Neutrosophic Extension of Maxwell Length-Biased Distribution and Its Application in Energy Sector

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Abstract: The length-biased distribution has important applications in modeling economic, reliability and energy data, especially when sampling the data preferentially toward the larger values. In this paper, we introduce the neutrosophic extension of the Maxwell length-biased distribution so that it can take into account the uncertainty, indeterminacy, and imprecision involved in data analysis. Basic statistical properties of the new distribution are presented including those of its shape properties, moments, reliability function, hazard and reversed hazard functions, etc. Estimation procedure, including maximum likelihood, method (MLE) is presented in the context of neutrosophic statistics. To emphasize the practicality of the proposed solution, an application to the energy domain shows that it can deal with uncertainty and delivering more robust results than classical counterparts.

Keywords: Neutrosophic logic, uncertain data, neutrosophic probability, estimation

1. Introduction

A weighted distribution is a modified version of a regular probability distribution that assigns greater importance, or "weight," to some values of the variable of interest [1]. It is particularly meaningful when the likelihood of observing a value is proportional to the fact that it is larger, or occurs more frequently, in reality [2].
In statistical form, weighted distribution can be defined as [3]:

$$f_W(y) = \frac{w(y)\, f_Y(y)}{\int_{-\infty}^{\infty} w(y)\, f_Y(y)\, dy}, -\infty < y < \infty \quad (1)$$

The concept was initiated by Fisher in 1934 and then extended by C. R. Rao in 1965 [4]. Weighted distributions find various references in applications, such as reliability, medicine, ecology, life sciences, in which real phenomena may not be followed by classical statistical models perfectly [5]. For instance, in survival or lifetime events, an individual with a longer lifetime is more likely to be sampled, so the observed distribution will consequently be length biased (one type of weighted distribution) [6]. Through weighted distributions one can allow for bias in such scenarios and thereby can establish better modeling of situations and more practical inferences [7].

Weighted distributions are valuable in modeling and prediction in the energy sector, where data are frequently biased because of changes in demand, intermittent supply, or preferential sampling of high values [8]. By applying weights as per need, these distributions can enable researchers to represent the impact of high or frequent energy requirements so as to achieve a more realistic portrayal of consumption behavior and system security [9]. It is apparent that weighted distributions are beneficial in forecasting energy demand, reliability of power systems, and risk analysis for unpredictable or variable energy consumption.

The model is also referred to as a length-biased distribution in the case where the weight function is proportional to the variable (i.e. $w(y) = y$) [10]. In this case, the probability distribution function values at higher values of the variable carry more weight. The class of length-biased distributions was first identified by Cox (1962) in renewal theory context. comparisons between the general weighted distributions and the length-biased distributions.

Mathematically we can write length-biased probability distribution as:

$$f_L(y) = \frac{y\, f_Y(y)}{\int_{-\infty}^{\infty} y\, f_Y(y)\, dy}, -\infty < y < \infty \tag{2}$$

A length-biased distribution has importance in statistical modeling because they arise naturally in real life when the larger values have high probability to be observed [11]. It is common in lifetime studies, reliability study, environmental study and energy data that long or large magnitude units can be more probable to be seen in a sample. Since systematic length bias in the data are controlled by the model, length-biased models can have more realistic interpretations for data subject to length bias and lead to more accurate statistical inference, prediction and decision making. They also represent a special case of weighted distributions, which make them a useful tool to address issues of biased sampling, uncertainty, etc. in applied contexts [12]-[13].

Neutrosophic logic is a field of research generated by the weaknesses of classical logic and then fuzzy logic by handling cases of (truth and falsehood) including truth, falsehood and indeterminacy [14]. There are a lot of situations in practice, in which there is partial, vague or even contradictory information but it is not able to deal with by system based on traditional binary or fuzzy degrees. Neutrosophic logic has been proposed to solve this problem to explicitly represent indeterminacy in addition to certainty and uncertainty [15]. Organized on that basis, neutrosophic statistics was constructed to generalize classical statistical techniques to imprecise or ambiguous data and thus offer more viable analysis tools [16]-[17]. Neutrosophic probability distributions then turn out to be a significant generalization of classical distributions which can model random phenomena under uncertainty, indetermination, and inconsistency. These distributions are especially valuable in areas where easily-measured variables are imprecise or only partially known since they represent a richer and more flexible geometry for their capturing the complexity of real systems [18].

