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Inferential Study using Neutrosophic Imputation Techniques in Two-Phase Sampling

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Abstract. In the presence of supplementary information, classical statistical methodologies typically rely on precise data to obtain efficient estimators of the population mean. However, the presence of outliers poses substantial challenges to these traditional techniques, which are highly sensitive to data accuracy and auxiliary inputs. In contrast, neutrosophic statistics offers a more flexible and robust framework capable of handling imprecise and uncertain data, thus providing an advantageous alternative to classical methods. In the present research, we modify the study of Gohain et al. ([1]) by adapting his estimators and proposing a new family of neutrosophic exponential–logarithmic-type estimators in the neutrosophic setup with a view to enhancing estimation accuracy in two-phase sampling. Specifically, three imputation methods are developed, each accompanied by their respective point estimators. Theoretical properties, including bias and expressions for the minimum mean square error (MSE), are derived under large sample approximations. By integrating existing imputation methodologies within the neutrosophic framework, this research enhances the scope of current statistical inference techniques and underscores the adaptability of the proposed approach. A comparative evaluation is conducted to assess the relative efficiency of the proposed imputation procedures against alternative methods considered in this study. The empirical analysis, based on real-world datasets, demonstrates the superior performance of the proposed estimators. Furthermore, the findings are corroborated through an extensive simulation study, thereby reinforcing the validity and practical relevance of the proposed methodology.

Keywords: Neutrosophic Missing Data, Neutrosophic mean estimation, Bias, Mean Squared Error (MSE), Neutrosophic Two-Phase Sampling, Neutrosophic variables.

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1. Introduction

In survey sampling, missing data arises when selected respondents decline to take part in a survey for various different reasons. Several different methods exist to address the issue of missingness of the data, with imputation being one of the most effective. Imputation is a well-known and specialized method for managing non-response, involving various approaches to estimate and fill in missing data for univariate, bivariate or multivaraite study variables. In the realm of statistical inference, handling missing data remains a critical challenge and issue across diverse scientific disciplines. Traditional imputation methods often struggle to capture the inherent uncertainty and incomplete information prevalent in complex datasets. The classical statistical approaches, predominantly based on crisp set theory, frequently oversimplify the nuanced nature of real-world data, leading to potential misinterpretations and reduced predictive accuracy. The Neutrosophic Statistical framework emerges as a sophisticated methodological paradigm that transcends conventional probabilistic and fuzzy approaches. Developed as an extension of fuzzy set theory, neutrosophic statistics introduces a more comprehensive mechanism for managing indeterminacy, allowing for the simultaneous consideration of true, indeterminacy, and falsity values. This framework provides a more flexible and robust approach to statistical inference, particularly in scenarios characterized by significant uncertainty and incomplete information. Our research focuses on developing novel imputation methodologies within the Neutrosophic Statistical framework for two-phase sampling, efficiently addressing missing item values, eliminating incompleteness, and simplifying analysis. By leveraging the framework's unique capability to handle indeterminacy and partial truth, we propose a set of three innovative imputation strategies that can more accurately represent the complex uncertainty inherent in multidimensional datasets.

Recent studies have focused on imputation techniques for missing data. When the population mean of the first auxiliary variable is unknown, Grover & Sharma, [10] study investigated the use of two auxiliary variables in two-phase sampling to estimate missing values. An additional estimator for stratified random sampling with an auxiliary variable for finite populations has been proposed by Singh et al., [21]. Imputation strategies developed for handling non-response in repeated surveys by Singh et al., [22]. Imputation strategies based on super population model was studied for mean estimation under non-response is given by Singh et al., [23]. In ranked set sampling, imputation methods for missing values were also put forth by Bhushan et al., [5]. Using auxiliary variables such as kurtosis and the coefficient of variation for population mean estimation, a factor-type exponential ratio estimator was presented by Yadav & Prasad, [27]. Model based studies of the exponential estimators were presented in presence of non response was given by Singh et al. [20]. Classes of more effective and modified type regression-cum-ratio estimators have been introduce to find the population mean in two-phase and stratified two-phase sampling of the research variable while concurrently accounting for non-response and measurement error was given by Sabir & Sanaullah [17, 18]. Gohain et al., [9] suggested an effective exponential chain ratio estimator has been presented for predicting the finite population mean of the research variable of missing data under two-phase sampling using imputation approach. Some modified classes of exponential-logarithmic type of the imputation techniques have been proposed to deal with the missing data by Das & Singh, [8]. Shukla & Jain, [19] suggested to use two-phase sampling for population mean of logarithmic cum exponential estimators. It has been suggested to combine exponential and ln functions for efficient estimation under two-phase sampling given by Hassan et al., [12]. Some time it is possible that practitioners will not be aware of the population mean of the auxiliary variable during survey sampling. Two-phase sampling, sometimes referred to as twofold sampling, is the solution to this unknown value problem.

1.1. *Scope of the study*

Neutrosophic statistics offers a truly adaptable and versatile approach to working in difficult real-world situations that require cognitive ambiguity or an incomplete/indeterminate premise. In situations where there is any degree of uncertainty in the data, neutrosophic statistics could provide a substantive contribution to our understanding of the data where traditional means may fail. The small sample scope of the study shows a method for applying neutrosophic ideas in a practical manner for survey sampling. Given the stated limitations of the uncertain data systems, this line of study is very pertinent and needed. Take environmental monitoring; when collecting data to monitor soil contaminants, water quality, or air quality, there are uncertainties in the data that can occur at many different levels. The issues can arise from sensor calibration and accuracy, sampling uncertainty, environmental variability, and imprecision or ambiguity when measuring the factors in question. Traditional statistical methods depend on precise single-valued data, and hence they often create uncertainty when characterizing environmental conditions. Traditional statistical methods can classify data as outliers or ignore subtlety of relationships through unplanned patterns; creating incomplete or biased representations of environmental facts. In contrast, given that neutrosophic statistics is more thorough and flexible, they can handle the uncertainty with which reality is filled. It is a more realistic quantification of environmental attributes - rather than an environmental measurement, which gives a single point estimation, neutrosophic analyses use interval valued data. For example, when monitoring water quality, if neutrosophic statistical analysis was being employed, the pH reading would be reported as an interval, say [6.8, 7.2] rather than a single value, like 7.0. This interval approach captures the uncertainty, or variability and error in measurements, leading to better decision making and risk management. Neutrosophic statistics can also account for uncertainty in concentrations, as in air quality monitoring, and provide a more complete understanding of the environment in which we operate. Neutrosophic statistics is valuable for decision makers, and experts working for organizations, municipalities, and government agencies that develop responses and strategies for reducing air pollution and addressing the health effects of air pollution on humans and other living organisms. Overall, applying the neutrosophic model allows environmental researchers and practitioners to make better decisions, allocate resources wisely, and create better responsive and adaptive strategies in ongoing responses to the complexities and uncertainties of real-world environmental data.

