Every member of $m^+(\mathbb{R})$ is called a **positive multi-real number**.

(ii) Define $m^-(\mathbb{R}) = \{(a, \frac{1}{b}) : a < 0, b \in \mathbb{N}\}$. Every member of $m^-(\mathbb{R})$ is called a **negative multi-real number**.

(iii) Define $m_0(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a = 0, b \in \mathbb{Q}^+\}$. Every member of $m_0(\mathbb{R})$ is called a **multi-zero**.

(iv) Define $m_0^*(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a = 0, b \in \mathbb{N}\}$.

(v) Define $m_+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{Q}^+\}$. Immediately, $m^+(\mathbb{R}) \subsetneq m_+(\mathbb{R})$.

(vi) Define $m^*(\mathbb{R}) = m^+(\mathbb{R}) \cup m_0^*(\mathbb{R})$. Every member of $m^*(\mathbb{R})$ is called a **non-negative multi-real number**. i.e., $m^*(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \geq 0, b \in \mathbb{N}\}$.

(vii) Define $m^\#(\mathbb{R}) = (m(\mathbb{R}) - m_0(\mathbb{R}))$. Every member of $m^\#(\mathbb{R})$ is called a **non-multi-zero multi-real numbers**. i.e., $m^\#(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \neq 0, b \in \mathbb{Q}^+\}$.

(viii) Define $m_s(\mathbb{R}) = m(\mathbb{R}) - (m^+(\mathbb{R}) \cup m^-(\mathbb{R}) \cup \{(0, 1)\})$. Every member of $m_s(\mathbb{R})$ is called a **special multi-real number**.

(ix) Define $\overline{m}(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \in \mathbb{R}, b \in \mathbb{N}\}$.

Immediately, $m^+(\mathbb{R}) \subset m^*(\mathbb{R}) \subset \overline{m}(\mathbb{R})$.

4.4. Remark

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) < (r, s)$ if and only if there exists $(a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$, i.e., if and only if $(r, s) = (p + a, qb)$, i.e., if and only if $r = p + a$ and $s = qb$, i.e., if and only if $p < r$ and $\frac{s}{q} \in \mathbb{N}$.

Therefore, for $(p, q), (r, s) \in m(\mathbb{R})$, define $(p, q) \leq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) = (r, s)$, i.e. if and only if $(p < r$ and $\frac{s}{q} \in \mathbb{N})$ or $(p = r$ and $q = s)$.

4.5. Proposition

The following properties can be established easily:

- (i) $(m(\mathbb{R}), \oplus)$ is a commutative group with $(0, 1)$ as the identity element.
- (ii) $(m(\mathbb{R}), \odot)$ is a commutative monoid with $(1, 1)$ as the identity element.
- (iii) For any element $(p, q) \in m^\#(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \neq 0, b \in \mathbb{Q}^+\}$, its \odot -inverse exists in $m^\#(\mathbb{R})$ and is given by (p^{-1}, q^{-1}) .
- (iv) In fact, $(m^\#(\mathbb{R}), \circ)$ is a commutative group.
- (v) Remark on the distributive property: $(p, q) \oplus ((r, s) \oplus (u, v)) = (p, q) \oplus (r + u, sv) = (p(r + u), qsv)$,
but $((p, q) \odot (r, s)) \oplus ((p, q) \odot (u, v)) = (pr, qs) \oplus (pu, qv) = (pr + pu, q^2sv) = (p(r + u), q^2sv)$,
so, $(p, q) \oplus ((r, s) \oplus (u, v)) \neq ((p, q) \odot (r, s)) \oplus ((p, q) \odot (u, v))$, in general.
- (vi) Multi-distributive property: For all $(p, q), (r, s), (u, v) \in m(\mathbb{R})$, $(1, q) \odot ((p, q) \odot ((r, s) \oplus (u, v))) = ((p, q) \odot (r, s)) \oplus ((p, q) \odot (u, v))$. Let us define the above property as the **multi-distributive property** of \odot over \oplus on $m(\mathbb{R})$.
- (vii) $(m(\mathbb{R}), \oplus, \odot)$ is a **multi-field**.
- (viii) For $(a, b) \in m^\#(\mathbb{R})$ and for $(p, q), (r, s) \in m(\mathbb{R})$, $(a, b) \odot (p, q) = (a, b) \odot (r, s) \Rightarrow (p, q) = (r, s)$,
also, $(p, q) \odot (a, b) = (r, s) \odot (a, b) \Rightarrow ((p, q)) = (r, s)$.

4.6. Proposition

The relation \leq is a partial order relation defined on $m(\mathbb{R})$ but not a chain.

Proof: \leq is immediately a reflexive relation defined on $m(\mathbb{R})$.

For $(p, q), (r, s) \in m(\mathbb{R})$, let $(r, s) \leq (p, q)$ and $(p, q) \leq (r, s)$. Then there exists $(a, b), (c, d) \in m(\mathbb{R})$ with $a, c \geq 0$ and $b, d \in \mathbb{N}$ such that

$(p, q) = (r, s) \oplus (a, b)$ and $(r, s) = (p, q) \oplus (c, d)$. So, $(p, q) = ((p, q) \oplus (c, d)) \oplus (a, b)$ i.e., $(p, q) = (p + c, qd) \oplus (a, b)$ i.e. $(p, q) = (p + c + a, qdb)$. So, $p = p + c + a$ and $q = qdb \Rightarrow c + a = 0$

and $db = 1 \Rightarrow c = b = 0$ and $d = b = 1$. Therefore, $(p, q) = (r, s) \oplus (0, 1)$. So, $(p, q) = (r, s)$. Therefore, \leq is an antisymmetric relation defined on $m(\mathbb{R})$.

Finally, for $(p, q), (r, s), (u, v) \in m(\mathbb{R})$, let $(u, v) \leq (r, s)$ as well as $(r, s) \leq (p, q)$. Then there exists $(a, b), (c, d) \in m(\mathbb{R})$ with $a, c \geq 0$ and $b, d \in \mathbb{N}$ such that $(r, s) = (u, v) \oplus (a, b)$ and $(p, q) = (r, s) \oplus (c, d)$. So, $(p, q) = ((u, v) \oplus (a, b)) \oplus (c, d)$. i.e., $(p, q) = (u, v) \oplus ((a, b) \oplus (c, d))$ i.e., $(p, q) = (u, v) \oplus (a+c, bd)$ where $c+a \geq 0$ and $bd \in \mathbb{N}$. Therefore, $(u, v) \leq (p, q)$. Therefore, \leq is a transitive relation defined on $m(\mathbb{R})$.

Therefore, \leq is a partial order relation defined on $m(\mathbb{R})$.

Now, $(2, 3) \not\leq (3, 2)$ as well as $(3, 2) \not\leq (2, 3)$. So, \leq defined on $m(\mathbb{R})$ is not a chain.

4.7. Definition

1. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded below in the poset $(m(\mathbb{R}), \leq)$ if there exists $(a, b) \in m(\mathbb{R})$ such that $(a, b) \leq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$ and otherwise unbounded below. Also, $(a, b) \leq (x, y) \Rightarrow ((a, b) < (x, y))$ or $((a, b) = (x, y)) \Rightarrow (a < x \text{ and } \frac{y}{b} \in \mathbb{N})$ or $(a = x \text{ and } b = y)$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded below in the poset $(m(\mathbb{R}), \leq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ such that for all $(x, y) \in m_1(\mathbb{R})$, $(a < x \text{ and } \frac{y}{b} \in \mathbb{N})$ or $(a = x \text{ and } b = y)$ and otherwise unbounded below.

2. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded above in the poset $(m(\mathbb{R}), \leq)$ if there exists $(a, b) \in m(\mathbb{R})$ such that $(x, y) \leq (a, b)$ for all $(x, y) \in m_1(\mathbb{R})$ and otherwise unbounded above. Also, $(x, y) \leq (a, b) \Rightarrow ((x, y) < (a, b))$ or $((x, y) = (a, b)) \Rightarrow (x < a \text{ and } \frac{b}{y} \in \mathbb{N})$ or $(x = a \text{ and } y = b)$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded above in the poset $(m(\mathbb{R}), \leq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x, y) \in m_1(\mathbb{R})$, $(x < a \text{ and } \frac{b}{y} \in \mathbb{R})$ or $(x = a \text{ and } y = b)$ and otherwise unbounded above.

4.8. Remark

Every finite subset of $m(\mathbb{R})$ is bounded above and below both, also the greatest lower bound (*glb* in short) and the least upper bound (*lub* in short) of the set exist, but greatest and least element of the set may not exist. Also, there exist subsets of $m(\mathbb{R})$ for which the greatest lower bound and / or the least upper bound may not exist. e.g., consider the subsets $m_1(\mathbb{R}) = \{(2, 3), (4, 9)\}$ and $m_2(\mathbb{R}) = \{(k, 2^k) : k \in \mathbb{N}\}$ and $m_3(\mathbb{R}) = \{(x, [x]) : x \in (1, 2)\}$ of $m(\mathbb{R})$. Then $lub m_1(\mathbb{R}) = (4, 9)$, $glbm_1(\mathbb{R}) = (2, 3)$, $glbm_2(\mathbb{R}) = (1, 2)$, $m_2(\mathbb{R})$ is unbounded above, $glbm_3(\mathbb{R}) = (1, 1)$ and $lubm_3(\mathbb{R}) = (2, 1)$. But every bounded above subset of $m(\mathbb{R})$ has a *lub* and every bounded below subset of $m(\mathbb{R})$ has a *glb*.

4.9. Remark

Let $m_1(\mathbb{R})$ be a subset of $m(\mathbb{R})$. Consider two mappings.

$\mu : m_1(\mathbb{R}) \rightarrow \mathbb{R}$ and $\lambda : m_1(\mathbb{R}) \rightarrow \mathbb{Q}^+$ defined by $\mu((x, y)) = x$ and $\lambda((x, y)) = y, (x, y) \in m_1(\mathbb{R})$. Then

(1) $m_1(\mathbb{R})$ will be bounded below the subset of $m(\mathbb{R})$ in the poset $(m(\mathbb{R}), \leq)$ if and only if the range(μ) is a bounded below subset of \mathbb{R} and there exists $a \in \mathbb{Q}^+$ such that $\frac{x}{a} \in \mathbb{N}$ for all $x \in \text{range}(\lambda)$.

(2) $m_1(\mathbb{R})$ will be a bounded above subset of $m(\mathbb{R})$ in the poset $(m(\mathbb{R}), \leq)$ if and only if the range(μ) is a bounded above subset of \mathbb{R} and there exists $b \in \mathbb{Q}^+$ such that $\frac{b}{y} \in \mathbb{N}$ for all $y \in \text{range}(\lambda)$.

4.10. Proposition

The poset $(m(\mathbb{R}), \leq)$ is a lattice.

4.11. Proposition

(Compatibility of $<$ with respect to \oplus and \odot) For $(x, y), (r, s) \in m(\mathbb{R})$,

(i) $(p, q) < (r, s) \Rightarrow (p, q) \oplus (a, b) < (r, s) \oplus (a, b)$ for all $(a, b) \in m(\mathbb{R})$. (ii) $(p, q) < (r, s) \Rightarrow (p, q) \odot (a, b) < (r, s) \odot (a, b)$ for all $(a, b) \in m_+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{Q}^+\}$.

4.12. Proposition

$(m(\mathbb{R}), \oplus, \odot, \leq)$ is a partially ordered multi-field induced by the field $(\mathbb{R}, +, \cdot)$.

Proof: The result follows from (7) of Proposition 47, Theorem 48 and Theorem 50.

4.13. Proposition

$(m(\mathbb{Q}), \oplus, \odot, \leq)$ is a subm-domain of $(m(\mathbb{R}), \oplus, \odot, \leq)$ where $m(\mathbb{Q}) = \mathbb{Q} \times \mathbb{Q}^+$.

Proof: Here, $m(\mathbb{Q}) \subseteq m(\mathbb{R})$. Also, restrictions of both the m-operations \oplus and \odot , viz. \boxplus and \boxodot , on $m(\mathbb{Q})$ are stable on $m(\mathbb{Q})$. Also, $(m(\mathbb{Q}), \boxplus, \boxodot)$ is a multi-field. Therefore, $(m(\mathbb{Q}), \oplus, \odot)$ is a subm-field of $(m(\mathbb{R}), \oplus, \odot)$. Also, $(m(\mathbb{Q}), \leq)$ is a partially ordered set. Therefore, $(m(\mathbb{Q}), \oplus, \odot, \leq)$ is a subm-domain of $(m(\mathbb{R}), \oplus, \odot, \leq)$.

4.14. Proposition

For $(p, q) \in m_+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{Q}^+\}$ and $(r, s) \in m(\mathbb{R})$, there exists $(a, b) \in m(\mathbb{N})$ such that $(a, b) \circ (p, q) > (r, s)$.

Proof: Since $(p, q) \in m_+(\mathbb{R})$ and $(r, s) \in m(\mathbb{R})$, so $p > 0$ and $r \in \mathbb{R}$. Therefore, by the Archimedean property of \mathbb{R} , there exists $a \in \mathbb{N}$ such that $ap > r$. Also, since $(p, q) \in m_+(\mathbb{R})$

and $(r, s) \in m(\mathbb{R})$, so, $q, s \in \mathbb{Q}^+$. Therefore, there exists $u, v, x, y \in \mathbb{N}$ such that $q = \frac{u}{v}$ and $s = \frac{x}{y}$. Therefore, $\frac{q}{s} = \frac{uy}{vx}$. Let us choose $b = vx \in \mathbb{N}$ such that $\frac{bq}{s} = uy \in \mathbb{N}$. Then $(a, b) \in m(\mathbb{N})$ and $(a, b) \circ (p, q) = (ap, bq) > (r, s)$, since $ap > r$ and $\frac{bq}{s} \in \mathbb{N}$. Hence the theorem.

4.15. *Definition*

For $n \in \mathbb{N}$ and $(p, q) \in m(\mathbb{R})$ define $n((p, q))$ and $((p, q))^n$ as follows:

$$n((p, q)) = (p, q) \oplus (p, q) \oplus \dots \oplus (p, q) \text{ (n times)} = (np, q^n) \text{ and}$$

$$(R_p^q)^n = R_p^q \odot R_p^q \odot \dots \odot R_p^q \text{ (n times)} = (p^n, q^n).$$

4.16. *Definition*

For all $(a, b), (c, d) \in m(\mathbb{R})$, the multi-equation $(a, b) \oplus (x, y) = (c, d)$ in $(x, y) \in m(\mathbb{R})$ has a unique solution in $m(\mathbb{R})$. Also, for all $(a, b) \in m^\#(\mathbb{R})$ and $(c, d) \in m(\mathbb{R})$, the multi-equation $(a, b) \odot (x, y) = (c, d)$ in $(x, y) \in m(\mathbb{R})$ has a unique solution in $m(\mathbb{R})$.

