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Ideal Convergence in neutrosophic 2-normed space via the parameter μ and Zweier Operator

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Abstract. This paper introduces a novel convergence concept termed $\mu-$ Zweier ideal convergence within neutrosophic 2-normed spaces (briefly, N2NS). The parameter $\mu = (\mu_n)$ represents a non-decreasing sequence of positive real numbers, with each μ_n tending to infinity. We investigate the behavior of μ-Zweier ideal convergence in N2NS, and further study μ-Zweier ideal Cauchy sequences, establishing a clear relationship between these sequences and their convergence properties. This novel concept extends the traditional ideal convergence in normed spaces to the more generalized structure of neutrosophic 2-normed spaces, where uncertainty and indeterminacy are inherent components.

Keywords: Ideal convergence, Zweier Operator, $\lambda-$ statistical convergence, Neutrosophic 2-normed spaces.

1. Introduction

Lotfi A. Zadeh [26] introduced fuzzy sets, establishing a foundational framework for handling uncertainty in various domains in 1965. Atanassov [2] extended the concept in 1986 by initiating intuitionistic fuzzy sets, which include both non-membership and membership degrees. Smarandache [24] further generalized the intuitionistic fuzzy sets and developed neutrosophic logic by introducing three independent components: truth, indeterminacy, and falsehood. This structure provides a strong framework for handling complicated uncertainties. Subsequently, Kirişçi and Şimşek [13] introduced neutrosophic normed spaces, which is a generalization of normed space, provides a structure where vectors with values representing truth, indeterminacy, and falsehood instead of real numbers. Murtaza et al. [19] have recently expanded upon the 2-norm concept introduced by Gähler [7] by developing neutrosophic 2-normed spaces and extending the idea of summability.

Mursaleen [17] familiarized the theory of $\lambda-$ statistical convergence as a generalization of $[\mathcal{V}, \lambda]-$ summability originally proposed by Leindler [16]. The idea of ideal convergence, developed by Kostyrko et al. [14], extends the classical notion of statistical convergence, which had been independently studied by Fast [5] and Steinhaus [25]. This concept has also been explored from the perspective of sequence spaces by Ahmad et al. [1], Savas and Das [20], Ŝalát et al. [21] and Mursaleen et al. [18].

Şengönül [22] introduced the notion of Zweier sequence spaces, and later extended this work in fuzzy numbers [23]. Hazarika et al. [8,9] further developed Şengönül framework by examining ideal convergence within this setting. Kumar and Mursaleen [15] introduced $(\lambda, \mu)-$ statistical convergence, further Khan and Ahmad [10] generalized their work and investigated Zweier ideal convergence via intuitionistic fuzzy normed space. Since then, numerous researchers have expanded this line of research into a variety of other fields [3, 4, 6, 11, 12].

2. **Prelimineries**

This section recalls key concepts, definitions remarks that will use as the foundation for our main discussions.

Definition 2.1. [14] A collection of subsets of the power set $\mathcal{V}$, denoted $\mathcal{J} \subseteq P(\mathcal{V})$ is termed an ideal in $\mathcal{V}$ subject to the satisfaction of the following requirements:

(a) $\emptyset \in \mathcal{J}$,

(b) For every $\mathcal{J}_1, \mathcal{J}_2 \in \mathcal{J},\ \mathcal{J}_1 \cup \mathcal{J}_2 \in \mathcal{J}$,

(c) For every $\mathcal{J}_1 \in \mathcal{J}$ and $\mathcal{J}_2 \subseteq \mathcal{J}_1,\ \mathcal{J}_2 \in \mathcal{J}$.

Remark 2.2. [14] If $\mathcal{J} \neq P(\mathcal{V})$, then the ideal $\mathcal{J}$ is termed nontrivial. A nontrivial ideal $\mathcal{J}$ is considered admissible if, $\{\{v\} : v \in \mathcal{V}\} \subseteq \mathcal{J}$.

Definition 2.3. [14] Let $\mathcal{J} \subseteq P(\mathcal{V})$ be a nontrivial ideal. Then the collection

$$\mathcal{F}(\mathcal{J}) = \{U \subset \mathcal{V} : U^c \in \mathcal{J}\}$$

is a filter on $\mathcal{V}$, known as the filter affiliated with ideal $\mathcal{J}$.

Definition 2.4. [14] Let $u = (u_i)$ be a sequence of real numbers. The sequence (u_i) is recognized as $\mathcal{J}-$ convergent to a real number ℓ if the set

$$\{i \leq \mathbb{N} :\mid u_i - \ell \mid \geq \epsilon\} \in \mathcal{J}.$$

Definition 2.5. [14] Let $u = (u_i)$ be a sequence of real numbers. The sequence (u_i) is recognized as $\mathcal{J}-$ Cauchy if for each ϵ, $k(\epsilon)$ there exist $k = k(\epsilon) \in \mathbb{N}$ so that, the set

$$\{i \leq \mathbb{N} :\mid u_i - u_k \mid \geq \epsilon\} \in \mathcal{J}.$$

Definition 2.6. [17] Let $\mu = (\mu_m)$ be a non-decreasing sequence of strictly positive real values in order that $\mu_m \to \infty$ as $m \to \infty$, and satisfying the condition:

$$\mu_{m+1} \leqslant \mu_m + 1, \ \mu_1 = 0.$$

Define the interval $I_m = [m - \mu_m + 1, m]$. For a set $N_1 \subseteq \mathbb{N}$, the $\mu-$ density of N_1 is given by:

$$\delta_\mu(K) = \lim_{m\to\infty} \frac{1}{\mu_m} |\{i \in I_m : i \in N_1\}|;$$

provided the limit exists. Generalized de la Valée-Poussin mean of $u = (u_i)$ is characterized by

$$t_m(u) = \frac{1}{\mu_m} \sum_{i \in J_m} u_i,$$

where $I_m = [m - \mu_m + 1, m]$. If $\mu_m = m$ for all m then μ- summability becomes equivalent to the classical Cesàro summability.