In many practical situations, especially in lifetime and reliability analysis, the samples are length-biased where large values are more observed than small ones. Although classical length-biased distributions give a mechanism for dealing with this bias still they have some shortcomings when the data suffers from uncertainty, incompletion, imprecision, etc., which are frequently encountered in practical fields like energy forecasting and consumption analysis [19]-[21]. This gap indicates the necessity of developing neutrosophic length-biased distribution having the property of length-biased and the advantage of flexibility of neutrosophic theory to support considerations of vagueness and hesitancy. Among the proposed models, the neutrosophic version of Maxwell length-biased distribution (MLBD) can be interesting due to the fact that it generalizes the known Maxwell distribution, works as a one-parameter distribution for convenience, and captures the information of the data under uncertainty all at once. This renders it a powerful and practical tool for energy sector data modeling, in which we consider the bias in observations as well as the uncertainty bound of measurements for reliable forecasting / analysis, too.

In this paper, the neutrosophic Maxwell length-biased distribution is introduced and it is an extension of the classical Maxwell distribution by utilizing the neutrosophic logic methodology. This form has the potential to treat uncertainty, indeterminacy, and the lack of complete information on the observed data which makes it tailored for real-world cases where exact measurements may be difficult to obtain. Introducing length-biased weight in the neutrosophic setting, the proposed model becomes a flexible and realistic tool for lifetime and energy-related data analysis which still has one parameter only.

The paper is organized as follows: In Section 2 the main findings concerning the classical marginal length-biased Maxwell distribution are stated. In section 3, we develop the neutrosophic structure of the proposed distribution and study some of its properties. Section 4 details the parameter estimation techniques: maximum likelihood and Bayesian. Section 5 applies the proposed model to energy sector data and illustrates its capability in dealing with uncertainty and bias. Last, major findings of this work is concluded in Section 6.

2 Classical Length biased Maxwell Distribution

To develop the LBMD ,it is important to write based line CDF and PDF of the Maxwell model which are given in Eq (3) and Eq (4) respectively:

$$F_Y(y) = \text{erf}\left(\frac{y}{\sqrt{2}\,\rho}\right) - \sqrt{\frac{2}{\pi}}\frac{y}{\rho}\exp\left(-\frac{y^2}{2\rho^2}\right), 0 < y < 1 \quad (3)$$

$$f_Y(y) = \sqrt{\frac{2}{\pi}}\frac{y^2}{\rho^3}\exp\left(-\frac{y^2}{2\rho^2}\right), 0 < y < \infty \quad (4)$$

Since the Maxwell distribution first time used in physical chemistry so its first applications relate with molecules speeds in any medium. The PDF of the Maxwell distribution is the distribution of the speeds of individual particles, or volume units, and gives is an indication of which velocity values are more likely to be found in an equilibrium system. It helps to build an understanding of the behavior of the variable of interest, presenting an estimate of where most of the observations are located and whether scatter is present, as well as how extreme values are more or less probable. The CDF provides this by measuring the probability that the speed is less than or equal to some level, enabling investigators to estimate the percentage of pools below some threshold speed. The PDF and CDF together are important for exploring the properties of the Maxwell distribution and helping analyze average speeds, fluctuation, and the probability of extreme events. In energy system optimization, models of these types provide tools to reflect the distribution of observed data, to assess the reliability of the system, and to perform forecasts, providing the basis for more advanced extensions such as the length-biased or the neutrosophic Maxwell model.

Now assuming the weight form $w(y) = y$). MLBD can be defined in PDF and CDF as:

$$f_L(y;\rho) = \frac{y^3\exp\left(-\frac{y^2}{2\rho^2}\right)}{2\rho^4}, \quad 0 < y < \infty, \rho > 0 \quad (5)$$

$$F_L(y;\rho) = 1 - \exp\left(-\frac{y^2}{2\rho^2}\right)\left(\frac{y^2}{2\rho^2} + 1\right), 0 < y < 1 \quad (6)$$

The PDF and CDF curves of Maxwell model can be seen in Figure 1.