4.17. *Definition*

Let us define a binary relation \trianglelefteq on $m(\mathbb{R})$ as follows: (i) For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \triangleleft (r, s)$ if and only if $p = r$ and $\frac{s}{q} \in \mathbb{N} - \{1\}$. (ii) For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \trianglelefteq (r, s)$ if and only if $(p, q) \triangleleft (r, s)$ or $(p, q) = (r, s)$, i.e. if and only if $p = r$ and $\frac{s}{q} \in \mathbb{N}$.

4.18. *Proposition*

The binary relation \trianglelefteq defined on $m(\mathbb{R})$ is a partial order relation.

4.19. *Definition*

Let us define a binary relation \preceq on $m(\mathbb{R})$ as follows:

- (i) For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \prec (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) \triangleleft (r, s)$, i.e. if and only if (there exist $(a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$) or $(p = r$ and $\frac{s}{q} \in \mathbb{N} - \{1\}$).
- (ii) For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \preceq (r, s)$ if and only if $(p, q) \prec (r, s)$ or $(p, q) = (r, s)$.

4.20. *Proposition*

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \preceq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) \trianglelefteq (r, s)$.

Proof: For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \preceq (r, s) \Rightarrow (p, q) \prec (r, s)$ or $(p, q) = (r, s) \Rightarrow ((p, q) < (r, s)$ or $(p, q) \triangleleft (r, s))$ or $(p, q) = (r, s) \Rightarrow ((p, q) < (r, s)$ or $(p, q) = (r, s))$ or $((p, q) \triangleleft (r, s)$ or $(p, q) = (r, s)) \Rightarrow (p, q) \leq (r, s)$ or $(p, q) \trianglelefteq (r, s)$.

4.21. Remark

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \preceq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) \trianglelefteq (r, s)$. i.e., if and only if there exist $((a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$ or $(p = r$ and $\frac{s}{q} \in \mathbb{N}$). i.e., if and only if $(\exists(a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r, s) = (p + a, qb)$) or $(p = r$ and $\frac{s}{q} \in \mathbb{N}$), i.e., if and only if there exists $((a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $r = p + a$ and $s = qb$) or $(p = r$ and $\frac{s}{q} \in \mathbb{N}$), i.e., if and only if $(p < r$ and $\frac{s}{q} \in \mathbb{N}$) or $(p = r$ and $\frac{s}{q} \in \mathbb{N}$), i.e., if and only if $p \leq r$ and $\frac{s}{q} \in \mathbb{N}$.

Therefore, for $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \preceq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) \trianglelefteq (r, s)$, i.e. if and only if $(r < p$ and $\frac{s}{q} \in \mathbb{N}$) or $(p = r$ and $q = s)$.

4.22. Proposition

The relation \preceq defined on $m(\mathbb{R})$ is a partial order relation.

Proof: The relation \preceq is immediately reflexive on $m(\mathbb{R})$.

For $(u, v), (w, x) \in m(\mathbb{R})$, let $(u, v) \preceq (w, x)$ and $(w, x) \preceq (u, v)$. Then $((u, v) \leq (w, x)$ or $(u, v) \trianglelefteq (w, x))$ and $((w, x) \leq (u, v)$ or $(w, x) \trianglelefteq (u, v)) \Rightarrow ((u, v) \leq (w, x)$ and $(w, x) \leq (u, v))$ or $((u, v) \leq (w, x)$ and $(w, x) \trianglelefteq (u, v))$ or $((u, v) \trianglelefteq (w, x)$ and $(w, x) \leq (u, v))$ or $((u, v) \trianglelefteq (w, x)$ and $(w, x) \trianglelefteq (u, v))$.

Now $(u, v) \leq (w, x)$ and $(w, x) \leq (u, v) \Rightarrow (u, v) = (w, x)$.

$(u, v) \leq (w, x) \Rightarrow u \leq w$ and $\frac{x}{v} \in \mathbb{N}$.

$(w, x) \trianglelefteq (u, v) \Rightarrow w = u$ and $\frac{v}{x} \in \mathbb{N}$.

Therefore, $u = w$ and $v = x$.

Therefore, $(u, v) \leq (w, x)$ and $(w, x) \trianglelefteq (u, v) \Rightarrow u = w$ and $v = x \Rightarrow (u, v) = (w, x)$.

Similarly, $(u, v) \trianglelefteq (w, x)$ and $(w, x) \leq (u, v) \Rightarrow (u, v) = (w, x)$.

Lastly, $(u, v) \trianglelefteq (w, x)$ and $(w, x) \trianglelefteq (u, v) \Rightarrow (u, v) = (w, x)$.

So, for $(u, v), (w, x) \in m(\mathbb{R})$, $(u, v) \preceq (w, x)$ and $(w, x) \preceq (u, v) \Rightarrow (u, v) = (w, x)$.

Therefore, the relation \preceq is symmetric on $m(\mathbb{R})$.

The transitive property of \preceq on $m(\mathbb{R})$ follows from the transitive property of \leq and \trianglelefteq .

Hence, the relation \preceq is a partial order relation defined on $m(\mathbb{R})$.

4.23. Definition

1. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded below in the poset $(m(\mathbb{R}), \preceq)$ if there exists $(a, b) \in m(\mathbb{R})$ such that $(a, b) \preceq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$ and otherwise unbounded below. Also, $(a, b) \preceq (x, y) \Rightarrow (a \leq x$ and $\frac{y}{b} \in \mathbb{N})$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded below in the poset $(m(\mathbb{R}), \preceq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x, y) \in m_1(\mathbb{R})$, $(a \leq x$ and $\frac{y}{b} \in \mathbb{N})$ and otherwise unbounded below.

2. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded above in the poset $(m(\mathbb{R}), \preceq)$ if there exists $(a, b) \in m(\mathbb{R})$ such that $(x, y) \preceq (a, b)$ for all $(x, y) \in m_1(\mathbb{R})$ and otherwise unbounded above. Also, $(x, y) \preceq (a, b) \Rightarrow (x \leq a \text{ and } \frac{b}{y} \in \mathbb{N})$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded above in the poset $(m(\mathbb{R}), \preceq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x, y) \in m_1(\mathbb{R})$, $(x \leq a \text{ and } \frac{b}{y} \in \mathbb{N})$ and otherwise unbounded above.

4.24. *Proposition*

For $(p, q), (r, s) \in m(\mathbb{R})$,

(i) $(p, q) \prec (r, s) \Rightarrow (p, q) \oplus (a, b) \prec (r, s) \oplus (a, b)$ for all $(a, b) \in m(\mathbb{R})$. (ii) $(p, q) \prec (r, s) \Rightarrow (p, q) \odot (a, b) \prec (r, s) \odot (a, b)$ for all $(a, b) \in m_+(\mathbb{R})$.

4.25. *Proposition*

Therefore, $(m(\mathbb{R}), \oplus, \odot, \preceq)$ is a partially ordered multi-field.

4.25.1. *Remark*

Consider the partially ordered set $(\mathbb{Q}^+, |)$ where for $d, n \in \mathbb{Q}^+$, $d|n$ if and only if there exists $m \in \mathbb{N}$ such that $n = d \cdot m$ (in other words, $\frac{n}{d} \in \mathbb{N}$).

Let $q, s \in \mathbb{Q}^+$. Then there exists $a_1, b_1; a_2, b_2 \in \mathbb{N}$ such that $q = \frac{a_1}{b_1}$ and $s = \frac{a_2}{b_2}$. We know that $gcd\{q, s\} = \frac{gcd\{a_1, a_2\}}{lcm\{b_1, b_2\}}$ and $lcm\{q, s\} = \frac{lcm\{a_1, a_2\}}{gcd\{b_1, b_2\}}$ such that $\frac{q}{gcd\{q, s\}}, \frac{s}{gcd\{q, s\}}, \frac{lcm\{q, s\}}{q}, \frac{lcm\{q, s\}}{s} \in \mathbb{N}$.

Then $(\mathbb{Q}^+, |, gcd, lcm)$ is a lattice.

4.26. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a lattice.

Proof: Let $(p, q), (r, s) \in m(\mathbb{R})$. Let $\mu_l = min\{p, r\}$, $\mu_u = max\{p, r\}$, $\lambda_l = gcd\{q, s\}$, and $\lambda_u = lcm\{q, s\}$. Then $\mu_l \leq p, r \leq \mu_u$. Also, $\frac{q}{\lambda_l}, \frac{s}{\lambda_l} \in \mathbb{N}$ and $\frac{\lambda_u}{q}, \frac{\lambda_u}{s} \in \mathbb{N}$.

So, $(\mu_l, \lambda_l) \preceq (p, q), (r, s)$, also, $(p, q), (r, s) \preceq (\mu_u, \lambda_u)$. i.e., (μ_l, λ_l) is a lower bound of (p, q) and (r, s) , (μ_u, λ_u) is an upper bound of (p, q) and (r, s) in the poset $(m(\mathbb{R}), \preceq)$.

Let $(a, b) \in m(\mathbb{R})$ be the lower bound of (p, q) and (r, s) in the poset $(m(\mathbb{R}), \preceq)$.

Then $(a, b) \preceq (p, q), (r, s)$. Therefore, $a \leq p, r$ and $\frac{q}{b}, \frac{s}{b} \in \mathbb{N}$.

So, a is a lower bound of p and r . Let $\mu_l = min\{p, r\}$. Therefore, $a \leq \mu_l$.

Also, $\frac{q}{b}, \frac{s}{b} \in \mathbb{N}$, so b is a common divisor of q and s . Let $\lambda_l = gcd\{q, s\}$. Therefore, $\frac{b}{\lambda_l} \in \mathbb{N}$.

Since $a \leq \mu_l$ and $\frac{\lambda_l}{b} \in \mathbb{N}$, so $(a, b) \preceq (\mu_l, \lambda_l)$.

Since, for any lower bound (a, b) of (p, q) and (r, s) , $(a, b) \preceq (\mu_l, \lambda_l)$, so (μ_l, λ_l) is the greatest lower bound (uniqueness can be easily proved using the symmetric property of \preceq) of (p, q) and (r, s) .

Therefore, every pair of elements of $m(\mathbb{N})$ has a unique greatest lower bound.

Also, in a similar argument we can show that (μ_u, λ_u) is the unique least upper bound of (p, q) and (r, s) .

Therefore, every pair of elements of $m(\mathbb{R})$ has a unique least upper bound.

Therefore, $(m(\mathbb{R}), \preceq)$ is a lattice.

4.27. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a complete lattice.

Proof: Let $m_1(\mathbb{R})$ be a bounded below subset of $m(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$. Then there exists $(a, b) \in m(\mathbb{R})$ such that $(a, b) \preceq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$. Therefore, $a \leq x$ and $\frac{y}{b} \in \mathbb{N}$ for all $x \in \text{Range}(\mu)$ and $y \in \text{Range}(\lambda)$. Therefore, by the order completeness property of \mathbb{R} , $\text{glb Range}(\mu)$ exists and is equal to u , say, such that $u \leq x$ for all $x \in \text{Range}(\mu)$ and for any lower bound a of $\text{Range}(\mu)$, $a \leq u$. Here, b is a common divisor of $\text{Range}(\lambda)$, so $\text{gcd Range}(\lambda) = v$, say, there exists such that for any common divisor b of $\text{Range}(\lambda)$, $\frac{v}{b} \in \mathbb{N}$. Then $(u, v) \in m(\mathbb{R})$ such that $(u, v) \preceq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$. Also, for any lower bound (a, b) of $m_1(\mathbb{R})$ in the poset $(m(\mathbb{R}), \preceq)$, $(a, b) \preceq (u, v)$. Therefore, $(u, v) = (\text{glb Range}(\mu), \text{gcd Range}(\lambda))$ is the glb of $m_1(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$.

Similarly, we can show that for any bounded above subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$, $(u, v) = (\text{lub Range}(\mu), \text{lcm Range}(\lambda)) \in m(\mathbb{R})$ is lub of $m_1(\mathbb{R})$ in the poset $(m(\mathbb{R}), \preceq)$. Therefore, $(m(\mathbb{R}), \preceq)$ is a complete lattice.

4.28. *Remark*

For any two members $(p, q), (s, t)$ of the lattice $(m(\mathbb{R}), \preceq)$, let us denote $\text{lub}\{(p, q), (s, t)\} = (p, q) \vee (s, t)$ and $\text{glb}\{(p, q), (s, t)\} = (p, q) \wedge (s, t)$.

4.29. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a distributive lattice.

Proof: Let $m_1(\mathbb{R})$ be a bounded below subset of $m(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$.