Definition 2.7. [17] Let $u = (u_i)$ be a sequence of real numbers. The sequence is recognized as $\mu-$ statistically convergent to a real number ℓ provided

$$\delta_\mu(E) = 0,$$

where

$$E = \{i \in I_m :\mid u_i - \ell \mid \geq \epsilon\},$$

and $\delta_\mu(E)$ represent the $\mu-$ density of the set E. This condition implies that for each $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\mu_m} \mid \{i \in I_m :\mid u_i - \ell \mid \geq \epsilon\} \mid = \ 0.$$

If $\mu_m = m, \ \forall m$, then it reduces to statistical convergence, which is defined by:

$$\lim_{m\to\infty} \frac{1}{m} \mid \{i \leq m :\mid u_i - \ell \mid \geq \epsilon\} \mid = \ 0.$$

Definition 2.8. [7] The 2- norm is a function $\|.,.\|$ defined on finite dimentional real vector space $\mathcal{V}$, where dimension $d \geq 2$ such that

$$\|.,.\| : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$$

fulfilling the below listed requirements: For all $w_1, w_2 \in \mathcal{V}$, and scalar a, one have

(1) $||w_1, w_2|| = 0$ iff w_1 and w_2 are linearly dependent;
(2) $\|w_1, w_2\| = \|w_2, w_1\|$;
(3) $\|aw_1, w_2\| = |a|\|w_1, w_2\|$ and
(4) $\|w_1, w_2 + w_3\| \leq \|w_1, w_2\| + \|w_1, w_3\|$.

The pair $(\mathcal{V}, ||.,.||)$ is known as 2 normed space in this case. Let $\mathcal{V} = \mathbb{R}^2$ and for $w_1 = (a_0, a_2)$ and $w_2 = (b_1, b_2)$ we define $\|w_1, w_2\| = |a_0 b_2 - a_2 b_1|,$, then $\|w_1, w_2\|$ is a 2− norm on $\mathcal{V} = \mathbb{R}^2$

Definition 2.9. [19] Take $\mathcal{V}$ as a vector soace. A six tuple $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ is known as a neutrosophic 2− normed spaces (briefly, N2NS) if $\alpha, \beta, \gamma : \mathcal{V} \times \mathcal{V} \times (0, \infty) \to [0, 1]$ are fuzzy sets, o is $t-$ conorm and $*$ is a $t-$ norm and fulfilling the below listed requirements: For each $a \neq 0$, $t, s > 0$ and for all $w, v, u \in \mathcal{V}$:

(1) $0 \leq \alpha(w, v; t),\ \beta(w, v; t),\ \gamma(w, v; t) \leq 1$;

(2) $\alpha(w, v; t) + \beta(w, v; t) + \gamma(w, v; t) \leq 3$;

(3) w and v are linearly dependent if and only if $\alpha(w, v; t) = 1$;

(4) $\alpha(aw, v; t) = \alpha(w, v; \frac{t}{|a|})$;

(5) $\alpha(w, v; t) * \alpha(w, u; s) \leq \alpha(w, v + u; t + s)$;

(6) $\alpha(w, v; .) : (0, \infty) \to [0, 1]$ is continuous non-decreasing function;

(7) $\lim\limits_{t\to\infty} \alpha(w, v; t) = 1$ and $\lim\limits_{t\to 0} \alpha(w, v; t) = 0$;

(8) $\alpha(v, w; t) = \alpha(w, v; t)$

(9) w and v are linearly dependent if and only if $\beta(w, v; t) = 0$;

(10) $\beta(aw, v; t) = \beta(w, v; \frac{t}{|a|})$;

(11) $\beta(w, v; t)$ o $\beta(w, u; s) \geq \beta(w, v + u; t + s)$;

(12) $\beta(w, v; .) : (0, \infty) \to [0, 1]$ is continuous non-increasing function;

(13) $\lim\limits_{t\to\infty} \beta(w, v; t) = 0$ and $\lim\limits_{t\to 0} \beta(w, v; t) = 1$;

(14) $\beta(v, w; t) = \beta(w, v; t)$;

(15) w and v are linearly dependent if and only if $\gamma(w, v; t) = 0$;

(16) $\gamma(aw, v; t) = \gamma(w, v; \frac{t}{|a|})$;

(17) $\gamma(w, v; t)$ o $\gamma(w, u; t) \geq \gamma(w, v + u; t + s)$;

(18) $\gamma(w, v; .) : (0, \infty) \rightarrow [0, 1]$ is continuous non-increasing function;

(19) $\lim_{t\rightarrow\infty} \gamma(w, v; t) = 0$ and $\lim_{t\rightarrow 0} \gamma(w, v; t) = 1$;

(20) $\gamma(w, v; t) = \gamma(v, w; t)$;

(21) If $s \leq 0$, then $\alpha(w, v; s) = 0$, $\beta(w, v; s) = 1$ and $\gamma(w, v; s) = 1$.