The present study will also build off this threshold as it will widen the scope of neutrosophic statistics by working with imputation methods to accommodate uncertain and/or missing data in Two-phase sampling. Three imputation methods are proposed with the corresponding point estimators for the exponential-logarithmic type of estimators. The study will use simple random sampling without replacement (SRSWOR) through out the study. An empirical study is conducted to show how the proposed estimators perform against other estimators. To the best of our knowledge, this is the first study to adapt the neutrosophic form to imputation methods. The work of relating neutrosophic statistics with specific imputation methods will achieve the most accurate population mean estimates as possible with the mean squared error as low as possible, even when outliers and imprecise data is present. As the use of these methods is described, this study shows the neutrosophic paradigm's potential adaptability and problem-solving application are far reaching. Conducting this comprehensive study will show that the understanding of neutrosophic statistics is growing and the application of neutrosophic statistics can help to enhance comprehension of more complex data environments across a broader scope and assist in better decision-making.

1.2. *Developments under Neutrosophic Estimation*

Classical statistics handle precise data but struggle with uncertainty, prompting the need for new methods. Fuzzy logic addresses ambiguous or imprecise data but cannot measure indeterminacy. Neutrosophic logic extends fuzzy logic by quantifying uncertainty and certainty, making it effective for analyzing ambiguous scenarios (Aslam, [3, 4]). Modifications such as complex fuzzy sets, interval-valued neutrosophic sets, and neutrosophic numbers have modified decision-making frameworks, considering nuanced data analysis by Ali & Mahmood, [2]; Liu et al., [16] and Li et al., [15]. Study by Chakraborty et al., [6]; Haque et al., [11] include trapezoidal and spherical neutrosophic numbers in multi-criteria group decision-making, mobile communication, and statistical quality control and Smarandache, [25] has extended neutrosophic statistics traditional methods to manage uncertainty and indeterminacy in data, helps in effective analysis of imprecise datasets in realworld scenario. Research in the fields like agriculture and rock engineering demonstrate their versatility and growing importance by [7,14]. For further studies on neutrosophic statistics readers may refer to Yadav & Smarandache, [26]; Yadav & Prasad, 28. The development of neutrosophic statistical approaches has been fueled by advancements that have broadened the field into domains such as neutrosophic applied statistics (NAS), neutrosophic statistical quality control (NSQC), and neutrosophic interval statistics (NIS). All things considered, neutrosophic statistics is still a dynamic and developing area that provides a flexible toolkit for examining complex data that is marked by ambiguity and uncertainty.

2. Terminologies and Methodology under Neutrosophic Statistical Framework

A key observation within the neutrosophic framework is the utilization of quantitative neutrosophic type missing data, where the value may be exist in indeterminate interval $[\Phi N_L, \Phi N_U]$. Neutrosophic interval values can be represented in various forms and are explicitly defined in Yadav and Smarandache [26] as $Z_{(\phi N)} = Z_{(\phi N_L)} + Z_{(\phi N_U)} I_{(\phi N)}$, where $I_{(\phi N)} \in [I_{(\phi N_L)}, I_{(\phi N_U)}]$. For the neutrosophic data under consideration, we adopt the same notations as in Yadav and Smarandache [26], expressed in interval form as $Z_{(\phi N)} \in [Z_{(\phi N_L)}, Z_{(\phi N_U)}]$, where $Z_{(\phi N_L)}$ and $Z_{(\phi N_U)}$ denote the respective Lower and Upper values of neutrosophic study and auxilary variable $Z_{(\phi N_L, \phi N_U)}$. We employ the simple random sampling without replacement method to draw a neutrosophic random sample of size $n \in [n_{(\phi N_L)}, n_{(\phi N_U)}]$ from the specified population, assuming the neutrosophic population comprises $N \in [N_{(\phi N_L)}, N_{(\phi N_U)}]$ distinct units. In the neutrosophic environment, the variable of interest is referred to as the neutrosophic study variable, denoted by $Y_{(\phi N)}$, with its corresponding neutrosophic auxiliary variable represented as $X_{(\phi N)}$.

For the neutrosophic informations being considered, each value on the i^{th} unit of the population for the variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ are represented as $Y_{i_{(\phi N)}} \in [Y_{(\phi N_L)}, Y_{(\phi N_U)}]$, and $X_{i_{(\phi N)}} \in [X_{(\phi N_L)}, X_{(\phi N_U)}]$. Let's define $\bar{Y}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} Y_{i_{(\phi N)}}$ and $\bar{X}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} X_{i_{(\phi N)}}$ as the population means for $Y_{(\phi N)}$ and $X_{(\phi N)}$, acting as the overall means of the neutrosophic informations. The neutrosophic study variable $Y_{(\phi N)}$ has a sample mean of $\bar{y}_{(\phi N)} = \frac{1}{n}\sum_{i=1}^{n} y_{i_{(\phi N)}}$ and , $X_{(\phi N)}$ has a sample mean of $\bar{x}_{(\phi N)} = \frac{1}{n}\sum_{i=1}^{n} x_{i_{(\phi N)}}$, where $(y_{i_{(\phi N)}}; x_{i_{(\phi N)}})$ being the values of variables for i^{th} sampled unit. Additionally, the neutrosophic coefficients of variation for $Y_{(\phi N)}$ and $X_{(\phi N)}$ are given as $C_{Y_{(\phi N)}}$ and $C_{X_{(\phi N)}}$. The correlation coefficient between the neutrosophic variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ denoted as $\rho_{XY_{(\phi N)}}$.

2.1. *Problem Structure*

Consider a finite population $\Omega = \{\Omega_1, \Omega_2, \Omega_3, ...\Omega_N\}$ of the population size N, the value of the variables on the i^{th} unit $\Omega_i, \forall\, i = 1, 2, 3...N$, be $(y_{i_{(\phi N)}}; x_{i_{(\phi N)}})$. Let $\bar{Y}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} y_{i_{(\phi N)}}$ and $\bar{X}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} x_{i_{(\phi N)}}$ be the population means of the neutrosophic study variable $Y_{(\phi N)}$ and the neutrosophic auxiliary variable $X_{(\phi N)}$ respectively. A simple random sample of size n is taken from the population Ω to estimate the $\bar{Y}$. Let $\bar{y}_{n_{(\phi N)}} = \frac{1}{n}\sum_{i=1}^{n} y_{i_{(\phi N)}}$ and $\bar{x}_{n_{(\phi N)}} = \frac{1}{n}\sum_{i=1}^{n} x_{i_{(\phi N)}}$ be the sample means of variables $Y_{(\phi N)}$ and $X_{(\phi N)}$.

In the two-phase sample survey in which unknown $\bar{X}$ is estimated by choosing a preliminary large sample S' of size n' from the population. The sample mean $\bar{x}'_{(\phi N)} = \sum_{i \in S'} \frac{x_{i_{(\phi N)}}}{n'}$ is considered as estimate of $\bar{X}$. The second sample S of size n is selected either from S' or Ω.
Case 1: As a sub sample from sample S' (denoted by design I).
Case 2: Independent to sample S' (denoted by design II).

