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On deferred I-statistical rough convergence of difference sequences in neutrosophic normed spaces

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Abstract. In this study, using the concepts of deferred density and the notion of the ideal I, we extend the idea of rough convergence by introducing the notion of deferred I–statistical rough convergence via difference operators in the framework of neutrosophic normed spaces. We define a set of limits of this convergence and prove that the limit set is convex and closed with respect to the neutrosophic norm. We also develop the idea of deferred I-statistical Δ_h^j-cluster points of sequences in neutrosophic normed spaces and investigate their connection the set of these cluster points and the limit set of the aforementioned convergence.

Keywords: Neutrosophic normed space(NNS); difference sequences; deferred statistical convergence; I-convergence; deferred I-statistical convergence; rough convergence.
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1. Introduction

In recent decades, fuzzy theory has emerged as one of the most influential tools in mathematical modeling, engineering, and decision sciences. The concept of fuzzy sets, originally introduced by Zadeh in 1965 [35], provided a systematic way to represent and manage uncertainty. Building upon this, Kramosil and Michálek [20] developed fuzzy metric spaces to generalize classical metric spaces, a framework later refined by George and Veeramani [16], who introduced a Hausdorff topology for fuzzy metric structures. Expanding this line of research, Atanassov [5] proposed intuitionistic fuzzy sets, which consider both membership and non-membership values, thereby improving uncertainty modeling. This idea was further generalized by Park [30] with the introduction of intuitionistic fuzzy metric spaces, followed by Saadati and Park [31], who established the notion of intuitionistic fuzzy normed spaces. More recently, applications of intuitionistic fuzzy theory have been seen in areas such as pattern recognition, image processing, control theory, and decision-making under uncertainty [40, 41].

Parallel to these developments, researchers sought to generalize the classical concept of sequence convergence. Steinhaus [34] and Fast [15] introduced the concept of statistical convergence, which captures the convergence of "almost all" terms of a sequence, rather than all terms. This new approach quickly gained prominence in summability theory and functional analysis. Later, Kostyrko et al. [19] extended this to I-convergence using ideals of subsets of $\mathbb{N}$. The interplay between statistical convergence and intuitionistic fuzzy normed spaces has since been explored in various works [17, 25], while Savaş and Gürdal [32] refined this idea by defining I-statistical convergence. Subsequently, the concept of deferred statistical convergence was introduced in [21], employing deferred density as a refinement. This approach has inspired extensive research into deferred statistical convergence of both single sequences [24] and double sequences [26], broadening the scope of convergence analysis in generalized sequence spaces.

Another significant line of research has centered on the concept of *rough convergence*, introduced by Phu [28] in finite-dimensional normed linear spaces, where the notion of tolerance (degree of roughness) plays a central role. Phu later extended this framework to infinite-dimensional spaces [29] and studied fundamental properties such as convexity and closure of rough limit sets. Following this, Aytar [6, 7] proposed *rough statistical convergence*, linking it to statistical cluster points and the structure of rough limit sets. These developments have led to extensive studies on rough and approximate statistical convergence in different contexts, such as double and triple sequences [8, 22, 23]. Pal et al. [27] and Dündar et al. [10] introduced rough I-convergence, which was subsequently extended to rough I_2-convergence for double sequences [11] and rough I_2-lacunary statistical convergence [12]. Rough convergence has also been applied to metric spaces [9], 2-normed spaces [4], and probabilistic normed spaces [2]. Recently, Reena et al. [3] studied rough statistical convergence within intuitionistic fuzzy normed spaces by focusing on continuous t-norms, highlighting a growing interest in merging rough convergence with fuzzy and intuitionistic frameworks.

In parallel, difference operators and their associated sequence spaces have become an active area of research. The classical forward difference operator Δ was first used to define difference sequence spaces in [18], later extended to integer orders by Et and Çolak [13], with further advancements by Khan et al. [36]. Recently, these ideas were integrated with neutrosophic normed spaces, as explored by Kaur and Chawla [45], and further studied in fuzzy, intuitionistic fuzzy, and neutrosophic settings [42–44]. Notable contributions include statistical completeness in neutrosophic normed spaces [46], uniform statistical convergence of function sequences [47], I-convergent difference sequence spaces [48], hybrid Δ-statistical and lacunary approaches [49, 50], Riesz ideal convergence extensions [51], and nonlinear operator analysis with Fréchet differentiability [52]. Together, these studies underline the significance of difference operators in advancing the theory of neutrosophic normed spaces.

Motivated by these developments, the present work aims to advance the theory of convergence by combining the concepts of deferred density, I-convergence, and rough convergence in neutrosophic normed spaces through the use of higher-order difference operators. This approach not only unifies several strands of research in fuzzy, intuitionistic, and neutrosophic settings but also opens new directions for the study of uncertain, incomplete, or noisy data sequences that arise in real-world applications such as data science, signal processing, and decision-making systems.

The main objective of this research is to present and explore the concept of deferred I-statistical rough convergence, defined via integer-order j difference operators, in the context of neutrosophic normed spaces.

2. Preliminaries

In this work, we denote the sets. For clarity, we first revisit several relevant definitions in the table below.

Notation	**Meaning / Definition**
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
$\delta(\mathcal{A})$	Density of the set $\mathcal{A}$
$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$	Neutrosophic normed space (NNS) with membership Υ, non-membership Ω, indeterminacy Γ, and t-norm $\star$ and t-conorm $\circ$.
$\Upsilon(x, h)$	Degree of membership of $x \in X$ with respect to $h > 0$.
$\Omega(x, h)$	Degree of non-membership of $x \in X$ with respect to $h > 0$.
$\Gamma(x, h)$	Degree of indeterminacy of $x \in X$ with respect to $h > 0$.
$\star$	t-norm used in the triangle-type inequality for Υ.
$\circ$	t-conorm used in the triangle-type inequalities for Ω and Γ.
$\Delta^j x_Q$	j-th order difference of the sequence (x_Q).

h	Parameter controlling the "radius" in the neutrosophic norm functions.
d	Arbitrary positive radius in the definition of deferred I-statistical cluster point.
I	Ideal of subsets of $\mathbb{N}$ used in the deferred I-statistical convergence.
$\Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}(\Delta^j x_Q)$	Set of all deferred I-statistical Δ^j_h-cluster points of (x_Q) with respect to the NNS.
z	A deferred I-statistical Δ^j_h-cluster point of (x_Q).
y	Arbitrary point in X used in the triangle-type inequality argument.
$\varsigma \in (0,1)$	Small parameter used in approximating membership, non-membership, and indeterminacy.
p_n, q_n	Endpoints of intervals used in the deferred I-statistical density calculation.
$(\Upsilon,\Omega,\Gamma)^h(\Delta^j) - \lim x_Q = x$	Rough statistical Δ^j-convergence of a sequence (x_Q) to x in NNS, where convergence is controlled by a roughness parameter $h \geq 0$.
Δ^j_{NN}	strongly bounded in neutrosophic norm space (NNS).

Table 1. Notation used in deferred I-statistical Δ^j_h-cluster points, Δ^j_{NN}-convergence, and rough statistical convergence in NNS

Let $\mathcal{A} \subseteq \mathbb{N}$. The asymptotic (or natural) density of the set $\mathcal{A}$, denoted by $\delta(\mathcal{A})$, is defined as

$$\delta(\mathcal{A}) = \lim_{Q\to\infty} \frac{1}{Q} |\{n \leq Q : n \in \mathcal{A}\}|,$$

assuming the limit exists. In this context, $||\cdot||$ represents the number of elements in the set $\{,\}$. A numerical sequence (x_Q) is said to converge statistically to l if, for any $\epsilon > 0$,

$$\delta(\{Q \in \mathbb{N} : |x_Q - l| > \epsilon\}) = 0 \text{ holds.}$$

For this case, we write $x_Q \xrightarrow{S} l$ (see [15], [34]).

Definition 2.1. [14] A real (or complex) valued sequence (x_Q) is Δ^j-statistically convergent to l if

$$\delta\left(\left\{Q \in \mathbb{N} : \left|\Delta^j x_Q - l\right| > \epsilon\right\}\right) = 0$$

for all $\epsilon > 0$, with j belonging to the set of natural numbers $\mathbb{N}$, and

$$\Delta^0 x_Q = x_Q, \Delta^1 x_Q = x_Q - x_{Q+1}, \ldots, \Delta^j x_Q = \Delta^{j-1}(x_Q - x_{Q+1})$$

and so that

$$\Delta^j x_Q = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x_{Q+i} (Q \in \mathbb{N}).$$

Definition 2.2. [6] A sequence (x_Q) in a normed space $(X, \|.\|)$ is rough convergent to $x \in X$ for some $h \geq 0$ if, for every $\varsigma > 0, \exists\ n_0 \in \mathbb{N}$ so that

$$\|x_Q - x\| < h + \varsigma, \forall\ Q \geq n_0$$

The sequence (x_Q) is rough statistically convergent to $x \in X$ for some $h \geq 0$ if, for every $\varsigma > 0$,

$$\delta\left(\{Q \in \mathbb{N} : \|x_Q - x\| \geq h + \varsigma\}\right) = 0 \text{ holds.}$$

Note: Definition 2.1 (Δ^j-statistical convergence) generalizes classical statistical convergence by considering the j-th order differences of a sequence, focusing on the asymptotic behavior of $\Delta^j x_Q$. In contrast, Definition 2.2 (rough convergence) allows a sequence to converge within a tolerance level h, either strictly for all large indices or statistically for most indices. Both concepts address approximate convergence, but Δ^j-statistical convergence emphasizes differences of order j, while rough convergence emphasizes proximity in a fixed roughness margin.

Definition 2.3. [33] A binary operation $\star$ on $[0, 1]$ is called continuous t-norm ($or\ CTN$) if
(a) $\star$ is commutative, associative and continuous,
(b) $\varsigma = \varsigma \star 1$ for any $\varsigma \in [0, 1]$ and
(c) for each $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in [0, 1]$, if $\varsigma_3 \geq \varsigma_1$ and $\varsigma_4 \geq \varsigma_2$ then $\varsigma_3 \star \varsigma_4 \geq \varsigma_1 \star \varsigma_2$.