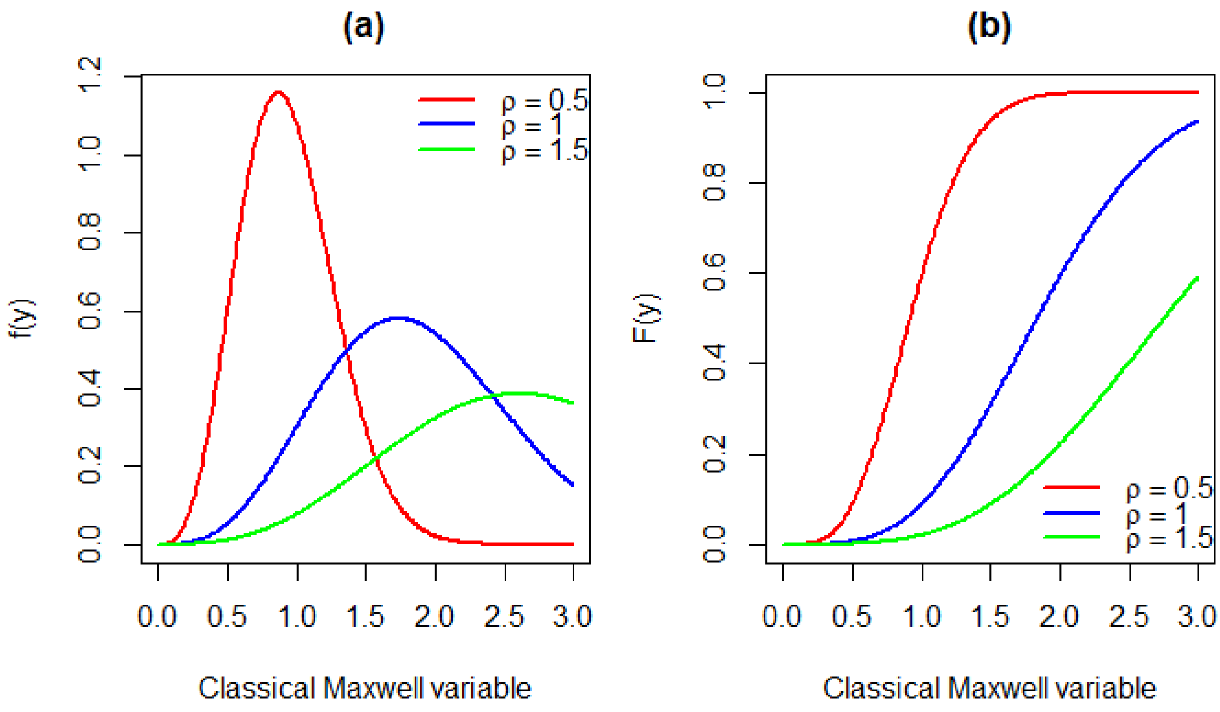

Figure 1 PDF and CDF of the LBMD for various ρ values

Figure 1 displays PDFs and CDFs of the LBMD, for three different ρ values. In the left panel, we plot the PDF, which shows how different values are likely to alternate with ρ. The one on the right displays the CDF. In both the subplots, the influence of the scale parameter ρ on the distribution form and cumulative behavior is evident.

To derive other classical properties of the LBMD we first see the rth moment of the distribution which can be established as:

$$\mu_r = E(Y^r) = \int_0^\infty y^r \frac{y^3 \exp\left(-\frac{y^2}{2\rho^2}\right)}{2\rho^4} dy = \mu_r = E(Y^r) = \sqrt{2^r}\, \rho^r\, \Gamma\left(\frac{4+r}{2}\right) \quad (7)$$

Now we can easily write from Eq (7):

$$\mu_1' = \sqrt{2^1}\, \rho^1\, \Gamma\left(\frac{4+1}{2}\right) = \sqrt{2}\, \rho\, \Gamma\left(\frac{5}{2}\right)$$
$$\mu_2' = \sqrt{2^2}\, \rho^2\, \Gamma\left(\frac{4+2}{2}\right) = 2\, \rho^2\, \Gamma(3)$$
$$\mu_3' = \sqrt{2^3}\, \rho^3\, \Gamma\left(\frac{4+3}{2}\right) = 2\sqrt{2}\, \rho^3\, \Gamma\left(\frac{7}{2}\right)$$
$$\mu_4' = \sqrt{2^4}\, \rho^4\, \Gamma\left(\frac{4+4}{2}\right) = 4\, \rho^4\, \Gamma(4)$$