Definition 2.10. [19] Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS. A sequence (u_i) in $\mathcal{V}$ is known as convergent to $u \in \mathcal{V}$ if, for every $\epsilon,\ t > 0$ and for all $w \in \mathcal{V}$, there exits $n_0 \in \mathbb{N}$ the following conditions hold

$$\alpha(u_i - u, w; t) > 1 - \epsilon, \beta(u_i - w, v; t) < \epsilon \ \text{ and } \ \gamma(u_i - u, w; t) < \epsilon, \forall\ i \geq n_0.$$

Definition 2.11. [19] Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS. A sequence (u_i) in $(\mathcal{V})$ is known as Cauchy if, for every $\epsilon,\ t > 0$ and for all $w \in \mathcal{V}$, there exits $n_0 \in \mathbb{N}$ the following conditions hold

$$\alpha(u_i - u_k, w; t) > 1 - \epsilon,\ \beta(u_i - u_k, w; t) < \epsilon,\ \gamma(u_i - u_k, w; t) < \epsilon\ \forall\ i, k \geq n_0.$$

Definition 2.12. [22] Consider $u = (u_i)$ be a sequence, and define a new sequence $v = (v_i)$ by the recurrence relation:

$$v_i = qu_i + (1 - q)u_{i-1},\ \text{with } u_{-1} = 0 \text{ and } 1 < q < \infty.$$

This transformation is known as the Z^q- transformation of u, where the associated matrix $Z^q = (z_{il})$ defined as:

$$z_{il} = \begin{cases} 1 - q, & (i - 1 = l)\ (i, l \in \mathbb{N}) \\ q, & (i = l) \\ 0, & \text{otherwise.} \end{cases}$$

The Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ introduced by Şengönül [22] as follows:

$$\mathcal{Z} = \{u = (u_k) \in \omega : Z^q u \in c\};$$

$$\mathcal{Z}_0 = \{u = (u_k) \in \omega : Z^q u \in c_0\},$$

where ω, c and c_0 represent the space of all real or complex valued sequences, spaces of convergent and null sequences, respectively.

3. **Main Results**

For the sake of convenience, we denote the Z^q- transformation of the sequence $u = (u_i) \in \mathcal{V}$ by $Z^q u = Z^q u_i$, and we assume throughout the article the nontrivial admissible ideal on $\mathbb{N}$ is denoted by $\mathcal{J}$.

Definition 3.1. Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a $N2NS$, and let $\mathcal{J}$ be an admissible ideal on $\mathbb{N}$. A sequence $u = (u_i)$ in $\mathcal{V}$ is known as $\mu-$ Zweier ideal convergent to ℓ, if for each $0 < \epsilon$ and for all $0 < t$, the set

$$\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \alpha(Z^q u_i - \ell, v; t) \leq 1 - \epsilon \text{ or } \frac{1}{\mu_n} \sum_{i \in J_n} \beta(Z^q u_i - \ell, v; t) \geq \epsilon, \\ \frac{1}{\mu_n} \sum_{i \in J_n} \gamma(Z^q u_i - \ell, v; t) \geq \epsilon\Big\}$$

belongs to the ideal $\mathcal{J}$, for all $v \in \mathcal{V}$.
One write $\mathcal{J}_\mu - \lim_{i \to \infty} Z^q u_i = \ell$

The proof of the next Lemma is direct and thus omitted.

Lemma 3.2. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and let $\mathcal{J}$ be an admissible ideal. Consider a sequence $u = (u_i)$ in $\mathcal{V}$. Then for each $t, \epsilon > 0$, the subsequent statements are identical:*

(1) $\mathcal{J}_\mu - \lim_{i \to \infty} Z^q u_i = \ell$;

(2) $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \alpha(Z^q u_i - \ell, v; t) \leq 1 - \epsilon\Big\} \in \mathcal{J}$, $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \beta(Z^q u_i - \ell, v; t) \geq \epsilon\Big\} \in \mathcal{J}$ *and* $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \gamma(Z^q v_i - \ell, v; t) \geq \epsilon\Big\} \in \mathcal{J}$;

(3) $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \alpha(Z^q u_i - \ell, v; t) > 1 - \epsilon\Big\} \in \mathcal{F}(\mathcal{J})$, $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \beta(Z^q u_i - \ell, v; t) < \epsilon\Big\} \in \mathcal{F}(\mathcal{J})$ *and* $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \gamma(Z^q u_i - \ell, v; t) < \epsilon\Big\} \in \mathcal{F}(\mathcal{J})$;

(4) $\Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i \in J_n} \alpha(Z^q u_i - \ell, v; t) \leq 1 - \epsilon$ *or* $\frac{1}{\mu_n} \sum_{i \in J_n} \beta(Z^q u_i - \ell, v; t) \geq \epsilon, \frac{1}{\mu_n} \sum_{i \in J_n} \gamma(Z^q u_i - \ell, v; t) \geq \epsilon\Big\} \in \mathcal{J}$;

(5) $\mathcal{J}_\mu - \lim\limits_{i\to\infty} \alpha(Z^q u_i - \ell, v; t) = 1$ *and* $\mathcal{J}_\mu - \lim\limits_{i\to\infty} \beta(Z^q u_i - \ell, v; t) = 0$, $\mathcal{J}_\mu - \lim\limits_{i\to\infty} \gamma(Z^q u_i - \ell, v; t) = 0$.