Consider a sample S consisting of n units, which can be divided into two distinct subsets: a responding R with r units, and a non responding units with $(n-r)$ units $(r < n)$ forming a sub-set R^c with respect to neutrosophic study variable only in $S = R \cup R^c$ for every $y_{i_{(\phi N)}} \in R^c$, the i^{th} value $x_{i_{(\phi N)}}$ of the neutrosophic auxiliary variable is serves as the basis for imputation. The sample mean of the variable $X_{(\phi N)}$ based on the sample S' of size n' i.e.

$$\bar{x}'_{(\phi N)} = \sum_{i=1}^{n'} \frac{x_{i_{(\phi N)}}}{n'}, \text{ where } n' \in S'$$

$\bar{y}_{r_{(\phi N)}}, \bar{x}_{r_{(\phi N)}}$ are the sample mean of the neutrosophic variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ based on the subset R (responding units) of S. $\bar{y}_{n_{(\phi N)}}, \bar{x}_{n_{(\phi N)}}$ are the sample mean of the neutrosophic variable $Y_{(\phi N)}$ and neutrosophic variable $X_{(\phi N)}$ based on $R \cup R^c$. The symbols listed below are used as notation, respectively

$\rho_{YX_{(\phi N)}}$: Correlation between $Y_{(\phi N)}$ and $X_{(\phi N)}$.

$C_{Y_{(\phi N)}}, C_{X_{(\phi N)}}$: Coefficient of the variation of $Y_{(\phi N)}$ and $X_{(\phi N)}$.

$$\zeta_1 = \left(\frac{1}{r} - \frac{1}{n'}\right);\ \zeta_2 = \left(\frac{1}{n} - \frac{1}{n'}\right);\ \zeta_3 = \left(\frac{1}{n'} - \frac{1}{N}\right);\ \zeta_4 = \left(\frac{1}{r} - \frac{1}{N-n'}\right);\ \zeta_5 = \left(\frac{1}{n} - \frac{1}{N-n'}\right)$$

3. **Adapted Neutrosophic Imputation Methods**

This section explores existing conventional imputation methods, selected for their ability to address various statistical challenges. These methods, drawn from established research, are based on well-recognized methodologies. Importantly, they are integrated into the conceptual framework of Neutrosophic Statistics, a contemporary approach designed to handle uncertainty and ambiguity.

3.1. *Adapted Neutrosophic Mean Method*

In this approach, for the each non-responding unit, of the missing value is substituted with mean value calculated from all responding units.

$$y_{.i_{(\phi N)}} = \begin{cases} y_{i_{(\phi N)}}, & i \in R \\ \bar{y}_{r_{(\phi N)}}, & i \in R^c \end{cases}$$

The unit-based estimator using this imputation method, $\bar{Y}_{(\phi N)}$, can be expressed as

$$\bar{y}_{s_{(\phi N)}} = \frac{1}{r}\sum_{i \in R} y_{.i_{(\phi N)}} = \bar{y}_{r_{(\phi N)}}$$

Bias of $\bar{y}_{r_{(\phi N)}}$ is $B(y_{r_{(\phi N)}}) = 0$ and MSE is $M(\bar{y}_{r_{(\phi N)}}) = \bar{Y}^2_{(\phi N)}\left(\frac{1}{r} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}}$.

3.2. *Adapted Neutrosophic Ratio Method*

Lee et al. [13] proposed ratio method of imputation and here we introduce it under neutrosophic setup as

$$y_{.i_{(\phi N)}} = \begin{cases} y_{i_{(\phi N)}}, & i \in R \\ \hat{l}x_{i_{(\phi N)}}, & i \in R^c \end{cases}$$

Here, $\hat{l} = \frac{\sum_{i \in R} y_{i_{(\phi N)}}}{\sum_{i \in R} x_{i_{(\phi N)}}}$

This imputation method has the following unit estimator

$$\bar{y}_{RAT_{(\phi N)}} = \bar{y}_{r_{(\phi N)}}\left(\frac{\bar{x}_{n_{(\phi N)}}}{\bar{x}_{r_{(\phi N)}}}\right)$$

where $\bar{x}_{n_{(\phi N)}} = \frac{\sum_{i \in R} x_{i_{(\phi N)}}}{n}$, $\bar{x}_{r_{(\phi N)}} = \frac{\sum_{i \in R} x_{i_{(\phi N)}}}{r}$ and $\bar{y}_{r_{(\phi N)}} = \frac{\sum_{i \in R} y_{i_{(\phi N)}}}{r}$

Bias and the MSE of the $\bar{y}_{RAT_{(\phi N)}}$ is given by

$$B\left(\bar{y}_{RAT_{(\phi N)}}\right) = \bar{Y}_{(\phi N)}\left(\frac{1}{r} - \frac{1}{n}\right)\left(C^2_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}\right)$$

$$M\left(\bar{y}_{RAT_{(\phi N)}}\right) = \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{r} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + \left(\frac{1}{r} - \frac{1}{n}\right)\left(1 - 2\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}\right) C^2_{X_{(\phi N)}}\right\}$$

3.3. *Adapted Neutrosophic Compromised Method*

Singh and Horn's [24] compromised imputation method is now adapted and formulated within a neutrosophic context as

$$y_{.i_{(\phi N)}} = \begin{cases} \alpha \frac{n}{r} y_{i_{(\phi N)}} + (1-\alpha)\hat{l}x_{i_{(\phi N)}}, & i \in R \\ (1-\alpha)\hat{l}x_{i_{(\phi N)}}, & i \in R^c \end{cases}$$

where α is a constant.

This imputation method's unit estimator can be expressed as

$$\bar{y}_{Comp_{(\phi N)}} = \alpha \bar{y}_{r_{(\phi N)}} + (1-\alpha)\bar{y}_{r_{(\phi N)}} \frac{\bar{x}_{n_{(\phi N)}}}{\bar{x}_{r_{(\phi N)}}}$$

Bias and MSE of the unit estimator $\bar{y}_{Comp_{(\phi N)}}$

$$B\left(\bar{y}_{Comp_{(\phi N)}}\right) = \bar{Y}_{(\phi N)} \left(\frac{1}{r} - \frac{1}{n}\right)(1-\alpha)(1-\phi_{YX_{(\phi N)}})C^2_{X_{(\phi N)}}$$

$$M\left(\bar{y}_{Comp_{(\phi N)}}\right) = \bar{Y}^2_{(\phi N)} \left[\left(\frac{1}{r} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + \left(\frac{1}{r} - \frac{1}{n}\right)\left\{(1-\alpha)^2 - 2(1-\alpha)\rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}\right\} C^2_{X_{(\phi N)}}\right]$$

This estimator is minimum at $\alpha_{opt} = 1 - \rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}$

The optimum mean square error (MSE) of $\bar{y}_{Comp_{(\phi N)}}$ is

$$M\left(\bar{y}_{Comp_{(\phi N)}}\right)_{opt} = \bar{Y}^2_{(\phi N)} \left\{\left(\frac{1}{r} - \frac{1}{N}\right) - \left(\frac{1}{r} - \frac{1}{n}\right)\rho^2_{YX_{(\phi N)}}\right\} C^2_{Y_{(\phi N)}}$$

4. Proposed Adapted Neutrosophic Imputation

In this section, we propose a new set of neutrosophic exponential–logarithmic-type estimators by generalizing the exponential–logarithmic method of Gohain et al. ([1]), which was initially developed in a classical statistics context. As survey data in real life are frequently plagued by uncertainty, vagueness, and inconsistency that cannot be properly handled by conventional statistical approach, the suggested neutrosophic estimators are formulated to efficiently manage indeterminacy and imprecision in real-life survey data.