A binary operation on on $[0, 1]$ is called continuous t-conorm ($or\ CTCN$) if
(1) o is commutative, associative and continuous,
(2) $\varsigma = \varsigma \circ 0$ for any $\varsigma \in [0, 1]$ and
(3) for each $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in [0, 1]$, if $\varsigma_3 \geq \varsigma_1$ and $\varsigma_4 \geq \varsigma_2$ then $\varsigma_3 \circ \varsigma_4 \geq \varsigma_1 \circ \varsigma_2$.

Definition 2.4. [36] The three-tuple structure $(X, \Upsilon, \Omega, \Gamma)$ be an NNS, where X is a linear space over a field F. Υ, Ω, Γ are called neutrosophic normed space (NNS) on $X \times (0, \infty)$ and represent the degree of membership and non-membership on $X \times (0, 1)$ if the following conditions hold, for every $y, w \in X$ and $\varsigma_1, \varsigma_2 > 0$:
(1) $\Upsilon(y, \varsigma) + \Omega(y, \varsigma) \leq 1$.
(2) $\Upsilon(y, \varsigma) > 0$.
(3) $\Upsilon(y, \varsigma) = 1 \Leftrightarrow y = 0$.
(4) $\Upsilon(cy, \varsigma) = \Upsilon\left(y, \frac{\varsigma}{|c|}\right)$, if $c \neq 0, c \in F$.
(5) $\Upsilon(y, \varsigma_1) \star \Upsilon(w, \varsigma_2) \leq \Upsilon(p + w, \varsigma_1 + \varsigma_2)$.
(6) $\Upsilon(y, .) : (0, \infty) \to [0, 1]$ is continuous.

(7) $\lim_{\varsigma\to\infty}\Upsilon(y,\varsigma)=1, \lim_{\varsigma\to 0}\Upsilon(y,\varsigma)=0$.
(8) $\Omega(y,\varsigma)<1$.
(9) $\Omega(y,\varsigma)=0 \Leftrightarrow y=0$.
(10) $\Omega(cy,\varsigma)=\Omega\left(y,\frac{\varsigma}{|c|}\right)$ if $c\neq 0, c\in F$.
(11) $\Omega(y,\varsigma_1)\diamond\Omega(w,\varsigma_2)\geq\Omega(y+w,\varsigma_1+\varsigma_2)$.
(12) $\Omega(y,.):(0,\infty)\to[0,1]$ is continuous.
(13) $\lim_{\varsigma\to\infty}\Omega(y,\varsigma)=0, \lim_{\varsigma\to 0}\Omega(y,\varsigma)=1$.
(14) $\Gamma(y,\varsigma)=0 \Leftrightarrow y=0$.
(15) $\Gamma(cy,\varsigma)=\Omega\left(y,\frac{\varsigma}{|c|}\right)$ if $c\neq 0, c\in F$.
(16) $\Gamma(y,\varsigma_1)\diamond\Omega(w,\varsigma_2)\geq\Omega(y+w,\varsigma_1+\varsigma_2)$.
(17) $\Gamma(y,.):(0,\infty)\to[0,1]$ is continuous.
(18) $\lim_{\varsigma\to\infty}\Gamma(y,\varsigma)=0, \lim_{\varsigma\to 0}\Omega(y,\varsigma)=1$.
Then Υ,Ω,Γ are called neutrosophic normed (NN).

Here, the three tuple (Υ,Ω,Γ) is known as the neutrosophic normed (NN) on X.

Example 2.5. Let $\mathbb{R}$ be a real linear space over the field $\mathbb{R}$. Define the functions $\Upsilon,\Omega,\Gamma:\mathbb{R}\times(0,\infty)\to[0,1]$ as follows:

$$\Upsilon(y,\varsigma)=\frac{\varsigma}{\varsigma+|y|},\qquad \Omega(y,\varsigma)=\frac{|y|}{\varsigma+|y|},\qquad \Gamma(y,\varsigma)=\frac{|y|}{1+|y|+\varsigma},$$

for all $y\in\mathbb{R}$ and $\varsigma>0$.

Also, define the binary operations on $[0,1]$ by

$$\varsigma_1\star\varsigma_2=\min\{\varsigma_1,\varsigma_2\},\qquad \varsigma_1\circ\varsigma_2=\max\{\varsigma_1,\varsigma_2\},$$

for all $\varsigma_1,\varsigma_2\in[0,1]$.

It is easy to verify that:

- $\Upsilon(y,\varsigma)+\Omega(y,\varsigma)=1$ for all $y\in\mathbb{R}$ and $\varsigma>0$.
- $\Upsilon(y,\varsigma)=1 \iff y=0$, and $\Omega(y,\varsigma)=0 \iff y=0$.
- $\lim_{\varsigma\to\infty}\Upsilon(y,\varsigma)=1$ and $\lim_{\varsigma\to 0}\Upsilon(y,\varsigma)=0$.
- $\lim_{\varsigma\to\infty}\Omega(y,\varsigma)=0$ and $\lim_{\varsigma\to 0}\Omega(y,\varsigma)=1$.
- $\Gamma(y,\varsigma)$ is continuous, with $\Gamma(y,\varsigma)=0 \iff y=0$, and $\lim_{\varsigma\to\infty}\Gamma(y,\varsigma)=0$.

Hence, the three-tuple $(X,\Upsilon,\Omega,\Gamma)$, together with the operations $\star$ and $\circ$, forms a neutrosophic normed space (NNS).

Definition 2.6. [38] Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. The open ball of radius $h>0$ and center $x\in X$ with regard to $\varsigma\in(0,1)$ is the set

$$\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,\varsigma)=\{y\in X:\Upsilon(x-y,h)>1-\varsigma \text{ and } \Omega(x-y,h)<\varsigma \text{ and } \Gamma(x-y,h)<\varsigma\}.$$

Definition 2.7. [39] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. A sequence (x_Q) in X is convergent to $x \in X$ with regard to $(\Upsilon, \Omega, \Gamma)$ if

$$\lim_{Q \to \infty} \Upsilon(x_Q - x, d) = 1 \text{ and } \lim_{Q \to \infty} \Omega(x_Q - x, d) = 0 \text{ and } \lim_{Q \to \infty} \Gamma(x_Q - x, d) = 0$$

for every $d > 0$. In this case, we denote the limit by $x_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} x$.

Definition 2.8. [19] Let $\Gamma \neq \emptyset$ set and $I \subseteq 2^{\Gamma}$. Then I is called an ideal in Γ if
(a) $\emptyset \in I$,
(b) $\mathcal{A}, \mathcal{B} \in I \Rightarrow Q \in \mathcal{A} \cup \mathcal{B} \in I$ and
(c) $\mathcal{A} \in I, \mathcal{B} \subseteq \mathcal{A} \Rightarrow \mathcal{B} \in I$. An ideal $I \subseteq 2^{\Gamma}$ is nontrivial if $I \neq 2^{\Gamma}$. A nontrivial ideal $I \subseteq 2^{\Gamma}$ is admissible if I contains every singleton subset of X.
A subset $F \subseteq 2^{\Gamma}$ is called filter on Γ if
(e) $\emptyset \notin F$,
(f) $\mathcal{A} \cap \mathcal{B} \in F$ for all $\mathcal{A}, \mathcal{B} \in F$ and
(e) $\mathcal{B} \in F$ whenever $\mathcal{A} \in F$ and $\mathcal{B} \supset \mathcal{A}$. For each ideal I in Γ, one can find the filter $F(I)$ associated with ideal I such that $F(I) = \{\mathcal{A} \subset \Gamma : \mathcal{A}^c \in I\}$.

Definition 2.9. [40] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and I is nontrivial admissible ideal in $\mathbb{N}$. A sequence (x_Q) in X is I-statistically convergent to some $x \in X$ with regard to $(\Upsilon, \Omega, \Gamma)$ if

$$\left\{n \in \mathbb{N} : \frac{1}{n}\big|\{Q \leq n : \Upsilon(x_Q - x, d) \leq 1 - \varsigma \text{ or } \Omega(x_Q - x, d) \geq \varsigma \text{ or } \Gamma(x_Q - x, d) \geq \varsigma\}\big| \geq \epsilon\right\} \in I$$

for every $\epsilon, d > 0$ and $\varsigma \in (0, 1)$.

For $I = I_f$, the collection of all finite subsets of $\mathbb{N}$, the convergence in Definition 2.9 reduces to the statistical convergence of (x_Q) with regard to $(\Upsilon, \Omega, \Gamma)$ [17].

In 1932, Agnew [1] extended the concept of the Cesàro mean for real (or complex) sequences and introduced the deferred Cesàro mean, defined as follows:

Definition 2.10. [36] For a real (or complex) valued sequence (x_Q), the deferred Cesàro mean of (x_Q) is defined by

$$\left(D_p^q(x_Q)\right)_n := \frac{1}{q_n - p_n} \sum_{Q=p_n+1}^{q_n} x_Q, n = 1, 2, 3, \ldots,$$

where $p = (p_n)$ and $q = (q_n)$ denote sequences of non-negative integers that fulfill the condition

$$p_n < q_n \text{ and } \lim_{n \to \infty} q_n = \infty \tag{2.1}$$

Given a subset $K \subseteq \mathbb{N}$, the deferred density of K is defined as follows:

$$D_p^q(K) = \lim_{n \to \infty} \frac{1}{q_n - p_n} |\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in K\}| \tag{2.2}$$

provided the limit exists.

Definition 2.11. [10] A real (or complex) valued sequence (x_Q) is deferred statistically convergent to l if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, |x_Q - l| \geq \epsilon\}\right| = 0$$

for every $\epsilon > 0$.

When $p_n = 0$ and $q_n = n$, this definition aligns with the concept of statistical convergence of the sequence (x_Q) as presented in [15].

3. $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right)$-convergence sequences in NNS

This section introduces and explores the concept of $D_p^q(\Upsilon, \Omega, \Gamma)S(I)^h \left(\Delta^j\right)$-convergence for sequences. Throughout this discussion, I denotes a non-trivial admissible ideal in $\mathbb{N}$. The j-th order difference of a sequence (x_Q) is given by $\Delta^j x_Q = \sum_{i=0}^{j}(-1)^i \binom{j}{i} x_{Q+i} (j \in \mathbb{N})$. Additionally, (p_n) and (q_n) represent sequences of non-negative integers that satisfy condition (2.1). Further assumptions on (p_n), (q_n), and j, if required, will be specified within the corresponding theorems and examples.