Theorem 3.3. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and let $\mathcal{J}$ be an admissible ideal. Consider a sequence $u = (u_i)$ in $\mathcal{V}$. If u is $\mu-$ Zweier ideal convergent to ℓ, then the limit is unique.*

Proof. Assume $\mathcal{J}_\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_1$ and $\mathcal{J}_\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_2$.

Given any $\epsilon > 0$, choose a corresponding $\zeta > 0$ in order that $(1-\zeta) * (1-\zeta) > 1-\epsilon$ and $\zeta \diamond \zeta < \epsilon$. Subsequently, for each $t > 0$, define

$$\begin{aligned}
P_{\alpha,1} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \alpha\Big(Z^q u_i - \ell_1, v; \frac{t}{2}\Big) \le 1-\zeta\Big\},\\
P_{\alpha,2} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \alpha\Big(Z^q u_i - \ell_2, v; \frac{t}{2}\Big) \le 1-\zeta\Big\},\\
P_{\beta,1} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \beta\Big(Z^q u_i - \ell_1, v; \frac{t}{2}\Big) \ge \zeta\Big\},\\
P_{\beta,2} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \beta\Big(Z^q u_i - \ell_2, v; \frac{t}{2}\Big) \ge \zeta\Big\}\\
P_{\gamma,1} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \gamma\Big(Z^q u_i - \ell_1, v; \frac{t}{2}\Big) \ge \zeta\Big\},\\
P_{\gamma,2} &= \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i\in J_n} \gamma\Big(Z^q u_i - \ell_2, v; \frac{t}{2}\Big) \ge \zeta\Big\}.
\end{aligned}$$

Since $\mathcal{J}_\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_1$, one obtain $P_{\alpha,1}$, $P_{\beta,1}$ and $P_{\gamma,1} \in \mathcal{J}$.
Moreover, using $\mathcal{J}_\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_2$, one have $P_{\alpha,2}$, $P_{\beta,2}$ and $P_{\gamma,2} \in \mathcal{J}$. Now, suppose that

$$P(\zeta, t) = [P_{\alpha,1} \cup P_{\alpha,2}] \cap [P_{\beta,1} \cup P_{\beta,2}] \cap [P_{\gamma,1} \cup P_{\gamma,2}].$$

Thus, $P(\zeta, t) \in \mathcal{J}$, implies $P^c(\zeta, t)$ is an non-empty set in $\mathcal{F}(\mathcal{J})$.
If $n \in P^c(\zeta, t)$, then there are three possibilities arises :
Either $n \in P^c_{\alpha,1} \cap P^c_{\alpha,2}$ or $n \in P^c_{\beta,1} \cap P^c_{\beta,2}$ or $n \in P^c_{\gamma,1} \cap P^c_{\gamma,2}$. Firstly, we consider that $n \in P^c_{\alpha,1} \cap P^c_{\alpha,2}$. Then, one obtain

$$\frac{1}{\mu_n} \sum_{i\in J_n} \alpha\Big(Z^q u_i - \ell_1, v; \frac{t}{2}\Big) > 1-\zeta$$

and

$$\frac{1}{\mu_n} \sum_{i\in J_n} \alpha\Big(Z^q u_i - \ell_2, v; \frac{t}{2}\Big) > 1-\zeta.$$

Now, take $k \in \mathbb{N}$ in such a way that

$$\alpha\Big(Z^q u_k - \ell_1, v; \frac{t}{2}\Big) > \frac{1}{\mu_n} \sum_{i\in J_n} \alpha\Big(Z^q u_i - \ell_1, v; \frac{t}{2}\Big) > 1-\zeta$$

and

$$\alpha\Big(Z^q u_k - \ell_2, v; \frac{t}{2}\Big) > \frac{1}{\mu_n}\sum_{i\in J_n}\alpha\Big(Z^q u_i - \ell_2, v; \frac{t}{2}\Big) > 1-\zeta$$

e.g., consider the maximum value of the set $\Big\{i \in J_n : \alpha(Z^q u_i - \ell_1, v; \frac{t}{2}), \alpha(Z^q u_i - \ell_2, v; \frac{t}{2})\Big\}$. Let k denote the index at which this maximum occurs. Consequently, we obtain

$$\alpha(\ell_1 - \ell_2, v; t) \geq \alpha(Z^q u_k - \ell_1, v; \frac{t}{2}) * \alpha(Z^q u_k - \ell_2, v; \frac{t}{2}) > (1-\zeta) * (1-\zeta) > 1-\epsilon.$$

As $0 < \epsilon$ was arbitrary, for every $t > 0$, one get $\alpha(\ell_1 - \ell_2, v; t) = 1$, which provides $\ell_1 = \ell_2$. Under other conditions, if $n \in P^c_{\beta,1} \cap P^c_{\beta,2}$ then, on similar manner one can show that

$$\beta(\ell_1 - \ell_2, v; t) < \epsilon,\ \gamma(\ell_1 - \ell_2, v; t) < \epsilon,\ \forall\ 0 < t.$$

Therefore, one have

$$\beta(\ell_1 - \ell_2, v; t) = 0,\ \gamma(\ell_1 - \ell_2, v; t) = 0,\ \forall\ 0 < t,$$

which yields $\ell_1 = \ell_2$. Hence, in all cases, one achieve that $\mathcal{J}_\mu-$ limit is unique. □

Definition 3.4. Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and $u = (u_i)$ be a sequence in $\mathcal{V}$. The sequence $u = (u_i)$ is recognized as $\mu-$ Zweier convergent to ℓ, if for ecah $\epsilon > 0$ and for all $t > 0$,, there is an index $n_0 \in \mathbb{N}$ so that

$$\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell, v; t) > 1-\epsilon,\ \frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell, v; t) < \epsilon$$
$$\text{and } \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell, v; t) < \epsilon,\ \text{for all } n \geq n_0.$$

One write $\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell$.