Now it is easy to see basic characteristics of the distribution:

$$\mu = \mu_1' = \sqrt{2}\, \rho\, \Gamma\left(\frac{5}{2}\right) \quad (8)$$

$$\sigma^2 = \mu_2' - (\mu_1')^2 = 2\, \rho^2\, \Gamma(3) - \left(\sqrt{2}\, \rho\, \Gamma\left(\frac{5}{2}\right)\right)^2 \quad (9)$$

$$\text{skewness} = \gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(\sigma^2)^{3/2}}$$

$$= \frac{2\sqrt{2}\,\rho^3\,\Gamma\left(\frac{7}{2}\right)-3\left(2\,\rho^2\,\Gamma(3)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)+2\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^3}{(\sigma^2)^{3/2}} \tag{10}$$

$$\text{kurtosis} = \gamma_2 = \frac{\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4}{(\sigma^2)^2} \tag{11}$$

$$= \frac{4\,\rho^4\,\Gamma(4)-4\left(2\sqrt{2}\,\rho^3\,\Gamma\left(\frac{7}{2}\right)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)+6\left(2\,\rho^2\,\Gamma(3)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^2-3\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^4}{(\sigma^2)^2} \tag{12}$$

By assuming different values of scale parameter of LBMD, the basic characteristics are presented in Table 1.

Table 1 Statistical characteristics of LBMD for different values of scale parameter

ρ	Mean	Variance	Skewness	Kurtosis
0.25	0.4699928	0.02910677	0.4056951	3.059295
0.5	0.9399856	0.11642707	0.4056951	3.059295
1.5	2.8199568	1.0478436	0.4056951	3.059295
3.0	5.6399136	4.19137438	0.4056951	3.059295

Table 1 shows how the values of the LBMD change as the parameter ρ. increases. When ρ. is small, the average values are also small, and the variation around the average is quite limited. As ρ. get larger, both the mean and the variability of the data increase, showing that the distribution stretches out more. Interestingly, the shape-related measures skewness and kurtosis remain the same across all values of ρ.

3 Neutrosophic Length Biased Maxwell Distribution

A random variable Y said to follow NLBMD if it follows the following forms of PDF and CDF:

$$f_Y(y) = \sqrt{2/\pi}\frac{y^2}{\rho_n^3}\exp\left(-\frac{y^2}{2\rho_n^2}\right), 0 < y < \infty \tag{13}$$

$$F_Y(y;\rho_n) = 1 - \exp\left(-\frac{y^2}{2\rho_n^2}\right)\left(\frac{y^2}{2\rho_n^2}+1\right), 0 < y < 1 \tag{14}$$

where scale parameter $\rho_n = [\rho_l, \rho_u]$ is in interval form.

Eq (13) and Eq (14) show that length biased Maxwell distribution is extended to neutrosophic PDF and CDF, by involving the uncertainty on the model parameter through the neutrosophic framework. In this context, the scale parameter is not treated as a crisp or fixed value, it is considered like a neutrosophic number with interval type that represents the degree of truth, indeterminacy and falsity at the same time. This generalization allows the model to learn more expressive representations of inexact and ambiguous information typically present in real-world data, and in particular for complex domains such as the energy sector, which is influenced by environmental, economic and operational variables that introduce unavoidable uncertainty. The neutrosophic PDF offers a versatile way to model the observations distribution with indeterminacy and the associated neutrosophic CDF makes sure that more useful information for cumulative probability is characterized under uncertainty or vagueness. The neutrosophic form of the Maxwell length biased distribution indicates this distribution as a valuable model for probabilistic modeling when classical assumption of crisp data is not satisfied. The structure of neutrosophic PDF and CDF are given in Figure 2.