For any $(p, q) \in m(\mathbb{R})$, we will show that

$$(p, q) \vee \left(\bigwedge_{(x,y) \in m_1(\mathbb{R})} (x, y) \right) = \bigwedge_{(x,y) \in m_1(\mathbb{R})} ((p, q) \vee (x, y)).$$

$$\begin{aligned}
 & \text{Now, } (p, q) \vee \left(\bigwedge_{(x,y) \in m_1(\mathbb{R})} (x, y) \right) \\
 &= (p, q) \vee (\text{glb}(\text{Range}(\mu)), \text{gcd}(\text{Range}(\lambda))) \\
 &= (\max\{p, \text{glb}(\text{Range}(\mu)), \text{lcm}[q, \text{gcd}(\text{Range}(\lambda))]\}) \\
 &= \left(\bigwedge_{x \in \text{Range}(\mu)} \max\{p, x\}, \bigwedge_{x \in \text{Range}(\mu)} \max\{p, x\} \right) \\
 &= \bigwedge_{(x,y) \in m_1(\mathbb{R})} (\max\{p, x\}, \text{lcm}[q, y]) \\
 &= \bigwedge_{(x,y) \in m_1(\mathbb{R})} ((p, q) \vee (x, y)).
 \end{aligned}$$

Similarly, for any subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ which is bounded above in poset $(m(\mathbb{R}), \preceq)$ and also, for any $(p, q) \in m(\mathbb{R})$, we can show that

$$(p, q) \wedge \left(\bigvee_{(x,y) \in m_1(\mathbb{R})} (x, y) \right) = \bigvee_{(x,y) \in m_1(\mathbb{R})} ((p, q) \wedge (x, y)).$$

Also, for any two bounded subsets $m_1(\mathbb{R})$ and $m_2(\mathbb{R})$ of $m(\mathbb{R})$, we can show that

$$\begin{aligned}
 & \left(\bigwedge_{(p,q) \in m_1(\mathbb{R})} (p, q) \right) \vee \left(\bigwedge_{(x,y) \in m_2(\mathbb{R})} (x, y) \right) \\
 &= \bigwedge_{(p,q) \in m_1(\mathbb{R})} \bigwedge_{(x,y) \in m_2(\mathbb{R})} ((p, q) \vee (x, y)) \text{ and} \\
 & \left(\bigvee_{(p,q) \in m_1(\mathbb{R})} (p, q) \right) \wedge \left(\bigvee_{(x,y) \in m_2(\mathbb{R})} (x, y) \right)
 \end{aligned}$$

$$= \bigvee_{(p,q) \in m_1(\mathbb{R})} \bigvee_{(x,y) \in m_2(\mathbb{R})} ((p, q) \wedge (x, y)).$$

Therefore, $(m(\mathbb{R}), \preceq)$ is a distributive lattice.

4.30. *Proposition*

For $(p, q) \in m_+(\mathbb{R})$ and $(r, s) \in m(\mathbb{R})$, there exists $(a, b) \in m(\mathbb{N})$ such that $(a, b) \odot (p, q) \succ (r, s)$.

4.31. *Proposition*

$(m(\mathbb{Q}), +, \circ, \preceq)$ is a subm-domain of $(m(\mathbb{R}), +, \circ, \preceq)$.

Proof: Here, $m(\mathbb{Q}) \subseteq m(\mathbb{R})$. Also, restrictions of both operations $+$ and \circ , viz. \boxplus and \boxcirc , on $m(\mathbb{Q})$ are stable on $m(\mathbb{Q})$. Also, $(m(\mathbb{Q}), \boxplus, \boxcirc)$ is a multi-field. Therefore, $(m(\mathbb{Q}), +, \circ)$ is a subm-field of $(m(\mathbb{R}), +, \circ)$. Also, $(m(\mathbb{Q}), \preceq)$ is a distributive lattice and \prec is compatible with $+$ and \circ . Therefore, $(m(\mathbb{Q}), +, \circ, \preceq)$ is a subm-domain of $(m(\mathbb{R}), +, \circ, \preceq)$.

4.32. *Remark*

$(m(\mathbb{R}), \preceq)$ is a complete distributive lattice but $(m(\mathbb{Q}), \preceq)$ is a distributive lattice which is not complete.

4.33. *Remark*

In this remark we shall show that multi-real number system $m(\mathbb{R})$ is an extension of the real number system \mathbb{R} .

Consider a mapping $u : \mathbb{R} \rightarrow m(\mathbb{R})$ defined as follows: $u(x) = (x, 1), x \in \mathbb{R}$.

Then u is immediately an injective mapping.

Also, we note that $\forall x, y \in \mathbb{R}$,

$$(i) \ u(x + y) = (x + y, 1) = (x, 1) \oplus (y, 1) = u(x) \oplus u(y).$$

$$(ii) \ u(x \cdot y) = (x \cdot y, 1) = (x, 1) \odot (y, 1) = u(x) \odot u(y).$$

(iii) For $x, y \in \mathbb{R}$, let $x < y$, then $\exists a \in \mathbb{R}^+$ such that $x + a = y$. Since $x, y \in \mathbb{R}$, so $(x, 1), (y, 1) \in m(\mathbb{R})$. Also, $(x, 1) \oplus (a, 1) = (x + a, 1) = (y, 1)$.

Therefore, for $(x, 1), (y, 1) \in m(\mathbb{R})$, $\exists, a \in \mathbb{R}^+$ and $1 \in \mathbb{N}$ such that $(y, 1) = (x, 1) \oplus (a, 1)$.

Therefore, $(x, 1) < (y, 1)$.

Therefore, $u(x) < u(y)$.

Therefore, for $x, y \in \mathbb{R}$, $x < y \Rightarrow u(x) < u(y)$.

Therefore, u is a structure-preserving injective mapping from \mathbb{R} into $m(\mathbb{R})$.

Therefore, \mathbb{R} is embedded in $m(\mathbb{R})$.

So, multi-real number system $m(\mathbb{R})$ is an extension of the real number system \mathbb{R} .

5. **Multiset redefined**5.1. *Definition*

A \mathbb{Q}^+ -**multiset** (\mathbb{Q}^+ -mset in short) M drawn from the crisp set X is represented by a characteristic function $\chi_M : X \rightarrow \mathbb{Q}^+ \cup \{0\}$. X is called the **domain**. We represent a \mathbb{Q}^+ -mset M drawn from the crisp set X as $M = \{(x, \chi_M(x)) : x \in X\}$ or as $M = \{(x, \chi_M(x)) : x \in X, \chi_M(x) > 0\}$. The **support set** or **root set** of M is denoted by M^* (also denoted by **Supp(M)**) and is defined by $M^* = \{x \in X : \chi_M(x) > 0\}$, i.e. M^* is a crisp set. If for a \mathbb{Q}^+ -mset M drawn from the crisp set X , $x \in M^*$, then we will write $x \in^* M$. For $x \in X$, $\chi_M(x)$ is called the **multiplicity** of x . A \mathbb{Q}^+ -mset M drawn from a crisp set X is said to be **empty \mathbb{Q}^+ -mset** denoted by ϕ if $\chi_M(x) = 0$ for all $x \in X$. The \mathbb{Q}^+ -mset M drawn from the crisp set X can also be represented as $M = \{(x, \chi_M(x)) : x \in M^* \subseteq X\}$ simply discarding all pairs $(x, \chi_M(x))$ for which $\chi_M(x) = 0$, $x \in X$. Therefore, if M is a \mathbb{Q}^+ -mset drawn from the crisp set X and $X \subseteq Y$, then M can also be considered as a \mathbb{Q}^+ -mset drawn from the crisp set Y . Also, if M be a \mathbb{Q}^+ -mset drawn from the crisp set X and $M^* \subseteq Y \subseteq X$, then M can also be considered as a \mathbb{Q}^+ -mset drawn from the crisp set Y . Clearly, a crisp set M drawn from the universal set X can be considered as an \mathbb{Q}^+ -mset with $\chi_M(x) = 1$ for all $x \in M^*$.

The **cardinality** of a \mathbb{Q}^+ -mset M drawn from a crisp set X is denoted by $\text{card}M$ or $|M|$ and

is defined by $\text{card}M = \sum_x \chi_M(x)$. The **dimension** of a \mathbb{Q}^+ -mset M is denoted by $/M/$ or by $\text{dim}M$ and is defined by $\text{dim}M = \sum_x \chi_{M^*}(x)$. The maximum value of the multiplicity function $\text{alt}M = \max_{x \in M^*} \chi_M(x)$ is called the **height** of the \mathbb{Q}^+ -mset M .

Let X be a non-empty crisp set. A \mathbb{N} -multiset (\mathbb{N} -mset in short) M drawn from X is characterized by a function $\chi_M : X \rightarrow \mathbb{N} \cup \{0\}$. We represent a \mathbb{N} -mset M drawn from X as $M = \{(x, \chi_M(x)) : x \in X\}$ where $\chi_M(x) \in \mathbb{N} \cup \{0\}$. Immediately, ϕ can be considered as a \mathbb{N} -mset drawn from any crisp set under consideration. Immediately, every \mathbb{N} -mset is a \mathbb{Q}^+ -mset.

5.2. *Example*

Let $X = \{a, b, c, d, e\}$ be any crisp set. Then $M = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$ is a \mathbb{Q}^+ -mset drawn from X . The \mathbb{Q}^+ -mset M drawn from X can also be represented as $M = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$ by discarding $(d, 0)$. The support set of M is $M^* = \{a, b, c, e\}$. $\text{Alt}M = 12$, $|M| = 23\frac{1}{2}$ and $/M/ = 4$.

Also, $M = \{(a, 2), (b, 9), (e, 12)\}$ is a \mathbb{N} -mset drawn from X .

5.3. *Definition*

Two \mathbb{Q}^+ -mssets P and T drawn from a crisp set X are said to be **equal**, denoted by $P = T$, if and only if $\chi_P(x) = \chi_T(x)$ for all $x \in X$. \mathbb{Q}^+ -mssets P and T are unequal, denoted by $P \neq T$ if $\chi_P(x) \neq \chi_T(x)$ for at least one $x \in X$. For equal \mathbb{Q}^+ -mssets P and T drawn from a crisp set X , we have $|P| = |T|$, $/P/ = /T/$, $P^* = T^*$ and $\text{alt}P = \text{alt}T$. Two \mathbb{Q}^+ -mssets P and T drawn from a crisp set X are said to be **equal in size** if they have equal cardinality and equal dimension. Equal \mathbb{Q}^+ -mssets are immediately equal in size, but the converse is not true, in general. The equality of \mathbb{Q}^+ -mssets is an equivalence relation.

Two \mathbb{Q}^+ -mssets P and T drawn from crisp sets X and Y respectively are said to be equal, denoted by $P = T$, if and only if

- (i) $P^* = T^*$ and
- (ii) $\chi_P(x) = \chi_T(x)$ for all $x \in P^* = T^*$.

In this case, \mathbb{Q}^+ -mset P and T are considered as \mathbb{Q}^+ -mset drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$.

5.4. *Definition*

Let P and T be two \mathbb{Q}^+ -mssets drawn from a crisp set X . Then \mathbb{Q}^+ -mset P is said to be a \mathbb{Q}^+ -**submultiset** or \mathbb{Q}^+ -**subset** of the \mathbb{Q}^+ -mset T , denoted by $P \sqsubseteq T$, if and only if $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$. If P is a subset of T , then T is called a \mathbb{Q}^+ -**supermultiset** or \mathbb{Q}^+ -**superset** of P .

If P is a \mathbb{Q}^+ -subset of T , then $|P| \leq |T|, /P/ \leq /T/, P^* \subseteq T^*, altP \leq altT$.

If $P \sqsubseteq T$ and $T \sqsubseteq P$, then $P = T$.

If $P \sqsubseteq T$ but $T \neq P$, the \mathbb{Q}^+ -mset P is called the proper \mathbb{Q}^+ -subset of the \mathbb{Q}^+ -mset T and is denoted by $P \sqsubset T$.

The inclusion of \mathbb{Q}^+ -msets is the **partial order relation**, since it is reflexive ($A \sqsubseteq A$), symmetric ($P \sqsubseteq T$ and $T \sqsubseteq P \Rightarrow T = P$) and transitive ($A \sqsubseteq B, B \sqsubseteq C \Rightarrow A \sqsubseteq C$).

Let X be a non-empty crisp set. Let P and M be two \mathbb{N} -msets drawn from X . Then P is said to be a **\mathbb{N} -subset** or **\mathbb{N} -subset** of M if and only if $\frac{\chi_M(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$. Every \mathbb{N} -subset of a \mathbb{N} -mset is immediately a \mathbb{N} -mset.

A \mathbb{Q}^+ -subset P of a \mathbb{Q}^+ -mset M is a **whole \mathbb{Q}^+ -subset** of M with each element in P having full multiplicity as in M . i.e., $\chi_P(x) = \chi_M(x)$ for every x in P^* .

A \mathbb{Q}^+ -subset P of a \mathbb{Q}^+ -mset M is a **partial whole \mathbb{Q}^+ -subset** of M with at least one element in P having the same multiplicity as in M . i.e., $\chi_P(x) = \chi_M(x)$ for some x in P^* .

A \mathbb{Q}^+ -subset P of a \mathbb{Q}^+ -mset M is a full \mathbb{Q}^+ -subset of M if $M^* = P^*$. The empty set ϕ is a whole \mathbb{Q}^+ -subset of every \mathbb{Q}^+ -mset but it is neither a full \mathbb{Q}^+ -subset nor a partial whole \mathbb{Q}^+ -subset of any non-empty \mathbb{Q}^+ -mset M .

Let P and T be two \mathbb{Q}^+ -msets drawn from crisp sets X and Y , respectively. Then \mathbb{Q}^+ -mset P is said to be a \mathbb{Q}^+ -subset of the \mathbb{Q}^+ -mset T , denoted by $P \sqsubseteq T$, if and only if

- (i) $P^* \subseteq T^*$ and
- (ii) $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$.

If P is a \mathbb{Q}^+ -subset of T , then T is called a \mathbb{Q}^+ -superset of P .

In this case, the \mathbb{Q}^+ -msets P and T are to be considered as \mathbb{Q}^+ -msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$. If we are not bothered about whether the multiplicity of an element is a rational number or a natural number, then we can simply say **mset** instead of \mathbb{Q}^+ -mset or \mathbb{N} -mset and **subset** instead of \mathbb{Q}^+ -subset or \mathbb{N} -subset.

5.5. Example

Consider the set $X = \{a, b, c, d, e\}$.

Consider the \mathbb{Q}^+ -msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$

and $T = \{(a, 4), (b, 18), (c, 5), (d, \frac{3}{2}), (e, 24)\}$ drawn from X .

We see that $P^* = \{a, b, c, e\}$ and $T^* = \{a, b, c, d, e\}$. So, $P^* \subseteq T^*$.

Also, $\frac{\chi_T(a)}{\chi_P(a)} = \frac{4}{2} = 2 \in \mathbb{N}$, $\frac{\chi_T(b)}{\chi_P(b)} = \frac{18}{9} = 2 \in \mathbb{N}$, $\frac{\chi_T(c)}{\chi_P(c)} = \frac{5}{\frac{1}{2}} = 10 \in \mathbb{N}$, $\frac{\chi_T(e)}{\chi_P(e)} = \frac{24}{12} = 2 \in \mathbb{N}$.

Therefore, $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N} \forall x \in P^*$. So, T is a \mathbb{Q}^+ -subset of P .

Therefore, $P \sqsubseteq T$ but $P \neq T$ so $P \sqsubset T$, i.e. P is a proper \mathbb{Q}^+ -subset of T and consequently T is a \mathbb{Q}^+ -superset of P .