Theorem 3.5. *Every $\mu-$ Zweier convergent sequence of $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ N2NS has a unique limit.*

Proof. Assume $\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_1$ and $\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell_2$. Given any $\epsilon > 0$, choose a corresponding $\zeta > 0$ in order that $1 - \epsilon < (1-\zeta) * (1-\zeta)$ and $\epsilon > \zeta \diamond \zeta$. Subsequently, for each $0 < t$, there exists $n_1 \in \mathbb{N}$ in order that

$$\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell_1, v; t) > 1-\epsilon,\ \frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell_1, v; t) < \epsilon$$
$$\text{and } \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell_1, v; t) < \epsilon,\ \text{for all } n \geq n_1.$$

Moreover, there exits $n_2 \in \mathbb{N}$ so that

$$\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i-\ell_2,v;t)>1-\epsilon,\ \frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i-\ell_2,v;t)<\epsilon$$

$$\text{and } \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i-\ell_2,v;t)<\epsilon, \forall\ n\geq n_2.$$

Assume $n_0=\max\{n_1,n_2\}$, thereafter for every $n_0\leq n_0$, there exists, a natural number k so that

$$\alpha(Z^q u_k-\ell_1,v;t)>\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i-\ell_1,v;t)>1-\zeta$$

and

$$\alpha(Z^q u_k-\ell_2,v;t)>\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i-\ell_2,t)>1-\zeta.$$

Hence, we get

$$\begin{aligned}\alpha(\ell_1-\ell_2,v;t)&\geq\alpha(Z^q u_k-\ell_1,t)*\alpha(Z^q u_k-\ell_2,v;t)\\&>(1-\zeta)*(1-\zeta)>1-\epsilon.\end{aligned}$$

As $0<\epsilon$ is arbitrary, one conclude that $\alpha(\ell_1-\ell_2,v;t)=1,\ \forall\ t>0$.

Similarly, it follows that $\beta(\ell_1-\ell_2,v;t)=0,\ \gamma(\ell_1-\ell_2,v;t)=0,\ \forall\ t>0$. Therefore, one deduce that $\ell_1=\ell_2$. □

Theorem 3.6. *Let $(\mathcal{V},\alpha,\beta,\gamma,*,o)$ be a N2NS, and let $u=(u_i)$ be a sequence in $\mathcal{V}$ such that $\mu-$ Zweier convergence of $Z^q u_i$ to ℓ holds. Then, the sequence $u=(u_i)$ also $\mu-$ Zweier ideal convergence to ℓ, i.e., $\mathcal{J}_\mu-$ convergence.*

Proof. Let $\mu-\lim_{i\to\infty} Z^q u_i=\ell$. Given any $0<t$ and $0<\epsilon$, there exists $N_0\in\mathbb{N}$ so that $\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i-\ell,v;t)>1-\epsilon$, $\frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i-\ell,v;t)<\epsilon$ and $\frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i-\ell,v;t)<\epsilon$ for each $n\geq N_0$. Then the set

$$\begin{aligned}Q(\epsilon,t)=\Big\{n\in\mathbb{N}:\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i-\ell,v;t)\leq1-\epsilon \text{ or}\\ \frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i-\ell,v;t)\geq\epsilon,\ \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i-\ell,v;t)\geq\epsilon\Big\}\end{aligned}$$

$Q(\epsilon,t)\subseteq\{1,2,3,...,N_0-1\}$. Consequently, the set $Q(\epsilon,t)$ contains at most finitely many elements. Given that the ideal $\mathcal{J}$ is admissible, it follows that $Q(\epsilon,t)\in\mathcal{J}$.

Therefore, we conclude that $\mathcal{J}_\mu-\lim_{i\to\infty} Z^q u_i=\ell$. □

Theorem 3.7. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and let a sequenc $u = (u_i)$ in $\mathcal{V}$ is $\mu-$ Zweier convergent to ℓ, the there exists a subsequence (u_{i_k}) of u is also convergent to same limit ℓ.*

Proof. Let $\mu - \lim\limits_{i\to\infty} Z^q u_i = \ell$. Hence, for any $t > 0$ and $0 < \epsilon$, there exists a $n_0 \in \mathbb{N}$ so that $\frac{1}{\mu_n} \sum_{i\in J_n} \alpha(Z^q u_i - \ell, v; t) > 1 - \epsilon$, $\frac{1}{\mu_n} \sum_{i\in J_n} \beta(Z^q u_i - \ell, v; t) < \epsilon$ and $\frac{1}{\mu_n} \sum_{i\in J_n} \gamma(Z^q u_i - \ell, v; t) < \epsilon$ for every $n_0 \le n$.
It is evident that for every $n_0 \le n$, one can select $i_k \in J_n$ such that

$$\alpha(Z^q u_{i_k} - \ell, v; t) > \frac{1}{\mu_n} \sum_{i\in J_n} \alpha(Z^q u_i - \ell, v; t) > 1 - \epsilon$$

,

$$\beta(Z^q u_{i_k} - \ell, v; t) < \frac{1}{\mu_n} \sum_{i\in J_n} \beta(Z^q u_i - \ell, v; t) < \epsilon$$

and

$$\gamma(Z^q u_{i_k} - \ell, v; t) < \frac{1}{\mu_n} \sum_{i\in J_n} \gamma(Z^q u_i - \ell, v; t) < \epsilon.$$