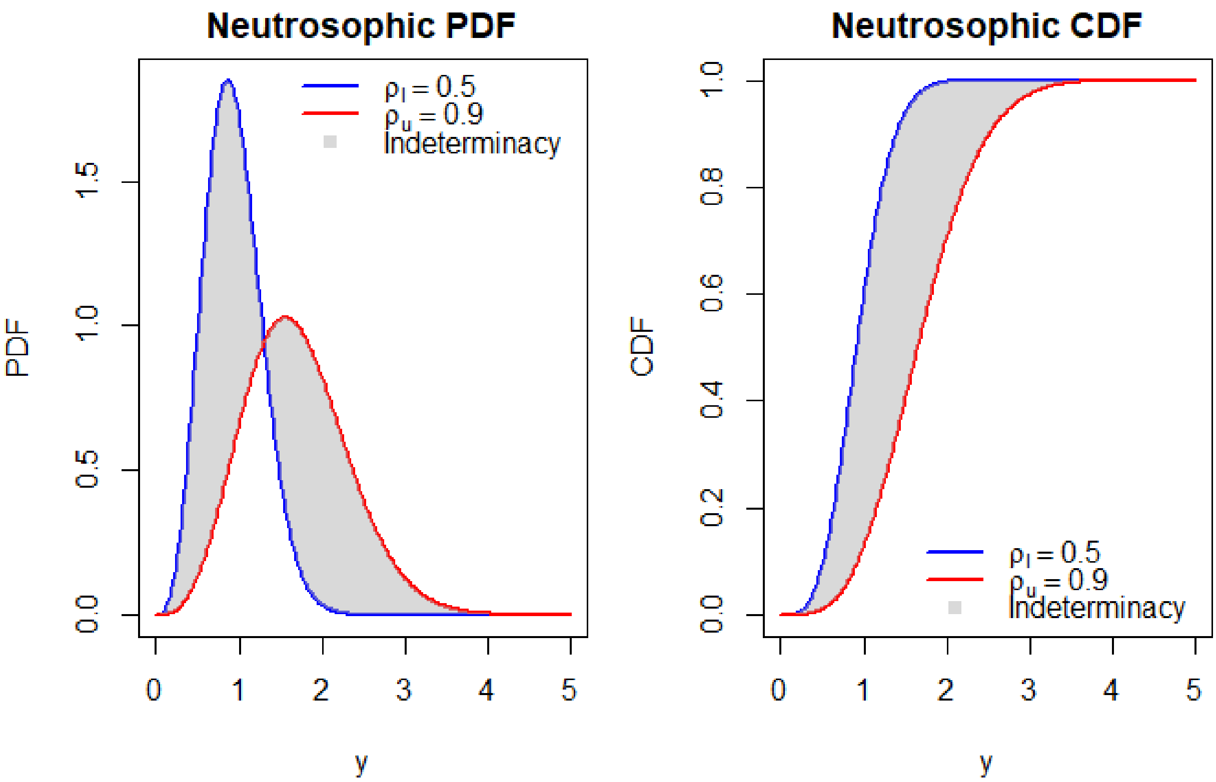

Figure 2 Neutrosophic PDF and CDF of the proposed distribution

The neutrosophic PDF and CDF of the length-biased Maxwell distribution are shown in Figure. 2. The extreme values of the parameter are the lower and upper curves and the region between them is the indeterminacy zone. This area of shading represents the uncertainty in the behavior of the distribution, that is the range of possible variation between the two bounds. The left panel is the neutrosophic PDF, and the right panel is the neutrosophic CDF as uncertainty over the entire distribution.

The quantile function is related to inverse of CDF which can be expressed as:

$$F_Y(Q(u;\rho_n);\rho_n) = u, 0 < u < 1 \tag{15}$$

Equivalently Eq (15) can be written as:

$$Q(u;\rho_n) = 1 - \exp\left(-\frac{Q(u;\rho_n)^2}{2\rho_n^2}\right)\left(\frac{Q(u;\rho_n)^2}{2\rho_n^2} + 1\right) = u$$

The quantile function of the NLBMD cannot be expressed in close form for direct computations. It is derived not analytically but by numerical methods by which the solution to the cumulative distribution function at a certain probability level is approximated. This can easily be solved using R software. This can help us to generate a random neutrosophic samples in the interval form where each interval represents the uncertainty between the lower and the upper parameter value and to provide a very flexible approach in order to grasp the indetermination of real-life data. The 40 random samples from the proposed model are given in Table 2.

Table 2 Random samples generated from proposed model

[0.729,1.313]	[1.208,2.174]	[0.837,1.507]	[1.359,2.445]	[1.505,2.71]
[0.411,0.739]	[0.941,1.693]	[1.378,2.48]	[0.961,1.73]	[0.878,1.581]
[1.569,2.824]	[0.876,1.576]	[1.081,1.946]	[0.98,1.765]	[0.52,0.936]
[1.394,2.51]	[0.689,1.241]	[0.402,0.723]	[0.766,1.379]	[1.559,2.805]
[1.372,2.469]	[1.097,1.974]	[1.044,1.879]	[1.907,3.433]	[1.059,1.906]
[1.114,2.004]	[0.955,1.719]	[1.0,1.8]	[0.731,1.315]	[0.581,1.046]
[1.598,2.876]	[1.4,2.52]	[1.095,1.97]	[1.217,2.191]	[0.347,0.624]
[0.897,1.614]	[1.17,2.107]	[0.659,1.187]	[0.757,1.363]	[0.675,1.215]

Table 2 presents a set of neutrosophic random samples generated from the NLBMD. Each entry is shown as an interval, where the lower value corresponds to the distribution with the smaller parameter ($\rho_l = 0.5$) and the upper value ($\rho_u = 0.9$) corresponds to the larger parameter.
The other important function that is related to CDF function is survival or reliability function which can be written as:

$$S_Y(y) = \exp\left(-\frac{y^2}{2\rho_n^2}\right)\left(\frac{y^2}{2\rho_n^2} + 1\right) \quad (16)$$

The survival function can be depicted in Figure 3.