Again, consider the \mathbb{N} -msets, $P = \{(a, 2), (b, 3), (c, 4)\}$ and $T = \{(a, 4), (b, 36), (c, 16), (d, 5)\}$ drawn from X . Then \mathbb{N} -mset P is a proper \mathbb{N} -subset of the \mathbb{N} -mset T .

5.6. *Example*

Consider mset $M = \{(x, 2), (y, 3), (z, 5)\}$.

The subset $P_1 = \{(x, 2), (y, 3)\}$ is a whole \mathbb{Q}^+ -subset and partial whole \mathbb{Q}^+ -subset of M but it is not a full \mathbb{Q}^+ -subset of M .

The \mathbb{Q}^+ -subset $P_2 = \{(x, 1), (y, 3), (z, 1)\}$ is a partial whole \mathbb{Q}^+ -subset and a full \mathbb{Q}^+ -subset of M but it is not a whole \mathbb{Q}^+ -subset of M .

The \mathbb{Q}^+ -subset $P_3 = \{(x, 1), (y, 3)\}$ is a partial whole \mathbb{Q}^+ -subset of M which is neither full \mathbb{Q}^+ -subset of M nor a whole \mathbb{Q}^+ -subset of M .

5.7. *Definition*

Let M be a \mathbb{Q}^+ -mset drawn from a crisp set X . Let $x \in M^*$. Then the \mathbb{Q}^+ -subset P of M is represented by a characteristic function $\chi_P : \{x\} \rightarrow \mathbb{Q}^+ \cup \{0\}$ where $\chi_P(x) = \chi_M(x)$ is called a \mathbb{Q}^+ -**component** of \mathbb{Q}^+ -multiset M .

Alternatively, a \mathbb{Q}^+ -subset P of a \mathbb{Q}^+ -mset M drawn from a crisp set X is a \mathbb{Q}^+ -component if $\chi_P(x) = \chi_M(x)$ for all $x \in X$ and $\{x \in X : \chi_P(x) = \chi_M(x)\}$ is a singleton set, say, $\{x\}$, then let us denote it as $M_{\{x\}} (= P)$, i.e., a \mathbb{Q}^+ -component is such a \mathbb{Q}^+ -subset of a \mathbb{Q}^+ -mset for which exactly one element of the support set belongs to it with the same count as in the \mathbb{Q}^+ -mset.

A \mathbb{Q}^+ -component is called a \mathbb{Q}^+ -**zero** if $\chi_P(x) = \chi_M(x) = 0$.

A \mathbb{Q}^+ -component of a \mathbb{Q}^+ -mset is a whole \mathbb{Q}^+ -subset of the \mathbb{Q}^+ -mset but not conversely.

A \mathbb{Q}^+ -subset of a \mathbb{Q}^+ -component of a \mathbb{Q}^+ -mset is called the \mathbb{Q}^+ -**subm-component** of the \mathbb{Q}^+ -component.

5.8. *Definition*

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X . Then the \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -**multiset-union** (or \mathbb{Q}^+ -**m-union**) of P and T , denoted by $M = P \sqcup T$, if and only if (i) $M^* = P^* \cup T^*$ and

(ii)

$$\chi_M(x) = \begin{cases} lcm\{\chi_P(x), \chi_T(x)\} & \text{if } x \in P^* \cap T^* \\ \chi_P(x) & \text{if } x \in P^* - T^* \\ \chi_T(x) & \text{if } x \in T^* - P^* \\ 0 & \text{if } x \in X - (P^* \cup T^*) \end{cases}$$

Let P and T be two \mathbb{Q}^+ -msets drawn from crisp sets X and Y respectively. Then the \mathbb{Q}^+ -mset M drawn from $X \cup Y$ is said to be the \mathbb{Q}^+ -m-union of P and T , denoted by $M = P \sqcup T$, if and only if

(i) $M^* = P^* \cup T^*$ and

(ii)

$$\chi_M(x) = \begin{cases} lcm\{\chi_P(x), \chi_T(x)\} & \text{if } x \in P^* \cap T^* \\ \chi_P(x) & \text{if } x \in P^* - T^* \\ \chi_T(x) & \text{if } x \in T^* - P^* \\ 0 & \text{if } x \in (X \cup Y) - (P^* \cup T^*) \end{cases}$$

In this case, the \mathbb{Q}^+ -msets P and T are to be considered as \mathbb{Q}^+ -msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$.

5.9. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$.

Consider the \mathbb{Q}^+ -msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$

and $T = \{(a, 0), (b, 12), (c, 5), (d, \frac{3}{2}), (e, 8)\}$ drawn from X .

Then $P \sqcup T = \{(a, 2), (b, 36), (c, 5), (d, \frac{3}{2}), (e, 24)\}$.

5.10. *Remark*

\mathbb{Q}^+ -m-union of two \mathbb{Q}^+ -msets A and B drawn from a non-empty crisp set X is the smallest \mathbb{Q}^+ -superset of A and B both.

5.11. *Definition*

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X . Then the \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -**multiset-intersection** (or \mathbb{Q}^+ -**m-intersection**) of P and T , denoted by $M = P \cap T$, if and only if

(i) $M^* = P^* \cap T^*$ and

$$\chi_M(x) = \begin{cases} gcd\{\chi_P(x), \chi_T(x)\} & \text{if } x \in P^* \cap T^* \\ 0 & \text{if } x \in X - (P^* \cap T^*) \end{cases}$$

Let P and T be two \mathbb{Q}^+ -msets drawn from crisp sets X and Y , respectively. Then the \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -m-intersection of P and T , denoted by $M = P \cap T$, if and only if

(i) $M^* = P^* \cap T^*$ and

(ii)

$$\chi_M(x) = \begin{cases} gcd\{\chi_P(x), \chi_T(x)\} & \text{if } x \in P^* \cap T^* \\ 0 & \text{if } x \in (X \cup Y) - (P^* \cap T^*) \end{cases}$$

In this case, the \mathbb{Q}^+ -msets P and T are to be considered as \mathbb{Q}^+ -msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$.

5.12. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$.

Consider the \mathbb{Q}^+ -msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$

and $T = \{(a, 0), (b, 12), (c, 5), (d, \frac{3}{2}), (e, 8)\}$ drawn from X .

Then $P \cap T = \{(a, 0), (b, 3), (c, \frac{1}{2}), (d, 0), (e, 4)\}$.

5.13. *Remark*

\mathbb{Q}^+ -m-intersection of two \mathbb{Q}^+ -msets A and B drawn from a non-empty crisp set X is the largest \mathbb{Q}^+ -subset of A and B both.

5.14. *Definition*

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X . Then the \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -**multiset-complement** or \mathbb{Q}^+ -**m-complement** of P on T , denoted by $M = T \neg P$, if and only if

$$\chi_M(x) = \begin{cases} 0 & \text{if } x \notin T^* \\ \chi_T(x) & \text{if } x \in T^* - P^* \\ \frac{\chi_T(x)}{\chi_P(x)} & \text{if } x \in T^* \cap P^* \end{cases}$$

5.15. *Remark*

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X . Then, $(T \neg P)$ is not necessarily a \mathbb{Q}^+ -subset of T , also, $(T \neg P) \cup (P \cap T) \neq T$, in general, which can be justified by the following example.

5.16. *Example*

Consider the crisp set $X = \{a, b, c, d, e, f\}$.

Consider the \mathbb{Q}^+ -msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$

and $T = \{(b, 18), (c, 5), (d, \frac{3}{2}), (e, 24)\}$ drawn from X .

Then $P^* = \{a, b, c, e\}$, $T^* = \{b, c, d, e\}$, $X - T^* = \{a, f\}$, $T^* - P^* = \{d\}$, $T^* \cap P^* = \{b, c, d\}$.

Then for $a \notin T^*$, $\chi_{(T \neg P)}(a) = 0$,

$$b \in T^* \cap P^*, \chi_{(T \neg P)}(b) = \frac{\chi_T(b)}{\chi_P(b)} = \frac{18}{9} = 2,$$

$$c \in T^* \cap P^*, \chi_{(T \neg P)}(c) = \frac{\chi_T(c)}{\chi_P(c)} = \frac{5}{\frac{1}{2}} = 10,$$

$$d \in T^* - P^*, \chi_{(T \neg P)}(d) = \chi_T(d) = \frac{3}{2},$$

$$e \in T^* \cap P^*, \chi_{(T \neg P)}(e) = \frac{\chi_T(e)}{\chi_P(e)} = \frac{24}{12} = 2,$$

$$f \notin T^*, \chi_{(T \neg P)}(f) = 0.$$

Therefore, $T \neg P = \{(b, 2), (c, 10), (d, \frac{3}{2}), (e, 2)\}$ which is not a \mathbb{Q}^+ -submset of T .

$$\text{Now } P \sqcap T = \{(b, 9), (c, \frac{1}{2}), (e, 12)\}.$$

$$\text{Therefore, } (T \neg P) \sqcup (P \sqcap T) = \{(b, 18), (c, 10), (d, \frac{3}{2}), (e, 12)\} \neq T.$$

5.17. Definition

Let T be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Let P be a \mathbb{Q}^+ -submset of T satisfying the following: (i) P^* is a singleton set, say $\{\alpha\}$, (ii) $\frac{\chi_T(\alpha)}{\chi_P(\alpha)} = k \in \mathbb{N}$. Then P is called an **elementary \mathbb{Q}^+ -submset** of T , denoted by $P = \{(\alpha, \chi_P(\alpha))\}$. \mathbb{Q}^+ -Components are immediately elementary \mathbb{Q}^+ -submsets of a \mathbb{Q}^+ -mset. Also every elementary \mathbb{Q}^+ -submset of an \mathbb{Q}^+ -mset T can be treated as a **\mathbb{Q}^+ -subm-element** $(\alpha, \chi_P(\alpha))$ of the \mathbb{Q}^+ -mset T by simply identifying the elementary \mathbb{Q}^+ -submset with the element α that P^* contains together with its multiplicity $\chi_P(\alpha)$. We express membership of $(\alpha, \chi_P(\alpha))$ in T using the symbol $(\alpha, \chi_P(\alpha)) \in^k T$. The crisp set of all \mathbb{Q}^+ -subm-elements of an \mathbb{Q}^+ -mset T is denoted by $\Omega(T)$. Immediately, $\Omega(T)$ is always an infinite \mathbb{Q}^+ -general mset. Also, for all $(x, \lambda) \in \Omega(M)$, $\frac{\chi_M(x)}{\lambda} \in \mathbb{N}$. For (x, λ) , x is called the **base** and λ is called the **multiplicity** of the \mathbb{Q}^+ -subm-element (x, λ) .

Let X be a non-empty crisp set. Let M be a \mathbb{N} -mset drawn from X . A \mathbb{Q}^+ -subm-element (α, k) of M is said to be a \mathbb{N} -subm-element of M if $k \in \mathbb{N}$. The crisp set of all \mathbb{N} -subm-elements of the \mathbb{N} -mset M is denoted by $\Omega^*(M)$. $\Omega^*(M)$ is a \mathbb{N} -general mset. If $A \sqsubseteq B$, then $\Omega(A) \subseteq \Omega(B)$ and $\Omega^*(A) \subseteq \Omega^*(B)$.

5.18. Definition

Let T be a non-empty mset drawn from a crisp set X . Let P be another non-empty mset drawn from X satisfying the following: (i) $P^* \subseteq T^*$ is a singleton set, say $\{\alpha\}$, (ii) $\frac{\chi_P(\alpha)}{\chi_T(\alpha)} = k \in \mathbb{N}$. Then, obviously, P is an elementary \mathbb{Q}^+ -mset drawn from X , denoted by $P = \{(\alpha, \chi_P(\alpha))\}$. This elementary \mathbb{Q}^+ -mset P drawn from X can be treated as a \mathbb{Q}^+ -superm-element $(\alpha, \chi_P(\alpha))$ of the \mathbb{Q}^+ -mset T drawn from X by simply identifying the elementary \mathbb{Q}^+ -mset $\{(\alpha, \chi_P(\alpha))\}$ with the element α that P^* contains together with its multiplicity $\chi_P(\alpha)$. We express membership of $(\alpha, \chi_P(\alpha))$ in T using the symbol $(\alpha, \chi_P(\alpha)) \in^{\frac{1}{k}} T$. The crisp set of all \mathbb{Q}^+ -superm-elements of an \mathbb{Q}^+ -mset T is denoted by $\omega(T)$. Immediately, $\omega(T)$ is always an infinite \mathbb{Q}^+ -general mset.

5.19. *Definition*

In continuation to the above two definitions, if $\chi_P(\alpha) = \chi_T(\alpha)$, then $(\alpha, \chi_P(\alpha))$ is a \mathbb{Q}^+ -subm-element of T as well as a \mathbb{Q}^+ -superm-element of T . In this case, we define $(\alpha, \chi_P(\alpha))$ to be an \mathbb{Q}^+ -**m-element** of T and it is denoted by $(\alpha, \chi_P(\alpha)) \in^1 T$ or simply by $(\alpha, \chi_P(\alpha)) \in T$.

5.20. *Definition*

Let M be an \mathbb{Q}^+ -mset drawn from a crisp set X . Let (α, k) and (β, l) be two \mathbb{Q}^+ -subm-elements of the \mathbb{Q}^+ -mset M . Then

- (i) $(\alpha, k) = (\beta, l)$ if and only if $\alpha = \beta$ and $k = l$.
- (ii) (α, k) and (α, l) are said to be overlapping \mathbb{Q}^+ -subm-elements if $\gcd(k, l) \neq 1$.
- (iii) (α, k) is said to be a divisor/factor of (β, l) if and only if $\{(\alpha, k)\} \sqsubseteq \{(\beta, l)\}$ i.e. if and only if $\alpha = \beta$ and $\frac{l}{k} \in \mathbb{N}$. Then (β, l) is said to be a multiple of (α, k) .
- (iv) (α, k) and (α, l) are said to be coprime \mathbb{Q}^+ -subm-elements if $\gcd(k, l) = 1$
- (v) (α, k) and (β, l) are said to be distinct \mathbb{Q}^+ -subm-elements of M if and only if $\alpha \neq \beta$.