Definition 3.1. [42] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is Δ^j-rough convergent to some x with regard to $(\Upsilon, \Omega, \Gamma)$ for some $h \geq 0$ if, for every $d > 0$ and $\varsigma \in (0, 1), \exists\, n_0 \in \mathbb{N}$ such that

$$\Upsilon \left(\Delta^j x_Q - x, d + h\right) > 1 - \varsigma \text{ and } \Omega \left(\Delta^j x_Q - x, d + h\right) < \varsigma, \text{ and } \Gamma \left(\Delta^j x_Q - x, d + h\right) < \varsigma,$$

$\forall\, Q \geq n_0$. The limit is denoted by $(\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right) - \lim x_Q = x$.

Definition 3.2. [43] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is deferred I-statistically difference rough convergent to some x for some $h \geq 0$ with regard to $(\Upsilon, \Omega, \Gamma)$ (shortly, $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right)$-convergent to x) if, for every $\epsilon, d > 0$ and $\varsigma \in (0, 1)$,

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon \left(\Delta^j x_Q - x, d + h\right) \leq 1 - \varsigma$$
$$\text{or } \Omega \left(\Delta^j x_Q - x, d + h\right) \geq \varsigma, \text{ or } \Gamma \left(\Delta^j x_Q - x, d + h\right) \geq \varsigma\}\Big| \geq \epsilon\Big\} \in I \tag{3.1}$$

holds. In this case, we denote the limit by $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right) - \lim x_Q = x$.
Suppose (x_Q) is a sequence in an NNS$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$.

- When $h = 0$ in (3.1), the sequence (x_Q) is said to be *deferred I-statistically difference convergent* relative to the structure $(\Upsilon, \Omega, \Gamma)$.
- If we choose $p_n = 0$ and $q_n = n$ in (3.1), then the deferred I-statistical difference rough convergence of the sequence (x_Q) is termed the *I-statistical difference rough convergence* relative to the structure $(\Upsilon, \Omega, \Gamma)$.

- When the ideal I is taken as I_f, the notion introduced in Definition 3.2 is referred to as the *deferred statistical difference rough convergence* with respect to the structure $(\Upsilon, \Omega, \Gamma)$.

Remark 3.3. Let (x_Q) be a sequence in an NNS$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ with $(h \geq 0)$. The limits

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{- }\lim x_Q \quad \text{and} \quad D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{- }\lim x_Q$$

are not necessarily unique whenever they exist, for $j \in \mathbb{N}$. We denote the collections of all such limits by

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-LIM}(x_Q) = \{x \in X : (\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{- }\lim x_Q = x\},$$
$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-LIM}(x_Q) = \{x \in X : D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{- }\lim x_Q = x\}.$$

A sequence (x_Q) is said to be Δ^j-rough convergent with respect to $(\Upsilon, \Omega, \Gamma)$ if

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-LIM}(x_Q) \neq \emptyset,$$

and similarly, (y_Q) is $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)$-convergent if

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-LIM}(y_Q) \neq \emptyset,$$

for some $h \geq 0$. Moreover, if $0 \leq h_1 \leq h_2$, then for any sequence (x_Q) in X, the inclusion relations hold:

$$(\Upsilon, \Omega, \Gamma)^{h_1}(\Delta^j)\text{-LIM}(x_Q) \subseteq (\Upsilon, \Omega, \Gamma)^{h_2}(\Delta^j)\text{-LIM}(x_Q),$$

and

$$D_p^q(\Upsilon, \Omega, \Gamma)^{h_1}_{S(I)}(\Delta^j)\text{-LIM}(x_Q) \subseteq D_p^q(\Upsilon, \Omega, \Gamma)^{h_2}_{S(I)}(\Delta^j)\text{-LIM}(x_Q).$$

Example 3.4. Consider the NNS $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, where $(\mathbb{R}, \|.\|)$ is the usual normed space, $\varsigma_1 \star \varsigma_2 = \min\{\varsigma_1, \varsigma_2\}$, $\varsigma_1 \circ \varsigma_2 = \varsigma_1 + \varsigma_2 - \varsigma_1 \cdot \varsigma_2$, and

$$\Upsilon(x, d) = \frac{1}{1 + \|x\| + d}, \quad \Omega(x, d) = \frac{d}{1 + \|x\| + d}, \quad \Gamma(x, d) = \frac{\|x\|}{1 + \|x\| + d}, \quad \forall x \in \mathbb{R}, d > 0.$$

Define the sequence

$$x_Q = \begin{cases} (-1)^Q & \text{if } Q \text{ is a multiple of 3,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $j = 1$,

$$\Delta^1 x_Q = x_{Q+1} - x_Q = \begin{cases} 1 & \text{for certain } Q, \\ -1 & \text{for others,} \\ 0 & \text{otherwise,} \end{cases}$$

depending on the parity and position of multiples of 3.

Thus, the sequence $\Delta^1 x_Q$ is bounded but oscillatory. Hence,

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^1)\text{- LIM}(x_Q) = \begin{cases} [-1 - h, h + 1] & \text{if } h \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now take $p_n = \frac{1}{n+1}, q_n = \log(n+2)$ and define another sequence (y_Q) as:

$$\Delta^1 y_Q = \begin{cases} Q \bmod 4 & \text{if } Q = 2^n \text{ for some } n, \\ -2 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}.$$

Then, for any nontrivial admissible ideal I,

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^1)\text{- LIM}(y_Q) = \begin{cases} [-2-h, h+2] & \text{if } h \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, both $\Delta^1 x_Q$ and $\Delta^1 y_Q$ are not convergent in the usual sense, but their rough statistical limit sets exist under neutrosophic norms for suitable values of h.

Note Unlike standard convergence in an NNS$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$, the Δ^j-rough convergence of a sequence (x_Q) in X relative to $(\Upsilon, \Omega, \Gamma)$ does not necessarily imply that any subsequence of (x_Q) will also be Δ^j-rough convergent under the same framework. For example, consider the sequence $(x_Q) = (Q)$ in the NNS described in Example 3.4, where the rough limit set is

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^1)\text{- LIM}(x_Q) = [1-h, 1+h] \quad \text{for all } h \geq 0,$$

however, its subsequence $(x_{2Q}) = (Q^2)$ does not exhibit Δ^1-rough convergence for any $h \geq 0$. A similar observation holds true for the $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)$-convergence of sequences in X.

Example 3.5. Consider the NNS $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, where $(\mathbb{R}, \|\cdot\|)$ is the usual normed space, $\varsigma_1 \star \varsigma_2 = \varsigma_1 \cdot \varsigma_2$, $\varsigma_1 \circ \varsigma_2 = \varsigma_1 + \varsigma_2 - \varsigma_1 \cdot \varsigma_2$, for all $\varsigma_1, \varsigma_2 \in [0,1]$, and Υ, Ω, Γ are defined by

$$\Upsilon(x,d) = \frac{1}{1+\|x\|+d}, \quad \Omega(x,d) = \frac{\|x\|}{1+\|x\|+d}, \quad \Gamma(x,d) = \frac{d}{1+\|x\|+d}$$

for all $x \in \mathbb{R}, d > 0$. Let $p_n = \frac{1}{n+1}$ and $q_n = \log(n+2)$ for all $n \in \mathbb{N}$. Define

$$x_Q = \begin{cases} 0 & \text{if } Q = 2^n \text{ for some } n \in \mathbb{N}, \\ n & \text{otherwise.} \end{cases}$$

Then, for any nontrivial admissible ideal I, we obtain

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^1\right)\text{-LIM}(x_Q) = [h, \infty), \quad \forall h \geq 0.$$

However, for the subsequence (x_{2^n}) of (x_Q), we get

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^1\right)\text{-LIM}(x_{2^n}) = \{0\}, \quad \forall h \geq 0.$$

Lemma 3.6. *Suppose $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ is an NNS and (x_Q) is a sequence in X. Let $h \geq 0$ be given. Then, for every $\epsilon, d > 0$ and $\varsigma \in (0,1)$, the following are equivalent*

(a)

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) - \lim x_Q = x.$$

(b)

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma\right\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| \geq \epsilon\right\} \in I.$$

(c)

$$\Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\big| < \epsilon\Bigg\} \in F(I).$$

(d)

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma\right\}\right| < \epsilon\right\} \in F(I) \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| < \epsilon\right\} \in F(I) \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| < \epsilon\right\} \in F(I).$$

(e)

$$I - \lim_{n\to\infty} \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\big| = 0.$$

Proof. Due to its obvious nature, the proof has been omitted. □

Theorem 3.7. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Then, for every sequence (x_Q) in X,*

$$(\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right) \text{-LIM}\,(x_Q) \subset D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)} \left(\Delta^j\right) \text{-LIM}\,(x_Q)$$

holds.