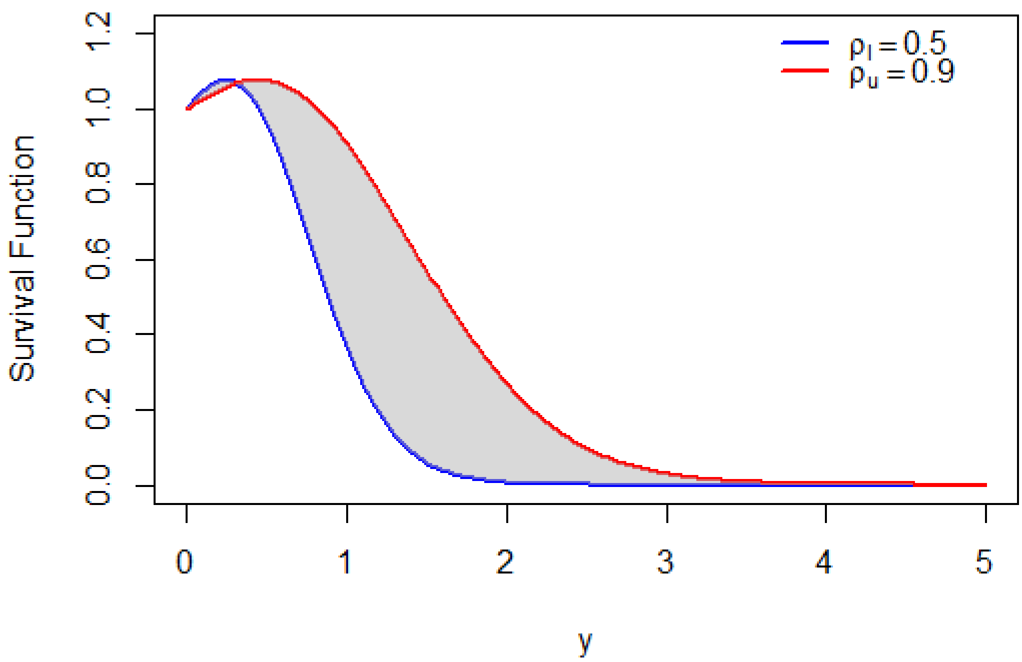

Figure 3 Survival function of the proposed distribution

The survival function of the neutrosophic length-biased Maxwell is depicted in Figure 3. The two curves [border curves] represent the lower and upper limits of the parameter, respectively, and the shaded region in between depicts the uncertain domain. This graph demonstrates the variability with which uncertainty is accumulated to prediction of survival, and how distribution behaves under various parameter values.

The neutrosophic mean and variance of the proposed model can be expressed as:

$$\mu_n = \sqrt{2}\,\rho_n\,\Gamma(5/2) \quad (17)$$

$$\sigma_n^2 = 2\rho_n^2\Gamma(3) - \left(\sqrt{2}\,\rho_n\,\Gamma(5/2)\right)^2 \quad (18)$$

The neutrosophic mean and variance of the proposed distribution will give the measures of location and scatter with the uncertainty of the parameters of model, respectively. Whereas, in the classical case, where the mean and variance are crisp values here, their neutrosophic forms are given as intervals to represent indeterminacy and vagueness. Such an assumption leads the results to be more realistic, since real data, especially those from complex systems such as the energy system, often are distorted by uncertainty, measurement errors or vague elements. Thus, the neutrosophic mean denotes not only a unique, average value, but also the range of different potential average values, while the neutrosophic variance measures the range within which the different data may spread around that mean. All together, they make the statistical model more robust by taking uncertainty instead of hiding it.
Based on neutrosophic mean and variance we can write the neutrosophic coefficient of variation as given below:

$$\mathrm{CV_n} = \frac{\sqrt{2\rho_n^2\Gamma(3)-\left(\sqrt{2}\,\rho_n\,\Gamma(5/2)\right)^2}}{\sqrt{2}\,\rho_n\,\Gamma(5/2)} \quad (19)$$