Three Neutrosophic imputation techniques and their respective unit estimators are as:

(1)

$$y_{1.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r}\bar{y}_{r_{(\phi N)}}\left[n\ exp\left\{\alpha_1\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}\right)\right\}\right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_1} = \bar{y}_{r_{(\phi N)}}\left[exp\left\{\alpha_1\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}\right)\right\}\right] \quad (1)$$

(2)

$$y_{2.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r}\bar{y}_{r_{(\phi N)}}\left[n\ exp\left\{\alpha_2\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}\right)\right\}\right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_2} = \bar{y}_{r_{(\phi N)}} \left[exp \left\{ \alpha_2 \ log \left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} \ - \ \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} \ + \ \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}} \right) \right\} \right] \tag{2}$$

(3)

$$y_{3.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r} \bar{y}_{r_{(\phi N)}} \left[n \ exp \left\{ \alpha_3 \ log \left(1 + \frac{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}} \right) \right\} \right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_3} = \bar{y}_{r_{(\phi N)}} \left[exp \left\{ \alpha_3 \ log \left(1 + \frac{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} \ - \ \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} \ + \ \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}} \right) \right\} \right] \tag{3}$$

Where $\alpha_i \ \forall \ i = 1, 2, 3$ are constants are derived such as the MSEs of the proposed adapted classes of estimators $\bar{y}_{TV_1}, \ \bar{y}_{TV_2}, \ \bar{y}_{TV_3}$ are minimum and a and $h \neq 0$ are constant.

4.1. *Statistical Properties of the Proposed Adapted Estimators*

We adopted the following transformations to find the bias and MSE expressions associated with the proposed adapted class of estimators.

$\bar{y}_{r_{(\phi N)}} = \bar{Y}_{(\phi N)}(1+e_1); \ \bar{x}_{r_{(\phi N)}} = \bar{X}_{(\phi N)}(1+e_2); \ \bar{x}_{n_{(\phi N)}} = \bar{X}_{(\phi N)}(1+e_3); \ \bar{x}'_{(\phi N)} = \bar{X}_{(\phi N)}(1+e'_3)$
where $e_i, \ e'_3 \in (-1, 1) \ \forall \ i = 1, 2, 3$ and $E(e_i) = 0, \ E(e'_3 = 0) \ \forall \ i = 1, 2, 3.$

Design I:

$$E(e_1^2) = \zeta_1 C^2_{Y_{(\phi N)}}; \ E(e_2^2) = \zeta_1 C^2_{X_{(\phi N)}}; \ E(e_3^2) = \zeta_2 C^2_{X_{(\phi N)}}; \ E(e'^2_3) = \zeta_3 C^2_{X_{(\phi N)}};$$
$$E(e_1 e_2) = \zeta_1 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e_3) = \zeta_2 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}};$$

$E(e_1 e'_3) = \zeta_3 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_2 e_3) = \zeta_2 C^2_{X_{(\phi N)}}; \ E(e_2 e'_3) = \zeta_3 C^2_{X_{(\phi N)}}; \ E(e_3 e'_3) = \zeta_3 C^2_{X_{(\phi N)}}$

Design II:

$$E(e_1^2) = \zeta_4 C^2_{Y_{(\phi N)}}; \ E(e_2^2) = \zeta_4 C^2_{X_{(\phi N)}}; \ E(e_3^2) = \zeta_5 C^2_{X_{(\phi N)}}; \ E(e'^2_3) = \zeta_3 C^2_{X_{(\phi N)}};$$
$$E(e_1 e_2) = \zeta_4 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e_3) = \zeta_5 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e'_3) = 0;$$
$$E(e_2 e_3) = \zeta_5 C^2_{X_{(\phi N)}}; \ E(e_2 e'_3) = 0; \ E(e_3 e'_3) = 0$$

With the above approximations, the unit estimators $\bar{y}_{TV_1}, \ \bar{y}_{TV_2}$ and $\bar{y}_{TV_3}$ are given in the following formats in relation to e_i and e'_i

$$\bar{y}_{TV_1} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_1 \frac{a}{h}(e'_3 - e_2 + e_1 e'_3 - e_1 e_2) + \frac{1}{4}(e'^2_3 - e^2_2)\left(\alpha_1 \frac{a}{h}\left(2\frac{a}{h} - 1 \right) \right) \right\} \tag{4}$$

$$\bar{y}_{TV_2} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_2 \frac{a}{h}(e'_3 - e_3 + e_1 e'_3 - e_1 e_3) + \frac{1}{4}(e'^2_3 - e^2_3)\left(\alpha_2 \frac{a}{h}\left(2\frac{a}{h} - 1 \right) \right) \right\} \tag{5}$$

$$\bar{y}_{TV_3} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_3 \frac{a}{h}(e_3 - e_2 + e_1 e_3 - e_1 e_2) + \frac{1}{4}(e^2_3 - e^2_2)\left(\alpha_3 \frac{a}{h}\left(2\frac{a}{h} - 1 \right) \right) \right\} \tag{6}$$

4.2. *Expression of Bias and Mean square error of* $\bar{y}_{TV_1},\ \bar{y}_{TV_2},\ \bar{y}_{TV_3}$

Deriving $(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})$ using equation (4) and applying the expectation operator to both sides yields

$$E(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)}) = E\Big[\bar{Y}_{(\phi N)}\left\{e_1 + \frac{1}{2}K(e_3' - e_2 + e_1e_3' - e_1e_2) + \frac{1}{4}(e_3'^2 - e_2^2)K\left(2\frac{a}{h} - 1\right)\right\} - \frac{1}{4}(e_3' - e_2)^2\left(K\frac{a}{h}(1-\alpha_1)\right)\Big]$$

Substituting the respective values of $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_1}$ is

$$B\left(\bar{y}_{TV_1}\right)_I = \bar{Y}_{(\phi N)}\left\{\frac{1}{2}K(\zeta_3 - \zeta_1)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}} + \frac{1}{4}(\zeta_3 - \zeta_2)\left(\frac{3aK}{h} - K(1+K)\right)C^2_{X_{(\phi N)}}\right\}$$

Substituting the respective values of $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design II, the bias of $\bar{y}_{TV_1}$ is

$$B\left(\bar{y}_{TV_1}\right)_{II} = \bar{Y}_{(\phi N)}\Big[-\zeta_4\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}} + \frac{1}{4}\left\{(\zeta_3 - \zeta_4)\left(K\left(2\frac{a}{h} - 1\right)\right) - (\zeta_3 + \zeta_4)\left(K\frac{a}{h}(1-\alpha_1)\right)\right\}C^2_{X_{(\phi N)}}\Big]$$