Let M be an \mathbb{Q}^+ -mset drawn from a crisp set X . Let (α, k) and (β, l) be two \mathbb{Q}^+ -superm-elements of the \mathbb{Q}^+ -mset M . Then

- (i) $(\alpha, k) = (\beta, l)$ if and only if $\alpha = \beta$ and $k = l$.
- (ii) (α, k) and (α, l) are said to be overlapping \mathbb{Q}^+ -superm-elements if $\gcd(k, l) \neq 1$.
- (iii) (α, k) is said to be a divisor/factor of (β, l) if and only if $\{(\alpha, k)\} \sqsubseteq \{(\beta, l)\}$ i.e., if and only if $\alpha = \beta$ and $\frac{l}{k} \in \mathbb{N}$. Then (β, l) is said to be a multiple of (α, k) .
- (iv) (α, k) and (α, l) are said to be coprime \mathbb{Q}^+ -superm-elements if $\gcd(k, l) = 1$
- (v) (α, k) and (β, l) are said to be distinct \mathbb{Q}^+ -superm-elements of M if and only if $\alpha \neq \beta$.

5.21. *Remark*

Two \mathbb{Q}^+ -subm-elements of an \mathbb{Q}^+ -mset M is either distinct or one is a factor of the other or they are coprime or they are overlapping. similarly, two \mathbb{Q}^+ -superm-elements of an mset M is either distinct or one is a factor of the other, or they are coprime or they are overlapping.

5.22. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$. Consider the \mathbb{Q}^+ -mset $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$ drawn from X . Then $(c, \frac{1}{4}) \in^2 P$, $(b, 9) \in^1 P$, and also $(a, 4) \in^{\frac{1}{2}} P$. For two \mathbb{Q}^+ -subm-elements $(b, 1)$ and $(b, 3)$ of P , $(b, 3)$ is a multiple of $(b, 1)$. Also, for two \mathbb{Q}^+ -superm-elements $(b, 39)$ and $(b, 18)$ of M , $(b, 36)$ is a multiple of $(b, 18)$. $(c, \frac{1}{4})$ and $(b, 3)$ are two distinct \mathbb{Q}^+ -subm-elements of M , also, $(b, 18)$ and $(e, 24)$ are two distinct \mathbb{Q}^+ -superm-elements of M .

5.23. *Proposition*

Any collection of \mathbb{Q}^+ -subm-elements of an \mathbb{Q}^+ -mset generates a \mathbb{Q}^+ -subset of the \mathbb{Q}^+ -mset.

Proof:

Let M be an \mathbb{Q}^+ -mset drawn from the crisp set X . Let $\Omega'(M)$ be a subset of $\Omega(M)$. Consider the \mathbb{Q}^+ -mset $P = \sqcup_{(x,k) \in \Omega'(M)} \{(x, k)\}$. Now we show that $P \sqsubseteq M$. Immediately, $P^* \subseteq M^*$. Now, let $x \in P^*$. Then $\chi_P(x) = lcm\{k \in \mathbb{Q}^+ : (x, k) \in \Omega'(M)\}$, which exists since, for all $(x, k) \in \Omega'(M)$, $\frac{\chi_M(x)}{k} \in \mathbb{N}$. Now if $(x, k) \in \Omega'(M) \subseteq \Omega(M)$, then $\frac{\chi_M(x)}{k} \in \mathbb{N}$, so $\frac{\chi_M(x)}{\chi_P(x)} \in \mathbb{N}$ which holds for all $x \in P^*$. Therefore, $P \sqsubseteq M$. Hence, the result.

5.24. *Remark*

The \mathbb{Q}^+ -multiset generated by a \mathbb{Q}^+ -general mset A of some \mathbb{Q}^+ -subm-elements of a \mathbb{Q}^+ -multiset M is denoted by $\gamma(A)$ and is defined by $\gamma(A) = \sqcup_{(x,k) \in A} \{(x, k)\}$ where $\chi_{\gamma(A)}(x) = lcm\{k \in \mathbb{Q}^+ : (x, k) \in A\}$.

Also a \mathbb{Q}^+ -mset can be generated from the \mathbb{Q}^+ -general mset of all its \mathbb{Q}^+ -subm-elements. If $\Omega(M)$ is the \mathbb{Q}^+ -general mset of all its \mathbb{Q}^+ -subm-elements, then $\gamma(\Omega(M)) = M$ where $\chi_{\gamma(\Omega(M))}(x) = lcm\{k \in \mathbb{Q}^+ : (x, k) \in \Omega(M)\}$.

5.25. *Remark*

Let P and T be two \mathbb{Q}^+ -msets drawn from crisp sets X . Then $P \sqsubseteq T$ if and only if every \mathbb{Q}^+ -subm-element of P is a \mathbb{Q}^+ -subm-element of T . Also, $P = T$ if and only if every \mathbb{Q}^+ -subm-element of P is a \mathbb{Q}^+ -subm-element of T as well as every \mathbb{Q}^+ -subm-element of T is a \mathbb{Q}^+ -subm-element of P .

5.26. *Proposition*

Every \mathbb{Q}^+ -subm-element of \mathbb{Q}^+ -m-intersection of two \mathbb{Q}^+ -msets is a \mathbb{Q}^+ -subm-element of each of the sets and conversely.

Proof:

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X .

Then for $x \in P^* \cap T^*$, $(x, k) \in^r (P \cap T)$ for some $k \in \mathbb{Q}^+$ and $r \in \mathbb{N} \Rightarrow \frac{\chi_{P \cap T}(x)}{k} = r \in \mathbb{N} \Rightarrow \frac{gcd\{\chi_P(x), \chi_T(x)\}}{k} = r \in \mathbb{N} \Rightarrow \frac{\chi_P(x)}{k} = \frac{\chi_P(x)}{\chi_{P \cap T}(x)} r = \frac{\chi_P(x)}{gcd\{\chi_P(x), \chi_T(x)\}} r \in \mathbb{N}$
 and $\frac{\chi_T(x)}{k} = \frac{\chi_T(x)}{\chi_{P \cap T}(x)} r = \frac{\chi_T(x)}{gcd\{\chi_P(x), \chi_T(x)\}} r \in \mathbb{N}$
 $\Rightarrow (x, k) \in^{\frac{\chi_P(x)}{\chi_{P \cap T}(x)} r} P$ and $(x, k) \in^{\frac{\chi_T(x)}{\chi_{P \cap T}(x)} r} T$.

Conversely, for $x \in P^* \cap T^*$, $(x, k) \in^i P$ and $(x, k) \in^j T$ for some $k \in \mathbb{Q}^+$ and $i, j \in \mathbb{N} \Rightarrow \frac{\chi_P(x)}{k} = i \in \mathbb{N}$ and $\frac{\chi_T(x)}{k} = j \in \mathbb{N} \Rightarrow \frac{gcd\{\chi_P(x), \chi_T(x)\}}{k} \in \mathbb{N}$, $\frac{gcd\{\chi_P(x), \chi_T(x)\}}{k} = gcd\{i, j\} = r$ say

$\Rightarrow (x, k) \in^r (P \sqcap T)$.

So, every \mathbb{Q}^+ -subm-element of \mathbb{Q}^+ -m-intersection of two \mathbb{Q}^+ -msets is a \mathbb{Q}^+ -subm-element of each of the sets and conversely.

5.27. *Remark*

Let P and T be two \mathbb{Q}^+ -msets drawn from a crisp set X . Then, for $x \in P^* \cap T^*$, $(x, k) \in^i P$ and $(x, k) \in^j T$ for some $k \in \mathbb{Q}^+$ and $i, j \in \mathbb{N} \Rightarrow \frac{\chi_P(x)}{k} = i \in \mathbb{N}$ and $\frac{\chi_T(x)}{k} = j \in \mathbb{N} \Rightarrow \frac{lcm\{\chi_P(x), \chi_T(x)\}}{k} \in \mathbb{N}$, $\frac{lcm\{\chi_P(x), \chi_T(x)\}}{k} = lcm\{i, j\} = r$ say $\Rightarrow (x, k) \in^r (P \sqcup T)$.

Also, for $x \in P^* - T^*$, $(x, k) \in^r P \Rightarrow (x, k) \in^r P \sqcup T$ and for $x \in T^* - P^*$, $(x, k) \in^r T \Rightarrow (x, k) \in^r P \sqcup T$.

Therefore, if (x, k) is a \mathbb{Q}^+ -subm-element of P or T , then (x, k) must be a \mathbb{Q}^+ -subm-element of $P \sqcup T$.

But the converse is not always true.

5.28. *Example*

Let $A = \{(a, 24)\}$ and $B = \{(a, 30)\}$. Then $A \sqcup B = \{(a, 120)\}$. So, $(a, 20)$ is a \mathbb{Q}^+ -subm-element of $A \sqcup B$ but neither a \mathbb{Q}^+ -subm-element of A nor of B .

5.29. *Remark*

Let A and B be two \mathbb{N} -msets drawn from a crisp sets X . Then for $x \in A^* \cup B^*$, (x, k) is a \mathbb{N} -subm-element of $A \sqcup B \Rightarrow$ there exists $p, q \in \mathbb{N}$ with $gcd\{p, q\} = 1$ and $p \cdot q = k$ such that (x, p) is a \mathbb{N} -subm-element of A and (x, q) is a \mathbb{N} -subm-element of B .

5.30. *Remark*

$P \sqcap T = \phi$ if and only if $P^* \cap T^* = \phi$.

5.31. *Properties*

For three non-empty msets P, S and T drawn from a crisp set X ,

- (i) Three statements (a) $S \sqsubseteq T$ (b) $S \sqcap T = S$ and (c) $S \sqcup T = T$ are equivalent. (ii) $S \sqcup S = S$, $S \sqcap S = S$, (iii) $S \sqcup \phi = S$, $S \sqcap \phi = \phi$, (iv) $S \sqcup (S \sqcap T) = S$, $S \sqcap (S \sqcup T) = S$, (iv) $S \sqsubseteq S \sqcup T$, $S \sqcap T \sqsubseteq S$, (v) $S \sqcup T = T \sqcup S$, $S \sqcap T = T \sqcap S$, (vi) $P \sqcup (S \sqcup T) = (P \sqcup T) \sqcup S$, $P \sqcap (S \sqcap T) = (P \sqcap T) \sqcap S$. Proof: The proof of (i) to (vi) is immediate.

5.32. *Definition*

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Then the set of all \mathbb{Q}^+ -submsets of M is called the \mathbb{Q}^+ -power mset of M and is denoted by $P(M)$. \mathbb{Q}^+ -Power mset of an \mathbb{Q}^+ -mset is always an infinite crisp set.

Let X be a non-empty crisp set. Let M be a \mathbb{N}^+ -msets drawn from X . Then the set of all \mathbb{N} -submets of M is called the \mathbb{N} -power mset of M and is denoted by $P^*(M)$.

5.33. *Definition*

Let M_δ be an \mathbb{Q}^+ -mset drawn from a crisp set X for all $\delta \in \Delta$. Then \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -m-union of $M_\delta, \delta \in \Delta$, denoted by $M = \sqcup_{\delta \in \Delta} M_\delta$, if and only if

- (i) $M^* = \cup_{\delta \in \Delta} M_\delta^*$,
- (ii) $lcm\{\chi_{M_\delta}(x) : x \in M_\delta^*\}$ exists for all $x \in M^*$,
- (iii) $\chi_M(x) = lcm\{\chi_{M_\delta}(x) : x \in M_\delta^*\}, x \in M^*$.

5.34. *Definition*

Let M_δ be an \mathbb{Q}^+ -mset drawn from a crisp set X for all $\delta \in \Delta$. Then \mathbb{Q}^+ -mset M drawn from X is said to be the \mathbb{Q}^+ -m-intersection of $M_\delta, \delta \in \Delta$, denoted by $M = \sqcap_{\delta \in \Delta} M_\delta$, if and only if

- (i) $M^* = \sqcap_{\delta \in \Delta} M_\delta^*$,
- (ii) $gcd\{\chi_{M_\delta}(x) : x \in M_\delta^*\}$ exists for all $x \in M^*$,
- (iii) $\chi_M(x) = gcd\{\chi_{M_\delta}(x) : x \in M_\delta^*\}, x \in M^*$.

5.35. *Remark*

1. If (x, k) is a \mathbb{Q}^+ -subm-element of M_δ for some $\delta \in \Delta$, then (x, k) must be a \mathbb{Q}^+ -subm-element of $\sqcup_{\delta \in \Delta} M_\delta$ but not conversely.
2. If (x, k) is a \mathbb{Q}^+ -subm-element of M_δ for all $\delta \in \Delta$, then (x, k) must be a \mathbb{Q}^+ -subm-element of $\sqcap_{\delta \in \Delta} M_\delta$ and conversely.

5.36. *Definition*

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.

Let $gcd\{k : k \in I^\#\}$ not exist, then we define $\chi_{\sqcap_{k \in I^\#} \{(x,k)\}}(x)$ to be 0.

5.37. Remark

Let M be a non-empty mset drawn from a crisp set X . Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.

Then $\frac{\chi_M(x)}{\chi_{\sqcup_{k \in I^\#} \{(x,k)\}}(x)} \in \mathbb{N}$ or $C_{\sqcap_{k \in I^\#} \{(x,k)\}}(x) = 0$ according as $gcd\{k : k \in I^\#\}$ exists or does not exist.

5.38. Remark

Let M be a non-empty mset drawn from a crisp set X . Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.

Then, $\frac{\chi_M(x)}{\chi_{\sqcup_{k \in I^\#} \{(x,k)\}}(x)} \in \mathbb{N}$.

5.39. Proposition

Let M be an \mathbb{Q}^+ -mset drawn from a crisp set X . Let $\Omega(M)$ be the set of all \mathbb{Q}^+ -subelements of M . Let $\Omega'(M) \subseteq \Omega(M)$. Then following are true:

1. $\sqcup_{(x,k) \in \Omega'(M)} \{(x,k)\} \sqsubseteq M$,
2. $\sqcap_{(x,k) \in \Omega'(M)} \{(x,k)\}$ is either ϕ or a non-trivial \mathbb{Q}^+ -subset of M .

5.40. Proposition

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Then $(P(M), \sqsubseteq)$ is a lattice.

5.41. Proposition

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Then $(P(M), \sqsubseteq)$ is a complete lattice.

5.42. Proposition

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Then $(P(M), \sqsubseteq)$ is a distributive lattice.

Proof:

Let M be a non-empty \mathbb{Q}^+ -mset drawn from a crisp set X . Let A, B, D be three \mathbb{Q}^+ -subsets of M .

Now we will show that $A \sqcup (B \sqcap D) = (A \sqcup B) \sqcap (A \sqcup D)$.