The proposed distribution coefficient of variation (CV) is an indicator to express the vagueness or indeterminacy associated with the individual possible spread of the data compared with its average proposal. Unlike classical CV, the neutrosophic one represents this relationship as an interval construct, thus the model can grasp imprecision and indeterminacy that fits more the nature of uncertain or incomplete data in practice.
Now skewness and kurtosis coefficients in terms of neutrosophic logic can be expressed as:

$$\gamma_n = \frac{=\{2\sqrt{2}\,\rho_n^3\,\Gamma(7/2)-3\left(2\rho_n^2\Gamma(3)\right)\left(\sqrt{2}\,\rho_n\Gamma(5/2)\right)+2\left(\sqrt{2}\,\rho_n\Gamma(5/2)\right)^3\}}{\{\left(\sigma_n^2\right)^{3/2}\}} \quad (20)$$

$$\kappa_n = \frac{4\rho_n^4\Gamma(4)-4\left(2\sqrt{2}\rho_n^3\Gamma(7/2)\right)\left(\sqrt{2}\rho_n\Gamma(5/2)\right)+6\left(2\rho_n^2\Gamma(3)\right)\left(\sqrt{2}\rho_n\Gamma(5/2)\right)^2-3\left(\sqrt{2}\rho_n\Gamma(5/2)\right)^4}{\left(\sigma_n^2\right)^2} \quad (21)$$

The skewness and kurtosis parameters of the NLBMD in neutrosophic form, offer greater facility to understand its shape and tail characteristics under uncertainty. The bentness indicates how lopsided the source is, and in the neutrosophic framework it represents several possible values not a particular one, where one values indeterminacy of data. On the other hand, the kurtosis corresponds to the peakness or flatness of the distribution, with the neutrosophic version presenting an interval that works with uncertainty and partial knowledge. In combination, these steps enable a more liberal read of the distributional properties, particularly in situations where precise parameter values are not known with absolute certainty.

4. Estimation Approach

Neutrosophic Maximum Likelihood Estimation (MLE) method is generalization of classical MLE involving uncertainty and indeterminacy in the data. Rather than a point estimate, it generates interval-valued estimates that encode both the stochasticity and the neutrosophic uncertainty of the problem. In this way, the model is able to attain a wide range of reasonable choices for the parameters, which in turn contributes to a robust estimation in front of inaccurate and/or partially unknown data. It is especially handy in practical world systems with variable input and some lack of information.
The likelihood function of the NLBMD can be written as:

$$L(y_1,\dots,y_n;\rho_n)=\prod_{i=1}^{n}\frac{y_i^3}{2\rho_n^4}\exp!\left(-\frac{y_i^2}{2\rho_n^2}\right) \quad (22)$$

Eq (22) in the loglikelihood form can be obtained as:

$$\log L\,(y_1,\dots,y_n;\rho_n)=\sum_{i=1}^{n}3\ln y_i-n\ln 2-4n\ln\rho_n-\frac{1}{2\rho_n^2}\sum_{i=1}^{n}y_i^2 \quad (23)$$

Differentiating Eq (23) with respect to unknow yielded:

$$\frac{\partial\log L}{\partial\rho_n}=-\frac{4n}{\rho_n}+\frac{\sum_{i=1}^{n}y_i^2}{\rho_n^3} \quad (24)$$

Eq (24) equating to zero yielded:

$$\widehat{\rho_n}=\sqrt{\frac{\sum_{i=1}^{n}y_i^2}{4n}}. \quad (25)$$

Table 3 Estimated parameter with mean square error (MSE) of the proposed model

Sample Size (n)	$\widehat{\rho_n}$	Neutrosophic MSE
25	[0.631, 0.889]	[0.032, 0.044]
50	[0.625, 0.885]	[0.03, 0.042]
75	[0.623, 0.882]	[0.029, 0.041]
150	[0.617, 0.878]	[0.027, 0.039]
250	[0.615, 0.875]	[0.026, 0.038]
500	[0.612, 0.872]	[0.025, 0.036]

The estimated neutrosophic parameter ρ_n and corresponding mean squared errors for various sample sizes are given in Table 3. The estimated interval for ρ_n becomes more accurate when the sample size grows, indicating strong evidence of overlapping lower and upper bounds. Also, the mean squared errors become smaller as the sample size increases, which means, as one expects, that the estimates tend to be more reliable and robust when based on a higher number of points. This, in fact, emphasizes the superiority of using large datasets for accurate estimation of neutrosophic parameters.