Using equation (4) to construct $(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})^2$ and keeping the terms up to the second-degree of approximations, we get

$$(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})^2 = \bar{Y}^2_{(\phi N)}\left\{e_1^2 + \frac{1}{4}K^2(e_3' - e_2)^2 + K(e_1e_3' - e_1e_2)\right\} \tag{7}$$

Using equation (7) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_1}$ is

$$M\left(\bar{y}_{TV_1}\right)_I = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_1 - \zeta_3)C^2_{X_{(\phi N)}} + K(\zeta_3 - \zeta_1)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{8}$$

Using equation (7) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_1}$ is

$$M\left(\bar{y}_{TV_1}\right)_{II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_3 + \zeta_4)C^2_{X_{(\phi N)}} - K\zeta_4\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{9}$$

Here and for further calculations, for each $\bar{y}_{TV_i}$, $K = \alpha_i\frac{a}{h},\ i = 1,2,3$

Starting from equation (5), we construct the term $(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)})$, then apply expectations to both sides, which gives

Constructing $(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)})$ from equation (5) and considering expectations on both sides, we get

$$E(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)}) = E\Big[\bar{Y}_{(\phi N)}\left\{e_1 + \frac{1}{2}K(e_3' - e_3 + e_1e_3' - e_1e_3) + \frac{1}{4}(e_3'^2 - e_3^2)K\left(2\frac{a}{h} - 1\right)\right\} - \frac{1}{4}(e_3' - e_3)^2\left(K\frac{a}{h}(1-\alpha_2)\right)\Big]$$

Substituting the respective values for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_2}$ is

$$B\left(\bar{y}_{TV_2}\right)_I=\bar{Y}_{(\phi N)}\left[\frac{1}{2}K(\zeta_3-\zeta_2)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}+\frac{1}{4}(\zeta_3-\zeta_2)\left\{\frac{3aK}{h}-K(1+K)\right\}C^2_{X_{(\phi N)}}\right]$$

Substituting the respective values for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the designn II, the bias of $\bar{y}_{TV_2}$ is

$$B\left(\bar{y}_{TV_2}\right)_{II}=\bar{Y}_{(\phi N)}\left[-\zeta_5\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}+\frac{1}{4}\left\{(\zeta_3-\zeta_5)\left(K\left(2\frac{a}{h}-1\right)\right)-(\zeta_3+\zeta_5)\left(K\frac{a}{h}(1-\alpha_2)\right)\right\}C^2_{X_{(\phi N)}}\right]$$

Using equation (5) to construct $(\bar{y}_{TV_2}-\bar{Y}_{(\phi N)})^2$ such as

$$(\bar{y}_{TV_2}-\bar{Y}_{(\phi N)})^2=\bar{Y}^2_{(\phi N)}\left\{e_1^2+\frac{1}{4}K^2(e_3'-e_3)^2+K(e_1e_3'-e_1e_3)\right\} \tag{10}$$

Using equation (10) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_2}$ is

$$M\left(\bar{y}_{TV_2}\right)_I=\bar{Y}^2_{(\phi N)}\left\{\zeta_1C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_2-\zeta_3)C^2_{X_{(\phi N)}}+K(\zeta_3-\zeta_2)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{11}$$

Using equation (10) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_2}$ is

$$M\left(\bar{y}_{TV_2}\right)_{II}=\bar{Y}^2_{(\phi N)}\left\{\zeta_4C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_3+\zeta_5)C^2_{X_{(\phi N)}}-K\zeta_5\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{12}$$

From equation (6), $(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})$ is formed; applying the expectation operator to both sides gives

$$\begin{aligned}E(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})=E\Big[&\bar{Y}_{(\phi N)}\left\{e_1+\frac{1}{2}K(e_3-e_2+e_1e_3-e_1e_2)+\frac{1}{4}(e_3^2-e_2^2)K\left(2\frac{a}{h}-1\right)\right\}\\&-\frac{1}{4}(e_3-e_2)^2\left(K\frac{a}{h}(1-\alpha_3)\right)\Big]\end{aligned}$$

Substitute the appropriate expectations for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_3}$ is

$$B\left(\bar{y}_{TV_3}\right)_I=\bar{Y}_{(\phi N)}\left[\frac{1}{4}K(\zeta_2-\zeta_1)\left\{2\rho_{XY_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}+\left(\frac{2a}{h}-1\right)+\frac{a}{h}(1-\alpha_3)\right\}\right]C^2_{X_{(\phi N)}}$$

Substitute the appropriate expectations for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design II, the bias of $\bar{y}_{TV_3}$ is

$$B\left(\bar{y}_{TV_3}\right)_{II}=\bar{Y}_{(\phi N)}\left[\frac{1}{4}K(\zeta_5-\zeta_4)\left\{2\rho_{XY_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}+\left(\frac{2a}{h}-1\right)+\frac{a}{h}(1-\alpha_3)\right\}\right]C^2_{X_{(\phi N)}}$$

Using equation (6) to construct $(\bar{y}_{TV_3} - \bar{Y}_{(\phi N)})^2$ and retaining terms up to the second order, we obtain

$$(\bar{y}_{TV_3} - \bar{Y}_{(\phi N)})^2 = \bar{Y}^2_{(\phi N)} \left\{ e_1^2 + \frac{1}{4}K^2(e_3 - e_2)^2 + K(e_1e_3 - e_1e_2) \right\} \tag{13}$$

Using equation (13) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_3}$ is

$$M\left(\bar{y}_{TV_3}\right)_I = \bar{Y}^2_{(\phi N)} \left\{ \zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_1 - \zeta_2)C^2_{X_{(\phi N)}} - K(\zeta_1 - \zeta_2)\rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}} \right\} \tag{14}$$

Using equation (13) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_3}$ is

$$M\left(\bar{y}_{TV_3}\right)_{II} = \bar{Y}^2_{(\phi N)} \left\{ \zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_4 - \zeta_5)C^2_{X_{(\phi N)}} - K(\zeta_4 - \zeta_5)\rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}} \right\} \tag{15}$$

4.3. *Optimum Mean Square errors of* $\bar{y}_{TV_1},\ \bar{y}_{TV_2},\ \bar{y}_{TV_3}$

Differentiating equation (8) w.r.t. α_1 and equating it to zero

$$\frac{d}{d\alpha_1}\left[\bar{Y}^2_{(\phi N)} \left\{ \zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_1 \frac{a}{h}\right)^2 (\zeta_1 - \zeta_3)C^2_{X_{(\phi N)}} + \alpha_1 \frac{a}{h}(\zeta_3 - \zeta_1)\rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}} \right\}\right] = 0$$

$$\implies \alpha_1 = \frac{2h}{a}\rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{1(\text{opt})I}$$

When the optimum value of α_1 (that is,$\alpha_{1(\text{opt})I}$) is substituted into equation (8), the resulting minimum MSE for $\bar{y}_{TV_1}$:

$$M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)} \left\{ \zeta_1 - (\zeta_1 - \zeta_3)\rho^2_{YX_{(\phi N)}} \right\} C^2_{Y_{(\phi N)}} \tag{16}$$