Hence, $\mu - \lim Z^q u_{i_k} = \ell$. □

Definition 3.8. Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS. A sequence $u = (u_i)$ in $\mathcal{V}$ is known as $\mu-$ Zweier Cauchy, if for any $0 < t,\ \epsilon$, there exists $n_0 \in \mathbb{N}$ so that $\frac{1}{\mu_n} \sum_{i,j\in J_n} \alpha(Z^q u_i - Z^q u_j, v; t) > 1 - \epsilon$, $\frac{1}{\mu_n} \sum_{i,j\in J_n} \beta(Z^q u_i - Z^q u_j, v; t) < \epsilon$ and $\frac{1}{\mu_n} \sum_{i,j\in J_n} \gamma(Z^q u_i - Z^q u_j, v; t) < \epsilon,\ \forall\ i, j \ge n_0$.

Definition 3.9. Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS. A sequence $u = (u_i)$ in $\mathcal{V}$ is known as a $\mu-$ Zweier ideal Cauchy, if for each $0 < \epsilon,\ t$,there exists a $n_0 \in \mathbb{N}$ in order that $\left\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i,k\in J_n} \alpha(Z^q u_i - Z^q u_k, v; t) > 1 - \epsilon\right\} \in \mathcal{F}(\mathcal{J})$, $\left\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i,k\in J_n} \beta(Z^q u_i - Z^q u_k, v; t) < \epsilon,\right\} \in \mathcal{F}(\mathcal{J})$ and $\left\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i,k\in J_n} \gamma(Z^q u_i - Z^q u_k, v; t) < \epsilon,\right\} \in \mathcal{F}(\mathcal{J})$

Theorem 3.10. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS. A sequence $u = (u_i)$ in $\mathcal{V}$ is $\mu-$ Zweier ideal Cauchy sequence iff it is $\mu-$ Zweier ideal convergent.*

Proof. Assume the sequence $u = (u_i)$ is $\mu-$ Zweier ideal Cauchy sequence but not convergent. Then, there exist, $k \in \mathbb{N}$ so that if one select

$$R(\epsilon, t) = \Big\{n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i,j\in J_n} \alpha(Z^q u_i - Z^q u_j, v; t) \le 1 - \epsilon \text{ or}$$

$$\frac{1}{\mu_n}\sum_{i,j\in J_n}\beta(Z^q u_i - Z^q u_j, v; t) \geq \epsilon, \frac{1}{\mu_n}\sum_{i,j\in J_n}\gamma(Z^q u_i - Z^q u_j, v; t) \geq \epsilon\Big\}$$

and

$$S(\epsilon, t) = \Big\{n \in \mathbb{N} : \frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell, v; \frac{t}{2}) \leq 1-\epsilon \text{ or}$$

$$\frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell, v; \frac{t}{2}) \geq \epsilon \text{ and } \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell, v; \frac{t}{2}) \geq \epsilon\Big\}.$$

Then $R(\epsilon, t) \in \mathcal{J}$ implies $R^C(\epsilon, t) \in \mathcal{F}(\mathcal{J})$. Since

$$\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - Z^q u_k, v; t) \geq \frac{2}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell, v; \frac{t}{2}) > 1-\epsilon,$$

$$\frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - Z^q u_k, v; t) \leq \frac{2}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell, v; \frac{t}{2}) < \epsilon$$

and

$$\frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - Z^q u_k, v; t) \leq \frac{2}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell, v; \frac{t}{2}) < \epsilon.$$

If $\frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell, v; \frac{t}{2}) > \frac{1-\epsilon}{2}$, $\frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell, v; \frac{t}{2}) < \frac{\epsilon}{2}$ and $\frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell, v; \frac{t}{2}) < \frac{\epsilon}{2}$ respectively. Then, one obtain

$$\delta\Big(\Big\{n \in \mathbb{N} : \frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - Z^q u_k, v; t) > 1-\epsilon, \frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - Z^q u_k, v; t) < \epsilon \text{ and}$$

$$\frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - Z^q u_k, v; t) < \epsilon\Big\}\Big) = 0,$$

that is, $R(\epsilon, t) \in \mathcal{F}(\mathcal{J})$, which contadicts our assumption. Therefore, sequence $u = (u_i)$ is $\mu-$ Zweier ideal convergent.

Conversely, Assume $\mathcal{J}_\mu - \lim_{i\to\infty} Z^q u_i = \ell$. Take $\zeta > 0$, in such a way that $1-\epsilon < (1-\zeta)*(1-\zeta)$ and $\epsilon > \zeta \diamond \zeta$. For all $t > 0$, construe

$$T(\zeta, t) = \Big\{n \in \mathbb{N} : \frac{1}{\mu_n}\sum_{i\in J_n}\alpha(Z^q u_i - \ell, v; \frac{t}{2}) \leq 1-\zeta \text{ or}$$

$$\frac{1}{\mu_n}\sum_{i\in J_n}\beta(Z^q u_i - \ell, v; \frac{t}{2}) \geq \zeta,\ \frac{1}{\mu_n}\sum_{i\in J_n}\gamma(Z^q u_i - \ell, v; \frac{t}{2}) \geq \zeta\Big\} \in \mathcal{J}.$$