5 Real Data Applicability

In this section we have utilized our approach to analyze energy data related to renewable energy in Saudi Arabia [22]. Most of the electricity in Saudi Arabia is generated from fossil fuels, especially natural gas and oil, which has historically been the dominant source of power in the kingdom. However, the government knows that it needs diversity and sustainability, and as such has been quite committed to developing more renewable energy. Of these, solar photovoltaic (PV) is the most attractive due to the fact that the country enjoys an incredible amount of sunshine, the decreasing costs of solar technology, and the possibility to be deployed on a large or distributed scale. Solar power is in this sense considered as a mainstay of the national strategy to lessen dependence on fossil fuels, to decrease emissions, and to generate more sustainable electricity supply due to increasing demand. The expansion of renewables, especially solar PV, offers substantial potential, but evaluating the available data on energy production and use can be fraught with uncertainty as demand shifts, weather patterns change, and precise or complete records may be missing. Such indeterminate and inconsistent information can be difficult to manage with traditional techniques. The notion of neutrosophic logic is particularly useful in the present context since allows to consider the truth, the indeterminacy, and the falsity in data analysis. This methodology is capable of dealing with uncertainty and vagueness and thus allows for more accurate assessment of renewable energy trends and better decisions in planning and managing the Saudi Arabian transition to sustainable electricity generation. We have randomly generated samples from uniform distribution. This data becomes neutrosophic by representing each year's solar PV generation not as a single fixed number, but as an interval that captures both lower and upper possible values. Instead of relying only on exact figures, small variations are introduced through random fluctuations, which account for uncertainty and imprecision in real-world measurements. Intentionally generated data is given in Table 4.

Table 4 Neutrosophic Interval of solar PV generation in Saudi Arabia (2010–2023)

[3.28, 4.71]	[4.78, 5.21]	[4.41, 5.59]	[25.88,26.12]	[45.86,46.13]
[64.88, 65.12]	[64.64, 65.35]	[319.91,320.09]	[915.37,916.63]	[4319.11,4320.88]
[41.94, 42.06]	[926.47, 927.53]	[45.65,434]	[938.99,939.01]	

The lower and upper estimates of solar PV electricity generation for Saudi Arabia, 2010–2023 are shown in the Table 4. These ranges were computed by modifying the observed data from the actual data with a random uniform variance and sampling the solid black lines where the lower bound is the original 5% beyond the actual value and the upper bound the original 5% behind the actual value. In two rows and seven columns, the table shows 14 pairs of intervals that define potential deviations from recorded levels by generation that illustrate the range of uncertainty the actual numbers.
Now utilizing the MLE estimator given Eq (25), we obtain:

$$\widehat{\rho_n} = [617.43, 617.74]$$

The smaller uncertainty has been captured by estimated scale estimator due to assumed smaller variation in the actual solar PV values. The interval of the neutrosophic estimation produces a value in accordance with the uncertainty and also considering the variation of the solar PV data. The lower estimate bound represents a more conservative scenario, through lower values of electricity generation, and the upper bound an optimistic scenario, through higher values. Combining this interval provides a loose and more realistic interpretation of the parameter, demonstrating how it can change under uncertainty. This strategy illustrates the potential of neutrosophic logic in dealing with the vagueness in practical renewable energy datum and provides more accurate basis for planning and decision-making.

6 Conclusions

In this study, we established the neutrosophic extension of the Maxwell length-biased distribution that could capture uncertainty, indeterminacy, and imprecision. Theoretical properties of the proposed new model were also derived, as well as estimation procedures, including maximum likelihood estimation, in the neutrosophic setting. To illustrate its practical use a sample application of the proposed technique was implemented in renewable energy data from Saudi Arabia, particularly solar PV generation. Converting the observed values as neutrosophic intervals, has allowed the uncertainty on electricity production is small fluctuation of data, providing interval-parameter estimation as richer and realistic than those of a classical approach. The findings indicate that neutrosophic estimator can not only handle variability but also offers more reliable insights for energy planning and policy decisions. Overall, this study confirms the usefulness of neutrosophic statistics in addressing vagueness and imprecision in renewable energy applications and sets a foundation for further research in other domains where uncertainty plays a central role.

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