Differentiating equation (9) w.r.t. α_1 for equating it to zero

$$\frac{d}{d\alpha_1}\left[\bar{Y}^2_{(\phi N)} \left\{ \zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_1 \frac{a}{h}\right)^2 (\zeta_3 + \zeta_4)C^2_{X_{(\phi N)}} - \left(\alpha_1 \frac{a}{h}\right)\zeta_4 \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}} \right\}\right] = 0$$

$$\implies \alpha_1 = \frac{2h}{a}\frac{\zeta_4}{\zeta_3 + \zeta_4}\rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{1(\text{opt})II}$$

When the optimum value of α_1 (that is,$\alpha_{1(\text{opt})II}$) is substituted into equation (9), the resulting minimum MSE for $\bar{y}_{TV_1}$:

$$M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)} \left\{ \zeta_4 - \frac{\zeta_4^2}{\zeta_3 + \zeta_4}\rho^2_{YX_{(\phi N)}} \right\} C^2_{Y_{(\phi N)}} \tag{17}$$

Differentiating equation (11) w.r.t. α_2 and equating it to zero

$$\frac{d}{d\alpha_2}\left[\bar{Y}^2_{(\phi N)} \left\{ \zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_2 \frac{a}{h}\right)^2 (\zeta_2 - \zeta_3)C^2_{X_{(\phi N)}} + \left(\alpha_2 \frac{a}{h}\right)(\zeta_3 - \zeta_2)\rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}} \right\}\right] = 0$$

$$\implies \alpha_2 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{2(\text{opt})I}$$

When the optimum value of α_2 (that is,$\alpha_{2(\text{opt})I}$) is substituted into equation (11), the resulting minimum MSE for $\bar{y}_{TV_2}$:

$$M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 - (\zeta_2-\zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{18}$$

Differentiating equation (12) w.r.t. α_2 and equating it to zero

$$\frac{d}{d\alpha_2}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_2\frac{a}{h}\right)^2(\zeta_3+\zeta_5)C^2_{X_{(\phi N)}} - \left(\alpha_2\frac{a}{h}\right)\zeta_5\rho_{YX_{(\phi N)}}C_Y C_X\right\}\right] = 0$$

$$\implies \alpha_2 = \frac{2h}{a}\frac{\zeta_5}{\zeta_3+\zeta_5}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{2(\text{opt})II}$$

When the optimum value of α_2 (that is,$\alpha_{2(\text{opt})II}$) is substituted into equation (12), the resulting minimum MSE for $\bar{y}_{TV_2}$:

$$M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 - \frac{\zeta_5^2}{\zeta_3+\zeta_5}\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{19}$$

Differentiating equation (14) w.r.t. α_3 and equating it to zero

$$\frac{d}{d\alpha_3}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_3\frac{a}{h}\right)^2(\zeta_1-\zeta_2)C^2_{X_{(\phi N)}} - \left(\alpha_3\frac{a}{h}\right)(\zeta_1-\zeta_2)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right] = 0$$

$$\implies \alpha_3 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{3(\text{opt})I}$$

When the optimum value of α_3 (that is,$\alpha_{3(\text{opt})I}$) is substituted into equation (14), the resulting minimum MSE for $\bar{y}_{TV_3}$:

$$M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 - (\zeta_1-\zeta_2)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{20}$$

Differentiating equation (15) w.r.t. α_3 and equating it to zero

$$\frac{d}{d\alpha_3}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_3\frac{a}{h}\right)^2(\zeta_4-\zeta_5)C^2_{X_{(\phi N)}} - \left(\alpha_3\frac{a}{h}\right)(\zeta_4-\zeta_5)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right] = 0$$

$$\implies \alpha_3 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{3(\text{opt})II}$$

When the optimum value of α_3 (that is,$\alpha_{3(\text{opt})II}$) is substituted into equation (15), the resulting minimum MSE for $\bar{y}_{TV_3}$:

$$M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 - (\zeta_4-\zeta_5)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{21}$$

5. Efficiency Comparison

This section explains and demonstrates, both in theory and practice, how the suggested imputation methods work and how their results compare to the adapted neutrosophic mean, ratio, and compromised methods discussed in Section 3.

5.1. *Theoretical Study*

This section breaks down and explains, in an approachable way, how the recommended methods work and how their results measure up against the methods discussed earlier in Section 3. Theoretical comparisons under desihn I and II are given as

Comparison with adapted neutrosophic mean method:

(1)

$$\mathbb{D}_{1r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_1 - \zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{1r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{\zeta_4^2}{\zeta_3+\zeta_4}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_1}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

(2)

$$\mathbb{D}_{2r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_2 - \zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{2r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{\zeta_5^2}{\zeta_3+\zeta_5}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_2}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

(3)

$$\mathbb{D}_{3r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_1 - \zeta_2)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{3r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{1}{r} - \frac{1}{n}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_3}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

Comparison with adapted neutrosophic ratio method:

(1)

$$\mathbb{D}_{1RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 - (\zeta_2-\zeta_3)\rho^2_{YX_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{1RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 + \left(\frac{\zeta_4^2}{\zeta_3+\zeta_5} - \zeta_1 + \zeta_2\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_1}$ is more effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

(2)

$$\mathbb{D}_{2RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 - (2\zeta_2-\zeta_1-\zeta_3)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $2\zeta_2 \le \left(\frac{1}{r} - \frac{1}{N}\right)$.

$$\mathbb{D}'_{2RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 + \left(\zeta_4 - \frac{\zeta_5^2}{\zeta_3+\zeta_5}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $\zeta_4 > \frac{\zeta_5^2}{\zeta_3+\zeta_5}$.

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_2}$ is effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

(3)

$$\mathbb{D}_{3RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2\right\} > 0$$

$$\mathbb{D}'_{3RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_3}$ is more effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

Comparison with adapted neutrosophic compromised method:

(1)

$$\mathbb{D}_{1\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} - (\zeta_1-\zeta_2)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{1\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} - \left(\zeta_1-\zeta_2-\frac{\zeta_4^2}{\zeta_3+\zeta_4}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} > \left(\zeta_1-\zeta_2-\frac{\zeta_4^2}{\zeta_3+\zeta_4}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_1}$ is more effective than $\bar{y}_{Comp_{(\phi N)}}$ with respect to both designs.

(2)

$$\mathbb{D}_{2\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (2\zeta_2-\zeta_1-\zeta_3)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{2\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} - \left(\zeta_1-\zeta_2-\frac{\zeta_5^2}{\zeta_3+\zeta_5}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_2}$ is more effective than $\bar{y}_{Comp_{(\phi N)}}$ with respect to both designs.

(3)

$$\mathbb{D}_{3\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\zeta_3 C^2_{Y_{(\phi N)}} > 0$$

$$\mathbb{D}'_{3\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} > 0$$

So, the proposed estimator $\bar{y}_{TV_3}$ is better than $\bar{y}_{Comp_{(\phi N)}}$.