This implies $T(\zeta, t)^C \in \mathcal{F}(\mathcal{J})$. Assume $k \in T^c(\zeta, t)$. Then, one obtain

$$\frac{1}{\mu_n}\sum_{k\in J_n}\alpha(Z^q u_i - \ell, v; \frac{t}{2}) > 1-\zeta,\ \frac{1}{\mu_n}\sum_{k\in J_n}\beta(Z^q u_i - \ell, v; \frac{t}{2}) < \zeta$$

$$\text{and } \frac{1}{\mu_n}\sum_{k\in J_n}\gamma(Z^q u_i - \ell, v; \frac{t}{2}) < \zeta.$$

For every $\epsilon > 0$, one take

$$H(\epsilon,t) = \Big\{ n \in \mathbb{N} : \frac{1}{\mu_n} \sum_{i,k\in J_n} \alpha(Z^q u_i - Z^q u_k, t) \leq 1-\epsilon \text{ or}$$

$$\frac{1}{\mu_n} \sum_{i,k\in J_n} \beta(Z^q u_i - Z^q u_k, t) \geq \epsilon,\ \frac{1}{\mu_n} \sum_{i,k\in J_n} \gamma(Z^q u_i - Z^q u_k, t) \geq \epsilon \Big\}.$$

To show $H(\epsilon,t) \subset T(\zeta,t)$, let us suppose $l \in H(\epsilon,t)$, we have

$$\frac{1}{\mu_n} \sum_{l,k\in J_n} \alpha(Z^q u_l - Z^q u_k, v; \frac{t}{2}) \leq 1-\epsilon,\ \frac{1}{\mu_n} \sum_{l,k\in J_n} \beta(Z^q u_l - Z^q u_k, v; \frac{t}{2}) \geq \epsilon$$

$$\text{and } \frac{1}{\mu_n} \sum_{l,k\in J_n} \gamma(Z^q u_l - Z^q u_k, v; \frac{t}{2}) \geq \epsilon.$$

According to the above inequality, one characterize the next three illustration as:

Case 1: Let $\frac{1}{\mu_n} \sum_{l,k\in J_n} \alpha(Z^q u_l - Z^q u_k, v; t) \leq 1-\epsilon$. Then $\frac{1}{\mu_n} \sum_{u\in J_n} \alpha(Z^q u_l - \ell, v; \frac{t}{2}) \leq 1-\zeta$, therefore $l \in T(\zeta,t)$.

Otherwise, if $\frac{1}{\mu_n} \sum_{l\in J_n} \alpha(Z^q x_{uv} - \ell, v; \frac{t}{2}) > 1-\zeta$. Then, we have

$$\begin{aligned} 1-\epsilon &\geq \frac{1}{\mu_n} \sum_{l,k\in J_n} \alpha(Z^q u_l - Z^q u_k, v; t) \\ &\geq \frac{1}{\mu_n} \sum_{l\in J_n} \alpha\left(Z^q u_l - \ell, v; \frac{t}{2}\right) * \frac{1}{\mu_n} \sum_{k\in J_n} \alpha\left(Z^q u_k - \ell, v; \frac{t}{2}\right) \\ &> (1-\zeta) * (1-\zeta) \\ &> 1-\epsilon \end{aligned}$$

a contradiction. Therefore $H(\epsilon,t) \subset T(\zeta,t)$.

Case 2: Consider $\frac{1}{\mu_n} \sum_{l,k\in J_n} \beta(Z^q u_l - Z^q u_k, v; t) \geq \epsilon$. We get $\frac{1}{\mu_n} \sum_{l\in J_n} \beta(Z^q u_l - \ell, v; \frac{t}{2}) \geq \zeta$, hence $l \in T(\zeta,t)$. Otherwise, if $\frac{1}{\mu_n \sum_{l\in J_n}} \beta(Z^q u_l - \ell, v; \frac{t}{2}) < \zeta$. Then, we obtain

$$\begin{aligned} \epsilon &\leq \frac{1}{\mu_n} \sum_{l,k\in J_n} \beta(Z^q u_l - Z^q u_k, v; t) \\ &\leq \frac{1}{\mu_n} \sum_{l\in J_n} \beta(Z^q u_l - \ell, v; \frac{t}{2}) \diamond \frac{1}{\mu_n} \sum_{k\in J_n} \beta(Z^q u_k - \ell, v; \frac{t}{2}) \\ &< \zeta \diamond \zeta < \epsilon, \end{aligned}$$

a contradiction. Hence $H(\epsilon,t) \subset T(\zeta,t)$.

Case 3: Consider $\frac{1}{\mu_n}\sum_{l,k\in J_n}\gamma(Z^q u_l - Z^q u_k, v; t) \geq \epsilon$. We get $\frac{1}{\mu_n}\sum_{l\in J_n}\gamma(Z^q u_l - \ell, v; \frac{t}{2}) \geq \zeta$, hence $l \in T(\zeta, t)$. Otherwise, if $\frac{1}{\mu_n \sum_{l\in J_n}}\gamma(Z^q u_l - \ell, v; \frac{t}{2}) < \zeta$. Then, we obtain

$$\begin{aligned}\epsilon &\leq \frac{1}{\mu_n}\sum_{l,k\in J_n}\gamma(Z^q u_l - Z^q u_k, v; t)\\ &\leq \frac{1}{\mu_n}\sum_{l\in J_n}\gamma(Z^q u_l - \ell, v; \frac{t}{2}) \diamond \frac{1}{\mu_n}\sum_{k\in J_n}\gamma(Z^q u_k - \ell, v; \frac{t}{2})\\ &< \zeta \diamond \zeta < \epsilon,\end{aligned}$$

a contradiction. Hence $H(\epsilon, t) \subset T(\zeta, t)$.
Consequently, we deduce that $H(\epsilon, t) \subset T(\zeta, t)$, implying $H(\epsilon, t) \in \mathcal{J}$. Therefore, the sequence $u = (u_i)$ is a $\mu-$ Zweier ideal Cauchy. □