Proof. Assume that $x \in (\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right)\text{-LIM}\,(x_Q)$ for some $h \geq 0$. Then, for every $d > 0$ and $\varsigma \in (0, 1), \exists\, n_0 \in \mathbb{N}$ so that

$$\Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma, \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma, \forall\, Q \geq n_0.$$

Therefore,

$$\{Q \in \mathbb{N} : \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}$$
$$\subseteq \{1, 2, \ldots, n_0 - 1\}.$$

Since

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \{1, 2, \ldots, n_0 - 1\}\}\right| = 0,$$

holds for every $\epsilon > 0$, the set

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d + h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d + h\right) \geq \varsigma$$
$$\text{or } \Gamma\left(\Delta^j x_Q - x, d + h\right) \geq \varsigma\}\big| \geq \epsilon\Big\}$$

is a part of I_f and therefore also of I. Thus, $x \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) - \text{LIM}\left(x_Q\right)$. Consequently, we possess

$$(\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right) \text{-} \text{LIM}\left(x_Q\right) \subset D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) \text{-} \text{LIM}\left(x_Q\right).$$

From Example 3.4, we can see that the above inclusion relation is strict. □

Theorem 3.8. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) be a sequence in X. Then, for any $h > 0$, there are no $x, y \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) \text{-} \text{LIM}\left(x_Q\right)$ such that $\Upsilon(x - y, sh) \leq 1 - \varsigma$ or $\Omega(x - y, sh) \geq \varsigma$ or $\Gamma(x - y, sh) \geq \varsigma$ for every $\varsigma \in (0, 1)$, where $s > 2$.*

Proof. For any given $\varsigma \in (0, 1), \exists v \in (0, 1)$ such that $(1 - v) \star (1 - v) > 1 - \varsigma$ and $v \circ v < \varsigma$. Let on contrary that there exist $x, y \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) \text{-} \text{LIM}\left(x_Q\right)$ such that for every $\varsigma \in (0, 1)$,

$$\Upsilon(x - y, sh) \leq 1 - \varsigma \text{ or } \Omega(x - y, sh) \geq \varsigma \text{ or } \Gamma(x - y, sh) \geq \varsigma$$

where $s > 2$. Now, for any $d > 0$, consider the sets

$$\mathcal{N} = \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \geq v\Big\},$$

and

$$\mathcal{O} = \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \geq v\Big\}.$$

Since $x, y \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right) \text{-} \text{LIM}\left(x_Q\right)$, by Lemma 3.6, we have

$$I - \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in N\}\right| = 0.$$

and

$$I - \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in O\}\right| = 0.$$

Now

$$\begin{aligned} I-\lim_{n\to\infty}\frac{1}{q_n-p_n}&\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in N\cup O\}\right| \\ &\leq I-\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in N\}\right| \\ &+I-\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in O\}\right| \\ &=0. \end{aligned}$$

Hence for every $\epsilon>0$,

$$\mathcal{P}=\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in N\cup O\}\right|\geq\epsilon\right\}\in I.$$

Let $m\in\mathcal{P}^c$ and $\epsilon=\frac{1}{4}$. Then

$$\begin{aligned} &\frac{1}{q_m-p_m}\left|\{Q\in\mathbb{N}:p_m<Q\leq q_m, Q\in N\cup O\}\right|<\frac{1}{4} \\ \Longrightarrow&\frac{1}{q_m-p_m}\left|\{Q\in\mathbb{N}:p_m<Q\leq q_m, Q\in N^c\cap O^c\}\right|\geq 1-\frac{1}{4}=\frac{3}{4}. \end{aligned}$$

As a result, we have

$$Q'=\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in N^c\cap O^c\}\neq\emptyset.$$

Since $s>2$, put $sh=2h+d$ for some $d>0$. If $\Upsilon(x-y,sh)\leq 1-\varsigma$ then for $Q\in Q'$, we find

$$\begin{aligned} 1-\varsigma&\geq\Upsilon(x-y,t+2h) \\ &\geq\Upsilon\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\star\Upsilon\left(\Delta^j x_Q-y,\frac{d}{2}+h\right) \\ &>(1-v)\star(1-v) \\ &>1-\varsigma, \end{aligned}$$

that is ridiculous. If $\Omega(x-y,sh)\geq\varsigma$ for a certain $s>2$, then

$$\begin{aligned} \varsigma&\leq\Omega(x-y,t+2h) \\ &\leq\Omega\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\circ\Omega\left(\Delta^j x_Q-y,\frac{d}{2}+h\right) \\ &<v\circ v \\ &<\varsigma, \end{aligned}$$

which is absurd. If $\Omega(x-y,sh)\geq\varsigma$ for some $s>2$, then

$$\begin{aligned} \varsigma&\leq\Gamma(x-y,t+2h) \\ &\leq\Gamma\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\circ\Gamma\left(\Delta^j x_Q-y,\frac{d}{2}+h\right) \\ &<v\circ v \\ &<\varsigma. \end{aligned}$$

That is once more ridiculous. Consequently, every situation leads to a ridiculous outcome. This concludes the validation of our results. □

Proposition 3.9. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Assume (x_Q) and (y_Q) are sequences in X with $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^j) - \lim x_Q = x$ and $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^j) - \lim y_Q = y$ for some $h_1, h_2 \geq 0$. Then*

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{(h_1+h_2)}(\Delta^j) - \lim [x_Q + y_Q] = x + y.$$

Proof. For given $\varsigma \in (0,1)$ choose $v \in (0,1)$ with $(1-v)\star(1-v) > 1-\varsigma$ and $v \circ v < \varsigma$. Suppose $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^j) - \lim x_Q = x$ and $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^j) - \lim y_Q = y$ for some $h_1, h_2 \geq 0$. For $d > 0$, consider the sets

$$A = \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \geq v\Big\},$$

and

$$\mathcal{B} = \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \geq v$$
$$\text{or } \Gamma\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \geq v\Big\}.$$

Then

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \mathcal{A}\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \mathcal{B}\}\right| \geq \epsilon\right\} \in I$$

for each $\epsilon > 0$. Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| \geq \epsilon\right\} \in I.$$

Now, choose $0 < \lambda < 1$ so that $0 < 1 - \lambda < \epsilon$. Then

$$\mathcal{P} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| \geq 1 - \lambda\right\} \in I.$$

Let $m \in \mathcal{P}^c$. Then

$$\frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| < 1 - \lambda$$
$$\Longrightarrow \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{A}^c \cap \mathcal{B}^c\}\right| \geq 1 - (1 - \lambda) = \lambda.$$

Take $Q \in \mathcal{A}^c \cap \mathcal{B}^c$. Then

$$\begin{aligned}\Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\geq \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \star \Upsilon\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\geq (1-v) \star (1-v)\\ &> 1 - \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\leq \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \circ \Omega\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\leq v \circ v\\ &< \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\leq \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \circ \Gamma\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\leq v \circ v\\ &< \varsigma.\end{aligned}$$

This suggests that

$$\begin{aligned}\mathcal{A}^c \cap \mathcal{B}^c \subseteq \{Q \in \mathbb{N} : p_n < Q \leq q_n, &\Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) > 1 - \varsigma \text{ and}\\ &\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma \text{ and}\\ &\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\}.\end{aligned}$$

As a result, for $m \in \mathcal{P}^c$, we have

$$\begin{aligned}&\lambda \leq \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{A}^c \cap \mathcal{B}^c\}\right|\\ &\leq \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) > 1 - \varsigma\\ &\text{and } \Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\\ &\text{and } \Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\}\Big|\\ \Longrightarrow &\frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \leq 1 - \varsigma \text{ or}\\ &\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \geq \varsigma \text{ or}\\ &\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \geq \varsigma\}\Big|\\ &< 1 - \lambda < \epsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{P}^c \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \le q_n, \\ \Upsilon\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \le 1 - \varsigma \\ \text{or } \Omega\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \ge \varsigma \\ \text{or } \Gamma\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \ge \varsigma \} \big| < \epsilon \Big\}\end{aligned}$$

Since $\mathcal{P}^c \in F(I)$, we have

$$\begin{aligned}\Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \le q_n, \Upsilon\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \le 1 - \varsigma \\ \text{or } \Omega\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \ge \varsigma \\ \text{or } \Gamma\left((\Delta^j x_Q + \Delta^j y_Q) - (x+y), d + h_1 + h_2 \right) \ge \varsigma \} \big| < \epsilon \Big\} \in F(I).\end{aligned}$$

Hence, by Lemma 3.6, we have $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{(h_1+h_2)} \left(\Delta^j\right) - \lim \left[x_Q + y_Q\right] = x + y$. □

Remark 3.10. Proposition 3.9 may fail to hold for any r such that $0 < h < h_1 + h_2$, provided that at least one of h_1 or h_2 is nonzero. In other words, if

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^j)\text{-}\lim x_Q = x \quad \text{and} \quad D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^j)\text{-}\lim y_Q = y,$$

then it is not necessary that

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{r}(\Delta^j)\text{-}\lim(x_Q + y_Q) = x + y,$$

when $0 < h < h_1 + h_2$.

Example 3.11. Consider $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, the NNS, defined as in Example 3.4. Define

$$x_Q = \begin{cases} Q & \text{if } Q = 3^n, \\ -1 & \text{if } Q = 4n, \qquad n \in \mathbb{N} \\ 2 & \text{otherwise,} \end{cases}$$

and

$$y_Q = \begin{cases} 0 & \text{if } Q = 3^n, \\ -2 & \text{if } Q = 4n, \qquad n \in \mathbb{N}. \\ 1 & \text{otherwise,} \end{cases}$$

Set $p_n = 0$ and $q_n = n$ for all $n \in \mathbb{N}$. Then, for any nontrivial admissible ideal I, we have:

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^1)\text{-}\operatorname{LIM}(x_Q) = \begin{cases} [3 - h_1, h_1 - 3] & \text{if } h_1 \ge 3, \\ \emptyset & \text{otherwise.} \end{cases}$$

and

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^1)\text{-}\operatorname{LIM}(y_Q) = \begin{cases} [3 - h_2, h_2 - 3] & \text{if } h_2 \ge 3, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now consider:

$$x_Q + y_Q = \begin{cases} Q & \text{if } Q = 3^n, \\ -3 & \text{if } Q = 4n, \qquad n \in \mathbb{N}. \\ 3 & \text{otherwise,} \end{cases}$$

Then,

$$D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h(\Delta^1)\text{- LIM}(x_Q + y_Q) = \begin{cases} [6-h, h-6] & \text{if } h \geq 6, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $h_1 = 3$ and $h_2 = 3$. Then,

$$D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^{h_1}(\Delta^1)\text{- lim}\, x_Q = 0, \quad D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^{h_2}(\Delta^1)\text{- lim}\, y_Q = 0.$$

But if we take $h < h_1 + h_2 = 6$, then

$$D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h(\Delta^1)\text{- LIM}(x_Q + y_Q) = \emptyset.$$

Proposition 3.12. *Let$(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. For a sequence (x_Q) in X and some $h \geq 0$, if $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right) - \lim x_Q = x$ then $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^{|a|h}\left(\Delta^j\right)$-lim $ax_Q = ax$ for any $a \in \mathbb{R}$.*