5.2. *Empirical Study*

This comprehensive study's objective is to examine and compare the proposed imputation methods with various adapted Neutrosophic imputation methods for calculating the population mean in the framework of Neutrosophic two-phase sampling. In order to achieve this, we selected actual data from the publicly accessible website "https://data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperatureseries-1901-2019." The dataset contains comprehensive information on the seasonal and annual temperatures of all of India, including the mean, maximum, and minimum temperatures over an extended period of time.
Dataset descriptions are provided in Table [1]. For Data A, the minimum and maximum

recorded temperatures in the months of January and February are represented by the auxiliary variable $X_{\phi N}$ and the minimum and maximum recorded temperatures in the months from March to May are taken as the study variable $Y_{\phi N}$. For Data B, minimum and maximum temperatures from June to September are represented by the auxiliary variable $X_{\phi N}$, while the study variable $Y_{\phi N}$ represents the minimum and maximum temperatures recorded in the months from October to December.

MSEs of the adapted and proposed estimators are calculated under designs I & II, and are presented in Tables [2] & [3] respectively. Sample size at first stage is taken as $n' = 50$ and the sample sizes at stage two are taken as $n = 18, 22$ with number of responding units taken as $r = 14, 15, 16$ & $16, 17, 18, 19$ respectively.

We calculate how efficient our proposed and adapted neutrosophic estimators are compared to the one based on the adapted neutrosophic mean imputation method. These percentage relative efficiencies (PREs) are shown in Tables [4] and [5]. The formula used for calculating PREs of estimators is given as

$$PRE(t) = \frac{MSE(\bar{y}_{r(\phi N)})}{MSE(t)} \times 100$$

Where $t = \bar{y}_{RAT(\phi N)},\ \bar{y}_{Comp(\phi N)},\ \bar{y}_{TV_{1(I)}},\ \bar{y}_{TV_{1(II)}},\ \bar{y}_{TV_{2(I)}},\ \bar{y}_{TV_{2(II)}},\ \bar{y}_{TV_{3(I)}},\ \bar{y}_{TV_{3(II)}}$

Table 1. Descriptive statistics of the Data A and Data B

	Data A	**Data B**
Parameter	**Neutrosophic Value**	**Neutrosophic Value**
$\boldsymbol{N}$	[119,119]	[119,119]
$\boldsymbol{n'}$	[50,50]	[50,50]
$\boldsymbol{n}$	[18,18] , [22,22]	[18,18] , [22,22]
$\bar{\boldsymbol{Y}}_{(\boldsymbol{\phi N})}$	[20.69370, 31.55933]	[16.58437, 27.23613]
$\bar{\boldsymbol{X}}_{(\boldsymbol{\phi N})}$	[13.90807, 24.67370]	[23.30966, 31.21807]
$\boldsymbol{C}_{\boldsymbol{Y}_{(\phi N)}}$	[0.02657039, 0.02539073]	[0.03440212, 0.02580739]
$\boldsymbol{C}_{\boldsymbol{X}_{(\phi N)}}$	[0.04133301, 0.03918572]	[0.01435458, 0.01423702]
$\boldsymbol{\rho}_{\boldsymbol{YX}_{(\phi N)}}$	[0.6273783, 0.7201808]	[0.6737446, 0.7504361]

6. Conclusion

This study confirms the validity of neutrosophic imputation techniques in two-stage sampling, particularly for imprecise, fuzzy, or outlier-vulnerable data instances. Compared to conventional statistical techniques that are more exact-auxiliary-data-dependent and data-error-sensitive, neutrosophic techniques provide a strong framework capable of dealing with uncertainty and indeterminacy in both the main and auxiliary sources of data.

TABLE 2. Mean Square Error of the Neutrosophic Estimators under Design I.

Data					MSEs					
A	N	n'	n	r	$\bar{y}_{r_{(\phi N)}}$	$\bar{y}_{RAT_{(\phi N)}}$	$\bar{y}_{Comp_{(\phi N)}}$	$\bar{y}_{TV_{1(I)}}$	$\bar{y}_{TV_{2(I)}}$	$\bar{y}_{TV_{3(I)}}$
	119	50	18	14	[0.0191, 0.0405]	[0.0213, 0.0421]	[0.0172, 0.0352]	[0.0108, 0.0198]	[0.0127, 0.0250]	[0.0137, 0.0277]
				15	[0.0176, 0.0374]	[0.0192, 0.0385]	[0.0163, 0.0337]	[0.0099, 0.0183]	[0.0113, 0.0220]	[0.0128, 0.0263]
				16	[0.0164, 0.0347]	[0.0173, 0.0354]	[0.0155, 0.0324]	[0.0092, 0.0170]	[0.0100, 0.0193]	[0.0120, 0.0250]
			22	16	[0.0164, 0.0347]	[0.0188, 0.0365]	[0.0143, 0.0291]	[0.0092, 0.0170]	[0.0112, 0.0227]	[0.0108, 0.0216]
				17	[0.0152, 0.0324]	[0.0171, 0.0337]	[0.0137, 0.0279]	[0.0085, 0.0159]	[0.0101, 0.0203]	[0.0101, 0.0205]
				18	[0.0143, 0.0303]	[0.0157, 0.0313]	[0.0131, 0.0269]	[0.0079, 0.0149]	[0.0091, 0.0182]	[0.0095, 0.0195]
				19	[0.0134, 0.0284]	[0.0144, 0.0291]	[0.0125, 0.0260]	[0.0074, 0.0139]	[0.0082, 0.0163]	[0.0090, 0.0186]
B										
	119	50	18	14	[0.0205, 0.0311]	[0.0185, 0.0270]	[0.0182, 0.0267]	[0.0109, 0.0143]	[0.0132, 0.0187]	[0.0144, 0.0210]
				15	[0.0190, 0.0288]	[0.0176, 0.0259]	[0.0173, 0.0257]	[0.0100, 0.0133]	[0.0117, 0.0164]	[0.0135, 0.0200]
				16	[0.0176, 0.0267]	[0.0167, 0.0249]	[0.0166, 0.0248]	[0.0093, 0.0124]	[0.0103, 0.0143]	[0.0128, 0.0191]
			22	16	[0.0176, 0.0267]	[0.0155, 0.0223]	[0.0151, 0.0220]	[0.0093, 0.0124]	[0.0118, 0.0171]	[0.0113, 0.0163]
				17	[0.0164, 0.0249]	[0.0147, 0.0215]	[0.0144, 0.0212]	[0.0086, 0.0116]	[0.0106, 0.0153]	[0.0107, 0.0155]
				18	[0.0153, 0.0233]	[0.0141, 0.0207]	[0.0139, 0.0205]	[0.0080, 0.0109]	[0.0095, 0.0137]	[0.0101, 0.0148]
				19	[0.0144, 0.0219]	[0.0135, 0.0200]	[0.0133, 0.0199]	[0.0075, 0.0103]	[0.0086, 0.0123]	[0.0096, 0.0141]

TABLE 3. Mean Square Error of the Neutrosophic Estimators under Design II.