We present the subsequent theorems without proof, as their proofs are straightforward:

Theorem 3.11. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and let $u = (u_i)$ be a sequence in $\mathcal{V}$. If u is a $\mu-$ Zweier Cauchy sequence, then u is also a $\mu-$ Zweier ideal Cauchy sequence.*

Theorem 3.12. *Let $(\mathcal{V}, \alpha, \beta, \gamma, *, o)$ be a N2NS, and let $u = (u_i)$ be a sequence in $\mathcal{V}$. If u is a $\mu-$ Zweier Cauchy sequence, then there exists a subsequence u_{i_j} of u that is an ordinary Zweier Cauchy.*

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References

1. Ahmad, M., Idrisi, M.I., Sirohi, A.K. Generalized neutrosophic ideal convergent sequence spaces. Journal of Mathematics and Computer Science **2023**, 30(4), 332-339.
2. Atanassov, K. T. Intuitionistic fuzzy sets. Fuzzy Sets and Systems **1986**, 20, 87-96.
3. Esi, A., Sapsizoglu, A. On Some Lacunary sigma Strong Zweier Convergent Sequence Spaces. ROMAI J. **2012**, 8(2), 61-70.
4. Et, M., Karakas, M., Cinar, M. Some geometric properties of a new modular space defined by Zweier operator. Fixed point Theory Appl. **2013**, 165, 1-10.
5. Fast, H. Sur la convergence statistique. Colloq. Math. **1951**, 2, 241-244.
6. Fadile, K. Y., Esi, A. On some strong Zweier convergent sequence spaces. Acta Universitatis Apulensis **2012**, 29, 9-15.
7. Gähler, S. Lineare 2-normierte Räume. Mathematische nachrichten **1964**, 28, 1-43.
8. Hazarika, B., Tamang, K., Singh, B.K. On Paranorm Zweier ideal convergent sequence spaces defined by Orlicz function. Journal of the Egyptian Mathematical Society **2014**, 22(3), 413-419.
9. Hazarika, B., Tamang, K., Singh, B.K. Zweier ideal convergent sequence spaces defined by Orlicz function. The Journal of Math. Comp. Sci. **2014**, 8(3), 307-318.

10. Khan, V. A., Ahmad, M. On (λ, μ) Zweier Ideal Convergence in Intuitionistic Fuzzy Normed Space, Yugoslav Journal of Operations Research **2020**, 30(4), 413-427.
11. Khan, V. A., Khan, N.On Zweier I-convergent Double Sequence Spaces. Filomat **2016**, 30(12), 3361-3369.
12. Khan, V. A., Fatima, H., Ahmad, A. Intuitionistic fuzzy zweier I-convergent double sequence spaces defined by Orlicz function. European Journal of Pure and Applied Mathematics **2017**, 10(3), 574-585.
13. Kirişçi, M., Şimşek, N. Neutrosophic normed spaces and statistical convergence, The Journal of Analysis **2020**, 28(4), 1059–1073.
14. Kostyrko, P., Salat, T., Wilczynski, W. I-convergence. Real Analysis Exchange **2000**, 26(2), 669-686.
15. Kumar, V., Mursaleen, M. On (λ, μ)-statistical convergence of double sequence on intuitionistic fuzzy normed spaces. Filomat **2011**, 25(2), 109-120.
16. Leindler, L. "Über die de la Vallêe-Pousnsche Summierbarkeit allge meiner orthogonalreihen. Acta Math. Acad. Sci. Hungarica **1965**, 16(3), 375-387.
17. Mursaleen, M. λ-Statistical convergence. Math. Slovaca **2000**, 50, 111-115.
18. Mursaleen, M., Mohiuddine, S.A., Edely, O.H.H. On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces. Computers and Mathematics with Applications **2010**, 59, 603-611.
19. Murtaza, S., Sharma, A., Kumar, V. Neutrosophic 2-normed spaces and generalized summability, Neutrosophic Sets and Systems **2023**, Vol. 55, 415-426.
20. Savas, E., Das, P. A generalized statistical convergence via ideals. Applied Mathematics Letters **2011**, 24, 826-830.
21. Ŝalát, T., Tripathy, B.C., Ziman, M. On I-convergence field. Ital. J. Pure Appl. Math. **2005**, 17, 45-54.
22. Şengönül, M. On the Zweier sequence space. Demonstratio Mathematica **2007**, 40(1), 181-196.
23. Şengönül, M. On the Zweier Sequence Spaces of Fuzzy Number. Hindawi Publishing Corporation Inter. J. of Math. and Math. Sci. **2014**, Article ID 439169, 1-9.
24. Smarandache, F. Neutrosophic set—a generalization of the intuitionistic fuzzy set. International journal of pure and applied mathematics **2005**, 24(3), 287–297.
25. Steinhaus, H.Sur la Convergence Ordinaire et la Convergence Asymptotique. Colloquium Mathematicum **1951**, 2, 73-74.
26. Zadeh, L.A. Fuzzy sets. Inform Control **1965** (8), 338-353.

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