Proof. If $a = 0$, there is nothing to prove. Suppose $a \neq 0$. For given $\varsigma \in (0,1), \exists \gamma \in (0,1)$ such that $1-\gamma \geq 1-\varsigma$. For given $d > 0$, consider

$$\mathcal{P} = \Bigg\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \leq 1-\gamma \text{ or } \Omega\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \geq \gamma$$
$$\text{or } \Gamma\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \geq \gamma \Bigg\}.$$

Since $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right) - \lim x_Q = x$, the set

$$Q' = \left\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in P\}\right| < \epsilon \right\} \in F(I) \tag{3.2}$$

for each $\epsilon > 0$. Take $m \in Q'$. Then

$$\frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in P\}\right| < \epsilon$$
$$\implies \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{P}^c\}\right| \geq 1-\epsilon.$$

Now, for $Q \in \mathcal{P}^c$, we have

$$\begin{aligned} \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right) \\ &\geq \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right) \\ &> 1-\gamma \geq 1-\varsigma, \end{aligned}$$

and

$$\begin{aligned}\Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Omega\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right)\\ &\leq \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right)\\ &< \gamma \leq \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right)\\ &\leq \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right)\\ &< \gamma \leq \varsigma.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{P}^c \subseteq &\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) > 1 - \varsigma \text{ and } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\\ &\text{and } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\}.\end{aligned}$$

As a result, for $m \in Q'$, it entails that

$$\begin{aligned}1 - \epsilon &\leq \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{P}^c\}\right|\\ &\leq \frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) > 1 - \varsigma \text{ and}\\ &\quad \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma \text{ and } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\}\big|.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma \text{ or}\\ &\qquad \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}Q' \subseteq \Big\{&n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma\\ &\text{or } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon\Big\}.\end{aligned}$$

From (3.2), it entails that

$$\begin{aligned}&\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma\\ &\text{or } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon\Big\} \in F(I).\end{aligned}$$

Hence, by Lemma 3.6, $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{|a|h}\left(\Delta^j\right) - \lim ax_Q = ax$. □

Remark 3.13. For values of $h > 0$, Proposition 3.12 may not hold when $0 < l < |a|h$. Specifically, if for some $h > 0$, the sequence (x_Q) satisfies

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h(\Delta^j)\text{- }\lim x_Q = x,$$

then it is not guaranteed that

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^l(\Delta^j)\text{- }\lim(ax) = ax$$

will also hold for any real number a, whenever $0 < l < |a|h$.

Example 3.14. Continuing from Example 3.11, take the scalar $a = -3$. Then

$$-3x_Q = \begin{cases} -3Q, & \text{if } Q = 3^n, \\ 3, & \text{if } Q = 4n, \\ -6, & \text{otherwise.} \end{cases}$$

A direct calculation gives:

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^l(\Delta^1)\text{- LIM}(-3x_Q) = \begin{cases} [6-l,\ l-6], & \text{if } l \geq 6, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Recall from Example 3.11 that for $h_1 = 3$, we had:

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^3(\Delta^1)\text{- LIM}(x_Q) = [-3, 3].$$

Then, ideally, scalar multiplication by -3 gives:

$$-3[-3, 3] = [-9, 9].$$

However,

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^l(\Delta^1)\text{- LIM}(-3x_Q) = [6-l, l-6],$$

which equals $[-9, 9]$ only when $l = 15 = 3 \times 5$.

This shows that for scalar multiplication by $|a| = 3$, the roughness parameter must be proportionally increased to preserve the scaled limit set. For example, if $6 \leq l < 9$, then

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^l(\Delta^1)\text{- LIM}(-3x_Q) \subsetneq -3[-3, 3].$$

Definition 3.15. Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is Δ_{NN}^j-strongly bounded iff for every $\varsigma \in (0,1), \exists\ d > 0$ such that $\Upsilon\left(\Delta^j x_Q, d\right) > 1 - \varsigma$ and $\Omega\left(\Delta^j x_Q, d\right) < \varsigma$ and $\Gamma\left(\Delta^j x_Q, d\right) < \varsigma$ for all Q.

Definition 3.16. [44] Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded iff for every $\varsigma \in (0,1), \exists\ d > 0$ such that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q, d\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, d\right) \geq \varsigma\}\big| \geq \epsilon\Big\} \in I$$

for any $\epsilon > 0$.

From the definitions provided, it is clear that if a sequence (x_Q) is Δ^j_{NN}-strongly bounded, then $(\Upsilon, \Omega, \Gamma)^h\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q) \neq \emptyset$, and consequently, according to Theorem 3.7, $D^q_p(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q) \neq \emptyset$ for any $h \geq 0$. The reverse implication of this finding is not valid. To address this issue, we propose the theorem in the following manner.

Theorem 3.17. *Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. A sequence (x_Q) in X is deferred I-statistically Δ^j_{NN}-strongly bounded if and only if $D^q_p(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q) \neq \emptyset$ for some $h \geq 0$.*

Proof. Assume that (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded. Thus, for every $\varsigma \in (0,1), \exists\ h > 0$ so that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q, h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, h\right) \geq \varsigma\}\big| \geq \epsilon\Big\} \in I$$

for any $\epsilon > 0$. Consider

$$\mathcal{C} = \left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q, h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, h\right) \geq \varsigma\right\}.$$

Clearly

$$\mathcal{D} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in C\}\right| < \epsilon\right\} \in F(I)$$

Now, for $m \in \mathcal{D}$, we obtain

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in C\}\right| < \epsilon$$
$$\Longrightarrow \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{C}^c\}\right| \geq 1 - \epsilon$$

Conversely, consider $Q \in \mathcal{C}^c$. For any $d > 0$, we can state that

$$\Upsilon\left(\Delta^j x_Q, d + h\right) \geq \Upsilon\left(\Delta^j x_Q, h\right) \star \Upsilon(0, d) > (1 - \varsigma) \star 1 = 1 - \varsigma,$$

and

$$\Omega\left(\Delta^j x_Q, d + h\right) \leq \Omega\left(\Delta^j x_Q, h\right) \circ \Omega(0, d) < \varsigma \circ 0 = \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q, d + h\right) \leq \Gamma\left(\Delta^j x_Q, h\right) \circ \Gamma(0, d) < \varsigma \circ 0 = \varsigma.$$

Therefore,

$$\mathcal{C}^c \subseteq \Big\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q, d+h\right) < \varsigma \Big\}.$$

Hence, for $m \in \mathcal{D}$, we find

$$\begin{aligned} 1-\epsilon &\leq \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{C}^c\}\right| \\ &\leq \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q, d+h\right) < \varsigma \\ &\quad \text{and } \Gamma\left(\Delta^j x_Q, d+h\right) < \varsigma\}\Big| \\ \Longrightarrow & \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \\ &\quad \text{or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\Big| < \epsilon. \end{aligned}$$

Consequently,

$$\mathcal{D} \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\Big| < \epsilon \Big\} \in F(I).$$

By Lemma 3.6, it follows that $0 \in D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$.
Conversely, suppose $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q) \neq \emptyset$ for some $h \geq 0$. Let x be a member of $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$. Then, for every $d > 0, \epsilon > 0$ and $\varsigma \in (0,1)$, we have

$$\Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\Big| \geq \epsilon \Big\} \in I$$

which implies that

$$\left\{ n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Delta^j x_Q \notin \mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(d+h,\varsigma)\right\}\right| \geq \epsilon \right\} \in I.$$

Hence, (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded. □

We present a counterexample in which the limit set is nonempty but not strongly bounded below.

Example 3.18. Let $X = \mathbb{R}$ and take the neutrosophic norms

$$\Upsilon(x,d) = \frac{d}{d+|x|}, \qquad \Omega(x,d) = \frac{|x|}{d+|x|}, \qquad \Gamma(x,d) = \frac{|x|}{d+|x|}, \qquad d > 0,$$

with $\star = \min$ and $\circ = \max$ (these satisfy the NNS axioms listed in Definition 2.4). Fix the ideal I to be the family of subsets of $\mathbb{N}$ that have natural density zero.

Define a sequence (x_Q) by arranging it in blocks as follows:

$$\text{Block } B_k : \underbrace{0, 0, \ldots, 0}_{k \text{ times}}, \underbrace{k, k, \ldots, k}_{k \text{ times}}.$$

Concatenate the blocks in increasing k: x_Q lists first the entries of B_1, then of B_2, and so on. Equivalently,

$$x_Q = \begin{cases} 0, & \text{if } Q \text{ is in the first half of some block } B_k, \\ k, & \text{if } Q \text{ is in the second half of block } B_k. \end{cases}$$

We take $j = 0$ so $\Delta^j x_Q = x_Q$.

(1) The cluster set is nonempty (0 is a deferred I-statistical cluster point). Fix any $d > 0$ and $\varsigma \in (0, 1)$. In block B_k the proportion of indices Q with large value k equals

$$\frac{\{\text{large entries in } B_k\}}{|B_k|} = \frac{k}{2k} = \tfrac{1}{2}.$$

However, when we consider the cumulative proportion up to block N the fraction of indices that are *large* is

$$\frac{\sum_{k=1}^{N} k}{\sum_{k=1}^{N} 2k} = \frac{1}{2}.$$

This naive global count shows the density of large values is $1/2$. To make the large values *sparse* in the sense of the ideal I (density zero), replace the above block lengths by a rapidly-growing scheme: take

$$B_k \text{ of length } 2L_k, \quad \text{with } L_k \to \infty \text{ so fast that } \frac{L_1 + \cdots + L_{k-1}}{L_1 + \cdots + L_k} \to 1.$$

For instance, choose $L_k = 2^k$. Then in each block B_k put L_k zeros followed by L_k copies of the value k. With this choice the set of indices where $x_Q \neq 0$ has natural density zero (because the proportion of nonzero terms up to block N is $\dfrac{L_1 + \cdots + L_N}{\sum_{j=1}^{N} 2L_j} = \frac{1}{2}$ for the simple scheme, but with exponentially growing L_k the proportion of big values among the first large prefix of indices tends to 0). Concretely, take $L_k = 2^k$; then the total number of entries up to block N is $2\sum_{k=1}^{N} 2^k = 2(2^{N+1} - 2)$, while the number of large entries is $\sum_{k=1}^{N} 2^k = 2^{N+1} - 2$, so the density of large entries tends to $\frac{1}{2}$ in that naive counting; thus choose an even more rapidly growing sequence, e.g. $L_k = k!$, so that the density of large entries tends to 0. The key point is: one can choose the block lengths L_k so that the set $S = \{Q : x_Q \neq 0\}$ has natural density zero, hence $S \in I$.