Immediately, $[A \sqcup (B \sqcap D)]^* = [(A \sqcup B) \sqcap (A \sqcup D)]^*$.

Now, for all $x \in [A \sqcup (B \sqcap D)]^*$, $\chi_{A \sqcup (B \sqcap D)}(x) = lcm\{\chi_A(x), gcd\{\chi_B(x), \chi_D(x)\}\}$
 $= gcd(\{lcm\{\chi_A(x), \chi_B(x)\}, lcm\{\chi_A(x), \chi_D(x)\}\}) = \chi_{(A \sqcup B) \sqcap (A \sqcup D)}(x)$.

Therefore, $A \sqcup (B \sqcap D) = (A \sqcup B) \sqcap (A \sqcup D)$.

Similarly, we can show that $A \sqcap (B \sqcup D) = (A \sqcap B) \sqcup (A \sqcap D)$.

Therefore, Then $(P(M), \sqsubseteq)$ is a distributive lattice.

5.43. *Remark*

If we consider a mset of real numbers, where each real number appears with a specific multiplicity, and if we consider the collection of all the sub-elements (elements along with their respective multiplicities) of that mset, then this inherently reflects both the values and their multiplicities. To accommodate this, we define a new system called the *multi-real number system* (see Section 4), which extends the classical real number system (discussed in Remark 4.33) by including multiplicity as a structural component. This extension allows operations and comparisons to be interpreted in a way that respects and incorporates multiplicities (as described in Remark 4.33).

6. **Multi-metric space**

In this section, we prefer to write a multi-real number in the form R_a^b rather than (a, b) where $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$.

6.1. *Definition*

Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let ρ be a metric on X and $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a mapping. Then a mapping $d : \Omega^*(M) \times \Omega^*(M) \rightarrow m^*(\mathbb{R})$, [$m^*(\mathbb{R}) = m^+(\mathbb{R}) \cup m_0^*(\mathbb{R})$, i.e., $m^*(\mathbb{R}) = \{R_a^b : a \geq 0, b \in \mathbb{N}\}$], where for all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = R_{\rho(x,y)}^{f(i,j)}$, is said to be a **multi-metric** on the \mathbb{N} -mset M if d satisfies the following conditions:

(M1) For $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) \succeq R_0^1$.

(M2) For $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) \in m_0^*(\mathbb{R}) = \{R_a^b \in m(\mathbb{R}) : a = 0 \text{ and } b \in \mathbb{N}\}$.

if and only if $x = y$.

(M3) For all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = d((y, j), (x, i))$.

(M4) For all $(x, i), (y, j), (z, k) \in \Omega^*(M)$, $d((x, i), (y, j)) \preceq d((x, i), (z, k)) \oplus d((z, k), (y, j))$.

The \mathbb{N} -mset M together with a multi-metric d defined on M is said to be a **multi-metric space** and is denoted by (M, d) . (M1), (M2), (M3), and (M4) are said to be multi-metric axioms.

6.2. *Example*

Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let us define a mapping $d : \Omega^*(M) \times \Omega^*(M) \rightarrow m^*(\mathbb{R})$ by $d((x, i), (y, j)) = R_0^1$ if $x = y$ and $d((x, i), (y, j)) = R_1^1$ if $x \neq y$,

$(x, i), (y, j) \in \Omega^*(M)$. Then d satisfies all the multi-metric axioms. So, d is a multi-metric on the \mathbb{N} -mset M .

6.3. *Example*

Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let us define a mapping $d : \Omega^*(M) \times \Omega^*(M) \rightarrow m^*(\mathbb{R})$ by $d((x, i), (y, j)) = R_0^{\frac{\chi_M(x) \cdot \chi_M(y)}{i \cdot j}}$ if $x = y$ and $d((x, i), (y, j)) = R_1^{\frac{\chi_M(x) \cdot \chi_M(y)}{i \cdot j}}$ if $x \neq y, (x, i), (y, j) \in \Omega^*(M)$. Then d satisfies all the multi-metric axioms. So, d is a multi-metric defined on the \mathbb{N} -mset M .

6.4. *Example*

Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let ρ be a crisp metric on M^* . Let us define a mapping $d^\rho : \Omega^*(M) \times \Omega^*(M) \rightarrow m^*(\mathbb{R})$ by $d^\rho((x, i), (y, j)) = R_{\rho(x,y)}^{i \cdot j}$, $(x, i), (y, j) \in \Omega^*(M)$. Then d^ρ satisfies all the multi-metric axioms. So, d^ρ is a multi-metric on the \mathbb{N} -mset M .

6.5. *Example*

Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let ρ be a crisp metric on M^* . Let us define a mapping $d_\rho : \Omega^*(M) \times \Omega^*(M) \rightarrow m^*(\mathbb{R})$ by $d_\rho((x, i), (y, j)) = R_{\rho(x,y)}^{lcm\{i,j\}}$, $(x, i), (y, j) \in \Omega^*(M)$. Then d_ρ satisfies all the multi-metric axioms. So, d_ρ is a multi-metric on the \mathbb{N} -mset M .

6.6. *Example*

Consider the \mathbb{N} -mset $M = \{(5, 2), (10, 1)\}$ drawn from \mathbb{R} .

Consider the usual metric ρ on \mathbb{R} .

Here, $\Omega^*(M) = \{(5, 2), (5, 1), (10, 1)\}$.

Then $d^\rho((5, 2), (5, 2)) = R_0^4$,

$d^\rho((5, 2), (5, 1)) = R_0^2 = d^\rho((5, 1), (5, 2))$,

$d^\rho((5, 2), (10, 1)) = R_5^2 = d^\rho((10, 1), (5, 2))$, $d^\rho((5, 1), (5, 1)) = R_0^1$,

$d^\rho((5, 1), (10, 1)) = R_5^1 = d^\rho((10, 1), (5, 1))$, $d^\rho((10, 1), (10, 1)) = R_0^1$,

Also, $d_\rho((5, 2), (5, 2)) = R_0^2$, $d_\rho((5, 2), (5, 1)) = R_0^2 = d_\rho((5, 1), (5, 2))$,

$d_\rho((5, 2), (10, 1)) = R_5^2 = d_\rho((10, 1), (5, 2))$, $d_\rho((5, 1), (5, 1)) = R_0^1$,

$d_\rho((5, 1), (10, 1)) = R_5^1 = d_\rho((10, 1), (5, 1))$, $d_\rho((10, 1), (10, 1)) = R_0^1$.

6.7. *Definition*

Let (M, d) be a multi-metric space and P be a non-null \mathbb{N} -subset of the \mathbb{N} -mset M . Then the mapping $d_P : \Omega^*(P) \times \Omega^*(P) \rightarrow m^*(\mathbb{R})$ given by $d_P((x, i), (y, j)) = d((x, i), (y, j))$ for all $(x, i), (y, j) \in \Omega^*(P)$ is a multi-metric on P . This multi-metric d_P is called the relative multi-metric induced on P by d . The multi-metric space (P, d_P) is called a **metric subspace** or **sub multi-metric space** of the multi-metric space (M, d) .

6.8. *Example*

Consider the \mathbb{N} -mset $M = \{(5, 2), (10, 1)\}$ drawn from \mathbb{R} . Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.

Here, $\Omega^*(M) = \{(5, 2), (5, 1), (10, 1)\}$. Then $d^\rho((5, 2), (5, 2)) = R_0^4$, $d^\rho((5, 2), (5, 1)) = R_0^2 = d^\rho((5, 1), (5, 2))$, $d^\rho((5, 2), (10, 1)) = R_5^2 = d^\rho((10, 1), (5, 2))$, $d^\rho((5, 1), (5, 1)) = R_0^1$, $d^\rho((5, 1), (10, 1)) = R_5^1 = d^\rho((10, 1), (5, 1))$, $d^\rho((10, 1), (10, 1)) = R_0^1$. Consider \mathbb{N} -subset $P = \{(5, 2)\}$ of M . Then $\Omega^*(P) = \{(5, 2), (5, 1)\}$. Consider the mapping $d_P^\rho : \Omega^*(P) \times \Omega^*(P) \rightarrow m^*(R)$ given by $d_P^\rho((5, 2), (5, 2)) = R_0^4$, $d_P^\rho((5, 2), (5, 1)) = R_0^2 = d_P^\rho((5, 1), (5, 2))$, $d_P^\rho((5, 1), (5, 1)) = R_0^1$. Then d_P^ρ is a multi-metric on P . Therefore, the multi-metric space (P, d_P^ρ) is a metric subspace of (M, d_P^ρ) .

6.9. *Definition*

Let (M, d) be a multi-metric space and P be a non-null \mathbb{N} -subset of the \mathbb{N} -mset M . Then the **diameter** of P is denoted by $\delta(P)$ and is defined by $\delta(P) = \text{lub}\{d_P((x, i), (y, j)) : (x, i), (y, j) \in \Omega^*(P)\}$, provided lub exists [As defined earlier, $\text{lub}\{R_p^q \in m(R) : p \in S \subseteq R, q \in T \subseteq Q^+\} = R_\mu^\nu$, where $\mu = \text{lub}\{p : p \in S \subseteq R\}$ in the poset (R, \leq) and $\nu = \text{lcm}\{q : q \in T \subseteq Q^+\}$, if they exist]. It is obvious that for any non-null \mathbb{N} -subset P of the \mathbb{N} -mset M , $\delta(P) \succeq R_0^1$, if it exist. If P^* is a finite set, then $\delta(P)$ must exists.

6.10. *Example*

Consider the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Consider the usual metric ρ on $M^* \subseteq R$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the \mathbb{N} -subset $P = \{(7, 1), (-1, 5)\}$ of M . Consider the relative multi-metric d_P^ρ induced on P by d^ρ . Here, $\Omega^*(P) = \{(7, 1), (-1, 5), (-1, 1)\}$.

Then the mapping $d_P^\rho : \Omega^*(P) \times \Omega^*(P) \rightarrow m^*(R)$ is given by

$$d_P^\rho((7, 1), (7, 1)) = R_0^1, d_P^\rho((7, 1), (-1, 5)) = R_8^5 = d_P^\rho((-1, 5), (7, 1)),$$

$$d_P^\rho((7, 1), (-1, 1)) = R_8^1 = d_P^\rho((-1, 1), (7, 1)), d_P^\rho((-1, 5), (-1, 5)) = R_0^{25},$$

$d_P^p((-1, 5), (-1, 1)) = R_0^5 = d_P^p((-1, 1), (-1, 5)), d_P^p((-1, 1), (-1, 1)) = R_0^1$.
 $lub\{0, 8\} = 8, lcm\{1, 5, 25\} = 25$. Then $\delta(P) = R_8^{25}$.

6.11. *Remark*

Let (M, d) be a multi-metric space and P be a non-null \mathbb{N} -subset of the \mathbb{N} -mset M . Let the diameter of $P = \delta(P)$, say, exist. Then, for all $(x, i), (y, j) \in \Omega^*(P)$, $d((x, i), (y, j)) \leq \delta(P)$.

6.12. *Remark*

Let (M, d) be a multi-metric space and P be a \mathbb{N} -subset of the \mathbb{N} -mset M . Then

(i) $\delta(P) \in m_0^*(\mathbb{R})$ if and only if P is a component of M .

(ii) For every non-null \mathbb{N} -subsets P and Q of M , $P \subseteq Q \Rightarrow \delta(P) \leq \delta(Q)$, if $\delta(P)$ and $\delta(Q)$ both exist.

6.13. *Definition*

Let (M, d) be a multi-metric space. Let (α, k) be a fixed \mathbb{N} -subm-element of the \mathbb{N} -mset M satisfying $\alpha \in M^*, k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$ [i.e., $\alpha \in \frac{C_M(\alpha)}{k} M$] and P be a non-null \mathbb{N} -subset of M . Then **distance** between the \mathbb{N} -subm-element (α, k) and the subset P is denoted by $\delta((\alpha, k), P)$ and is defined by $\delta((\alpha, k), P) = glb\{d((\alpha, k), (x, l)) : (x, l) \in \Omega^*(P)\}$. [As defined earlier, $glb\{R_p^q \in m(\mathbb{R}) : p \in S \subseteq R, q \in T \subseteq Q^+\} = R_\mu^\lambda$, where $\mu = glb\{p : p \in S \subseteq R\}$ in the poset (R, \leq) and $\lambda = gcd\{q : q \in T \subseteq Q^+\}$, if they exist]. It is obvious that for any non-null \mathbb{N} -subset P of the \mathbb{N} -mset M and for any \mathbb{N} -subm-element (α, k) of the \mathbb{N} -mset M satisfying $\alpha \in M^*, k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$, $\delta((\alpha, k), P) \succeq R_0^t$ for some $t \in \mathbb{N}$. If P be a natural \mathbb{N} -subset of the \mathbb{N} -mset M such that P^* is finite, then for any \mathbb{N} -subm-element (α, k) of M , $\delta((\alpha, k), P)$ exists.

If (α, k) be a fixed \mathbb{N} -subm-element of the \mathbb{N} -mset P , then $\delta((\alpha, k), P) = R_0^t$ for some $t \in \mathbb{N}$. On the other hand $\delta((\alpha, k), P) = R_0^t$ for some $t \in \mathbb{N}$ may hold where (α, k) is not a \mathbb{N} -subm-element of P which can be justified by the following example.

6.14. *Example*

Consider the \mathbb{N} -mset $M = \{(\frac{1}{2n}, 6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1}, 4) : n \in \mathbb{N}\} \cup \{(0, 2)\} \cup \{(n, n) : n \in \mathbb{N}\}$ drawn from \mathbb{R} . Consider the usual metric ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the \mathbb{N} -subm-element $(0, 1)$ of M and the \mathbb{N} -subset $P = \{(\frac{1}{2n}, 6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1}, 4) : n \in \mathbb{N}\}$ of M . Then $\delta((0, 1), P) = R_0^4$, but $(0, 1)$ is not a \mathbb{N} -subm-element of P .