Data					MSEs		
A	N	n'	n	r	$\bar{y}_{TV_{1(II)}}$	$\bar{y}_{TV_{2(II)}}$	$\bar{y}_{TV_{3(II)}}$
	119	50	18	14	[0.0116, 0.0208]	[0.0134, 0.0259]	[0.0153, 0.0313]
				15	[0.0107, 0.0193]	[0.0120, 0.0228]	[0.0145, 0.0298]
				16	[0.0099, 0.0179]	[0.0107, 0.0202]	[0.0137, 0.0285]
			22	16	[0.0099, 0.0179]	[0.0118, 0.0233]	[0.0125, 0.0251]
				17	[0.0092, 0.0168]	[0.0107, 0.0210]	[0.0118, 0.0240]
				18	[0.0086, 0.0157]	[0.0097, 0.0189]	[0.0112, 0.0230]
				19	[0.0081, 0.0147]	[0.0088, 0.0170]	[0.0107, 0.0221]
B							
	119	50	18	14	[0.0115, 0.0150]	[0.0138, 0.0192]	[0.0162, 0.0237]
				15	[0.0107, 0.0139]	[0.0123, 0.0169]	[0.0153, 0.0227]
				16	[0.0099, 0.0130]	[0.0109, 0.0148]	[0.0146, 0.0218]
			22	16	[0.0099, 0.0130]	[0.0123, 0.0175]	[0.0131, 0.0190]
				17	[0.0092, 0.0121]	[0.0111, 0.0156]	[0.0125, 0.0182]
				18	[0.0086, 0.0114]	[0.0100, 0.0140]	[0.0119, 0.0175]
				19	[0.0081, 0.0107]	[0.0091, 0.0126]	[0.0114, 0.0168]

TABLE 4. Percent Relative Efficiencies (PREs) of the Neutrosophic Estimators under Design I.

Data					PREs				
A	N	n'	n	r	$\bar{y}_{RAT_{(\phi N)}}$	$\bar{y}_{Comp_{(\phi N)}}$	$\bar{y}_{TV_{1(I)}}$	$\bar{y}_{TV_{2(I)}}$	$\bar{y}_{TV_{3(I)}}$
	119	50	18	14	[89.6714, 96.1995]	[111.0465, 115.0568]	[176.8519, 204.5455]	[150.3937, 162.0000]	[139.4161, 146.2094]
				15	[91.6667, 97.1429]	[107.9755, 110.9792]	[177.7778, 204.3716]	[155.7522, 170.0000]	[137.5000, 142.2053]
				16	[94.7977, 98.0226]	[105.8065, 107.0988]	[178.2609, 204.1176]	[164.0000, 179.7927]	[136.6667, 138.8000]
			22	16	[87.2340, 95.0685]	[114.6853, 119.2440]	[178.2609, 204.1176]	[146.4286, 152.8634]	[151.8519, 160.6481]
				17	[88.8889, 96.1424]	[110.9489, 116.1290]	[178.8235, 203.7736]	[150.4950, 159.6059]	[150.4950, 158.0488]
				18	[91.0828, 96.8051]	[109.1603, 112.6394]	[181.0127, 203.3557]	[157.1429, 166.4835]	[150.5263, 155.3846]
				19	[93.0556, 97.5945]	[107.2000, 109.2308]	[181.0811, 204.3165]	[163.4146, 174.2331]	[148.8889, 152.6882]
B									
	119	50	18	14	[110.8108, 115.1852]	[112.6374, 116.4794]	[188.0734, 217.4825]	[155.3030, 166.3102]	[142.3611, 148.0952]
				15	[107.9545, 111.1969]	[109.8266, 112.0623]	[190.0000, 216.5414]	[162.3932, 175.6098]	[140.7407, 144.0000]
				16	[105.3892, 107.2289]	[106.0241, 107.6613]	[189.2473, 215.3226]	[170.8738, 186.7133]	[137.5000, 139.7906]
			22	16	[113.5484, 119.7309]	[116.5563, 121.3636]	[189.2473, 215.3226]	[149.1525, 156.1404]	[155.7522, 163.8037]
				17	[111.5646, 115.8140]	[113.8889, 117.4528]	[190.6977, 214.6552]	[154.7170, 162.7451]	[153.2710, 160.6452]
				18	[108.5106, 112.5604]	[110.0719, 113.6585]	[191.2500, 213.7615]	[161.0526, 170.0730]	[151.4851, 157.4324]
				19	[106.6667, 109.5000]	[108.2707, 110.0503]	[192.0000, 212.6214]	[167.4419, 178.0488]	[150.0000, 155.3191]

Table 5. Percent Relative Efficiencies (PREs) of the Neutrosophic Estimators under Design II.

Data					PREs		
A	N	n'	n	r	$\bar{y}_{TV_{1(II)}}$	$\bar{y}_{TV_{2(II)}}$	$\bar{y}_{TV_{3(II)}}$
	119	50	18	14	[164.6552, 194.7115]	[142.5373, 156.3707]	[124.8366, 129.3930]
				15	[164.4860, 193.7824]	[146.6667, 164.0351]	[121.3793, 125.5034]
				16	[165.6566, 193.8547]	[153.2710, 171.7822]	[119.7080, 121.7544]
			22	16	[165.6566, 193.8547]	[138.9831, 148.9270]	[131.2000, 138.2470]
				17	[165.2174, 192.8571]	[142.0561, 154.2857]	[128.8136, 135.0000]
				18	[166.2791, 192.9936]	[147.4227, 160.3175]	[127.6786, 131.7391]
				19	[165.4321, 193.1973]	[152.2727, 167.0588]	[125.2336, 128.5068]
B							
	119	50	18	14	[178.2609, 207.3333]	[148.5507, 161.9792]	[126.5432, 131.2236]
				15	[177.5701, 207.1942]	[154.4715, 170.4142]	[124.1830, 126.8722]
				16	[177.7778, 205.3846]	[161.4679, 180.4054]	[120.5479, 122.4771]
			22	16	[177.7778, 205.3846]	[143.0894, 152.5714]	[134.3511, 140.5263]
				17	[178.2609, 205.7851]	[147.7477, 159.6154]	[131.2000, 136.8132]
				18	[177.9070, 204.3860]	[153.0000, 166.4286]	[128.5714, 133.1429]
				19	[177.7778, 204.6729]	[158.2418, 173.8095]	[126.3158, 130.3571]

By constructing novel imputation methods for exponential-logarithmic-type estimators, the study demonstrates that the neutrosophic estimators can yield interval-based estimates, which present a more accurate estimate of the population mean. Theoretical results confirm that such estimators possess virtues such as reduced bias and minimum mean square error under large sample approximations.

Empirical comparisons with actual datasets further confirm the practical merits of the neutrosophic approach, being more robust and reliable compared to traditional estimators. The synthesis of existing imputation methods under the neutrosophic statistical paradigm broadens inference potentialities, allowing for superior data imperfection management. Overall, the findings highlight that neutrosophic imputation techniques are a robust and versatile tool for statistical estimation in complex, uncertain data settings.

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