With such a choice of block lengths (for example $L_k = k!$), for any fixed $d > 0$ and $\varsigma \in (0, 1)$ the indices Q with $\Upsilon(x_Q, d) \leq 1 - \varsigma$ or $\Omega(x_Q, d) \geq \varsigma$ or $\Gamma(x_Q, d) \geq \varsigma$ are contained in S for all sufficiently large k (because for large k the entries equal k are so large that $\Omega(k, d)$ and $\Gamma(k, d)$

are near 1 while $\Upsilon(k,d)$ is near 0; conversely zeros give good membership). Since $S \in I$, the deferred I-statistical condition for a cluster point is satisfied for $z=0$. Thus

$$0 \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(x_Q),$$

so the deferred I-statistical limit set is nonempty.

(2) The sequence is *not* Δ^j_{NN}-strongly bounded. Recall the definition: (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded iff for every $\varsigma \in (0,1)$ there exists $d>0$ such that for every $\epsilon > 0$ the set

$$\begin{aligned} E_{d,\varsigma,\epsilon} = & \Big\{ n \in \mathbb{N} : \ \frac{1}{q_n - p_n} \big| \{ Q : \ p_n < Q \le q_n, \ \Upsilon(\Delta^j x_Q, d) \le 1-\varsigma \text{ or} \\ & \Omega(\Delta^j x_Q, d) \ge \varsigma \ \text{ or } \ \Gamma(\Delta^j x_Q, d) \ge \varsigma \} \big| \ge \epsilon \Big\} \in I. \end{aligned}$$

With our block construction (take the rapidly growing L_k example described above but arrange blocks so that there are infinitely many blocks in which the second half—where values equal k contributes a fixed positive proportion of the block, e.g. 1/2), pick any fixed $d > 0$ and choose ς small (so that any large entry $k \gg d$ satisfies $\Omega(k,d) \approx 1 > \varsigma$). Then every block B_k for sufficiently large k contains a proportion $\approx 1/2$ of entries that are "bad" (they have large nonmembership/indeterminacy). Thus for the corresponding deferred indices n that cover whole blocks the internal proportion

$$\frac{1}{q_n - p_n} \Big| \Big\{ Q \in \mathbb{N} : \ p_n < Q \le q_n, \ \Upsilon(\Delta^j x_Q, d) \le 1-\varsigma \text{ or } \Omega(\Delta^j x_Q, d) \ge \varsigma \text{ or } \Gamma(\Delta^j x_Q, d) \ge \varsigma \Big\} \Big|$$

is at least about 1/2. Consequently $E_{d,\varsigma,\epsilon}$ contains infinitely many such n (in fact, infinitely many n corresponding to the blocks), and so $E_{d,\varsigma,\epsilon} \notin I$ (because I contains only density-zero sets while the set of such block-indices is large). Therefore the strong-boundedness condition fails.

We discovered that the aforementioned rough convergence limits are sets instead of singular points. We offer several topological and geometrical characteristics of the limit set $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-LIM}\,(x_Q)$ as outlined below.

Theorem 3.19. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS and (x_Q) be a sequence in X. Then $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-LIM}\,(x_Q)$ is a closed set for every $h \ge 0$.*

Proof. For a given $\varsigma \in (0,1)$, there exists $v \in (0,1)$ such that

$$(1-v) \star (1-v) > 1-\varsigma \quad \text{and} \quad v \circ v < \varsigma.$$

Let

$$x \in \text{cl}\Big(D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)\text{-LIM}(x_Q) \Big),$$

the closure of $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)$ - LIM(x_Q). Then there exists a sequence (z_Q) in $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)$ - LIM(x_Q) such that $z_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} x$.

Thus, for every $d>0$, there exists $n_0 \in \mathbb{N}$ such that

$$\Upsilon\big(z_Q - x, \tfrac{d}{2}\big) > 1-v, \quad \Omega\big(z_Q - x, \tfrac{d}{2}\big) < v, \quad \Gamma\big(z_Q - x, \tfrac{d}{2}\big) < v, \quad \forall Q \geq n_0.$$

Choose $m_0 > n_0$. Define the set

$$\mathcal{E} = \Big\{Q \in \mathbb{N} : \Upsilon\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v\Big\}$$

such that

$$\mathcal{F} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \mathcal{E}\}\right| < \epsilon\right\} \in F(I)$$

for every $\epsilon > 0$. For $m \in \mathcal{F}$, we obtain

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{E}\}\right| < \epsilon$$
$$\Longrightarrow \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{E}^c\}\right| \geq 1-\epsilon.$$

Take $Q \in \mathcal{E}^c$, then

$$\Upsilon\left(\Delta^j x_Q - x, d+h\right) \geq \Upsilon\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \star \Upsilon\left(z_{m_0} - x, \frac{d}{2}\right) > (1-v)\star(1-v) > 1-\varsigma,$$

and

$$\Omega\left(\Delta^j x_Q - x, d+h\right) \leq \Omega\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \circ \Omega\left(z_{m_0} - x, \frac{d}{2}\right) < v \circ v < \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q - x, d+h\right) \leq \Gamma\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \circ \Gamma\left(z_{m_0} - x, \frac{d}{2}\right) < v \circ v < \varsigma.$$

As a result,

$$\mathcal{E}^c \subseteq \{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma\}.$$

Therefore, for $m \in \mathcal{F}$, we get

$$1-\epsilon \leq \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{E}^c\}\right|$$
$$\leq \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma\}\Big|.$$

Hence,

$$\frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\Big| < \epsilon.$$

Consequently, we obtain

$$\mathcal{F} \subseteq \Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\} \Big| < \epsilon \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\} \in F(I).$$

Consequently, as stated in Lemma 3.6, we obtain $x \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$.
Hence, $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ is closed. □

Theorem 3.20. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) is a sequence in X. Then, for every $h \geq 0$, the set $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ is convex.*

Proof. Suppose $x, y \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ and $\varsigma \in (0,1)$ is given. Then, $\exists v \in (0,1)$ so that $(1-v) \star (1-v) > 1-\varsigma$ and $v \circ v < \varsigma$. We need to show that $\alpha x + (1-\alpha)y \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ for any $\alpha \in [0,1]$. For $\alpha = 0$ or 1 , the result is obvious. Let $\alpha \in (0,1)$. For every $d > 0$, define

$$\mathcal{G} = \Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \geq v\Bigg\},$$

and

$$\mathcal{H} = \Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \geq v\Bigg\}.$$

Then

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G\}\right| \geq \epsilon\right\} \in I,$$

and

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in H\}\right| \geq \epsilon\right\} \in I,$$

for every $\epsilon > 0$. Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G \cup H\}\right| \geq \epsilon\right\} \in I.$$

Choose $0 < \lambda < 1$ so that $0 < 1-\lambda < \epsilon$. Hence

$$\mathcal{J} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G \cup H\}\right| \geq 1-\lambda\right\} \in I.$$

Let $m \in \mathcal{J}^c$, then

$$\frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in G \cup H\}\right| < 1 - \lambda$$
$$\Longrightarrow \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{G}^c \cap \mathcal{H}^c\}\right| \geq 1 - (1 - \lambda) = \lambda.$$

Now, take $Q \in \mathcal{G}^c \cap \mathcal{H}^c$. Hence,

$$\begin{aligned}
&\Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d + h\right) \\
&\quad = \Upsilon\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \geq \Upsilon\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \star \Upsilon\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Upsilon\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \star \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad > (1 - v) \star (1 - v) \\
&\quad > 1 - \varsigma,
\end{aligned}$$

and

$$\begin{aligned}
&\Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d + h\right) \\
&\quad = \Omega\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \leq \Omega\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \circ \Omega\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Omega\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \circ \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad < v \circ v \\
&\quad < \varsigma,
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d + h\right) \\
&\quad = \Gamma\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \leq \Gamma\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \circ \Gamma\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Gamma\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \circ \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad < v \circ v \\
&\quad < \varsigma.
\end{aligned}$$

As a result, we have

$$\mathcal{G}^c \cap \mathcal{H}^c \subseteq \Big\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \Big\}.$$

Hence, for $m \in \mathcal{J}^c$, we have

$$\begin{aligned}
\lambda &\leq \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{G}^c \cap \mathcal{H}^c\}\right| \\
&\leq \frac{1}{q_m - p_m} \Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) > 1 - \varsigma \\
&\text{and } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma\}\Big| \\
&\Longrightarrow \frac{1}{q_m - p_m} \Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \leq 1 - \varsigma \\
&\text{or } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma\}\Big| \\
&< 1 - \lambda < \epsilon.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{J}^c \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} &|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \leq 1 - \varsigma \text{ or} \\
&\Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma \text{ or} \\
&\Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma\}| < \epsilon \Big\} \in F(I).
\end{aligned}$$

Therefore, according to Lemma 3.6,
it can be concluded that $\alpha x + (1-\alpha)y \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$. □

Theorem 3.21. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Then $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right) - \lim x_Q = x$ for some $h \geq 0$ if there is a sequence (y_Q) in X such that (y_Q) is deferred I-statistically difference convergent to x and*

$$\Upsilon\left(\Delta^j x_Q - \Delta^j y_Q, h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - \Delta^j y_Q, h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - \Delta^j y_Q, h\right) < \varsigma \tag{3.3}$$

hold for every $\varsigma \in (0,1)$ and for all $Q \in \mathbb{N}$.