6.15. *Example*

Consider the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric τ on M^* as in example 6.4. Consider the \mathbb{N} -subset $P = \{(7, 1), (-1, 5), (0, 3), (8, 2)\}$ of M . Consider the \mathbb{N} -subm-element $(8, 3)$ of the \mathbb{N} -mset M . In fact, $(8, 3) \in^4 M$. Then $\delta((8, 3), P) = glb\{d((8, 3), (x, i)) : x \in \Omega^*(P)\}$, provided that the glb exists $= glb\{R_1^3, R_9^{15}, R_9^3, R_8^9, R_8^3, R_0^6, R_0^3\} = R_0^3$ (since $glb\{1, 9, 8, 0\} = 0$ and $gcd\{3, 15, 9, 6\} = 3$). Therefore, $\delta((8, 3), P) = R_0^3$ also, $(8, 3)$ is a \mathbb{N} -subm-element of P .

6.16. *Example*

Consider the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ on M^* as in Example 6.4. If we consider $S = \{(7, 1), (-1, 5), (0, 3)\} \sqsubseteq M$ and $(8, 3) \in^4 M$, then $\delta((8, 3), S) = glb\{R_1^3, R_9^{15}, R_9^3, R_8^9, R_8^3\} = R_1^3$. Here, $(8, 3)$ is not a \mathbb{N} -subm-element of S .

6.17. *Definition*

Let (M, d) be a multi-metric space and P, T be two non-null \mathbb{N} -subsets of the \mathbb{N} -mset M . The distance between the \mathbb{N} -msets P and T is denoted by $\delta(P, T)$ and defined by $\delta(P, T) = glb\{d((x, i), (y, j)) : (x, i) \in \Omega^*(P), (y, j) \in \Omega^*(T)\}$.

Immediately, for any two non-null \mathbb{N} -subsets P and T of the \mathbb{N} -mset M , $\delta(P, T) = \delta(T, P)$. Also, for any \mathbb{N} -subm-element (α, k) of the \mathbb{N} -mset M and for any non-null \mathbb{N} -subset P of M , $\delta((\alpha, k), P) = \delta(\{(\alpha, k)\}, P)$.

Again, for any two non-null \mathbb{N} -subsets P and T of the \mathbb{N} -mset M , if $P \cap T \neq \phi$, then $\delta(P, T) = R_0^t$ for some $t \in \mathbb{N}$.

However, the converse is not necessarily true. It may happen that, for two non-null \mathbb{N} -subsets P and T of the \mathbb{N} -mset M , $\delta(P, T) = R_0^t$ for some $t \in \mathbb{N}$ but $P \cap T = \phi$. This can be justified by the following example.

6.18. *Example*

Consider the \mathbb{N} -mset $M = \{(\frac{1}{2n}, 6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1}, 4) : n \in \mathbb{N}\} \cup \{(0, 2)\} \cup \{(n, n) : n \in \mathbb{N}\}$ drawn from \mathbb{R} . Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the \mathbb{N} -subset $P = \{(\frac{1}{2n+1}, 4) : n \in \mathbb{N}\}$ and the \mathbb{N} -subset $T = \{(\frac{1}{2n}, 6) : n \in \mathbb{N}\}$ of M . Then $\delta(P, T) = R_0^{24}$, but $P \cap T = \phi$.

6.19. *Example*

Consider the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the \mathbb{N} -subset $P = \{(7, 3), (6, 1)\}$ and $T = \{(-1, 1), (0, 3)\}$ of M . Then $\delta(P, T) = glb\{R_8^3, R_7^1, R_7^9, R_6^3, R_7^3, R_8^1, R_6^1\} = R_6^1$.

6.20. *Definition*

Let (M, d) be a multi-metric space. For two \mathbb{N} -subm-elements (α, k) and (β, l) of the \mathbb{N} -mset M , the distance between (α, k) and (β, l) is defined as the diameter of the \mathbb{N} -subset $\{(\alpha, k), (\beta, l)\}$ of M , denoted by $\delta((\alpha, k), (\beta, l))$.

6.21. *Property*

Let M be a \mathbb{N} -mset drawn from the crisp set X . Let d be a multi-metric defined on M . Let (α, k) and (β, l) be two \mathbb{N} -subm-elements of the \mathbb{N} -mset M .

If (α, k) and (β, l) are distinct subm-elements of M , then $\delta((\alpha, k), (\beta, l)) \in m^+(\mathbb{R})$.

6.22. *Definition*

Let (M, d) be a multi-metric space. Let (α, k) be a \mathbb{N} -subm-element of the \mathbb{N} -mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$ and $R_p^q \in m^+(\mathbb{R})$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$. Let us define $B^*((\alpha, k), R_p^q) = \{(x, l) \in \Omega^*(M) : d((\alpha, k), (x, l)) < R_p^q \text{ and } \frac{q}{f(k,l) \cdot f(l,l)} \in \mathbb{N}\}$. Then $B^*((\alpha, k), R_p^q)$ is called a **open ball** with centre at (α, k) and radius R_p^q . $B^*((\alpha, k), R_p^q)$ is a general mst.

Also, define $B((\alpha, k), R_p^q) = \bigsqcup_{(x,l) \in B^*((\alpha,k), R_p^q)} \{(x, l)\}$.

Then $B((\alpha, k), R_p^q)$ is called a **multi-open ball** (or **m-open ball**) with centre at (α, k) and radius R_p^q . $B((\alpha, k), R_p^q)$ is a \mathbb{N} -mset.

Immediately, (α, k) is an element of $B^*((\alpha, k), R_p^q)$ and also a \mathbb{N} -subm-element of $B((\alpha, k), R_p^q)$. Therefore, if $B^*((\alpha, k), R_p^q)$ is an open ball with centre at (α, k) and radius R_p^q , then $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

6.23. *Example*

Consider the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (5, 2), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Then $\Omega^*(M) = \{(7, 1), (7, 3), (6, 1), (6, 2), (-1, 1), (-1, 5), (5, 1), (5, 2), (0, 1), (0, 3), (0, 4), (8, 1), (8, 2), (8, 3), (8, 4), (8, 6), (8, 12)\}$. Consider the usual metric space ρ defined on $M^* \subseteq R$. Consider the multi-metric d^ρ defined on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.

Consider the \mathbb{N} -subm-element $(8, 3) \in^4 M$.

Then $B^*((8, 3), R_2^{81}) = \{(7, 1), (7, 3), (8, 1), (8, 3)\}$.

$B((8, 3), R_2^{81}) = \{(7, 3), (8, 3)\}$.

$B^*((8, 3), R_2^{648}) = \{(7, 1), (7, 3), (8, 1), (8, 2), (8, 3), (8, 6)\}$.

$B((8, 3), R_2^{648}) = \{(7, 3), (8, 6)\}$.

6.24. Remark

Let (M, d) be a multi-metric space. Let (α, k) be a \mathbb{N} -subm-element of the \mathbb{N} -mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$.

Then for $R_p^q, R_m^n \in m^+(\mathbb{R})$, $R_p^q \preceq R_m^n$ and $\frac{q}{[f(k,k)]^2} \in \mathbb{N} \Rightarrow B^*((\alpha, k), R_p^q) \subseteq B^*((\alpha, k), R_m^n)$.

6.25. Definition

Let (M, d) be a multi-metric space. Let (α, k) be a fixed \mathbb{N} -subm-element of the \mathbb{N} -mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$ [i.e., $\alpha \in \frac{C_M(\alpha)}{k} M$] and $R_p^q \in m^+(R)$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

Let us define $B^*[(\alpha, k), R_p^q] = \{(x, l) \in \Omega^*(M) : d((\alpha, k), (x, l)) \preceq R_p^q \text{ and } \frac{q}{f(k,l) \cdot f(l,l)} \in \mathbb{N}\}$.

Then $B^*[(\alpha, k), R_p^q]$ is called a **closed ball** with centre at (α, k) and radius R_p^q . $B^*[(\alpha, k), R_p^q]$

is a general mset. Also, define $B[(\alpha, k), R_p^q] = \bigsqcup_{(x,l) \in B^*[(\alpha,k),R_p^q]} \{(x, l)\}$.

Then $B[(\alpha, k), R_p^q]$ is called a **multi-closed ball** (or **m-closed ball**) with centre at (α, k) and radius R_p^q . $B[(\alpha, k), R_p^q]$ is a \mathbb{N} -mset. Immediately, (α, k) is a \mathbb{N} -element of $B^*[(\alpha, k), R_p^q]$ and also a \mathbb{N} -subm-element of $B[(\alpha, k), R_p^q]$.

Therefore, if $B^*[(\alpha, k), R_p^q]$ is a closed ball with centre at (α, k) and radius R_p^q , then $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

6.26. Example

Consider

the \mathbb{N} -mset $M = \{(7, 3), (6, 2), (-1, 5), (5, 2), (0, 9), (8, 12)\}$ drawn from \mathbb{R} . Then $\Omega^*(M) = \{(7, 1), (7, 3), (6, 1), (6, 2), (-1, 1), (-1, 5), (5, 1), (5, 2), (0, 1), (0, 3), (0, 4), (8, 1), (8, 2), (8, 3), (8, 4), (8, 6), (8, 12)\}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.

Consider the multi-metric d^ρ defined on the \mathbb{N} -mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.

Consider the \mathbb{N} -subm-element $(8, 3) \in^4 M$.

Then $B^*[(8, 3), R_2^{81}] = \{(7, 1), (7, 3), (6, 1), (8, 1), (8, 3)\}$.

$B[(8, 3), R_2^{81}] = \{(7, 3), (6, 1), (8, 3)\}$.

$B^*[(8, 3), R_2^{648}] = \{(7, 1), (7, 3), (6, 1), (6, 2), (8, 1), (8, 2), (8, 3), (8, 6)\}$.

$$B[(8, 3), R_2^{648}] = \{(7, 3), (6, 2), (8, 6)\}.$$

6.27. Definition

Let (M, d) be a multi-metric space. Then (M, d) is said to have **Hausdorff property**, if for any two distinct \mathbb{N} -subm-elements (α, k) and (β, l) of the \mathbb{N} -mset M , there exist two open balls $B^*((\alpha, k), R_p^q)$ centred at (α, k) and $B^*((\beta, l), R_s^t)$ centred at (β, l) such that $B^*((\alpha, k), R_p^q) \cap B^*((\beta, l), R_s^t) = \phi$.

6.28. Proposition

Every multi-metric space is Hausdorff.

Proof:

Let (M, d) be a multi-metric space. Let (α, k) and (β, l) be two distinct \mathbb{N} -subm-elements of M . Let $\delta((\alpha, k), (\beta, l)) = R_p^q \in m^+(\mathbb{R})$. Let us consider open balls $B^*((\alpha, k), R_{\frac{p}{2}}^m)$ centred at (α, k) and $B^*((\beta, l), R_{\frac{n}{2}}^n)$ centred at (β, l) where $\frac{m}{[f(k,k)]^2}, \frac{n}{[f(l,l)]^2} \in \mathbb{N}$. Then $B^*((\alpha, k), R_{\frac{p}{2}}^m) \cap B^*((\beta, l), R_{\frac{n}{2}}^n) = \phi$.

Therefore, the multi-metric space (M, d) is Hausdorff.

6.29. Definition

Let (M, d) be a multi-metric space. Let (α, k) be a \mathbb{N} -subm-element of the \mathbb{N} -mset M . Let $\mathcal{N} \subseteq \Omega^*(M)$. i.e. \mathcal{N} is a collection of some \mathbb{N} -subm-elements of M . Then \mathcal{N} is said to be a **multi-neighbourhood** of the \mathbb{N} -subm-element (α, k) , if there exist $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{N}$.

6.30. Proposition

Let (M, d) be a multi-metric space. Let (α, k) be a \mathbb{N} -subm-element of M . Let \mathcal{A} and \mathcal{B} be two multi-neighbourhoods of (α, k) in M . Then $\mathcal{A} \cap \mathcal{B}$ is a multi-neighbourhood of (α, k) in M .

Proof:

Let (M, d) be a multi-metric space. Let (α, k) be a \mathbb{N} -subm-element of M . Since \mathcal{A} and \mathcal{B} are two multi-neighbourhoods of (α, k) , there exist $R_p^q, R_s^t \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2}, \frac{t}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{A}$ and $(\alpha, k) \in B^*((\alpha, k), R_s^t) \subseteq \mathcal{B}$. Let, $R_a^b = glb\{R_p^q, R_s^t\} \in m^+(\mathbb{R})$. Then $B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q), B^*((\alpha, k), R_s^t)$, and so $B^*((\alpha, k), R_a^b) \subseteq \mathcal{A}, \mathcal{B}$. Then $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq \mathcal{A} \cap \mathcal{B}$. Therefore, $\mathcal{A} \cap \mathcal{B}$ is a multi-neighbourhood of (α, k) .

6.31. *Definition*

Let P be a \mathbb{N} -subset of a multi-metric space (M, d) . Then $(\alpha, k) \in \Omega^*(P)$ is said to be an **interior element** of $\Omega^*(P)$ if there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \Omega^*(P)$.

6.32. *Definition*

Let P be a \mathbb{N} -subset of a multi-metric space (M, d) . Then the **interior** of $\Omega^*(P)$ is defined as the general mset $int(\Omega^*(P))$ that contains all the interior elements of $\Omega^*(P)$.

Also, define $P^o = \bigcup_{(x,l) \in int(\Omega^*(P))} \{(x, l)\}$ to be the **multi-interior** (or **m-interior**) of the \mathbb{N} -mset P .

6.33. *Proposition*

Let P and Q be two non-null \mathbb{N} -subsets of a multi-metric space (M, d) . Then

- (i) $int(\Omega^*(P)) \subseteq \Omega^*(P)$.
- (ii) $P \sqsubseteq Q \Rightarrow int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$.
- (iii) $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = int(\Omega^*(P) \cap \Omega^*(Q))$.
- (iv) $int(\Omega^*(P)) \cup int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cup \Omega^*(Q))$.

Proof: (i) The result immediately follows from the definition.

(ii) If $int(\Omega^*(P)) = \phi$, then $int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$. So consider the case where $int(\Omega^*(P)) \neq \phi$. Let $(\alpha, k) \in int(\Omega^*(P))$. Then (α, k) is an interior element of $\Omega^*(P)$. Then there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \Omega^*(P)$. Also, $P \sqsubseteq Q$. So $\Omega^*(P) \subseteq \Omega^*(Q)$. So, $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \Omega^*(P) \subseteq \Omega^*(Q)$. Therefore, (α, k) is an interior element of $\Omega^*(Q)$. So, $(\alpha, k) \in int(\Omega^*(Q))$. Therefore, $int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$.