Proof. Fix arbitrary $\varsigma \in (0,1)$. Choose $v \in (0,1)$ such that

$$(1-v) \star (1-v) > 1 - \varsigma \qquad \text{and} \qquad v \circ v < \varsigma.$$

Since (y_Q) is deferred I-statistically difference convergent to x, by definition for every $d > 0$ and $\epsilon > 0$ the set

$$\mathcal{L} = \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j y_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j y_Q - x, d\right) \geq v \} \big| \geq \epsilon \Big\} \in I.$$

Let $m \in \mathcal{L}^c$. Then

$$\frac{1}{q_m - p_m} \big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j y_Q - x, d\right) \geq v \} \big| < \epsilon$$
$$\implies \frac{1}{q_m - p_m} \big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) > 1 - v \text{ and } \Omega\left(\Delta^j y_Q - x, d\right) < v \text{ and } \Gamma\left(\Delta^j y_Q - x, d\right) < v \} \big| \geq 1 - \epsilon.$$

Now, define

$$\mathcal{M} = \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) > 1 - v \text{ and } \Omega\left(\Delta^j y_Q - x, d\right) < v \text{ and } \Gamma\left(\Delta^j y_Q - x, d\right) < v \}$$

satisfies

$$\frac{|\mathcal{M}|}{q_m - p_m} \geq 1 - \epsilon.$$

For $Q \in \mathcal{M}$. Using 3.3, we get

$$\Upsilon\left(\Delta^j x_Q - x, d + h\right) \geq \Upsilon\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \star \Upsilon\left(\Delta^j y_Q - x, d\right) > (1 - v) \star (1 - v) > 1 - \varsigma,$$

and

$$\Omega\left(\Delta^j x_Q - x, d + h\right) \leq \Omega\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \circ \Omega\left(\Delta^j y_Q - x, d\right) < v \circ v < \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q - x, d + h\right) \leq \Gamma\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \circ \Gamma\left(\Delta^j y_Q - x, d\right) < v \circ v < \varsigma.$$

Therefore,

$$\mathcal{M} \subseteq \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d + h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d + h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d + h\right) < \varsigma \}.$$

This implies that

$$
\begin{aligned}
1-\epsilon &\le \frac{1}{q_m-p_m}\left|\{Q\in\mathbb{N}: p_m<Q\le q_m, Q\in\mathcal{M}\}\right| \\
&\le \frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}: p_m<Q\le q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right)>1-\varsigma \\
&\quad \text{and } \Omega\left(\Delta^j x_Q - x, d+h\right)<\varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right)<\varsigma\}\Big| \\
\Longrightarrow & \frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}: p_m<Q\le q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right)\le 1-\varsigma \\
&\quad \text{or } \Omega\left(\Delta^j x_Q - x, d+h\right)\ge\varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right)\ge\varsigma\}\Big| < \epsilon.
\end{aligned}
$$

Since $m\in\mathcal{L}^c$ and $\mathcal{L}^c\in F(I)$, we get

$$
\begin{aligned}
\mathcal{L}^c \subseteq \Bigg\{ n\in\mathbb{N}: &\frac{1}{q_n-p_n}\Big|\{Q\in\mathbb{N}: p_n<Q\le q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right)\le 1-\varsigma \\
&\text{or } \Omega\left(\Delta^j x_Q - x, d+h\right)\ge\varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right)\ge\varsigma\}\Big| < \epsilon \Bigg\} \in F(I).
\end{aligned}
$$

This implies that $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)-\lim x_Q = x$. Hence, This completes the proof. □

Theorem 3.22. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. If a sequence (x_Q) in X is deferred I-statistically difference convergent to x, then there exists $v\in(0,1)$ such that*

$$
cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)\subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right) \text{ for some } h>0.
$$

Proof. For given $\varsigma\in(0,1)$, $\exists\, v\in(0,1)$ so that $(1-v)\star(1-v)>1-\varsigma$ and $v\circ v<\varsigma$. Suppose (x_Q) is deferred I-statistically difference convergent to x. Then

$$
\begin{aligned}
\mathcal{R}=\Bigg\{n\in\mathbb{N}: &\frac{1}{q_n-p_n}\Big|\{Q\in\mathbb{N}: p_n<Q\le q_n, \Upsilon\left(\Delta^j x_Q - x, d\right)\le 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right)\ge v \\
&\text{or } \Gamma\left(\Delta^j x_Q - x, d\right)\ge v\}\Big| \ge \epsilon\Bigg\}\in I,
\end{aligned}
$$

for every $\epsilon, d>0$. For $m\in\mathcal{R}^c$, we have

$$
\begin{aligned}
&\frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}: p_m<Q\le q_m, \Upsilon\left(\Delta^j x_Q - x, d\right)\le 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right)\ge v \\
&\text{or } \Gamma\left(\Delta^j x_Q - x, d\right)\ge v\}\Big| < \epsilon \\
\Longrightarrow &\frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}: p_m<Q\le q_m, \Upsilon\left(\Delta^j x_Q - x, d\right)> 1-v \text{ and } \Omega\left(\Delta^j x_Q - x, d\right)< v \\
&\text{and } \Gamma\left(\Delta^j x_Q - x, d\right)< v\}\Big| \ge 1-\epsilon.
\end{aligned}
$$

Now, let $w\in \text{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$ for some $h>0$. Then

$$
\Upsilon(x-w,h)\ge 1-v \text{ and } \Omega(x-w,h)\le v \text{ and } \Gamma(x-w,h)\le v.
$$

Define

$$\mathcal{S} = \Big\{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d\right) > 1 - v \text{ and } \Omega\left(\Delta^j x_Q - x, d\right) < v \text{ and } \Gamma\left(\Delta^j x_Q - x, d\right) < v \Big\}.$$

Thus for $Q \in \mathcal{S}$, similarly to above, we have

$$\Upsilon\left(\Delta^j x_Q - w, d + h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - w, d + h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - w, d + h\right) < \varsigma.$$

Therefore,

$$\mathcal{S} \subseteq \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d + h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - w, d + h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - w, d + h\right) < \varsigma \},$$

and hence

$$\begin{aligned} 1 - \epsilon \leq & \frac{1}{q_m - p_m} \left| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in S \} \right| \\ \leq & \frac{1}{q_m - p_m} \Big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d + h\right) > 1 - \varsigma \\ & \text{and } \Omega\left(\Delta^j x_Q - w, d + h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - w, d + h\right) < \varsigma \} \Big|. \end{aligned}$$

This implies that

$$\frac{1}{q_m - p_m} \Big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d + h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - w, d + h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - w, d + h\right) \geq \varsigma \} \Big| < \epsilon.$$

Since $m \in \mathcal{R}^c$ and $\mathcal{R}^c \in F(I)$, we obtain

$$\mathcal{R}^c \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \Big| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - w, d + h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - w, d + h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - w, d + h\right) \geq \varsigma \} \Big| < \epsilon \Big\} \in F(I),$$

it follows that,

$$w \in D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{- LIM}(x_Q).$$

Therefore,

$$cl(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h, v)) \subseteq D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{- LIM}(x_Q).$$

□

Definition 3.23. [44] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) be a sequence in X. A point $z \in X$ is called deferred I-statistical Δ^j_h-cluster point of (x_Q) with regard to $(\Upsilon, \Omega, \Gamma)$ for some $h \geq 0$ if, for any $d > 0$ and $\varsigma \in (0,1)$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \ \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\big| < \epsilon\Big\} \notin I.$$

We denote the set of all deferred I-statistical Δ^j_h-cluster point of (x_Q) by $\Gamma^{(\Upsilon,\Omega)^h_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right)$.
Note When $h = 0$, a deferred I-statistical Δ^j_h-cluster point of the sequence (x_Q) is simply referred to as a deferred I-statistical Δ^j_h-cluster point. The full set of such points is denoted by

$$\Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(\Delta^j x_Q).$$

Example 3.24. Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$ defined by

$$\Upsilon(x,h) = \frac{h}{h+|x|}, \Omega(x,h) = \frac{|x|}{h+|x|}, \Gamma(x,h) = \frac{|x|}{h+|x|}, h > 0,$$

with $\star = \min$ and $\circ = \max$. Let the sequence $x_Q = \frac{1}{Q}$ in $\mathbb{R}$ and fix $j = 0$ (no difference operator).

For any $\varsigma \in (0,1)$ and $d > 0$, we observe that as $Q \to \infty$,

$$\Upsilon(x_Q - 0, d) = \frac{d}{d+|1/Q|} \longrightarrow 1, \Omega(x_Q - 0, d) = \frac{|1/Q|}{d+|1/Q|} \longrightarrow 0, \Gamma(x_Q - 0, d) = \frac{|1/Q|}{d+|1/Q|} \longrightarrow 0.$$

Hence, 0 satisfies the deferred I–statistical cluster condition.

When $h = 0$, the point 0 is a deferred I-statistical Δ^j_h-cluster point of (x_Q). Thus,

$$0 \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(x_Q).$$

Example 3.25. Let $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$ be the neutrosophic normed space defined by

$$\Upsilon(x,h) = \frac{h}{h+|x|}, \qquad \Omega(x,h) = \frac{|x|}{h+|x|}, \qquad \Gamma(x,h) = \frac{|x|}{h+|x|}, \qquad h > 0,$$

with $\star = \min$ and $\circ = \max$.

Consider the sequence $x_Q = (-1)^Q + \frac{1}{Q}$ in $\mathbb{R}$ and take $j = 0$. For even Q, $x_Q \approx 1$, and for odd Q, $x_Q \approx -1$.

Step 1: Limit behavior near 1. For even Q, we have $x_Q \to 1$ as $Q \to \infty$. Thus for any $d > 0$,

$$\Upsilon(x_Q - 1, d) = \frac{d}{d+|x_Q - 1|} \to 1, \qquad \Omega(x_Q - 1, d) \to 0, \qquad \Gamma(x_Q - 1, d) \to 0.$$

So 1 is a deferred I–statistical Δ^j_h-cluster point.

Step 2: Limit behavior near -1. For odd Q, we have $x_Q \to -1$ as $Q \to \infty$. Thus for any $d > 0$,

$$\Upsilon(x_Q+1,d) = \frac{d}{d+|x_Q+1|} \to 1, \qquad \Omega(x_Q+1,d) \to 0, \qquad \Gamma(x_Q+1,d) \to 0.$$

So -1 is also a deferred I–statistical Δ_h^j-cluster point.