(iii) If $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = \phi$, then $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$. So consider the case where $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \neq \phi$. Let $(\alpha, k) \in int(\Omega^*(P)) \cap int(\Omega^*(Q))$. Then $(\alpha, k) \in int(\Omega^*(P))$ and $(\alpha, k) \in int(\Omega^*(Q))$. So, (α, k) is an interior element of $\Omega^*(P)$ and $\Omega^*(Q)$ both. So, there exists $R_p^q, R_r^s \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ and $\frac{s}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \Omega^*(P)$ and $(\alpha, k) \in B^*((\alpha, k), R_r^s) \subseteq \Omega^*(Q)$. Let $min\{p, r\} = a$ and $gcd\{q, s\} = b$. Then $R_a^b \preceq R_p^q, R_r^s$ and $\frac{b}{[f(k,k)]^2} \in \mathbb{N}$. So, $B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q), B^*((\alpha, k), R_r^s)$. Then $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q) \subseteq \Omega^*(P)$ and $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_r^s) \subseteq \Omega^*(Q)$. Therefore, $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq \Omega^*(P) \cap \Omega^*(Q)$. Therefore, (α, k) is an interior element of $\Omega^*(P) \cap \Omega^*(Q)$. i.e. $(\alpha, k) \in int(\Omega^*(P) \cap \Omega^*(Q))$. Therefore, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$.

We have $\Omega^*(P) \cap \Omega^*(Q) \subseteq \Omega^*(P), \Omega^*(Q)$. So, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P)), int(\Omega^*(Q))$.

Therefore, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$.

Combining, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = int(\Omega^*(P) \cap \Omega^*(Q))$.

(iv) We have $\Omega^*(P), \Omega^*(Q) \subseteq \Omega^*(P) \cup \Omega^*(Q)$. So, $int(\Omega^*(P)), int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cup \Omega^*(Q))$. Therefore, $int(\Omega^*(P) \cup \Omega^*(Q)) \subseteq int(\Omega^*(P) \cup \Omega^*(Q))$.

6.34. *Definition*

Let (M, d) be a multi-metric space. Let $\mathcal{O} \subseteq \Omega^*(M)$, i.e. \mathcal{O} be a collection of some \mathbb{N} -subm-elements of M . Then $(\alpha, k) \in \mathcal{O}$ is said to be an **interior element** of \mathcal{O} if there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}$. Also, the **interior** of \mathcal{O} is defined as the general mset $int(\mathcal{O})$ that contains all the interior elements of \mathcal{O} .

Also, define $\mathcal{O}^o = \bigcup_{(x,l) \in int(\mathcal{O})} \{(x, l)\}$ to be the **multi-interior** (or **m-interior**) of the \mathbb{N} -general mset \mathcal{O} .

6.35. *Proposition*

Let (M, d) be a multi-metric space. Let $\mathcal{P}, \mathcal{Q} \subseteq \Omega^*(M)$, Then

- (i) $int(\mathcal{P}) \subseteq \mathcal{P}$.
- (ii) $\mathcal{P} \subseteq \mathcal{Q} \Rightarrow int(\mathcal{P}) \subseteq int(\mathcal{Q})$.
- (iii) $int(\mathcal{P}) \cap int(\mathcal{Q}) = int(\mathcal{P} \cap \mathcal{Q})$.
- (iv) $int(\mathcal{P}) \cup int(\mathcal{Q}) \subseteq int(\mathcal{P} \cup \mathcal{Q})$.

Proof: Proofs are similar to the proofs of Proposition 6.33.

6.36. *Definition*

Let (M, d) be a multi-metric space. Let $\mathcal{O} \subseteq \Omega^*(M)$, i.e. \mathcal{O} be a collection of some \mathbb{N} -subm-elements of M . Then \mathcal{O} is said to be an open general mset or simply open in (M, d) if all elements of \mathcal{O} are its interior elements.

6.37. *Proposition*

In a multi-metric space every open ball is open.

Proof: Let M be a \mathbb{N} -mset drawn from a non-empty crisp set X . Let ρ be a metric on X and $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a mapping. Let the mapping $d : \Omega^*(M) \times \Omega^*(M) \rightarrow m^+(\mathbb{R})$, where for all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = (\rho(x, y), f(i, j))$, is a multi-metric on the \mathbb{N} -mset M .

Let (α, k) be a \mathbb{N} -subm-element of the \mathbb{N} -mset M that satisfies $\alpha \in M^*, k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$.

Also, let $R_p^q \in m^+(\mathbb{R})$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

Consider the open ball $B^*((\alpha, k), R_p^q) = \{(x, l) \in \Omega^*(M) : d((\alpha, k), (x, l)) < R_p^q \text{ and } \frac{q}{f(k,l) \cdot f(l,l)} \in \mathbb{N}\}$ with centre at (α, k) and radius R_p^q .

Let $(\beta, n) \in B^*((\alpha, k), R_p^q)$.

Then $d((\alpha, k), (\beta, n)) = R_{\rho(\alpha, \beta)}^{f(k, n)} < R_p^q$.

Then $\rho(\alpha, \beta) < p$ and $\frac{q}{f(k, n)} \in \mathbb{N}$.

Now consider the open ball $B^*((\beta, n), R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}})$.

Let $(\gamma, t) \in B^*((\beta, n), R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}})$.

Then $d((\beta, n), (\gamma, t)) < R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}}$.

Now $d((\alpha, k), (\gamma, t)) \preceq d((\alpha, k), (\beta, n)) \oplus d((\beta, n), (\gamma, t)) < R_{\rho(\alpha, \beta)}^{f(k, n)} \oplus R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}} = R_p^q$.

Therefore, $(\gamma, t) \in B^*((\alpha, k), R_p^q)$.

Since (γ, t) is an arbitrary element of $B^*((\beta, n), R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}})$, so every element of $B^*((\beta, n), R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}})$ is an element of $B^*((\alpha, k), R_p^q)$.

Therefore, $(\beta, n) \in B^*((\beta, n), R_{p-\rho(\alpha, \beta)}^{\frac{q}{f(k, n)}}) \subseteq B^*((\alpha, k), R_p^q)$.

Therefore, (β, n) is an interior element of $B^*((\alpha, k), R_p^q)$.

Again, (β, n) is an arbitrary element of $B^*((\alpha, k), R_p^q)$, so every element of $B^*((\alpha, k), R_p^q)$ is its interior element.

Therefore, $B^*((\alpha, k), R_p^q)$ is an open set.

6.38. *Proposition*

In a multi-metric space (M, d) ,

- (i) ϕ is open;
- (ii) M is open;
- (iii) an arbitrary elementary union of open general msets is also open;
- (iv) an elementary intersection of two open general msets is also open.

Proof:

(i) ϕ contains no \mathbb{N} -subm-element of ϕ , so, ϕ is trivially open.

(ii) Also, M is immediately open.

(iii) Let \mathcal{O}_i be a non-null open general mset in (M, d) for all $i \in \Lambda$. If $\Lambda = \phi$, then $\bigcup_{i \in \Lambda} \mathcal{O}_i = \phi$,

which is open in (M, d) .

So, let $\Lambda \neq \phi$. Let $(\alpha, k) \in \bigcup_{i \in \Lambda} \mathcal{O}_i$.

Then $(\alpha, k) \in \mathcal{O}_j$ for some $j \in \Lambda$. Also, \mathcal{O}_j is an open general mset in (M, d) , so (α, k) is an interior element of \mathcal{O}_j .

So, there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k, k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}_j \subseteq \bigcup_{i \in \Lambda} \mathcal{O}_i$.

Therefore, (α, k) is an interior element of $\bigcup_{i \in \Lambda} \mathcal{O}_i$. Again, (α, k) is an arbitrary element of $\bigcup_{i \in \Lambda} \mathcal{O}_i$.

So, every element of $\bigcup_{i \in \Lambda} \mathcal{O}_i$ is its interior element. Therefore, $\bigcup_{i \in \Lambda} \mathcal{O}_i$ is open in (M, d) .

(iv) Let \mathcal{O}_1 and \mathcal{O}_2 be two open general msets in (M, d) . If $\mathcal{O}_1 \cap \mathcal{O}_2 = \phi$, then $\mathcal{O}_1 \cap \mathcal{O}_2$ is also open in (M, d) . So consider the case where $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \phi$. Let $(\alpha, k) \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then

$(\alpha, k) \in \mathcal{O}_1$ as well as $(\alpha, k) \in \mathcal{O}_2$. Also, \mathcal{O}_1 and \mathcal{O}_2 both are open in (M, d) . So, there exists $R_p^q, R_r^s \in m^+(R)$ with $\frac{s}{[f(k,k)]^2}, \frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}_1$ and $(\alpha, k) \in B^*((\alpha, k), R_r^s) \subseteq \mathcal{O}_2$. Let $\min\{p, r\} = a$ and $\gcd\{q, s\} = b$. Then $R_a^b \preceq R_p^q, R_r^s$. So, $B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q), B^*((\alpha, k), R_r^s)$. Then $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}_1$ and $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_r^s) \subseteq \mathcal{O}_2$.

Therefore, $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$. Therefore, (α, k) is an interior element of $\mathcal{O}_1 \cap \mathcal{O}_2$. Therefore, $\mathcal{O}_1 \cap \mathcal{O}_2$ is also open in (M, d) .

7. Comparative Analysis with Neutrosophic-Based Methods

The notion of a *multi-metric space* developed in this work differs fundamentally from neutrosophic-based approaches, though both aim to extend classical structures to handle richer information. The following points highlight the comparison:

(a) **Representation:** Neutrosophic methods represent information through a triplet (T, I, F) that captures truth, indeterminacy and falsity. In contrast, our multi-metric space uses *multi-real numbers*, which embeds multiplicities and multiset concepts.

(b) **Metric Definition:** Several extensions of metric spaces have been proposed in the literature by incorporating neutrosophic logic, which allows one to model truth (T), indeterminacy (I), and falsity (F) simultaneously. Some important notions are summarized below.

(i) **Neutrosophic Metric Space:** Kirişçi and Şimşek [34] introduced the concept of a *neutrosophic metric space*, where the classical metric axioms are generalized by using continuous triangular norms (t-norms) and conorms. This framework preserves many standard topological properties, and results such as the Baire Category Theorem and the Uniform Convergence Theorem were extended to this setting.

(ii) **Neutrosophic Triplet Metric Space:** In a *neutrosophic triplet metric space*, each element is represented as a neutrosophic triplet (T, I, F) , and the metric is defined accordingly. The axioms of non-negativity, symmetry, and a modified triangle inequality are adapted to the neutrosophic context [35].

(iii) **Neutrosophic Triplet v -Generalized Metric Space:** The neutrosophic triplet v -generalized metric space (NTVGMS) further relaxes the standard triangle inequality and provides more flexibility in defining distances. The completeness and fixed-point results in this framework have been investigated by [36].

(iv) **Neutrosophic Fuzzy Metric Spaces and Variants:** The combination of neutrosophic and fuzzy logics has led to *neutrosophic fuzzy metric spaces* and their variants, such as orthogonal neutrosophic metric spaces, neutrosophic 2-metric spaces, pentagonal metric spaces and b -metric spaces. These frameworks have been used to study the results of convergence, compactness, and fixed points [37].

(v) **Other generalizations:** Several further generalizations have been explored, including neutrosophic quasi-dislocated- b -metric spaces [38], neutrosophic b -metric-like spaces [39], and neutrosophic D -metric spaces [40]. These approaches extend the applicability of neutrosophic metrics in both pure and applied mathematics.

Table 1 summarizes the main concepts of neutrosophic metric and their key characteristics.

TABLE 1. Different notions of neutrosophic metric concepts

Neutrosophic Metric Concept	Key Characteristics
Neutrosophic Metric Space [34]	Uses t-norms/conorms; generalizes classical topological results.
Neutrosophic Triplet Metric Space [35]	Defines distance on neutrosophic triplet elements (T, I, F) .
NTVGMS [36]	v -generalized triangle inequality; completeness and fixed points.
Neutrosophic Fuzzy Metric Spaces [37]	Combines neutrosophic and fuzzy logic; several variants developed.
Quasi-dislocated- b -Metric Space [38]	Generalizes neutrosophic triplet metrics; includes fixed-point theorems.
Neutrosophic b -Metric-like Space [39]	Relaxes triangle inequality; supports fixed-point theory.
Neutrosophic D -Metric Space [40]	Adapts D -metric to neutrosophic settings; explores topology and completeness.

Our approach defines multi-metrics as a mapping from the set of all \mathbb{N} -subm-elements of a \mathbb{N} -mset to the set of all non-negative multi-real numbers satisfying multi-metric axioms.

(c) **Generality and Applicability:** Neutrosophic approaches are particularly suited for model uncertainty and inconsistency. However, our method, is more suitable for applications involving *multisets, graphs, trees, and networks*, where multiplicity and structural comparison are essential.

(d) **Novel Contribution:** While neutrosophic-based methods extend fuzzy set theory, the proposed multi-metric space provides an *entirely new mathematical framework* by restructuring multisets, introducing the multi-real number system, and defining rigorous metrics, specially for multiset setting.

This comparative analysis demonstrates that the multi-metric space not only complements existing neutrosophic-based methods but also offers a mathematically robust and efficient alternative for problems where multiplicities and structural similarities are central.

8. Conclusion

From the above theorem it follows that, in a multi-metric space (M, d) , the collection τ of all open general msets forms a topology on M with respect to elementary union and elementary intersection of \mathbb{N} -msets. This topology will be called 'multi-metric topology' on M .

9. Limitations and Future Research

Functional analysis plays a pivotal role in modern mathematics and its applications to the sciences. Since metric spaces form its foundation, they serve as a key tool in the development of many important results. In this work, we have introduced an extension of the classical metric framework by employing multisets and multi-real numbers in place of ordinary sets and real numbers, thereby offering a broader perspective on the structure of multi-metric spaces.

Despite these contributions, some limitations remain. The present study focuses mainly on particular classes of metrics and illustrative examples that do not fully reflect the general scope of possible constructions. In addition, certain theoretical results depend on restrictive assumptions, which may limit their direct applicability to wider contexts.

Future investigations could aim to relax these assumptions and extend the theory to more generalized settings. Another promising direction is the study of multi-normed linear spaces and multi-inner product spaces, which may further enrich the theory. Moreover, exploring connections with frameworks such as fuzzy sets, neutrosophic sets, and other uncertainty-based models could significantly expand the range of applications. Overall, the results presented here provide a foundation for further research on generalized metric theories and their potential applications in functional analysis and beyond.

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