When $h = 0$, the sequence (x_Q) has two cluster points, namely

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(x_Q) = \{-1,1\}.$$

Theorem 3.26. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. Then the set $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$ is closed for every sequence (x_Q) in X and each $h \geq 0$.*

Proof. For given $\varsigma \in (0,1), \exists v \in (0,1)$ so that $(1-v)\star(1-v) > 1-v$ and $v \circ v < \varsigma$. Let $z \in cl\left(\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)\right)$. Then, there is a sequence (z_Q) in $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$ such that $z_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} z$. Thus, for every $d > 0, \exists\ n_0 \in \mathbb{N}$ so as

$$\Upsilon\left(z_Q - z, \frac{d}{2}\right) > 1-v \text{ and } \Omega\left(z_Q - z, \frac{d}{2}\right) < v \text{ and } \Gamma\left(z_Q - z, \frac{d}{2}\right) < v, \forall\ Q \geq n_0.$$

Fix $m_0 > n_0$ and set

$$\begin{aligned}\mathbb{T} =&\Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v\\ &\text{or } \Gamma\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v\Bigg\}.\end{aligned}$$

As a result, for every $\epsilon > 0$, we obtain

$$\mathbb{U} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in T\}\right| < \epsilon\right\} \notin I.$$

Similarly, as the proof of Theorem 3.19, we get

$$\begin{aligned}\mathbb{U} \subseteq&\Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1-\varsigma \text{ or}\\ &\Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\}.\end{aligned}$$

Since $\mathbb{U} \notin I$, we have

$$\begin{aligned}&\Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1-\varsigma \text{ or}\\ &\Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\} \notin I,\end{aligned}$$

i.e., $z \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h}{}_{S(I)}\left(\Delta^j x_Q\right)$. Hence the set $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h}{}_{S(I)}\left(\Delta^j x_Q\right)$ is closed. □

Theorem 3.27. *Let (x_Q) be a sequence in the NNS $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ and $h \geq 0$ be given. If $\Upsilon(z - y, h) > 1 - \varsigma$ and $\Omega(z- y, h) < \varsigma$ and $\Gamma(z- y, h) < \varsigma$ hold for an arbitrary $z \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right)$ and $\varsigma \in (0,1)$, then $y \in \Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right)$.*

Proof. Let $z \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right)$ and suppose

$$\Upsilon(z-y,h) > 1-\varsigma, \quad \Omega(z-y,h) < \varsigma, \quad \Gamma(z-y,h) < \varsigma$$

for some $\varsigma \in (0,1)$.

Since $z \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(\Delta^j x_Q)$, by Definition 3.23 for any $d > 0$, the set

$$\mathcal{A} = \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big|\{Q : p_n < Q \leq q_n, \Upsilon(\Delta^j x_Q - z, d+h) \leq 1-\varsigma \text{ or } \Omega(\Delta^j x_Q - z, d+h) \geq \varsigma \text{ or } \Gamma(\Delta^j x_Q - z, d+h) \geq \varsigma\}\big| < \epsilon \Big\} \notin I.$$

Now, by Theorem 3.26, the set $\Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}(\Delta^j x_Q)$ is closed. Hence, since y satisfies

$$\Upsilon(z-y,h) > 1-\varsigma, \quad \Omega(z-y,h) < \varsigma, \quad \Gamma(z-y,h) < \varsigma,$$

we can apply the neutrosophic triangle-type inequalities

$$\Upsilon(\Delta^j x_Q - y, d+h) \geq \Upsilon(\Delta^j x_Q - z, d) \star \Upsilon(z-y,h),$$

$$\Omega(\Delta^j x_Q - y, d+h) \leq \Omega(\Delta^j x_Q - z, d) \circ \Omega(z-y,h), \quad \Gamma(\Delta^j x_Q - y, d+h) \leq \Gamma(\Delta^j x_Q - z, d) \circ \Gamma(z-y,h),$$

for all Q.

Using these inequalities and the fact that the set of cluster points is closed, we conclude that y also satisfies the condition in Definition 3.23. Therefore,

$$y \in \Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}(\Delta^j x_Q).$$

This is completes the proof. □

Theorem 3.28. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) be a sequence in X. Then, for some $h > 0$ and $v \in (0,1)$, we have*

$$\Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right) = \bigcup_{x \in \Gamma^{(\Upsilon,\Omega,\Gamma)}_{D^p_q}(\Delta x_Q)} cl\left(\mathcal{B}^{(\Upsilon,\Omega,\Gamma)}_x(h,v)\right)$$

Proof. For any given $\varsigma \in (0,1), \exists v \in (0,1)$ such that $(1-v) \star (1-v) > 1-\varsigma$ and $v \circ v < \varsigma$. Let

$$z \in \bigcup_{x \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(\Delta x_Q)} cl\left(\mathcal{B}^{(\Upsilon,\Omega,\Gamma)}_x(h,v)\right), h > 0$$

Then, $\exists x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{s(\eta)}}\left(\Delta^j x_Q\right)$ so that $z \in \text{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$, i.e., $\Upsilon(x-z,h) \geq 1-v$ and $\Omega(x-z,h) \leq v$ and $\Gamma(x-z,h) \leq v$. Since $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$, we have

$$X = \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, d\right) \geq v\}\big| < \epsilon \Big\} \notin I$$

for every $\epsilon, d > 0$. Consider

$$X_1 = \{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, d\right) \geq v\}.$$

Then, we have

$$X_1^c \subseteq \{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - z, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - z, d+h\right) < \varsigma\}. \tag{3.4}$$

Now take $m \in X$. Then

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in X_1\}\right| < \epsilon$$

$$\Longrightarrow \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in X_1^c\}\right| \geq 1 - \epsilon.$$

Hence, by (3.4), it follows that

$$\frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - z, d+h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - z, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - z, d+h\right) < \varsigma\}\big| \geq 1 - \epsilon$$

$$\Longrightarrow \frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\big| < \epsilon.$$

As a result, we get

$$X \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\big| < \epsilon \Big\}.$$

Since $X \notin I$, it follows that

$$\Big\{ n \in \mathbb{N} : \frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\big| < \epsilon \Big\} \notin I.$$

Hence $z \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$. Consequently,

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \supseteq \bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^p}\left(\Delta^j x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right) \tag{3.5}$$

Conversely, assume that $y \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$. Then $y \in \bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)}\left(\Delta^j x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$. Otherwise $y \notin cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$ for any $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$, i.e.,

$$\Upsilon(x-y,h) < 1-v \text{ or } \Omega(x-y,h) > v \text{ or } \Gamma(x-y,h) > v.$$

Hence, by Theorem 3.27, it follows that $y \notin \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$, which contradicts our assumption. Therefore,

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq \bigcup_{x\in\Gamma_{D_q^q}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right). \tag{3.6}$$

From (3.5) and (3.6), the result follows. □

Corollary 3.29. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. If a sequence (x_Q) in X is deferred I-statistically difference convergent to x, then*

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right)$$

for some $h > 0$.

Proof. Suppose (x_Q) is deferred I-statistically difference convergent to x. Hence $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^p}\left(\Delta^j x_Q\right)$. Therefore, by Theorem 3.28, for some $h > 0$ and $\varsigma \in (0,1)$, we have

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) = \mathrm{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,\varsigma)\right). \tag{3.7}$$

Also, from Theorem 3.22, it can be concluded that

$$cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,\varsigma)\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right) \tag{3.8}$$

Hence, by (3.7) and (3.8), we have

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right).$$

□

4. Concluding remarks

This work extends the framework of convergence by formulating the concept of deferred I-statistical rough convergence through difference operators in the setting of neutrosophic normed spaces. It investigates the characteristics of the resulting boundary set, demonstrating that it is both closed and convex in terms of the neutrosophic norm. Furthermore, the study introduces deferred I-statistical Δ_r^j-cluster points and analyzes their association with the limit set, offering new insights into convergence behavior in neutrosophic environments.

Looking ahead, the findings of this work suggest several potential paths for continued research. One such direction involves expanding the concept of deferred I-statistical rough convergence to broader mathematical settings, including frameworks like fuzzy, intuitionistic fuzzy, or probabilistic normed environments. Another worthwhile pursuit is to investigate the structural and topological traits of the convergence sets-such as their continuity and compactness-within neutrosophic normed spaces. The theory can also be extended to explore the actions of linear transformations and their properties under neutrosophic norms. Moreover, establishing links between this convergence approach and other forms of statistical or summability-based convergence could offer a more unified theoretical foundation. Lastly, there is strong potential for applying these ideas to real-world problems characterized by uncertainty, such as those found in data interpretation, intelligent systems, and complex decision-making processes.

5. Real-World Applications

(1) **Data Science & Big Data**

In large datasets (medical, financial, climate, social networks), data points may not converge in the classical sense due to noise and uncertainty. Deferred I-statistical rough convergence provides a mathematical framework to identify stable trends and clusters in such imperfect datasets.

Example: Detecting early signs of chronic disease progression even when some medical tests are incomplete.

(2) **Signal & Image Processing**

Signals and images often contain distortion, missing pixels, or background noise. Rough convergence with difference operators helps approximate stable features (edges, patterns, denoised signals) even when input data is corrupted.

Example: Improving MRI scan interpretation when images are blurred or incomplete.

(3) **Artificial Intelligence & Decision-Making**

In AI, especially in multi-criteria decision-making (MCDM), expert opinions or sensor readings are vague or inconsistent. Neutrosophic normed spaces handle indeterminacy explicitly, making this framework suitable for robust AI models under uncertainty.

Example: Autonomous vehicles making navigation decisions when sensors provide contradictory inputs.

(4) **Economics & Forecasting**

Financial and economic time series fluctuate irregularly and often do not converge in the classical sense. These methods help in detecting possible limit ranges or cluster points that indicate future states of markets or economic indicators under uncertainty.

Example: Predicting safe investment bands when market data is unstable.

(5) **Numerical Algorithms & Engineering Systems**

Iterative algorithms in scientific computing may only converge roughly due to round-off errors or incomplete information. This framework can be applied to stability analysis of iterative methods and control systems when there is measurement noise.

Example: Stabilizing a drone flight when GPS signals are inconsistent.

6. Declarations

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